Problem Sheet 4: Free Quantum Field Theory

Comments on these questions are always welcome. For instance if you spot any typos or feel the wording is unclear, drop me an email at `T.Evans` at the usual Imperial address.

Note: problems marked with a $*$ are the most important to do and are core parts of the course. Those without any mark are recommended. It is likely that the exam will draw heavily on material covered in these two types of question. Problems marked with a ! are harder and/or longer. Problems marked with a $\sharp$ are optional. For the exam it will be assumed that material covered in these optional $\sharp$ questions has not been seen before and such optional material is unlikely to be used in an exam.

Here operators are written without 'hats' so you will need to deduce what is an operator from the context.

$*$1. Heisenberg picture free real scalar field

A free real scalar field in the Heisenberg picture, $\phi_{\mathrm{H}}(t,\boldsymbol{x})$, is defined by

$$\phi_{\mathrm{H}}(t,\boldsymbol{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}\mathrm{e}^{-i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{p}}\mathrm{e}^{i\omega_{\boldsymbol{p}}t-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \tag{1}$$

We can find the Schrödinger picture equivalent by looking at the $t=0$ special case.

(i) Show that $\phi^\dagger_{\mathrm{H}} = \phi_{\mathrm{H}}$. This hermitian property is the equivalent of the classical field being real.

(ii) Show that the conjugate momentum is

$$\Pi_{\mathrm{H}}(t,\boldsymbol{x}) = -i\int \frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}}\left(a_{\boldsymbol{p}}\mathrm{e}^{-i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}} - a^\dagger_{\boldsymbol{p}}\mathrm{e}^{i\omega_{\boldsymbol{p}}t-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \tag{2}$$

(iii) Assume that the annihilation and creation operators obey their usual commutation relations

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\,, \qquad [a_{\boldsymbol{p}}, a_{\boldsymbol{q}}] = \left[a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = 0\,. \tag{3}$$

Show that the ϕ_{H} field and its conjugate Π_{H} obey their canonical commutation relations

$$[\phi_{\mathrm{H}}(t,\boldsymbol{x}), \Pi_{\mathrm{H}}(t,\boldsymbol{y})] = i\delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \qquad [\phi_{\mathrm{H}}(t,\boldsymbol{x}), \phi_{\mathrm{H}}(t,\boldsymbol{y})] = [\Pi_{\mathrm{H}}(t,\boldsymbol{x}), \Pi_{\mathrm{H}}(t,\boldsymbol{y})] = 0\,. \tag{4}$$

(iv) Find expressions for $a_{\boldsymbol{p}}$ and $a^\dagger_{\boldsymbol{p}}$ in terms of $\phi_{\mathrm{H}}(t=0,\boldsymbol{x})$ and $\Pi_{\mathrm{H}}(t=0,\boldsymbol{x})$.

(v) Now assume the equal time commutation relations for the field and its conjugate (4). Starting from these field commutators, derive $[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})$. In fact either set of commutations relations is an equally good starting point (fundamental axiom) for QFT.

(vi) The operator for momentum is P where

$$\boldsymbol{P} \equiv -\int d^3\boldsymbol{x}\;\pi(t,\boldsymbol{x})\boldsymbol{\nabla}\phi(t,\boldsymbol{x})\,. \tag{5}$$

Show that, up to a constant, that for all times

$$\boldsymbol{P} = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\;\boldsymbol{p}a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}}\,. \tag{6}$$

Interpret your result.

(vii) Show that if the operator for energy, the Hamiltonian H is at any one time

$$H = \int d^3\boldsymbol{x} \left(\frac{1}{2}\Pi^2 + \frac{1}{2}(\boldsymbol{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \right) \tag{7}$$

then

$$H = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \left(\omega_k a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}} + (\text{constant}) \right) , \qquad \omega_k = \left| \sqrt{\boldsymbol{k}^2 + m^2} \right| \geq 0 \,. \tag{8}$$

(viii) Show that for this free theory, the number of particles for each momentum $\boldsymbol{k}$ is conserved, i.e. show $[H, a^\dagger_{\boldsymbol{k}} a_{\boldsymbol{k}}] = 0$ where H is defined at any one time t.

2. Time evolution of annihilation operator

(i) Given the Hamiltonian of a free field (8) and that the annihilation and creation operators obey (3) show that

$$[H, a_{\boldsymbol{p}}] = -\omega_{\boldsymbol{p}} a_{\boldsymbol{p}} \,. \tag{9}$$

(ii) Using the commutator (9), prove by induction that $(itH)^n a_{\boldsymbol{p}} = a_{\boldsymbol{p}}[it(H - E_{\boldsymbol{p}})]^n$. You may also use the fact that if A and B are two operators that commute, then $e^A e^B = e^{A+B}$. This is a special case of the Baker-Campbell-Hausdorf identity or you can check this by expanding out the left- and the right-hand sides.

(iii) Show that

$$e^{iHt} a_{\boldsymbol{p}} e^{-iHt} = a_{\boldsymbol{p}} e^{-i\omega_{\boldsymbol{p}} t} \,. \tag{10}$$

Hence show that

$$e^{iHt} a^\dagger_{\boldsymbol{p}} e^{-iHt} = a^\dagger_{\boldsymbol{p}} e^{+i\omega_{\boldsymbol{p}} t} \,. \tag{11}$$

3. Delta Functions

The Dirac delta function is defined through

$$\int_{-\infty}^{+\infty} dx\, \delta(x - x_0) f(x) = f(x_0) \,. \tag{12}$$

You should always start from this equation when using a delta function.

(i) Show that

$$\int dy\, \delta(g(y)) f(y) = \sum_{y_0 \in \mathcal{Z}} \frac{f(y_0)}{|g'(y_0)|} \,, \tag{13}$$

for any function g and f. Here $g'(y) = \partial g / \partial y$ and $\mathcal{Z} = \{y_0 | g(y_0) = 0\}$ is the set values for y where $g(y) = 0$. *Hint:* Assume the zero's of g are widely spaced and $g'(y_0) \neq 0$ then consider what happens near one of the zeros. Rearrange the expression into a form similar to the definition of the delta function (12).

(ii) Why is

$$I = \int \frac{d^4k}{(2\pi)^4}\, \delta\!\!\!{}^{-}(k^2 - m^2) f(k^2, p^2, (k-p)^2) \tag{14}$$

Lorentz invariant if k and p are four vectors?

Hence deduce that $I = I(p^2, m)$.

Show that

$$I = \sum_{k_0=\pm\omega} \int \frac{đ^3 k}{2\omega}\, f(m^2, p^2, (k-p)^2)\,, \qquad \omega = |\sqrt{\boldsymbol{k}^2 - m^2}|\,. \tag{15}$$

Hence deduce that $đ^3 k/(2\omega)$ is Lorentz invariant.

(iii) Show that

$$2\pi\delta(p_0 - \omega) = \frac{i}{p_0 - \omega + i\epsilon} - \frac{i}{p_0 - \omega - i\epsilon} \tag{16}$$

∗4. The Advanced Propagator

The advanced propagator (two-point Green function) for the real free scalar field $\hat{\phi}(x)$ (in any picture) is defined as

$$D_{\mathrm{A}}(x) = -\theta(-x^0)\langle 0| \left[\hat{\phi}(x), \hat{\phi}(0)\right] |0\rangle \tag{17}$$

We will find this propagator for free fields. For simplicity work in the Heisenberg picture.

(i) Show that advanced propagator $D_{\mathrm{A}}(x)$ can be written as

$$D_{\mathrm{A}}(x) = \int \frac{d^4 p}{(2\pi)^4} \frac{i}{(p_0 - i\epsilon)^2 - \boldsymbol{p}^2 - m^2} e^{-ip\cdot x} \qquad 1 \gg \epsilon > 0 \tag{18}$$

where the integrations are all along the real axes of the four p^μ components. The parameter ϵ is a small positive infinitesimal which is taken to zero (from the positive side) at the end of the calculation. Note that in you need to "close the contours" in the upper or lower half-plane depending on the value of the time coordinate. This needs to be specified in detail in the solution.

Optional: Sometimes rather than adding an $i\epsilon$ to the integrand, the p_0 curve is bent by infinitesimal amounts near the poles. The two view are completely equivalent as we take $\epsilon \to 0^+$. As an option you might like to consider the same calculation in this language. That is show that

$$D_{\mathrm{A}}(x) = \int \frac{d^3 p}{(2\pi)^3} \int_C \frac{dp_0}{2\pi} \frac{i}{(p_0)^2 - \boldsymbol{p}^2 - m^2} e^{-ip\cdot x} \tag{19}$$

for a suitable contour C in the complex p^0 plane which you must find.

(ii) Show that

$$(\partial^2 + m^2) D_{\mathrm{A}}(x) = -i\delta^4(x)\,. \tag{20}$$

*5. Time evolution and propagators of a complex scalar field

This is collecting up some EFS from the lectures.

(i) Consider two sets of annihilation and creation operators obeying

$$\left[\hat{a}_{i\boldsymbol{p}}, \hat{a}^\dagger_{j\boldsymbol{q}}\right] = (2\pi)^3 \delta_{ij}\delta^3(\boldsymbol{p}-\boldsymbol{q}) , \qquad [\hat{a}_{i\boldsymbol{p}}, \hat{a}_{j\boldsymbol{q}}] = \left[\hat{a}^\dagger_{i\boldsymbol{p}}, \hat{a}^\dagger_{j\boldsymbol{q}}\right] = 0 , \qquad i,j = 1,2 . \tag{21}$$

Show that if we define

$$\hat{b}_{\boldsymbol{p}} = \frac{1}{\sqrt{2}} (\hat{a}_{1\boldsymbol{p}} + i\hat{a}_{2\boldsymbol{p}}) , \qquad \hat{c}_{\boldsymbol{p}} = \frac{1}{\sqrt{2}} (\hat{a}_{1\boldsymbol{p}} - i\hat{a}_{2\boldsymbol{p}}) . \tag{22}$$

then b and c also obey the canonical commutations relations for annihilation and creation operators, i.e. they are equally good descriptions for the annihilation and creation of particles/quanta.

(ii) Show that

$$\hat{a}^\dagger_{1\boldsymbol{p}}\hat{a}_{1\boldsymbol{p}} + \hat{a}^\dagger_{2\boldsymbol{p}}\hat{a}_{2\boldsymbol{p}} = \hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}} + c^\dagger_{\boldsymbol{p}}\hat{c}_{\boldsymbol{p}} \tag{23}$$

Thus deduce that if

$$\widehat{H} = \sum_{i=1,2} \int đ^3\boldsymbol{p}\, \omega_{\boldsymbol{p}} \left(\hat{a}^\dagger_{i\boldsymbol{p}}\hat{a}_{i\boldsymbol{p}} + Z_{\boldsymbol{p}}\right) \tag{24}$$

(where $Z_{\boldsymbol{p}}$ is the zero point energy for mode $\boldsymbol{p}$) then

$$\widehat{H} = \int đ^3\boldsymbol{p}\, \omega_{\boldsymbol{p}} \left(\hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}} + c^\dagger_{\boldsymbol{p}}\hat{c}_{\boldsymbol{p}} + 2Z_{\boldsymbol{p}}\right) . \tag{25}$$

(iii) Show that if at $t=0$ we have

$$\hat{\Phi}_{\mathrm{H}}(t=0,\boldsymbol{x}) \quad = \quad \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}(\hat{b}_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + \hat{c}^\dagger_{\boldsymbol{p}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}) \tag{26}$$

then with the Hamiltonian given by (25) we have

$$\hat{\Phi}_{\mathrm{H}}(t,\boldsymbol{x}) \quad = \quad \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}(\hat{b}_{\boldsymbol{p}}e^{-ipx} + \hat{c}^\dagger_{\boldsymbol{p}}e^{+ipx}) \tag{27}$$

where $px \equiv p_\mu x^\mu$ and $p_0 = +\omega_p$.

(iv) Consider two real[1] scalar fields

$$\hat{\phi}_{\mathrm{H}i}(t,\boldsymbol{x}) \quad = \quad \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(\hat{a}_{i\boldsymbol{p}}\mathrm{e}^{-i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}} + \hat{a}^\dagger_{i\boldsymbol{p}}\mathrm{e}^{i\omega_{\boldsymbol{p}}t-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \tag{28}$$

where $\hat{a}_{i\boldsymbol{p}}$ and $\hat{a}^\dagger_{i\boldsymbol{p}}$ are the operators given in (21). Show that if (22) is true then the field operators obey

$$\hat{\Phi}_{\mathrm{H}}(t,\boldsymbol{x}) = \frac{1}{\sqrt{2}}\left(\hat{\phi}_{\mathrm{H}1}(t,\boldsymbol{x}) + i\hat{\phi}_{\mathrm{H}2}(t,\boldsymbol{x})\right) , \qquad \hat{\Phi}^\dagger_{\mathrm{H}}(t,\boldsymbol{x}) = \frac{1}{\sqrt{2}}\left(\hat{\phi}_{\mathrm{H}1}(t,\boldsymbol{x}) - i\hat{\phi}_{\mathrm{H}2}(t,\boldsymbol{x})\right) . \tag{29}$$

(v) For a complex field the Wightman function is defined as

$$D(x-y) := \langle 0|\hat{\Phi}(x)\hat{\Phi}^\dagger(y)|\, 0\rangle . \tag{30}$$

Find an expression for $D(x-y)$ for the complex free fields given by (27). This should be exactly the same function as we found for real scalar fields.

[1] These represent classical real scalar fields but their quantised versions are hermitian not real.

(vi) For these fields show that $\langle 0|\hat{\Phi}(x)\hat{\Phi}(y)|\,0\rangle = 0$ for any time separation.

(vii) The Feynman propagator for complex scalar fields is defined as

$$\Delta_{\rm F}(x-y) := \langle 0|{\rm T}\hat{\Phi}(x)\hat{\Phi}^\dagger(y)|\,0\rangle \tag{31}$$

where T is the time ordering operator. Using results already proved for the free field as needed (no need to repeat contour integration tricks explicitly), show that this has the same form as that found for free real fields, i.e. in momentum space it is equal to

$$\Delta_{\rm F}(p) = \frac{i}{p^2 - m^2 + i\epsilon}\,. \tag{32}$$

Thus deduce that these are also solutions of the Klein-Gordon equation.

Why must $\langle 0|{\rm T}\Phi^\dagger(y)\Phi(x)|\,0\rangle = \langle 0|{\rm T}\Phi(x)\Phi^\dagger(y)|\,0\rangle$ if the times are unequal?

6. Charge of a complex scalar field

Consider a free complex scalar field $\Phi(\boldsymbol{x})$ with momentum density $\Pi(\boldsymbol{x})$. We will work in the Schrödinger picture (or at $t = 0$ in the Heisenberg picture) for simplicity (it removes some exponentials in the calculations) where we have

$$\Phi(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}(b_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + c^\dagger_{\boldsymbol{p}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}) \tag{33}$$

$$\Pi(\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}}(c_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} - b^\dagger_{\boldsymbol{p}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}) \tag{34}$$

(i) Show that if the Noether charge Q (here for the U(1) phase symmetry of the complex scalar field) is defined as

$$Q \equiv i\int d^3x[\Phi^\dagger\Pi^\dagger - \Pi\Phi] \tag{35}$$

then it may be written as

$$Q = \int \frac{d^3p}{(2\pi)^3}[b^\dagger_{\boldsymbol{p}}b_{\boldsymbol{p}} - c^\dagger_{\boldsymbol{p}}c_{\boldsymbol{p}}] \tag{36}$$

after ignoring an infinite c-number contribution.

(ii) Show that the ETCR (equal time commutation relations) for complex scalar fields are [Tong (2.17), p.34]

$$[\Phi(x),\Pi(y)]_{x_0=y_0} = i\delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \qquad \left[\Phi^\dagger(x),\Pi^\dagger(y)\right]_{x_0=y_0} = i\delta^3(\boldsymbol{x}-\boldsymbol{y})\,. \tag{37}$$

with the ETCR between any other pair of Φ, $\Phi^\dagger$, Π or $\Pi^\dagger$ equal to zero.

Show that Q

$$[Q,\Phi(\boldsymbol{x})] = -\Phi(\boldsymbol{x}) \tag{38}$$

You can obtain this result using either of the two expressions for Q and you should do both.

(iii) In the language of group theory, if an operator Q is the generator of a continuous symmetry then $[Q, \eta(\boldsymbol{x})] = q\eta(\boldsymbol{x})$ provided that the operator η is a charge eigenstate, i.e. if η is an operator with a definite charge under this symmetry. If that is true then q is the c-number charge of the operator η under this symmetry.

The corresponding full symmetry transformation for an operator is then

$$\eta \to \eta' = e^{i\theta Q} \eta e^{-i\theta Q} \tag{39}$$

where θ is the real parameter associated with the generator Q of these continuous symmetry transformations. Show that if η is a charge eigenstate with charge q then $\eta' = e^{i\theta q}\eta$.