Solutions 4: Free Quantum Field Theory

1. Heisenberg picture free real scalar field

We have

$$\phi(t,\boldsymbol{x}) \;=\; \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}\mathrm{e}^{-i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{p}}\mathrm{e}^{i\omega_{\boldsymbol{p}}t-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \tag{1}$$

(i) By taking an explicit hermitian conjugation, we find our result that $\phi^\dagger = \phi$. You need to note that all the parameters are real: $t, \boldsymbol{x}, \boldsymbol{p}$ are obviously real by definition and if m^2 is real and positive semi-definite then ω_p is real for all values of $\boldsymbol{p}$. Also $(\hat{a}^\dagger)^\dagger = \hat{a}$ is needed.

(ii) Since $\pi = \dot{\phi} = \partial_t\phi$ classically, we try this on the operator (1) to find

$$\pi(t,\boldsymbol{x}) \;=\; -i\int \frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}}\left(a_{\boldsymbol{p}}\mathrm{e}^{-i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}} - a^\dagger_{\boldsymbol{p}}\mathrm{e}^{i\omega_{\boldsymbol{p}}t-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \tag{2}$$

(iii) In this question as in many others it is best to leave the expression in terms of commutators. That is exploit

$$[A+B, C+D] = [A,C] + [A,D] + [B,C] + [B,D] \tag{3}$$

as much as possible. Note that here we do not have a sum over two terms A and B but a sum over an infinite number, an integral, but the principle is the same.

The second way to simplify notation is to write $e^{-i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}} = e^{-ipx}$ so that we choose $p_0 = +\omega_p$.

The simplest way however is to write $\widehat{A}_{\boldsymbol{p}} = \hat{a}_{\boldsymbol{p}}e^{-i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}}$ and $\hat{A}^\dagger_{\boldsymbol{p}} = \hat{a}^\dagger_{\boldsymbol{p}}e^{-i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}}$. You can quickly check these satisfy the same commutation relations as $\hat{a}_{\boldsymbol{p}}$ and $\hat{a}^\dagger_{\boldsymbol{p}}$. In fact $\widehat{A}_{\boldsymbol{p}} = e^{iHt}\hat{a}_{\boldsymbol{p}}e^{-iHt} = \hat{a}_{\boldsymbol{p}}(t)$ for a free field, i.e. we have just applied time evolution to the free field Heisenberg picture operators.

We have the usual commutation relations for the annihilation and creation operators with continuous labels (here the three-momenta) [1]

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\,, \qquad [a_{\boldsymbol{p}}, a_{\boldsymbol{q}}] = \left[a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = 0\,. \tag{4}$$

The first field commutation relation is then

$$\begin{aligned}
[\phi(t,\boldsymbol{x}), \pi(t,\boldsymbol{y})] \;&=\; \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\int\frac{d^3q}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\boldsymbol{q}}}{2}} \\
&\quad \left[(a_{\boldsymbol{p}}e^{-i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{p}}e^{i\omega_{\boldsymbol{p}}t-i\boldsymbol{p}\cdot\boldsymbol{x}}), (a_{\boldsymbol{q}}e^{-i\omega_{\boldsymbol{q}}t+i\boldsymbol{q}\cdot\boldsymbol{y}} - a^\dagger_{\boldsymbol{q}}e^{i\omega_{\boldsymbol{q}}t-i\boldsymbol{q}\cdot\boldsymbol{y}})\right] & (5)\\
&=\; (-i)\int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\sqrt{\frac{\omega_{\boldsymbol{p}}}{4\omega_{\boldsymbol{q}}}} \\
&\quad \left(-e^{-i(\omega_{\boldsymbol{p}}-\omega_{\boldsymbol{q}})t+i\boldsymbol{p}\cdot\boldsymbol{x}-i\boldsymbol{q}\cdot\boldsymbol{y}}\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] + e^{i(\omega_{\boldsymbol{p}}-\omega_{\boldsymbol{q}})t-i\boldsymbol{p}\cdot\boldsymbol{x}+i\boldsymbol{q}\cdot\boldsymbol{y}}\left[a^\dagger_{\boldsymbol{p}}, a_{\boldsymbol{q}}\right]\right) & (6)\\
&=\; (-i)\int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\sqrt{\frac{\omega_{\boldsymbol{p}}}{4\omega_{\boldsymbol{q}}}}\left(-e^{-i(\omega_{\boldsymbol{p}}-\omega_{\boldsymbol{q}})t+i\boldsymbol{p}\cdot\boldsymbol{x}-i\boldsymbol{q}\cdot\boldsymbol{y}}(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\right. \\
&\qquad\qquad \left. +e^{i(\omega_{\boldsymbol{p}}-\omega_{\boldsymbol{q}})t-i\boldsymbol{p}\cdot\boldsymbol{x}+i\boldsymbol{q}\cdot\boldsymbol{y}}(2\pi)^3(-\delta^3(\boldsymbol{p}-\boldsymbol{q}))\right) & (7)\\
&=\; i\int\frac{d^3p}{(2\pi)^3}\frac{1}{2}\left(e^{+i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} + e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right) & (8)\\
&=\; \frac{i}{2}2\delta^3(\boldsymbol{x}-\boldsymbol{y}) & (9)
\end{aligned}$$

[1]Note that this gives the annihilation and creation operators units of energy^{-3} (in natural units). The usual $\left[a_i, a^\dagger_j\right] = i\delta_{ij}$ is for the case of discrete labels.

Thus we find the canonical equal-time commutation relation

$$[\phi(t,\boldsymbol{x}),\pi(t,\boldsymbol{y})] \;=\; i\delta^3(\boldsymbol{x}-\boldsymbol{y})\,. \tag{10}$$

For the commutators

$$\left[\hat{\phi}(t,\boldsymbol{x}),\hat{\phi}(t,\boldsymbol{y})\right] = [\hat{\pi}(t,\boldsymbol{x}),\hat{\pi}(t,\boldsymbol{y})] = 0\,. \tag{11}$$

we see that they are of similar form

$$\begin{aligned}\left[\hat{\phi}(t=0,\boldsymbol{x}),\hat{\phi}(t=0,\boldsymbol{y})\right] &= \Big[\int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}(\hat{a}_{\boldsymbol{p}}\mathrm{e}^{+i\boldsymbol{p}\cdot\boldsymbol{x}}+\hat{a}^\dagger_{\boldsymbol{p}}\mathrm{e}^{-i\boldsymbol{p}\cdot\boldsymbol{x}}),\\ &\qquad \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{q}}}}(\hat{a}_{\boldsymbol{q}}\mathrm{e}^{+i\boldsymbol{q}\cdot\boldsymbol{y}}+\hat{a}^\dagger_{\boldsymbol{q}}\mathrm{e}^{-i\boldsymbol{q}\cdot\boldsymbol{y}})\Big] \end{aligned}\tag{12}$$

$$\begin{aligned}[\hat{\pi}(t=0,\boldsymbol{x}),\hat{\pi}(t=0,\boldsymbol{y})] &= \Big[\int \frac{d^3p}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}}(\hat{a}_{\boldsymbol{p}}\mathrm{e}^{+i\boldsymbol{p}\cdot\boldsymbol{x}}-\hat{a}^\dagger_{\boldsymbol{p}}\mathrm{e}^{-i\boldsymbol{p}\cdot\boldsymbol{x}}),\\ &\qquad \int \frac{d^3q}{(2\pi)^3}(-i)\sqrt{\frac{\omega_{\boldsymbol{q}}}{2}}(\hat{a}_{\boldsymbol{q}}\mathrm{e}^{+i\boldsymbol{q}\cdot\boldsymbol{y}}-\hat{a}^\dagger_{\boldsymbol{q}}\mathrm{e}^{-i\boldsymbol{q}\cdot\boldsymbol{y}})\Big] \end{aligned}\tag{13}$$

Thus all we need to show that

$$C_s \;=\; \int \frac{d^3p\,d^3q}{(\omega_{\boldsymbol{p}}\omega_{\boldsymbol{q}})^{s/2}}\left[(\hat{a}_{\boldsymbol{p}}\mathrm{e}^{+i\boldsymbol{p}\cdot\boldsymbol{x}}+s\hat{a}^\dagger_{\boldsymbol{p}}\mathrm{e}^{-i\boldsymbol{p}\cdot\boldsymbol{x}}),(\hat{a}_{\boldsymbol{q}}\mathrm{e}^{-i\boldsymbol{q}\cdot\boldsymbol{y}}+s\hat{a}^\dagger_{\boldsymbol{q}}\mathrm{e}^{+i\boldsymbol{q}\cdot\boldsymbol{y}})\right]\,, \tag{14}$$

is zero where $s=\pm1$ as the field (momenta) commutator is proportional to C_+ (C_-). Using (3) and (4) we find exactly as before that

$$\begin{aligned} C_s &= \int \frac{d^3p\,d^3q}{(\omega_{\boldsymbol{p}}\omega_{\boldsymbol{q}})^{s/2}}\Big(\,[\hat{a}_{\boldsymbol{p}},\hat{a}_{\boldsymbol{q}}]\,\mathrm{e}^{+i\boldsymbol{p}\cdot\boldsymbol{x}+i\boldsymbol{q}\cdot\boldsymbol{y}} + s\left[\hat{a}_{\boldsymbol{p}},\hat{a}^\dagger_{\boldsymbol{q}}\right]\mathrm{e}^{+i\boldsymbol{p}\cdot\boldsymbol{x}-i\boldsymbol{q}\cdot\boldsymbol{y}}\\ &\qquad\qquad + s\left[\hat{a}^\dagger_{\boldsymbol{p}},\hat{a}_{\boldsymbol{q}}\right]\mathrm{e}^{-i\boldsymbol{p}\cdot\boldsymbol{x}+i\boldsymbol{q}\cdot\boldsymbol{y}} + \left[\hat{a}^\dagger_{\boldsymbol{p}},\hat{a}^\dagger_{\boldsymbol{q}}\right]\mathrm{e}^{-i\boldsymbol{p}\cdot\boldsymbol{x}-i\boldsymbol{q}\cdot\boldsymbol{y}}\Big) && (15)\\ &= \int \frac{d^3p\,d^3q}{(\omega_{\boldsymbol{p}})^s}\left(s\delta(\boldsymbol{p}-\boldsymbol{q})\mathrm{e}^{+i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} - s\delta(\boldsymbol{p}-\boldsymbol{q})\mathrm{e}^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right) && (16)\\ &= s\int \frac{d^3p}{(\omega_{\boldsymbol{p}})^s}\left(\mathrm{e}^{+i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} - \mathrm{e}^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}\right) && (17)\end{aligned}$$

If you now change integration variable in the second term to $\boldsymbol{p}'=-\boldsymbol{p}$ you will find it is of exactly the same form as the first term and these cancel giving zero as required.

Aside - Trying to be Careful

This is really rubbish isn't it! The integrals here look divergent as they scale as energy (=momentum=mass) to the power $(3-s)$. The oscillatory factor just means we are adding $+\infty$ and $-\infty$ as we integrate out at high energy scales, at large $|\boldsymbol{p}|$. The whole thing looks (and is) badly defined. QFT is not very reasonable about divergences.

Riemann-Lebesgue lemma[2] is the real answer here but unlikely to be helpful to Imperial Physics students who aren't given the relevant mathematical tools. If we set up the problem more carefully we could apply this lemma and you'd be happy. In practice I haven't set up the maths carefully. I think you will find most QFT text books are the same. We really should define everything carefully to make sure that such integrals are always defined properly.

[2] See `https://bit.ly/2KOsAkA`.

The quick and dirty way here is to remember that these integrals here are part of an **operator** expression coming from commutators even if it is a unit operator. To evaluate these expressions the operator *must* act on something. So you should let your expression act on a dummy function $f(\boldsymbol{p})$. Now this dummy function has to be of the right type, some well behaved function. Basically that has to be something that falls off for large $|\boldsymbol{p}|$ "nicely". The functions needed will always respect our space-time symmetries, so here will always be functions of $|\boldsymbol{p}|$. A suitable example might be something that falls off as $\exp(-\boldsymbol{p}.\boldsymbol{p}/(2\sigma^2))$ where you can choose sigma to be as big as you like (take it to infinity only after everything else is done). Now your integrals are of the right form for Riemann-Lebesgue lemma to apply. Alternatively, as each integral is now well behaved and finite, you can start to manipulate them. In our case we would need $f(\boldsymbol{p}) = f(-\boldsymbol{p})$ for our odd/even arguments to work but the space-time symmetries guarantee that any function we have in practice will have that symmetry.

To be more precise we should set everything up carefully. This is what the Axiomatic QFT[3] is very careful about.

(iv) Given $\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$ then $\omega_{\boldsymbol{p}} = \omega_{-\boldsymbol{p}}$ and hence

$$\begin{aligned}
\int \mathrm{d}^3x\, \phi(t=0,\boldsymbol{x})\mathrm{e}^{i\mathbf{q}\cdot\boldsymbol{x}} &= \int \mathrm{d}^3x\, \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{p}}\mathrm{e}^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right)\mathrm{e}^{i\mathbf{q}\cdot\boldsymbol{x}} && (18)\\
&= \int \mathrm{d}^3x\, \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}\mathrm{e}^{i(\boldsymbol{p}+\mathbf{q})\cdot\boldsymbol{x}} + a^\dagger_{\boldsymbol{p}}\mathrm{e}^{i(\mathbf{q}-\boldsymbol{p})\cdot\boldsymbol{x}}\right) && (19)\\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}(2\pi)^3\delta^{(3)}(\mathbf{q}+\boldsymbol{p}) + a^\dagger_{\boldsymbol{p}}(2\pi)^3\delta^{(3)}(\mathbf{q}-\boldsymbol{p})\right) && (20)\\
&= \frac{1}{\sqrt{2\omega_{\mathbf{q}}}}\left(a_{-\mathbf{q}} + a^\dagger_{\mathbf{q}}\right) && (21)
\end{aligned}$$

$$\begin{aligned}
\int \mathrm{d}^3x\, \pi(t=0,\boldsymbol{x})\mathrm{e}^{i\mathbf{q}\cdot\boldsymbol{x}} &= -i\int \mathrm{d}^3x\, \frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}}\left(a_{\boldsymbol{p}}\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}} - a^\dagger_{\boldsymbol{p}}\mathrm{e}^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right)\mathrm{e}^{i\mathbf{q}\cdot\boldsymbol{x}} && (22)\\
&= -i\int \mathrm{d}^3x\, \frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}}\left(a_{\boldsymbol{p}}\mathrm{e}^{i(\boldsymbol{p}+\mathbf{q})\cdot\boldsymbol{x}} - a^\dagger_{\boldsymbol{p}}\mathrm{e}^{i(\mathbf{q}-\boldsymbol{p})\cdot\boldsymbol{x}}\right) && (23)\\
&= -i\int \frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}}\left(a_{\boldsymbol{p}}(2\pi)^3\delta^{(3)}(\mathbf{q}+\boldsymbol{p}) - a^\dagger_{\boldsymbol{p}}(2\pi)^3\delta^{(3)}(\mathbf{q}-\boldsymbol{p})\right) && (24)\\
&= -i\sqrt{\frac{\omega_{\mathbf{q}}}{2}}\left(a_{-\mathbf{q}} - a^\dagger_{\mathbf{q}}\right) && (25)
\end{aligned}$$

Thus solving for $a_{-\mathbf{q}}$ and $a^\dagger_{\mathbf{q}}$ gives

$$\begin{aligned}
a_{\mathbf{q}} &= \int \mathrm{d}^3x\, \mathrm{e}^{-i\mathbf{q}\cdot\boldsymbol{x}}\left(\sqrt{\frac{\omega_{\mathbf{q}}}{2}}\phi(0,\boldsymbol{x}) + i\frac{1}{\sqrt{2\omega_{\mathbf{q}}}}\pi(0,\boldsymbol{x})\right) && (26)\\
a^\dagger_{\mathbf{q}} &= \int \mathrm{d}^3x\, \mathrm{e}^{i\mathbf{q}\cdot\boldsymbol{x}}\left(\sqrt{\frac{\omega_{\mathbf{q}}}{2}}\phi(0,\boldsymbol{x}) - i\frac{1}{\sqrt{2\omega_{\mathbf{q}}}}\pi(0,\boldsymbol{x})\right) && (27)\\
& && (28)
\end{aligned}$$

[3] See `https://bit.ly/2wrg1ID`.

Hence since $[\phi(\boldsymbol{x}),\phi(\boldsymbol{y})]=[\pi(\boldsymbol{x}),\pi(\boldsymbol{y})]=0$

$$\begin{aligned}
\left[a_{\boldsymbol{p}},a_{\mathbf{q}}^\dagger\right] &= \int \mathrm{d}^3x\,\mathrm{e}^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\mathrm{d}^3y\,\mathrm{e}^{i\mathbf{q}\cdot\boldsymbol{y}} &(29)\\
&\quad\times\left[\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}}\phi(0,\boldsymbol{x})+i\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\pi(0,\boldsymbol{x}),\sqrt{\frac{\omega_{\mathbf{q}}}{2}}\phi(0,\boldsymbol{y})-i\frac{1}{\sqrt{2\omega_{\mathbf{q}}}}\pi(0,\boldsymbol{y})\right] &(30)\\
&= \int \mathrm{d}^3x\,\mathrm{d}^3y\,\mathrm{e}^{-i\boldsymbol{p}\cdot\boldsymbol{x}+i\mathbf{q}\cdot\boldsymbol{y}}\left(-i\sqrt{\frac{\omega_{\boldsymbol{p}}}{4\omega_{\mathbf{q}}}}\left[\phi(0,\boldsymbol{x}),\pi(0,\boldsymbol{y})\right]+i\sqrt{\frac{\omega_{\mathbf{q}}}{4\omega_{\boldsymbol{p}}}}\left[\pi(0,\boldsymbol{x}),\phi(0,\boldsymbol{y})\right]\right) &(31)\\
&= \int \mathrm{d}^3x\,\mathrm{d}^3y\,\mathrm{e}^{-i\boldsymbol{p}\cdot\boldsymbol{x}+i\mathbf{q}\cdot\boldsymbol{y}}\left(\sqrt{\frac{\omega_{\boldsymbol{p}}}{4\omega_{\mathbf{q}}}}\delta^{(3)}(\boldsymbol{x}-\boldsymbol{y})+\sqrt{\frac{\omega_{\mathbf{q}}}{4\omega_{\boldsymbol{p}}}}\delta^{(3)}(\boldsymbol{y}-\boldsymbol{x})\right) &(32)\\
&= \int \mathrm{d}^3x\,\mathrm{e}^{i(\mathbf{q}-\boldsymbol{p})\cdot\boldsymbol{x}}\left(\sqrt{\frac{\omega_{\boldsymbol{p}}}{4\omega_{\mathbf{q}}}}+\sqrt{\frac{\omega_{\mathbf{q}}}{4\omega_{\boldsymbol{p}}}}\right) &(33)\\
&= (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\mathbf{q}) &(34)
\end{aligned}$$

(v) Taking the derivative we have

$$\boldsymbol{\nabla}\phi(\boldsymbol{x})=\int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{i\boldsymbol{p}}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}}-a_{\boldsymbol{p}}^\dagger\mathrm{e}^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \tag{35}$$

so[4]

$$\begin{aligned}
\boldsymbol{P} &= -\int \mathrm{d}^3x\,\pi(\boldsymbol{x})\boldsymbol{\nabla}\phi(\boldsymbol{x}) &(36)\\
&= -\int \mathrm{d}^3x\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}}\left(a_{\boldsymbol{p}}\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}}-a_{\boldsymbol{p}}^\dagger\mathrm{e}^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right)\frac{\mathbf{q}}{\sqrt{2\omega_{\mathbf{q}}}}\left(a_{\mathbf{q}}\mathrm{e}^{i\mathbf{q}\cdot\boldsymbol{x}}-a_{\mathbf{q}}^\dagger\mathrm{e}^{-i\mathbf{q}\cdot\boldsymbol{x}}\right) &(37)\\
&= -\int \mathrm{d}^3x\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\sqrt{\omega_{\boldsymbol{p}}}\mathbf{q}}{2\sqrt{\omega_{\mathbf{q}}}} &(38)\\
&\quad\times\left\{\mathrm{e}^{i(\boldsymbol{p}+\mathbf{q})\cdot\boldsymbol{x}}a_{\boldsymbol{p}}a_{\mathbf{q}}-\mathrm{e}^{i(\boldsymbol{p}-\mathbf{q})\cdot\boldsymbol{x}}a_{\boldsymbol{p}}a_{\mathbf{q}}^\dagger-\mathrm{e}^{-i(\boldsymbol{p}-\mathbf{q})\cdot\boldsymbol{x}}a_{\boldsymbol{p}}^\dagger a_{\mathbf{q}}+\mathrm{e}^{-i(\boldsymbol{p}+\mathbf{q})\cdot\boldsymbol{x}}a_{\boldsymbol{p}}^\dagger a_{\mathbf{q}}^\dagger\right\} &(39)\\
&= -\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\sqrt{\omega_{\boldsymbol{p}}}\mathbf{q}}{2\sqrt{\omega_{\mathbf{q}}}} &(40)\\
&\quad\times\left\{(2\pi)^3\delta^{(3)}(\boldsymbol{p}+\mathbf{q})\left(a_{\boldsymbol{p}}a_{\mathbf{q}}+a_{\boldsymbol{p}}^\dagger a_{\mathbf{q}}^\dagger\right)-(2\pi)^3\delta^{(3)}(\boldsymbol{p}-\mathbf{q})\left(a_{\boldsymbol{p}}a_{\mathbf{q}}^\dagger+a_{\boldsymbol{p}}^\dagger a_{\mathbf{q}}\right)\right\} &(41)\\
&= -\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\boldsymbol{p}}{2}\left(-a_{\boldsymbol{p}}a_{-\boldsymbol{p}}-a_{\boldsymbol{p}}^\dagger a_{-\boldsymbol{p}}^\dagger-a_{\boldsymbol{p}}a_{\boldsymbol{p}}^\dagger-a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}}\right) &(42)
\end{aligned}$$

Now $\boldsymbol{p}\left(a_{\boldsymbol{p}}a_{-\boldsymbol{p}}+a_{\boldsymbol{p}}^\dagger a_{-\boldsymbol{p}}^\dagger\right)$ is an *odd* function under $\boldsymbol{p}\to-\boldsymbol{p}$ and so integrates to zero. Thus

$$\begin{aligned}
\boldsymbol{p} &= \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\boldsymbol{p}}{2}\left(a_{\boldsymbol{p}}a_{\boldsymbol{p}}^\dagger+a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}}\right) &(43)\\
&= \int\frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\boldsymbol{p}}{2}\left(2a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}}+(2\pi)^3\delta^{(3)}(0)\right) &(44)\\
&= \int\frac{\mathrm{d}^3p}{(2\pi)^3}\boldsymbol{p}\,a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}} &(45)
\end{aligned}$$

since again $\boldsymbol{p}\delta^{(3)}(0)$ is an odd function (albeit it poorly defined!).

[4] NOTE THIS IS FOR $t=0$ should add in more exponentials for non-tirvial times!

The interpretation is that $a^\dagger_{\boldsymbol{p}} a_{\boldsymbol{p}}\, d^3p$ is the operator giving the number of quanta in a small volume d^3p centred at momentum $\boldsymbol{p}$. This will indeed contribute $\boldsymbol{p}$ to the total momenta. We see the same type of term for the energy, the Hamiltonian operator H, but we get a factor of $\omega_{\boldsymbol{p}}$ not $\boldsymbol{p}$ in that case.

(vi) From (2) we have

$$\begin{aligned}\int d^3\boldsymbol{x}\, \Pi^2 &= \int d^3\boldsymbol{x}\ -i\int \frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}}\left(a_{\boldsymbol{p}}\mathrm{e}^{-i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}} - a^\dagger_{\boldsymbol{p}}\mathrm{e}^{i\omega_{\boldsymbol{p}}t-i\boldsymbol{p}\cdot\boldsymbol{x}}\right)\\ &\quad .-i\int \frac{\mathrm{d}^3q}{(2\pi)^3}\sqrt{\frac{\omega_{\boldsymbol{q}}}{2}}\left(a_{\boldsymbol{q}}\mathrm{e}^{-i\omega_{\boldsymbol{q}}t+i\boldsymbol{q}\cdot\boldsymbol{x}} - a^\dagger_{\boldsymbol{q}}\mathrm{e}^{i\omega_{\boldsymbol{q}}t-i\boldsymbol{q}\cdot\boldsymbol{x}}\right) &(46)\\ &= -\int d^3\boldsymbol{x}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\sqrt{\frac{\omega_{\boldsymbol{p}}\omega_{\boldsymbol{q}}}{4}}\left(a_{\boldsymbol{p}}\mathrm{e}^{-i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}} - a^\dagger_{\boldsymbol{p}}\mathrm{e}^{i\omega_{\boldsymbol{p}}t-i\boldsymbol{p}\cdot\boldsymbol{x}}\right)\\ &\quad \left(a_{\boldsymbol{q}}\mathrm{e}^{-i\omega_{\boldsymbol{q}}t+i\boldsymbol{q}\cdot\boldsymbol{x}} - a^\dagger_{\boldsymbol{q}}\mathrm{e}^{i\omega_{\boldsymbol{q}}t-i\boldsymbol{q}\cdot\boldsymbol{x}}\right) &(47)\end{aligned}$$

Using that

$$\int d^3\boldsymbol{x}\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}} = (2\pi)^3\delta(\boldsymbol{p}) \qquad (48)$$

we apply the $\int d^3\boldsymbol{x}$ and use the resulting delta function of $\delta^3(\boldsymbol{p}\pm\boldsymbol{q})$ to eliminate the $\boldsymbol{q}$ integral. This gives us

$$\int d^3\boldsymbol{x}\, \Pi^2 = -\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\omega_{\boldsymbol{p}}}{2}\left(a_{\boldsymbol{p}}a_{-\boldsymbol{p}}\mathrm{e}^{-2i\omega_{\boldsymbol{p}}t} - a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}} - a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}}a^\dagger_{-\boldsymbol{p}}\mathrm{e}^{+2i\omega_{\boldsymbol{p}}t}\right) \qquad (49)$$

From (2) we have

$$\begin{aligned}\int d^3\boldsymbol{x}\,(\boldsymbol{\nabla}\phi)^2 &= \int d^3\boldsymbol{x}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(i\boldsymbol{p}a_{\boldsymbol{p}}\mathrm{e}^{-i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}} - i\boldsymbol{p}a^\dagger_{\boldsymbol{p}}\mathrm{e}^{i\omega_{\boldsymbol{p}}t-i\boldsymbol{p}\cdot\boldsymbol{x}}\right)\\ &\quad .\int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{q}}}}\left(i\boldsymbol{q}a_{\boldsymbol{q}}\mathrm{e}^{-i\omega_{\boldsymbol{q}}t+i\boldsymbol{q}\cdot\boldsymbol{x}} - i\boldsymbol{q}a^\dagger_{\boldsymbol{q}}\mathrm{e}^{i\omega_{\boldsymbol{q}}t-i\boldsymbol{q}\cdot\boldsymbol{x}}\right) &(50)\\ &= \int d^3\boldsymbol{x}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\int \frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\boldsymbol{p}.\boldsymbol{q}}{\sqrt{4\omega_{\boldsymbol{p}}\omega_{\boldsymbol{q}}}}\left(ia_{\boldsymbol{p}}\mathrm{e}^{-i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}} - ia^\dagger_{\boldsymbol{p}}\mathrm{e}^{i\omega_{\boldsymbol{p}}t-i\boldsymbol{p}\cdot\boldsymbol{x}}\right)\\ &\quad .\left(ia_{\boldsymbol{q}}\mathrm{e}^{-i\omega_{\boldsymbol{q}}t+i\boldsymbol{q}\cdot\boldsymbol{x}} - ia^\dagger_{\boldsymbol{q}}\mathrm{e}^{i\omega_{\boldsymbol{q}}t-i\boldsymbol{q}\cdot\boldsymbol{x}}\right) &(51)\end{aligned}$$

Again we apply the $\int d^3\boldsymbol{x}$ and use the resulting delta function of $\delta^3(\boldsymbol{p}\pm\boldsymbol{q})$ to eliminate the $\boldsymbol{q}$ integral to find

$$\int d^3\boldsymbol{x}\,(\boldsymbol{\nabla}\phi)^2 = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\boldsymbol{p}^2}{2\omega_{\boldsymbol{p}}}\left(a_{\boldsymbol{p}}a_{-\boldsymbol{p}}\mathrm{e}^{-2i\omega_{\boldsymbol{p}}t} + a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}}a^\dagger_{-\boldsymbol{p}}\mathrm{e}^{2i\omega_{\boldsymbol{p}}t}\right) \qquad (52)$$

Finally, in the same manner we find that

$$\int d^3\boldsymbol{x}\,(\phi)^2 = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}\left(a_{\boldsymbol{p}}a_{-\boldsymbol{p}}\mathrm{e}^{-2i\omega_{\boldsymbol{p}}t} + a_{\boldsymbol{p}}a^\dagger_{\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}}a_{\boldsymbol{p}} + a^\dagger_{\boldsymbol{p}}a^\dagger_{-\boldsymbol{p}}\mathrm{e}^{2i\omega_{\boldsymbol{p}}t}\right) \qquad (53)$$

Putting this together we find

$$
\begin{aligned}
H &= \frac{1}{2}\int d^3\boldsymbol{x}\, \Pi^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2 \qquad (54)\\
&= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\omega_p^2+\boldsymbol{p}^2+m^2}{2\omega_{\boldsymbol{p}}}\left(a_{\boldsymbol{p}}a_{\boldsymbol{p}}^\dagger + a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}}\right)\\
&\quad \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{-\omega_p^2+\boldsymbol{p}^2+m^2}{2\omega_{\boldsymbol{p}}}\left(a_{\boldsymbol{p}}a_{-\boldsymbol{p}}\mathrm{e}^{-2i\omega_{\boldsymbol{p}}t} + a_{\boldsymbol{p}}^\dagger a_{-\boldsymbol{p}}^\dagger \mathrm{e}^{2i\omega_{\boldsymbol{p}}t}\right) \qquad (55)\\
&= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\omega_p\left(a_{\boldsymbol{p}}a_{\boldsymbol{p}}^\dagger + a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}}\right) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\omega_p\left(a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}} + \frac{1}{2}(2\pi)^3\delta^3(0)\right) \qquad (56)
\end{aligned}
$$

with $\omega_p = +\sqrt{\boldsymbol{p}^2+m^2}$

(vii)

$$
\begin{aligned}
[H, a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}] &= \left[\int \frac{\mathrm{d}^3p}{(2\pi)^3}\omega_p\left(a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}} + \frac{1}{2}\right), a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}\right] \qquad (57)\\
&= \left[\int \frac{\mathrm{d}^3p}{(2\pi)^3}\omega_p a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}}, a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}\right] \qquad (58)\\
&= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\omega_p\left[a_{\boldsymbol{p}}^\dagger a_{\boldsymbol{p}}, a_{\boldsymbol{k}}^\dagger a_{\boldsymbol{k}}\right] = 0 \qquad (59)
\end{aligned}
$$

Note that this result is not true for any interesting i.e. interacting theory. The interactions mix the modes of different momenta leading to a lack of conservation of particle number. Only a continuous symmetry can guarantee conserved numbers and those are usually linked to total numbers of various particles, not the individual quanta of one particle at one momentum.

2. Time evolution of annihilation operator

(i) From (56) and using the commutation relation

$$
\left[a_{\boldsymbol{p}}, a_{\boldsymbol{q}}^\dagger\right] = (2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{q})\,, \qquad [a_{\boldsymbol{p}}, a_{\boldsymbol{q}}] = \left[a_{\boldsymbol{p}}^\dagger, a_{\boldsymbol{q}}^\dagger\right] = 0\,, \qquad (60)
$$

we have that

$$
\begin{aligned}
[H, a_{\boldsymbol{p}}] &= \int \frac{\mathrm{d}^3q}{(2\pi)^3}\left[\omega_q a_{\boldsymbol{q}}^\dagger a_{\boldsymbol{q}} + \frac{1}{2}, a_{\boldsymbol{p}}\right] \qquad (61)\\
&= \int \frac{\mathrm{d}^3q}{(2\pi)^3}\omega_q\left(a_{\boldsymbol{q}}^\dagger a_{\boldsymbol{q}} a_{\boldsymbol{p}} - a_{\boldsymbol{p}} a_{\boldsymbol{q}}^\dagger a_{\boldsymbol{q}}\right) \qquad (62)\\
&= \int \frac{\mathrm{d}^3q}{(2\pi)^3}\omega_q\left(a_{\boldsymbol{q}}^\dagger a_{\boldsymbol{q}} a_{\boldsymbol{p}} - (a_{\boldsymbol{q}}^\dagger a_{\boldsymbol{q}} a_{\boldsymbol{p}} + (2\pi)^3\delta(\boldsymbol{p}-\boldsymbol{q})a_{-\boldsymbol{p}})\right) \qquad (63)\\
&= -\int \mathrm{d}^3q\,\omega_q\delta(\boldsymbol{p}-\boldsymbol{q}))a_{\boldsymbol{p}} = -\omega_p a_{\boldsymbol{p}} \qquad (64)
\end{aligned}
$$

as required.

(ii) We want to prove that

$$
(itH)^n a_{\boldsymbol{p}} = a_{\boldsymbol{p}}\left[it(H-\omega_{\boldsymbol{p}})\right]^n \qquad (65)
$$

using $[H, a_{\boldsymbol{p}}] = -\omega_{\boldsymbol{p}} a_{\boldsymbol{p}}$. First note that trivially the relation holds for $n = 1$. Now if we assume the relation holds for $n = m$ then

$$\begin{aligned}(itH)^{m+1} a_{\boldsymbol{p}} &= (itH)(itH)^m a_{\boldsymbol{p}} & (66)\\ &= itH a_{\boldsymbol{p}} \left[it(H - \omega_{\boldsymbol{p}})\right]^m & (67)\\ &= it(a_{\boldsymbol{p}} H - \omega_{\boldsymbol{p}} a_{\boldsymbol{p}}) \left[it(H - \omega_{\boldsymbol{p}})\right]^m & (68)\\ &= a_{\boldsymbol{p}} \left[it(H - \omega_{\boldsymbol{p}})\right]^{m+1} . & (69)\end{aligned}$$

So the relation holds for $n = m+1$ if it holds for $n = m$. Thus, by induction, it holds for all $n \geq 0$.

(iii) Start by expanding the first exponential

$$\begin{aligned}\mathrm{e}^{iHt} a_{\boldsymbol{p}} \mathrm{e}^{-iHt} &= \left(\sum_{n=0}^{\infty} \frac{1}{n!} (iHt)^n\right) a_{\boldsymbol{p}} \mathrm{e}^{-iHt} & (70)\\ &= \left(\sum_n \frac{1}{n!} (iHt)^n a_{\boldsymbol{p}}\right) \mathrm{e}^{-iHt} & (71)\\ &= \left(\sum_n \frac{1}{n!} a_{\boldsymbol{p}} \left[it(H - \omega_{\boldsymbol{p}})\right]^n\right) \mathrm{e}^{-iHt} & (72)\\ &= a_{\boldsymbol{p}} \left(\sum_n \frac{1}{n!} \left[it(H - \omega_{\boldsymbol{p}})\right]^n\right) \mathrm{e}^{-iHt} & (73)\\ &= a_{\boldsymbol{p}} \mathrm{e}^{it(H-\omega_{\boldsymbol{p}})} \mathrm{e}^{-iHt} & (74)\\ &= a_{\boldsymbol{p}} \mathrm{e}^{it(H-\omega_{\boldsymbol{p}}-H)} & (75)\\ & & (76)\end{aligned}$$

Thus we see that[5]

$$\mathrm{e}^{iHt} a_{\boldsymbol{p}} \mathrm{e}^{-iHt} = a_{\boldsymbol{p}} \mathrm{e}^{-it\omega_{\boldsymbol{p}}} \tag{77}$$

where we have used the obvious facts that $[H, H] = 0$ and $[\omega_{\boldsymbol{p}}, H] = 0$.

The corresponding equation for $a_{\boldsymbol{p}}^\dagger$ follows from hermitian conjugation

$$\begin{aligned}\left(\mathrm{e}^{iHt} a_{\boldsymbol{p}} \mathrm{e}^{-iHt}\right)^\dagger &= \left(a_{\boldsymbol{p}} \mathrm{e}^{-it\omega_{\boldsymbol{p}}}\right)^\dagger & (78)\\ \Rightarrow \quad \left(\mathrm{e}^{-iHt}\right)^\dagger (a_{\boldsymbol{p}})^\dagger \left(\mathrm{e}^{iHt}\right)^\dagger &= \left(\mathrm{e}^{-it\omega_{\boldsymbol{p}}}\right)^\dagger (a_{\boldsymbol{p}})^\dagger & (79)\\ \Rightarrow \quad \mathrm{e}^{+iHt} a_{\boldsymbol{p}}^\dagger \mathrm{e}^{-iHt} &= \mathrm{e}^{+it\omega_{\boldsymbol{p}}} a_{\boldsymbol{p}}^\dagger & (80)\\ \Rightarrow \quad \mathrm{e}^{iHt} a_{\boldsymbol{p}}^\dagger \mathrm{e}^{-iHt} &= a_{\boldsymbol{p}}^\dagger \mathrm{e}^{+i\omega_{\boldsymbol{p}} t} . & (81)\end{aligned}$$

3. Delta Functions

The Dirac delta function is defined through

$$\int_{-\infty}^{+\infty} dx\, \delta(x - x_0) f(x) = f(x_0) . \tag{82}$$

You should always start from this equation when using a delta function.

[5] Another neat way to prove this by taking the derivative of the equation with respect to t and solving the resulting operator valued differential equation.

(i) Consider

$$J = \int dy\ \delta(g(y)) f(y)\,. \tag{83}$$

Assume that the zero's of g are at $\mathcal{Z} = \{y_0 | g(y_0) = 0\}$ and are widely spaced. Then the only places where the integral (83) has a non-zero contribution is in the region of one of these zeros as we need the argument of the delta function to be zero from (82). So we can write J as

$$J = \sum_{y_0 \in \mathcal{Z}} \int_{y_0-\epsilon}^{y_0+\epsilon} dy\ \delta(g(y)) f(y)\,. \tag{84}$$

So consider one of these zeros, say y_0, and expand the function g around this zero to find that $g(y) = g(y_0) + (y - y_0)g'(y_0) + O((y - y_0)^2)$. By definition $G(y_0) = 0$ so we have for small ϵ that

$$J = \sum_{y_0 \in \mathcal{Z}} \int_{y_0-\epsilon}^{y_0+\epsilon} dy\ \delta((y - y_0)g'(y_0)) f(y)\,. \tag{85}$$

Now change variable to $x = (y - y_0)g'(y_0)$ to match the form given in the definition of the delta function (82). The change of variables is gives us

$$J = \sum_{y_0 \in \mathcal{Z}} \int_{-\eta}^{\eta} \frac{dx}{g'(y_0)}\ \delta(x) f(y)\,, \qquad \eta = g'(y_0)\epsilon\,. \tag{86}$$

Now in order to apply the formula (82) we note that we must be running from below the zero of the argument of the delta function, from below x_0 in (82), to above it. For the case of $g'(y_0) > 0$ there is no problem as η is positive and we have the same form as (82). The range of integration can be extended to infinity without any problem. When $g'(y_0) < 0$ however, η is negative and we are running past the zero in the wrong direction. However easy to switch direction but we get an overall minus sign in this case. This then cancels the negative sign of $g'(y_0)$ in the denominator. Thus we have that

$$J = \sum_{y_0 \in \mathcal{Z}} \int_{-\eta'}^{\eta'} \frac{dx}{|g'(y_0|)}\ \delta(x) f(y)\,, \qquad \eta' = |g'(y_0)|\epsilon\,. \tag{87}$$

Now we apply (82) to find that

$$J = \int dy\ \delta(g(y)) f(y) = \sum_{y_0 \in \mathcal{Z}} \frac{f(y_0)}{|g'(y_0)|}\,, \tag{88}$$

(ii) By inspection

$$I = \int \frac{d^4k}{(2\pi)^4}\ \delta\!\!\!{}^{-}(k^2 - m^2) f(k^2, p^2, (k-p)^2) \tag{89}$$

must be Lorentz invariant if k and p are four vectors as the expression is made up of Lorentz scalars. The arguments k^2, p^2, $(k-p)^2$ and m^2 are all Lorentz scalars. The measure, d^4k is Lorentz invariant because any boost to new variables $k'^\mu = \Lambda^\mu_\nu k^\nu$ produces a Jacobian factor in the transformation $|\Lambda|$ but this is 1 by definition of the Lorentz transformations. Thus all elements of this integral are invariant.

There is only one variable p in this problem so we can only be a function of the only remaining variable p (and of course it can depend on m or other constant parameters). The only invariant we can build out of this is p^2 so the result must be a function of p^2.

The result

$$I = \sum_{k_0=\pm\omega} \int \frac{d\!\!{}^{-}\!{}^3k}{2\omega}\, f(m^2, p^2, (k-p)^2)\,, \qquad \omega = |\sqrt{\boldsymbol{k}^2 - m^2}|\,. \tag{90}$$

follows from (88). Consider the k_0 integration where we have that $g(k_0) = (k_0)^2 - \omega^2$. This has two zeros at $k_0 = \pm\omega$. We find that $g'(k_0) = 2k_0$ so at the zeros we have $g'(k_0) = 2|k_0| = 2\omega$.

Since I is Lorentz invariant as a whole, and all the other terms in the form (90) are Lorentz invariant, we deduce that $d\!\!{}^{-}\!{}^3k/(2\omega)$ is also Lorentz invariant. You could also prove this directly by changing variables to a boosted frame $k'^\mu = \Lambda^\mu_\nu k^\nu$.

(iii) Consider the integral

$$K = \int_{-\infty}^{+\infty} dp_0\, f(p_0) \left(\frac{i}{p_0 - \omega + i\epsilon} - \frac{i}{p_0 - \omega - i\epsilon} \right) \tag{91}$$

The poles and contours used in the two terms are shown in figure 1.

For the first term we can distort the integration path so that near the pole at $p_0 - \omega + i\epsilon$ we run along a semi circle centred on ω of radius $\eta > 0$ running above the axis (positive imaginary part), see figure 2. We will assume η is infinitesimal so that we do not encounter any poles in the function f. We run along the real axis for the rest of the path. We can now take the limit $\epsilon \to 0$ to place the pole on the real axis as for $\eta > 0$ the path does not run through the pole.

$$K_+ = \int_{C_+} dp_0\, f(p_0) \frac{i}{p_0 - \omega} \tag{92}$$

For the second term we move the contour of integration so near its pole at $p_0 - \omega - i\epsilon$ so that now this path runs along the real axis except for a semi-circle centred on ω of radius $\eta > 0$ but this time the semi-circle is below the axis (negative imaginary part), see figure 2.

$$K_- = -\int_{C_-} dp_0\, f(p_0) \frac{i}{p_0 - \omega} \tag{93}$$

If we reverse the direction of integration in this second case we absorb the overall minus sign. Since the whole result is $K = K_+ + K_-$ when we put the two together, the contributions coming from two integrations along the real axes now cancel. The only part remaining is an integration around a small circle of radius η centred on the pole at $k_0 = \omega$, see figure 3. Note we are running round this pole in a negative sense so that the residue there tells us this integral is equal to $-2\pi i$ times the residue. The residue is simply $if(p_0 = \omega)$ with the the factor of i coming from the numerator. Thus we find that we have

$$K = 2\pi f(p_0 = \omega) \tag{94}$$

However using the definition of the Dirac delta function (82) we also have that this may be written as

$$K = 2\pi \int_{-\infty}^{+\infty} dp_0 f(p_0) \delta(p_0 - \omega) \tag{95}$$

Now comparing (91) and (95) we can identify that

$$2\pi\delta(p_0 - \omega) = \frac{i}{p_0 - \omega + i\epsilon} - \frac{i}{p_0 - \omega - i\epsilon} \tag{96}$$

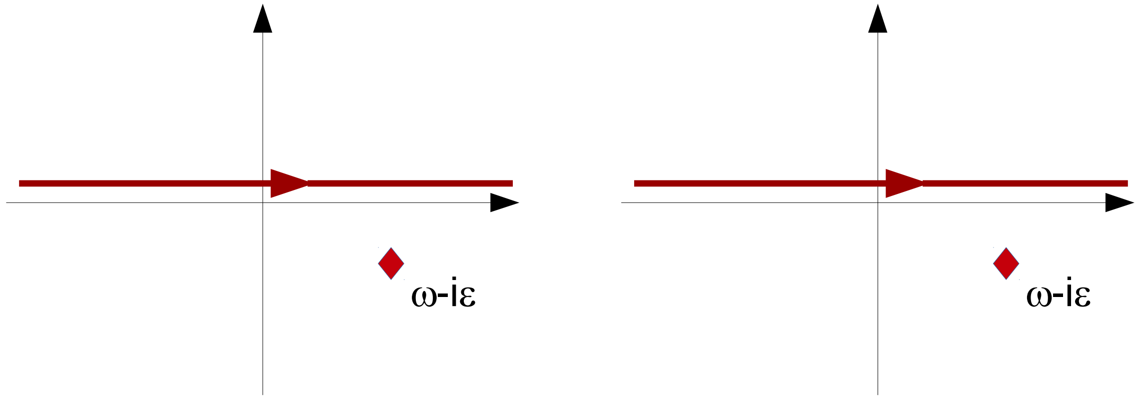


Figure 1: Figure showing two contours used for two terms in the delta function representation K. Here the poles are placed off the real axis and contours run along the real line.

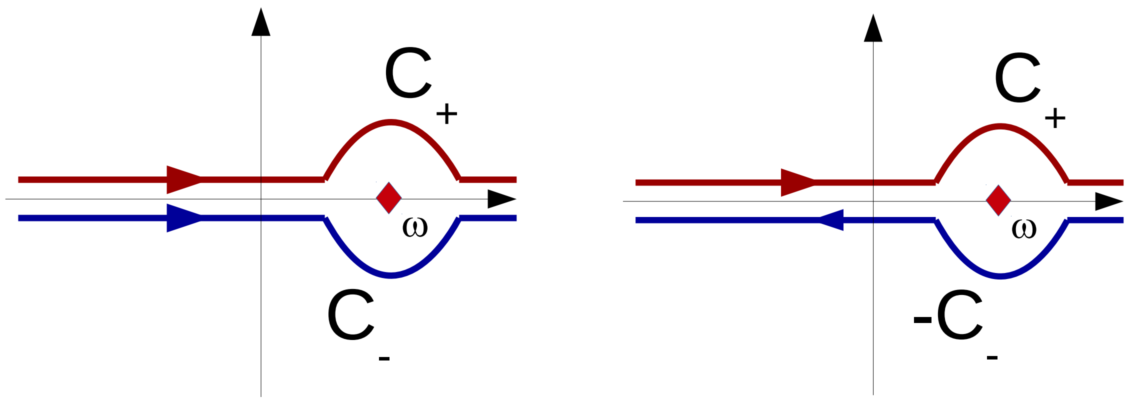


Figure 2: Figure showing two contours used for two terms in the delta function representation, K_+ of (92) uses C_+ shown on the left while on the right is C_- used by K_- of (93). The poles of the integrand are now on the real axis but the contours follow semicircles above or below the real line to avoid the the poles.

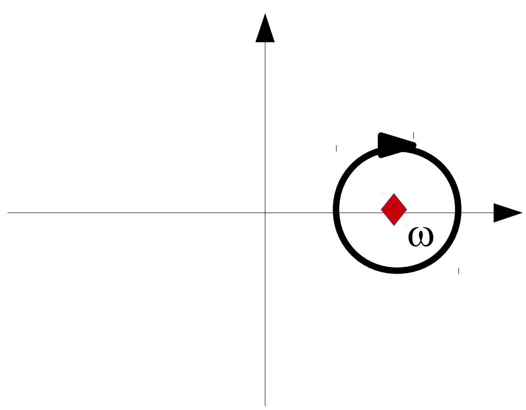


Figure 3: Absorbing the minus sign of the second term of (91) by reversing the direction of the countour C_-, the only non-zero contribution now comes from a small circle around the pole running in the negative direction..

4. The Advanced Propagator

We have

$$D_A(x) = -\theta(-x^0)\,\langle 0|[\phi(x),\phi(0)]|0\rangle \tag{97}$$

(i) We start by evaluating D_A. We have

$$\begin{aligned}
[\phi(x),\phi(0)] &= \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{4\omega_{\boldsymbol{p}}\omega_{\mathbf{q}}}}\left(\left[a_{\boldsymbol{p}},a^\dagger_{\mathbf{q}}\right]e^{-ip\cdot x}+\left[a^\dagger_{\boldsymbol{p}},a_{\mathbf{q}}\right]e^{ip\cdot x}\right)\Big|_{p^0=\omega_{\boldsymbol{p}}} && (98)\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{4\omega_{\boldsymbol{p}}\omega_{\mathbf{q}}}}(2\pi)^3\delta^{(3)}(\boldsymbol{p}-\mathbf{q})\left(e^{-ip\cdot x}-e^{ip\cdot x}\right)\Big|_{p^0=\omega_{\boldsymbol{p}}} && (99)\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}\left(e^{-ip\cdot x}-e^{ip\cdot x}\right)\Big|_{p^0=\omega_{\boldsymbol{p}}} && (100)
\end{aligned}$$

Hence since $\langle 0|0\rangle = 1$ we have[6]

$$\begin{aligned}
D_A(x) &= -\theta(-x^0)\,\langle 0|[\phi(x),\phi(0)]|0\rangle && (101)\\
&= -\theta(-x^0)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}\left(e^{-ip\cdot x}-e^{ip\cdot x}\right)\Big|_{p^0=\omega_{\boldsymbol{p}}} && (102)\\
&= -\theta(-x^0)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}\left(e^{-i\omega_{\boldsymbol{p}}x^0}e^{i\boldsymbol{p}\cdot\boldsymbol{x}}-e^{i\omega_{\boldsymbol{p}}x^0}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) && (103)\\
&= -\theta(-x^0)\int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}\left(e^{-i\omega_{\boldsymbol{p}}x^0}e^{i\boldsymbol{p}\cdot\boldsymbol{x}}-e^{i\omega_{\boldsymbol{p}}x^0}e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \qquad \boldsymbol{p}\leftrightarrow-\boldsymbol{p}\text{ in second term} && (104)\\
&= -\theta(-x^0)\int \frac{d^3p}{(2\pi)^3}\left(\frac{1}{2\omega_{\boldsymbol{p}}}e^{-ip\cdot x}\Big|_{p^0=\omega_{\boldsymbol{p}}}+\frac{1}{(-2\omega_{\boldsymbol{p}})}e^{-ip\cdot x}\Big|_{p^0=-\omega_{\boldsymbol{p}}}\right) && (105)
\end{aligned}$$

We now need to introduce a p_0 integration and rewrite the expression in terms of a contour integration. There are two standard ways to do this. In the first approach we shift the poles of the integrand, introducing a small positive infinitesimal ϵ into the integrand which is taken to zero (from the positive side) at the end of the calculation. This is the approach used in the lectures and it is common practice to use this notation, especially in the case of the time-ordered (Feynman) propagator. The second approach is to make small distortions in the contour away from the real p_0 axis near the poles. This is used by Tong in his derivation of the Feynman propagator (sec.2.7.1 page 38) though Tong reverts to the first and standard notation later on (see Tong equation (3.37)). Both methods are equivalent in the $\epsilon \to 0^+$ limit.

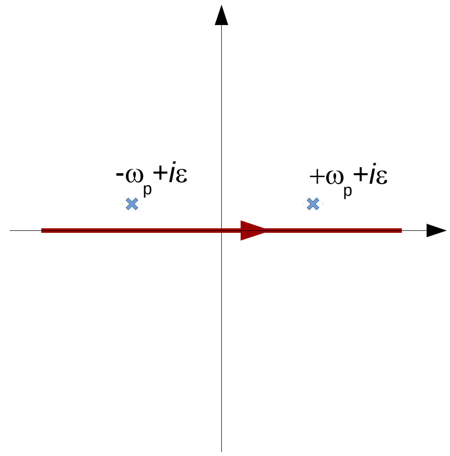
First approach

Consider

$$\begin{aligned}
I_1(x) &= \int \frac{d^4p}{(2\pi)^4}\frac{i}{(p_0-i\epsilon)^2-\boldsymbol{p}^2-m^2}\,e^{-ip\cdot x} && (106)\\
&= -\int \frac{d^3p}{(2\pi)^3}\int_{-\infty}^{\infty}\frac{dp^0}{2\pi i}\frac{1}{(p^0-\omega_{\boldsymbol{p}}-i\epsilon)(p^0+\omega_{\boldsymbol{p}}-i\epsilon)}\,e^{-ip\cdot x} && (107)
\end{aligned}$$

[6] For the $\boldsymbol{p}\leftrightarrow-\boldsymbol{p}$ in second term don't forget that in changing variable in each momentum component $\int_{-\infty}^{+\infty} dp_i$ means the range of integration changes from $-\infty$ to $+\infty$ to the other way round giving another minus sign.

where, as usual, $\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$. The p_0 integration is along the real axis with poles in the integrand as shown here



(108)

The $\mathrm{d}p^0$ integrand

$$f(p^0, \boldsymbol{p}) = \frac{1}{(p^0 - \omega_{\boldsymbol{p}} - i\epsilon)(p^0 + \omega_{\boldsymbol{p}} - i\epsilon)} \mathrm{e}^{-ip\cdot x} \tag{109}$$

has simple poles at

$$p^0 = \pm\omega_{\boldsymbol{p}} + i\epsilon. \tag{110}$$

Near these poles the integrand looks like $f \approx R_\pm/(p^0 \mp \omega_{\boldsymbol{p}} + i\epsilon)$ with residues $R_\pm$ given by

$$R_\pm = \pm\frac{1}{2\omega_{\boldsymbol{p}}} \, \mathrm{e}^{-ip\cdot x}\Big|_{p^0=\pm\omega_{\boldsymbol{p}}+i\epsilon} \tag{111}$$

The idea is that we think of our expression for the advanced propagator in (105) as being of the form

$$D_A(x) = -\theta(-x^0) \int \frac{\mathrm{d}^3p}{(2\pi)^3} \left(R_+ + R_-\right) . \tag{112}$$

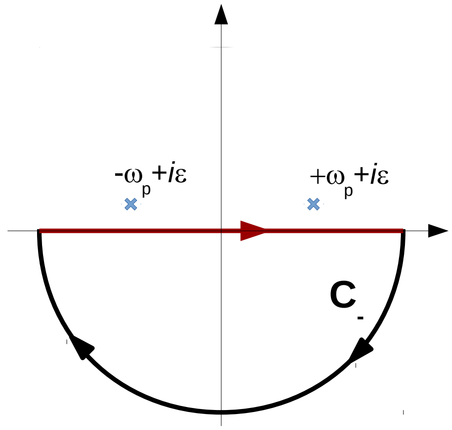
In order for this to match $I_1(x)$ of (107) we need to find a closed contour C such that by using the residue theorem we can deduce that

$$\int_C \frac{\mathrm{d}p^0}{2\pi i} f(p^0, \boldsymbol{p}) = \theta(-x^0) \left(R^+ + R^-\right) \tag{113}$$

If $x^0 > 0$ then $\mathrm{e}^{-ip^0x^0} \to 0$ as $\Im(p^0) = -i\infty$. This means that an integration of this integrand f round a large semi-circle running around the lower half plane is equal to zero

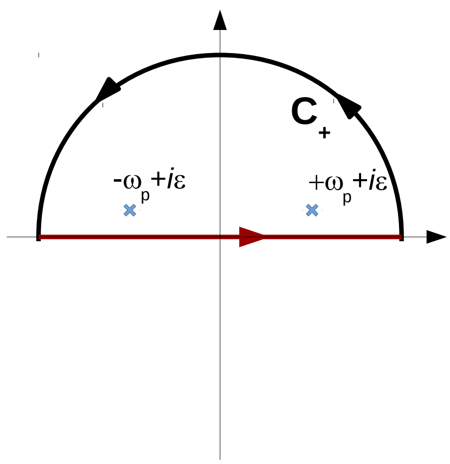
$$\int_{C_-} \frac{\mathrm{d}p^0}{2\pi i} f(p^0, \boldsymbol{p}) = 0 \quad \text{if} x_0 > 0 \text{ as } \Im(p^0) \to -i\infty\,. \tag{114}$$

We can therefore add this integration of f around the C_- semi-circle to our p_0 integration along the real axis in I_1 without changing the result for I_1. So we produce an expression for I_1 which uses a closed contour for the p_0 integration by adding this lower semi-circle. Now no poles are enclosed within this closed contour so the residue theorem tells us the result is zero



$$\int_C \frac{\mathrm{d}p^0}{2\pi i} f(p^0, \boldsymbol{p}) = 0 \qquad \text{if } x^0 > 0 \,. \tag{115}$$

If $x^0 < 0$ then $\mathrm{e}^{-ip^0 x^0} \to 0$ as $\Im(p^0) = +i\infty$. This means that an integration of this integrand f round a large semi-circle running around the upper half plane will give zero. We can therefore add this to our existing p_0 integration along the real axis in I_1 without changing the result. So we produce a closed contour by adding the semi-circle above and now the residue theorem tells us that we pick up contributions from both poles. This gives us



$$\int_C \frac{\mathrm{d}p^0}{2\pi i} f(p^0, \boldsymbol{p}) = R_+ + R_- \qquad \text{if } x^0 < 0 \,. \tag{116}$$

Putting the two cases together gives us the desired result

$$\int_C \frac{\mathrm{d}p^0}{2\pi i} f(p^0, \boldsymbol{p}) = \theta(-x^0) \left(R^+ + R^-\right) \,. \tag{117}$$

Alternative approach

The second approach to these types of problem is to distort the contour away from the real p_0 axis near the poles by a small amount. So now consider

$$I(x) = \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \int_C \frac{\mathrm{d}p^0}{2\pi} \frac{i}{p^2 - m^2} \mathrm{e}^{-ip\cdot x} \tag{118}$$

$$= -\int \frac{\mathrm{d}^3 p}{(2\pi)^3} \int_C \frac{\mathrm{d}p^0}{2\pi i} \frac{1}{(p^0 - \omega_{\boldsymbol{p}})(p^0 + \omega_{\boldsymbol{p}})} \mathrm{e}^{-ip\cdot x} \tag{119}$$

where, as usual, $\omega_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2}$. The $\mathrm{d}p^0$ integrand

$$f(p^0, \boldsymbol{p}) = \frac{1}{(p^0 - \omega_{\boldsymbol{p}})(p^0 + \omega_{\boldsymbol{p}})} \mathrm{e}^{-ip\cdot x} \tag{120}$$

has simple poles at

$$p^0 = \pm\omega_{\boldsymbol{p}} \tag{121}$$

with residues $f \approx R_\pm/(p^0 \mp \omega_{\boldsymbol{p}})$ near $p^0 = \pm\omega_{\boldsymbol{p}}$ given by

$$R_\pm = \pm\frac{1}{2\omega_{\boldsymbol{p}}}\, \mathrm{e}^{-ip\cdot x}\big|_{p^0=\pm\omega_{\boldsymbol{p}}} \tag{122}$$

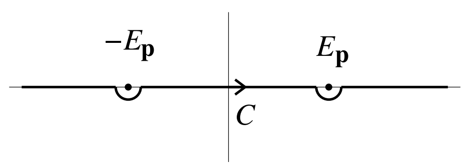
Observing that we can rewrite

$$D_A(x) = -\theta(-x^0)\int \frac{\mathrm{d}^3p}{(2\pi)^3}\,(R_+ + R_-) \tag{123}$$

in order for this to match $I(x)$ we need to find a contour C such that

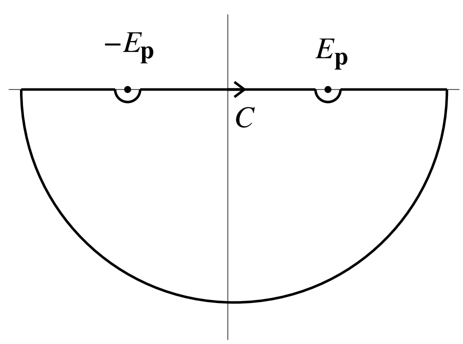
$$\int_C \frac{\mathrm{d}p^0}{2\pi i} f(p^0,\boldsymbol{p}) = \theta(-x^0)\left(R^+ + R^-\right) \tag{124}$$

Consider the following contour



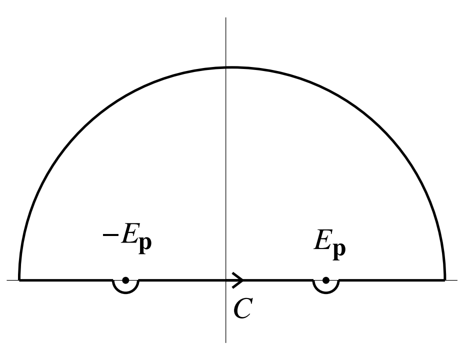
(125)

If $x^0 > 0$ then $\mathrm{e}^{-ip^0x^0} \to 0$ as $\Im(p^0) = -i\infty$ and we close the contour below and pick up no poles



$$\int_C \frac{\mathrm{d}p^0}{2\pi i} f(p^0,\boldsymbol{p}) = 0 \tag{126}$$

If $x^0 < 0$ then $\mathrm{e}^{-ip^0x^0} \to 0$ as $\Im(p^0) = +i\infty$ and we close the contour below and picking up both poles



$$\int_C \frac{\mathrm{d}p^0}{2\pi i} f(p^0,\boldsymbol{p}) = R_+ + R_- \tag{127}$$

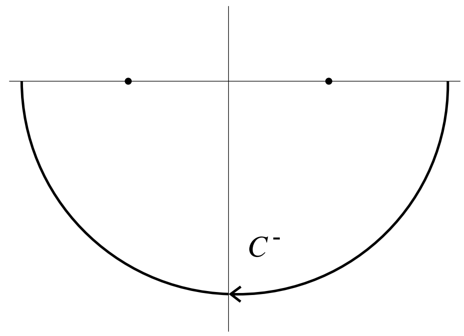
so that indeed

$$\int_C \frac{\mathrm{d}p^0}{2\pi i} f(p^0,\boldsymbol{p}) = \theta(-x^0)\left(R^+ + R^-\right) \tag{128}$$

To give a bit more detail on how we close the contour, consider $x^0 > 0$. We have (remember p^0 is complex)

$$\left| \frac{1}{(p^0)^2 - \omega_{\boldsymbol{p}}^2} e^{-ip^0 x^0} \right| = \frac{e^{\operatorname{Im} p^0 x^0}}{\left|(p^0)^2 - \omega_{\boldsymbol{p}}^2\right|} \leq \frac{e^{\operatorname{Im} p^0 x^0}}{|p^0|^2 - \omega_{\boldsymbol{p}}^2} \tag{129}$$

where the last step comes from writing $|p^0|^2 = |(p^0)^2 - \omega_{\boldsymbol{p}}^2 + \omega_{\boldsymbol{p}}^2| \leq |(p^0)^2 - \omega_{\boldsymbol{p}}^2| + |\omega_{\boldsymbol{p}}^2|$ using the triangle inequality. Closing the integral below we have $\operatorname{Im} p^0 \leq 0$ so that $e^{\operatorname{Im} p^0 x^0} \leq 1$. Thus, evaluating first at finite $|p^0|$ we have, for the infinite semi-circular path C_- on the lower half plane



(130)

we have

$$\begin{aligned}
\left| \int_{C^-} dp^0 \frac{1}{(p^0)^2 - \omega_{\boldsymbol{p}}^2} e^{-ip^0 x^0} \right| &\leq \int_{C^-} dp^0 \left| \frac{1}{(p^0)^2 - \omega_{\boldsymbol{p}}^2} e^{-ip^0 x^0} \right| && (131)\\
&\leq \int_{C^-} dp^0 \frac{e^{\operatorname{Im} p^0 x^0}}{|p^0|^2 - \omega_{\boldsymbol{p}}^2} && (132)\\
&\leq \int_{C^-} dp^0 \frac{1}{|p^0|^2 - \omega_{\boldsymbol{p}}^2} && (133)\\
&= \frac{\pi |p^0|}{|p^0|^2 - \omega_{\boldsymbol{p}}^2} \qquad \text{(integrate on } C^- \text{ with } dp^0 = |p^0| d\theta) && (134)\\
&\to 0 \qquad \text{as } p^0 \to \infty && (135)
\end{aligned}$$

Hence we see that the contribution from the integration along C^- is zero and we can close the contour C along C^- in order to evaluate $I(x)$ when $x^0 > 0$. A similar argument holds for $x^0 < 0$.

(ii) We have, taking the derivative of $e^{-ip\cdot x}$

$$\begin{aligned}
\left(\partial^2 + m^2\right) D_A(x) &= \int_C \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2} \left(-p^2 + m^2\right) e^{-ip\cdot x} && (136)\\
&= -\int_C \frac{d^4p}{(2\pi)^4} i e^{-ip\cdot x} && (137)\\
&= -i\delta^{(4)}(x) && (138)
\end{aligned}$$

(Note the integrand has no poles on the real axis, so none of the subtleties in the path C relevant when defining $D_A(x)$ appear.)

∗5. Time evolution and propagators of a complex scalar field

(i) Consider two sets of annihilation and creation operators $\hat{a}_1^\dagger$, $\hat{a}_2^\dagger$, a_1 and a_2 obeying

$$\left[\hat{a}_{i\boldsymbol{p}}, \hat{a}^\dagger_{j\boldsymbol{q}}\right] = (2\pi)^3\delta^3((\boldsymbol{p}-\boldsymbol{q}))\delta_{ij}\,, \qquad [\hat{a}_{i\boldsymbol{p}}, \hat{a}_{j\boldsymbol{q}}] = \left[\hat{a}^\dagger_{i\boldsymbol{p}}, \hat{a}^\dagger_{j\boldsymbol{q}}\right] = 0\,, \qquad i,j=1,2\,. \tag{139}$$

We define

$$\hat{b}_{\boldsymbol{p}} = \frac{1}{\sqrt{2}}\left(\hat{a}_{1\boldsymbol{p}} + i\hat{a}_{2\boldsymbol{p}}\right)\,, \qquad \hat{c}_{\boldsymbol{p}} = \frac{1}{\sqrt{2}}\left(\hat{a}_{1\boldsymbol{p}} - i\hat{a}_{2\boldsymbol{p}}\right)\,. \tag{140}$$

so that

$$\hat{b}^\dagger_{\boldsymbol{p}} = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger_{1\boldsymbol{p}} - i\hat{a}^\dagger_{2\boldsymbol{p}}\right)\,, \qquad \hat{c}^\dagger_{\boldsymbol{p}} = \frac{1}{\sqrt{2}}\left(\hat{a}^\dagger_{1\boldsymbol{p}} + i\hat{a}^\dagger_{2\boldsymbol{p}}\right)\,. \tag{141}$$

We require several commutators to be proved. The key identity here is that for any operators A, B, C, D we have

$$[A+B, C+D] = [A,C] + [A,D] + [B,C] + [B,D] \tag{142}$$

which you should prove if this is not familiar.

The commutators between a b or c annihilation and a $\hat{b}$ or $\hat{c}$ creation operator are all of the form

$$\frac{1}{2}\left[\left(\hat{a}_{1\boldsymbol{p}} + is_p\hat{a}_{2\boldsymbol{p}}\right), \left(\hat{a}^\dagger_{1\boldsymbol{q}} + is_q\hat{a}^\dagger_{2\boldsymbol{q}}\right)\right] \tag{143}$$

where $s_p, s_q = \pm 1$. Expanding we have that

$$\begin{aligned}
&\frac{1}{2}\left[\left(\hat{a}_{1\boldsymbol{p}} + is_p\hat{a}_{2\boldsymbol{p}}\right), \left(\hat{a}^\dagger_{1\boldsymbol{q}} + is_q\hat{a}^\dagger_{2\boldsymbol{q}}\right)\right] && (144)\\
&= \frac{1}{2}\left[\hat{a}_{1\boldsymbol{p}}, \hat{a}^\dagger_{1\boldsymbol{q}}\right] + is_q\frac{1}{2}\left[\hat{a}_{1\boldsymbol{p}}, \hat{a}^\dagger_{2\boldsymbol{q}}\right] + is_p\frac{1}{2}\left[\hat{a}_{2\boldsymbol{p}}, \hat{a}^\dagger_{1\boldsymbol{q}}\right] - s_ps_q\frac{1}{2}\left[\hat{a}_{2\boldsymbol{p}}, \hat{a}^\dagger_{2\boldsymbol{q}}\right] && (145)\\
&= \frac{1}{2}\bar{\delta}^3(\boldsymbol{p}-\boldsymbol{q}) + 0 + 0 - s_ps_q\frac{1}{2}\bar{\delta}^3(\boldsymbol{p}-\boldsymbol{q}) && (146)\\
&= \frac{1}{2}(1-s_ps_q)\bar{\delta}^3(\boldsymbol{p}-\boldsymbol{q})\,. && (147)
\end{aligned}$$

So we find

$$\begin{aligned}
[\hat{b}_{\boldsymbol{p}}, \hat{c}^\dagger_{\boldsymbol{q}}] &= 0 && s_p=+1, s_q=+1 && (148)\\
[\hat{b}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{q}}] &= \bar{\delta}^3(\boldsymbol{p}-\boldsymbol{q}) && s_p=+1, s_q=-1 && (149)\\
[\hat{c}_{\boldsymbol{p}}, \hat{c}^\dagger_{\boldsymbol{q}}] &= \bar{\delta}^3(\boldsymbol{p}-\boldsymbol{q}) && s_p=-1, s_q=+1 && (150)\\
[\hat{c}_{\boldsymbol{p}}, \hat{b}^\dagger_{\boldsymbol{q}}] &= 0 && s_p=-1, s_q=-1 && (151)
\end{aligned}$$

Note the last is the hermitian conjugate of the first so you could avoid calculating it for that reason.

The commutators between a pair of b or c annihilation operators are all of the form

$$\frac{1}{2}\left[\left(\hat{a}_{1\boldsymbol{p}} + is_p\hat{a}_{2\boldsymbol{p}}\right), \left(\hat{a}_{1\boldsymbol{q}} + is_q\hat{a}_{2\boldsymbol{q}}\right)\right] \tag{152}$$

where $s_p, s_q = \pm 1$. Since annihilation operators commute with each other even if they are of the same type and at the same momentum, all these commutators are clearly zero. Taking the hermitian conjugate, (or using a similar argument for the creation operators), we see that all the commutators between a pair of $\hat{b}^\dagger$ or $\hat{c}^\dagger$ creation operators are also zero.

(ii) Given the question below for Q we can save time by considering $\hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}} + s\hat{c}^\dagger_{\boldsymbol{p}}\hat{c}_{\boldsymbol{p}}$ where $s = \pm 1$. We then have that

$$\begin{aligned}
\hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}} + s\hat{c}^\dagger_{\boldsymbol{p}}\hat{c}_{\boldsymbol{p}} &= \frac{1}{2}\left(\hat{a}^\dagger_{1\boldsymbol{p}} - i\hat{a}^\dagger_{2\boldsymbol{p}}\right)(\hat{a}_{1\boldsymbol{p}} + i\hat{a}_{2\boldsymbol{p}}) \\
&\quad + \frac{s}{2}\left(\hat{a}^\dagger_{1\boldsymbol{p}} + i\hat{a}^\dagger_{2\boldsymbol{p}}\right)(\hat{a}_{1\boldsymbol{p}} - i\hat{a}_{2\boldsymbol{p}}) && (153) \\
&= \frac{1}{2}\left(\hat{a}^\dagger_{1\boldsymbol{p}}\hat{a}_{1\boldsymbol{p}} + i\hat{a}^\dagger_{1\boldsymbol{p}}\hat{a}_{2\boldsymbol{p}} - i\hat{a}^\dagger_{2\boldsymbol{p}}\hat{a}_{1\boldsymbol{p}} + \hat{a}^\dagger_{2\boldsymbol{p}}\hat{a}_{2\boldsymbol{p}}\right) \\
&\quad + \frac{s}{2}\left(\hat{a}^\dagger_{1\boldsymbol{p}}\hat{a}_{1\boldsymbol{p}} - i\hat{a}^\dagger_{1\boldsymbol{p}}\hat{a}_{2\boldsymbol{p}} + i\hat{a}^\dagger_{2\boldsymbol{p}}\hat{a}_{1\boldsymbol{p}} + \hat{a}^\dagger_{2\boldsymbol{p}}\hat{a}_{2\boldsymbol{p}}\right) && (154) \\
&= \frac{1}{2}\left((1+s)\left(\hat{a}^\dagger_{1\boldsymbol{p}}\hat{a}_{1\boldsymbol{p}} + \hat{a}^\dagger_{2\boldsymbol{p}}\hat{a}_{2\boldsymbol{p}}\right) + i(1-s)\left(\hat{a}^\dagger_{1\boldsymbol{p}}\hat{a}_{2\boldsymbol{p}} - \hat{a}^\dagger_{2\boldsymbol{p}}\hat{a}_{1\boldsymbol{p}}\right)\right) && (155)
\end{aligned}$$

When $s = +1$ we have the desired result that

$$\hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}} + \hat{c}^\dagger_{\boldsymbol{p}}\hat{c}_{\boldsymbol{p}} = \hat{a}^\dagger_{1\boldsymbol{p}}\hat{a}_{1\boldsymbol{p}} + \hat{a}^\dagger_{2\boldsymbol{p}}\hat{a}_{2\boldsymbol{p}}\,. \tag{156}$$

From this we see that (here $Z_{\boldsymbol{p}}$ is the zero point energy for mode $\boldsymbol{p}$)

$$\begin{aligned}
\widehat{H} &= \sum_{i=1,2}\int đ^3\boldsymbol{p}\,\omega_{\boldsymbol{p}}\left(\hat{a}^\dagger_{i\boldsymbol{p}}\hat{a}_{i\boldsymbol{p}} + Z_{\boldsymbol{p}}\right) && (157) \\
&= \int đ^3\boldsymbol{p}\,\omega_{\boldsymbol{p}}\left(\hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}} + \hat{c}^\dagger_{\boldsymbol{p}}\hat{c}_{\boldsymbol{p}} + 2Z_{\boldsymbol{p}}\right) && (158)
\end{aligned}$$

(iii) With the Hamiltonian given by (158), to show that

$$\hat{\Phi}_{\rm H}(t,\boldsymbol{x}) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}(\hat{b}_{\boldsymbol{p}}e^{-ipx} + \hat{c}^\dagger_{\boldsymbol{p}}e^{+ipx}) \tag{159}$$

given the form at $t - 0$ (where $px \equiv p_\mu x^\mu$ and $p_0 = +\omega_p$) we can use the same method as used for the real field. That is since like all operators[7]

$$\hat{\Phi}_{\rm H}(t,\boldsymbol{x}) = \exp\{i\widehat{H}t\}\hat{\Phi}_{\rm H}(t=0,\boldsymbol{x})\exp\{-i\widehat{H}t\} \tag{160}$$

all we need to do is look at the behaviour of

$$\exp\{i\widehat{H}t\}\hat{b}_{\boldsymbol{p}}\exp\{-i\widehat{H}t\} \qquad \text{and} \qquad \exp\{i\widehat{H}t\}\hat{c}^\dagger_{\boldsymbol{p}}\exp\{-i\widehat{H}t\}\,. \tag{161}$$

As the c and b operators commute, the only part that matter that matters is the $\hat{b}^\dagger\hat{b}$ term in the first case and the $\hat{c}^\dagger c$ term in the second case. More formally we can commute one part of the Hamitonian through the $\hat{b}_{\boldsymbol{p}}$ or the $\hat{c}^\dagger_{\boldsymbol{p}}$ operator to be cancelled. The problem reduces to that of a single type of operator, i.e. we are left with

$$\exp\{i\widehat{H}_bt\}\hat{b}_{\boldsymbol{p}}\exp\{-i\widehat{H}_bt\} \qquad \text{and} \qquad \exp\{i\widehat{H}_ct\}\hat{c}^\dagger_{\boldsymbol{p}}\exp\{-i\widehat{H}_ct\}\,, \tag{162}$$

where

$$\widehat{H}_b = \int đ^3\boldsymbol{p}\,\omega_{\boldsymbol{p}}\hat{b}^\dagger_{\boldsymbol{p}}\hat{b}_{\boldsymbol{p}}\,, \qquad \widehat{H}_c = \int đ^3\boldsymbol{p}\,\omega_{\boldsymbol{p}}\hat{c}^\dagger_{\boldsymbol{p}}\hat{c}_{\boldsymbol{p}}\,. \tag{163}$$

We have already derived these so from (77) we know that

$$\exp\{i\widehat{H}_bt\}\hat{b}_{\boldsymbol{p}}\exp\{-i\widehat{H}_bt\} = \exp\{-i\omega_{\boldsymbol{p}}t\} \qquad \text{and} \qquad \exp\{i\widehat{H}_ct\}\hat{c}^\dagger_{\boldsymbol{p}}\exp\{-i\widehat{H}_ct\} = \exp\{+i\omega_{\boldsymbol{p}}t\}\,. \tag{164}$$

and we find the desired time evolution for the free complex scalar field in the Heisenberg picture.

[7] As this is a free Hamiltonian it is the same in Heisenberg and Scroödinger pictures and so it needs no subscript.

(iv) (No answer for this part so far).

Consider two real[8] scalar fields

$$\hat{\phi}_{\mathrm{H}i}(t,\boldsymbol{x}) = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(\hat{a}_{i\boldsymbol{p}}\mathrm{e}^{-i\omega_{\boldsymbol{p}}t+i\boldsymbol{p}\cdot\boldsymbol{x}} + \hat{a}^{\dagger}_{i\boldsymbol{p}}\mathrm{e}^{i\omega_{\boldsymbol{p}}t-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \tag{165}$$

where $\hat{a}_{i\boldsymbol{p}}$ and $\hat{a}^{\dagger}_{i\boldsymbol{p}}$ are the operators given in (139). Show that if (**??**) is true then the field operators obey

$$\hat{\Phi}_{\mathrm{H}}(t,\boldsymbol{x}) = \frac{1}{\sqrt{2}}\left(\hat{\phi}_{\mathrm{H}1}(t,\boldsymbol{x}) + i\hat{\phi}_{\mathrm{H}2}(t,\boldsymbol{x})\right), \qquad \hat{\Phi}^{\dagger}_{\mathrm{H}}(t,\boldsymbol{x}) = \frac{1}{\sqrt{2}}\left(\hat{\phi}_{\mathrm{H}1}(t,\boldsymbol{x}) - i\hat{\phi}_{\mathrm{H}2}(t,\boldsymbol{x})\right). \tag{166}$$

(v) The Wightman function for a complex scalar field is defined as

$$D(x-y) := \langle 0|\hat{\Phi}(x)\hat{\Phi}^{\dagger}(y)|\,0\rangle\,. \tag{167}$$

Substituting (159) into (167) give us

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{q}}}}\langle 0|(\hat{b}_{\boldsymbol{p}}e^{-ipx} + \hat{c}^{\dagger}_{\boldsymbol{p}}e^{+ipx})(\hat{b}^{\dagger}_{\boldsymbol{q}}e^{+iqy} + \hat{c}_{\boldsymbol{q}}e^{-iqy})|\,0\rangle \tag{168}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{q}}}}\langle 0|\hat{b}_{\boldsymbol{p}}\hat{b}^{\dagger}_{\boldsymbol{q}}|\,0\rangle e^{-ipx+iqy} \tag{169}$$

as the c ($\hat{c}^{\dagger}$) operator annihilates the ket (bra) vacuum. Commuting the two b operators gives a non-zero term containing a delta function in momentum (which will kill one integral) plus a second term $\hat{b}^{\dagger}_{\boldsymbol{q}}\hat{b}_{\boldsymbol{p}}$ which again annihilates the vacuum states so does not contribute. This leaves us with

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}e^{-ip(x-y)} \tag{170}$$

exactly we found before for real scalar fields.

Note that from the definition we have that for any type of field

$$(D(x-y))^* := \langle 0|\hat{\Phi}(y)\hat{\Phi}^{\dagger}(x)|\,0\rangle = D(y-x)\,. \tag{171}$$

For the free field case in (170) we can see this explicitly.

(vi) For free complex scalar fields by substituting (159) into $\langle 0|\hat{\Phi}(x)\hat{\Phi}(y)|\,0\rangle$ gives us

$$\langle 0|\hat{\Phi}(x)\hat{\Phi}(y)|\,0\rangle = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{q}}}}\langle 0|(\hat{b}_{\boldsymbol{p}}e^{-ipx} + \hat{c}^{\dagger}_{\boldsymbol{p}}e^{+ipx})(\hat{b}_{\boldsymbol{q}}e^{-iqy} + \hat{c}^{\dagger}_{\boldsymbol{q}}e^{+iqy})|\,0\rangle \tag{172}$$

$$= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{q}}}}\langle 0|\hat{b}_{\boldsymbol{p}}\hat{c}^{\dagger}_{\boldsymbol{q}}|\,0\rangle e^{-ipx+iqy} \tag{173}$$

as the b ($\hat{c}^{\dagger}$) operator annihilates the ket (bra) vacuum. However the two remaining operators commute and again they then annihilate the vacuum so this case is zero for any time.

[8]These represent classical real scalar fields but their quantised versions are hermitian not real.

(vii) The Feynman propagator for complex scalar fields is defined as

$$\Delta_{\rm F}(x-y) := \langle 0|{\rm T}\hat{\Phi}(x)\hat{\Phi}^\dagger(y)|\,0\rangle \tag{174}$$

where T is the time ordering operator. We can express $\Delta_{\rm F}$ in terms of the Wightman functions for the complex field (167)

$$\Delta_{\rm F}(x-y) \;=\; \theta(x_0-y_0)\langle 0|\hat{\Phi}(x)\hat{\Phi}^\dagger(y)|\,0\rangle + \theta(y_0-x_0)\langle 0|\hat{\Phi}^\dagger(y)\hat{\Phi}(x)|\,0\rangle \tag{175}$$

Following the same steps as above we can show that

$$\begin{aligned}
\langle 0|\hat{\Phi}^\dagger(y)\hat{\Phi}(x)|\,0\rangle &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{q}}}}\langle 0|(\hat{b}^\dagger_{\boldsymbol{q}}e^{+iqy}+\hat{c}_{\boldsymbol{q}}e^{-iqy})(\hat{b}_{\boldsymbol{p}}e^{-ipx}+\hat{c}^\dagger_{\boldsymbol{p}}e^{+ipx})|\,0\rangle && (176)\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{q}}}}\langle 0|\hat{c}_{\boldsymbol{q}}\hat{c}^\dagger_{\boldsymbol{p}}|\,0\rangle e^{-iqy+ipx} && (177)\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{q}}}}(2\pi)^3\delta^3(\boldsymbol{q}-\boldsymbol{p})e^{-iqy+ipx} && (178)\\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega_{\boldsymbol{p}}}e^{-ip(y-x)} && (179)
\end{aligned}$$

By comparing with (170) we see that

$$\langle 0|\hat{\Phi}^\dagger(y)\hat{\Phi}(x)|\,0\rangle \;=\; D(y-x)\,. \tag{180}$$

This means that the Feynman propagator for complex scalar fields is

$$\Delta_{\rm F}(x-y) = \langle 0|{\rm T}\hat{\Phi}(x)\hat{\Phi}^\dagger(y)|\,0\rangle \;=\; \theta(x_0-y_0)D(x-y)+\theta(y_0-x_0)D(y-x)\,. \tag{181}$$

This is *exactly* the same form as we had for the real free field scalar propagator so the Feynman propagator is of exactly the same form i.e. in momentum space it is equal to

$$\Delta_{\rm F}(p) = \frac{i}{p^2-m^2+i\epsilon}\,. \tag{182}$$

Clearly these are also solutions of the Klein-Gordon equation as they were in the real free scalar field case. Again the poles at $p^2=m^2$ indicate that we have particle-like solutions of mass m dominating the behaviour. The presences of two distinct types of annihliation/creation operator in this complex field indicates there are two independent degrees of freedom with the same mass, here distinct particle and anti-particles.

Note that $\langle 0|{\rm T}\hat{\Phi}^\dagger(y)\hat{\Phi}(x)|\,0\rangle = \langle 0|{\rm T}\hat{\Phi}(x)\hat{\Phi}^\dagger(y)|\,0\rangle$ if the times are unequal because the T operator defines the order so how we write the fields under the T operator has no effect.

6. Charge of a complex scalar field

We have by definition

$$Q = i\int \mathrm{d}^3x \left(\Phi^\dagger\Pi^\dagger - \Pi\Phi\right) \tag{183}$$

(i) We have

$$\begin{aligned} i\int \mathrm{d}^3x\,\Pi(\boldsymbol{x})\Phi(\boldsymbol{x}) &= \int \mathrm{d}^3x \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\sqrt{\frac{\omega_{\boldsymbol{p}}}{4\omega_{\boldsymbol{q}}}}\left(c_{\boldsymbol{p}}\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}} - b^\dagger_{\boldsymbol{p}}\mathrm{e}^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right)\left(b_{\boldsymbol{q}}\mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{x}} + c^\dagger_{\boldsymbol{q}}\mathrm{e}^{-i\boldsymbol{q}\cdot\boldsymbol{x}}\right) & (184)\\ &= \int \mathrm{d}^3x \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\sqrt{\omega_{\boldsymbol{p}}}}{\sqrt{4\omega_{\boldsymbol{q}}}}\\ &\quad\times\left\{\mathrm{e}^{i(\boldsymbol{p}+\boldsymbol{q})\cdot\boldsymbol{x}}c_{\boldsymbol{p}}b_{\boldsymbol{q}} + \mathrm{e}^{i(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{x}}c_{\boldsymbol{p}}c^\dagger_{\boldsymbol{q}} - \mathrm{e}^{-i(\boldsymbol{p}-\boldsymbol{q})\cdot\boldsymbol{x}}b^\dagger_{\boldsymbol{p}}b_{\boldsymbol{q}} - \mathrm{e}^{-i(\boldsymbol{p}+\boldsymbol{q})\cdot\boldsymbol{x}}b^\dagger_{\boldsymbol{p}}c^\dagger_{\boldsymbol{q}}\right\} & (185)\\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{\sqrt{\omega_{\boldsymbol{p}}}}{\sqrt{4\omega_{\boldsymbol{q}}}}\\ &\quad\times\left\{(2\pi)^3\delta^{(3)}(\boldsymbol{p}+\boldsymbol{q})\left(c_{\boldsymbol{p}}b_{\boldsymbol{q}} - c^\dagger_{\boldsymbol{p}}b^\dagger_{\boldsymbol{q}}\right) + (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\left(c_{\boldsymbol{p}}c^\dagger_{\boldsymbol{q}} - b^\dagger_{\boldsymbol{p}}b_{\boldsymbol{q}}\right)\right\} & (186)\\ &= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(c_{\boldsymbol{p}}b_{-\boldsymbol{p}} - c^\dagger_{\boldsymbol{p}}b^\dagger_{-\boldsymbol{p}} + c_{\boldsymbol{p}}c^\dagger_{\boldsymbol{p}} - b^\dagger_{\boldsymbol{p}}b_{\boldsymbol{p}}\right) & (187)\end{aligned}$$

Now

$$\begin{aligned} i\int \mathrm{d}^3x\,\Phi^\dagger(\boldsymbol{x})\Pi^\dagger(\boldsymbol{x}) &= -\left(i\int \mathrm{d}^3x\,\Pi(\boldsymbol{x})\Phi(\boldsymbol{x})\right)^\dagger & (188)\\ &= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(c_{-\boldsymbol{p}}b_{\boldsymbol{p}} - c^\dagger_{-\boldsymbol{p}}b^\dagger_{\boldsymbol{p}} - c_{\boldsymbol{p}}c^\dagger_{\boldsymbol{p}} + b^\dagger_{\boldsymbol{p}}b_{\boldsymbol{p}}\right) & (189)\end{aligned}$$

and hence

$$\begin{aligned} Q &= \frac{1}{2}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(c_{-\boldsymbol{p}}b_{\boldsymbol{p}} - c_{\boldsymbol{p}}b_{-\boldsymbol{p}} + c^\dagger_{\boldsymbol{p}}b^\dagger_{-\boldsymbol{p}} - c^\dagger_{-\boldsymbol{p}}b^\dagger_{\boldsymbol{p}} + 2b^\dagger_{\boldsymbol{p}}b_{\boldsymbol{p}} - 2c_{\boldsymbol{p}}c^\dagger_{\boldsymbol{p}}\right) & (190)\\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(b^\dagger_{\boldsymbol{p}}b_{\boldsymbol{p}} - c_{\boldsymbol{p}}c^\dagger_{\boldsymbol{p}}\right) & (191)\\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(b^\dagger_{\boldsymbol{p}}b_{\boldsymbol{p}} - c^\dagger_{\boldsymbol{p}}c_{\boldsymbol{p}} - (2\pi)^3\delta^{(3)}(0)\right) & (192)\\ &= \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(b^\dagger_{\boldsymbol{p}}b_{\boldsymbol{p}} - c^\dagger_{\boldsymbol{p}}c_{\boldsymbol{p}}\right) + \text{infinite constant} & (193)\end{aligned}$$

where we have used the fact that odd functions integrate to zero.

(ii) Substitute in the form of the fields

$$\begin{aligned} \Phi(\boldsymbol{x}) &= \int \frac{d^3p}{(2\Pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}(b_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} + c^\dagger_{\boldsymbol{p}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}) & (194)\\ \Pi(\boldsymbol{x}) &= \int \frac{d^3p}{(2\Pi)^3}(-i)\sqrt{\frac{\omega_{\boldsymbol{p}}}{2}}(c_{\boldsymbol{p}}e^{i\boldsymbol{p}\cdot\boldsymbol{x}} - b^\dagger_{\boldsymbol{p}}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}) & (195)\end{aligned}$$

and hence confirm that the ETCR (equal time commutation relations) for complex scalar fields are [Tong (2.71), p.34]

$$[\Phi(x),\Pi(y)]_{x_0=y_0} = (2\pi)^3\delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \qquad \left[\Phi^\dagger(x),\Pi^\dagger(y)\right]_{x_0=y_0} = (2\pi)^3\delta^3(\boldsymbol{x}-\boldsymbol{y})\,, \tag{196}$$

along with the other zero commutators of the ETCR:

$$0 = [\Phi(x), \Phi(y)]_{x_0=y_0} = \left[\Phi(x), \Phi^\dagger(y)\right]_{x_0=y_0} \tag{197}$$

$$= [\Pi(x), \Pi(y)]_{x_0=y_0} = \left[\Pi(x), \Pi^\dagger(y)\right]_{x_0=y_0} \tag{198}$$

$$= \left[\Phi(x), \Pi^\dagger(y)\right]_{x_0=y_0} = 0\,, \tag{199}$$

and so forth.

Note that the ETCR for the complex fields should match what we found for the real field case exactly. These ETCR are true for ANY field. Since we are meant to treat Φ and $(\Phi)^\dagger$ as separate fields, they should each obey exactly the same equal time commutation relations. It is trivial to check the as we can take the hermitian conjugate of $[\Phi, \Pi]$ and should get the $[\Phi^\dagger, \Pi^\dagger]$ for free and it should look the same. The factor of i is critical for that. This is really $i\hbar\delta^3(x-y)$ but with $\hbar = 1$, the same factor as seen in the QM $[q, p] = i\hbar$ commutator.

Then we can prove the charge-field commutator by calculating (not all operators are at equal times)

$$[Q, \Phi(\boldsymbol{x})] = i\int d^3y \left[\Phi^\dagger(\boldsymbol{y})\Pi^\dagger(\boldsymbol{y}) - \Phi(\boldsymbol{y})\Pi(\boldsymbol{y}), \Phi(\boldsymbol{x})\right] \tag{200}$$

$$= -i\int d^3y\, [\Phi(\boldsymbol{y})\Pi(\boldsymbol{y}), \Phi(\boldsymbol{x})] \tag{201}$$

$$= -i\int d^3y \left(-i\Phi(\boldsymbol{y})\delta^3(\boldsymbol{x}-\boldsymbol{y})\right) \tag{202}$$

$$= -\Phi(\boldsymbol{x}) \tag{203}$$

Alternatively, we can prove this as follows:

$$[Q, \Phi(\boldsymbol{x})] = \int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{q}}}}\left(\left[c^\dagger_{\boldsymbol{p}}b_{\boldsymbol{p}} - b^\dagger_{\boldsymbol{p}}c_{\boldsymbol{p}}, b_{\boldsymbol{q}}\right]\mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{x}} + \left[c^\dagger_{\boldsymbol{p}}b_{\boldsymbol{p}} - b^\dagger_{\boldsymbol{p}}c_{\boldsymbol{p}}, c^\dagger_{\boldsymbol{q}}\right]\mathrm{e}^{-i\boldsymbol{q}\cdot\boldsymbol{x}}\right) \tag{204}$$

$$= -\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{\mathrm{d}^3q}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{q}}}}(2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})\left(b_{\boldsymbol{p}}\mathrm{e}^{i\boldsymbol{q}\cdot\boldsymbol{x}} + b^\dagger_{\boldsymbol{p}}\mathrm{e}^{-i\boldsymbol{q}\cdot\boldsymbol{x}}\right) \tag{205}$$

$$= -\int \frac{\mathrm{d}^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(b_{\boldsymbol{p}}\mathrm{e}^{i\boldsymbol{p}\cdot\boldsymbol{x}} + b^\dagger_{\boldsymbol{p}}\mathrm{e}^{-i\boldsymbol{p}\cdot\boldsymbol{x}}\right) \tag{206}$$

$$= -\Phi(\boldsymbol{x}) \tag{207}$$

where we have used $[ab, c] = a\,[b, c] + [a, c]\,b$ and hence

$$\left[b^\dagger_{\boldsymbol{p}}b_{\boldsymbol{p}}, b_{\boldsymbol{q}}\right] = \left[c^\dagger_{\boldsymbol{p}}, b_{\boldsymbol{q}}\right]b_{\boldsymbol{p}} = -(2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})b_{\boldsymbol{p}} \tag{208}$$

$$\left[b^\dagger_{\boldsymbol{p}}b_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{q}}\right] = 0 \tag{209}$$

$$\left[c^\dagger_{\boldsymbol{p}}c_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{q}}\right] = b^\dagger_{\boldsymbol{p}}\left[c_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\delta^{(3)}(\boldsymbol{p}-\boldsymbol{q})b^\dagger_{\boldsymbol{p}} \tag{210}$$

$$\left[c^\dagger_{\boldsymbol{p}}c_{\boldsymbol{p}}, b_{\boldsymbol{q}}\right] = 0 \tag{211}$$

(iii) We have that $[Q, \eta(\boldsymbol{x})] = q\eta(\boldsymbol{x})$ for some operator η with c-number charge q. Consider

$$\eta \to \eta' = e^{i\theta Q}\eta e^{-i\theta Q} \tag{212}$$

We proceed exactly as we did for this type of transformation for the Hamiltonian and annihilation operator. In fact as all we used was the commutation relation, a simple substitution into the answer given above is sufficient to convince that indeed $\eta' = e^{i\theta q}\eta$.

However let us illustrate this with a different proof.

$$\frac{d\eta'}{d\theta} = iQe^{i\theta Q}.\eta.e^{-i\theta Q} + Qe^{i\theta Q}.\eta.(-iQe^{-i\theta Q}) \tag{213}$$

$$= ie^{i\theta Q}.Q\eta.e^{-i\theta Q} - ie^{i\theta Q}.\eta Q.(e^{-i\theta Q}) \tag{214}$$

$$= ie^{i\theta Q}.[Q,\eta].e^{-i\theta Q} = ie^{i\theta Q}.qQ.e^{-i\theta Q} = iqe^{i\theta Q}Q.e^{-i\theta Q} \tag{215}$$

$$\Rightarrow \quad \frac{d\eta'}{d\theta} = iqQ \tag{216}$$

Integrating this and using the fact that if $\theta = 0$ we have $\eta = \eta'$ as a boundary condition, we see that we get our answer $\eta' = e^{i\theta q}\eta$. If integration of an operator equation worries you, note that if you substitute the solution into the differential equation, the operators are not involved in the differentiations, just the c-number parts with which you are familiar.