Problem Sheet 5: Generic Interacting Quantum Field Theory

Comments on these questions are always welcome. For instance if you spot any typos or feel the wording is unclear, drop me an email at `T.Evans` at the usual Imperial address.

Note: problems marked with a $*$ are the most important to do and are core parts of the course. Those without any mark are recommended. It is likely that the exam will draw heavily on material covered in these two types of question. Problems marked with a ! are harder and/or longer. Problems marked with a $\sharp$ are optional. For the exam it will be assumed that material covered in these optional $\sharp$ questions has not been seen before and such optional material is unlikely to be used in an exam.

Here operators are written without 'hats' so you will need to deduce what is an operator from the context. Unless otherwise stated, operators and states are in the interaction picture.

$*$1. The Interaction Picture Evolution Operator $U(t_2, t_1)$

The relationship between the Schrödinger and Interaction pictures is given by

$$|\psi, t\rangle_{\mathrm{I}} = \exp\{+iH_{0,\mathrm{S}}t\}\,|\psi, t\rangle_{\mathrm{S}}\,, \tag{1}$$

$$\mathcal{O}_{\mathrm{I}}(t) = \exp\{+iH_{0,\mathrm{S}}t\}\,\mathcal{O}_{\mathrm{S}}\exp\{-iH_{0,\mathrm{S}}t\}\,. \tag{2}$$

The evolution of interaction picture states is found through the interaction picture evolution operator $U(t_2, t_1)$ where

$$|\psi, t_2\rangle_{\mathrm{I}} = U(t_2, t_1)\,|\psi, t_1\rangle_{\mathrm{I}}\,. \tag{3}$$

(i) Show that the U operator satisfies the following conditions:

(a) $U(t, t) = 1$

(b) $U(t_2, t_1)U(t_1, t_0) = U(t_2, t_0)$

(c) It is unitary, i.e. $[U(t_2, t_1)]^\dagger\, U(t_2, t_1) = 1$

(d) $[U(t_2, t_1)]^\dagger = U(t_1, t_2)$ (note the order of arguments).

(ii) Starting from the Schrödinger equation

$$i\frac{d}{dt}|\psi, t\rangle_{\mathrm{S}} = H_{\mathrm{S}}\,|\psi, t\rangle_{\mathrm{S}} \tag{4}$$

show that

$$|\psi, t\rangle_{\mathrm{I}} = \exp\{+iH_{0,\mathrm{S}}t\}\exp\{-iH_{\mathrm{S}}t\}\,|\psi, t=0\rangle_{\mathrm{S}}\,. \tag{5}$$

Starting from (10) we might guess that

$$\text{(This is wrong!)} \qquad |\psi, t\rangle_{\mathrm{I}} = \exp\{-iH_{\mathrm{int},\mathrm{S}}t\}\,|\psi, t=0\rangle_{\mathrm{S}} \qquad \text{(This is wrong!)} \tag{6}$$

where $H_{\mathrm{int},\mathrm{S}} = H_{\mathrm{S}} - H_{0,\mathrm{S}}$. Why is is not generally true?

(iii) Instead, show that the Schrödinger equation (9) implies that

$$i\frac{d}{dt}|\psi, t\rangle_{\mathrm{I}} = H_{\mathrm{int},\mathrm{I}}(t)\,|\psi, t\rangle_{\mathrm{I}}\,, \tag{7}$$

$$i\frac{d}{dt}U(t, t_0) = H_{\mathrm{int},\mathrm{I}}(t)\,U(t, t_0)\,. \tag{8}$$

(iv) By writing the evolution of $|\psi, t\rangle_{\mathrm{I}}$ in (12) in terms of infinitesimal positive time steps ϵ, show that for $t_2 > t_1$

$$U(t_2, t_1) = \mathrm{T}\left(\exp\{-i\int_{t_1}^{t_2} dt'\ H_{\mathrm{int,I}}(t')\}\right)\ ,\ \text{if } t_2 > t_1\,. \tag{9}$$

The Baker-Campbell-Hausdorf identity (BCH) is

$$\exp\{\widehat{A}\}\exp\{\widehat{B}\} = \exp\{\widehat{A} + \widehat{B} + \frac{1}{2}[\widehat{A}, \widehat{B}] + \frac{1}{12}[\widehat{A}, [\widehat{A}, \widehat{B}]] - \frac{1}{12}[\widehat{B}, [\widehat{A}, \widehat{B}]] + \ldots\} \tag{10}$$

The additional terms in the $\ldots$ represent terms containing all possible combinations of $\widehat{A}$ and $\widehat{B}$ operators in all possible multiple commutators. For instance at the next order the terms have four operators in all possible triple commutators e.g. it contains $[\widehat{A}, [\widehat{A}, [\widehat{A}, \widehat{B}]]]$ and so forth. Each multiple commutator is multiplied by a known c-number.

Why do we not need to use this relation in the derivation of the solution (14)? That is show that

$$\mathrm{T}\left(\exp\{\widehat{A}\}\exp\{\widehat{B}\}\right) = \mathrm{T}\left(\exp\{\widehat{A} + \widehat{B}\}\right) \tag{11}$$

where $\widehat{A}$ and $\widehat{B}$ are some operators defined at a specific time. Show this to second order in the operators.

Optional: convince yourself that (16) works to all orders.

(v) *Optional*: What is the form for $U(t_2, t_1)$ when $t_2 < t_1$? This needs to be given in terms of some exponential of an integral $\int_{t_1}^{t_2} dt'\ H_{\mathrm{int,I}}(t')$ similar to (14). You can either repeat the proof above for this case or try to argue for any form you guess.

*2. Contractions

(i) For bosonic fields, a contraction is defined and denoted as

$$\overbracket{\phi_1\phi_2} = \Delta_{12} = \mathrm{T}(\phi_1\phi_2) - \mathrm{N}(\phi_1\phi_2) \tag{12}$$

where $\phi_1 = \phi_1(x_1)$ and $\phi_2 = \phi_2(x_2)$ are any two bosonic[1] fields. Here $\mathrm{N}(\ldots)$ is general normal ordering where, for a given split of fields $\phi_i = \phi_i^+ + \phi_i^-$, ϕ_i^+ are moved to the right of all ϕ_i^-.

More specifically the normal ordering operator N indicates the following algorithm. You replace any ϕ_i by $(\phi_i^+ + \phi_i^-)$ and expand until you have a sum of terms in which each term is a product of $\phi_i^\pm$. In each term, you switch the order of any pair of neighbouring fields so a $(\ldots\phi_i^+\phi_j^-\ldots)$ becomes $(\ldots\phi_j^-\phi_i^+\ldots)$. You repeat these swaps until there are no more $\phi_i^+\phi_j^-$ neighbouring pairs. Note this means that the order of the *plus* parts of the fields, ϕ_i^+, is unchanged relative to themselves, likewise the *minus* parts ϕ_i^-. The only changes are in the relative order of plus and minus parts of the fields.

Time ordering, $\mathrm{T}(\ldots)$, moves fields so that each field has later (earlier) time operators to the left (right)[2].

[1]Fermionic fields have some extra signs in these definitions.

[2]For simplicity it is best to set the times of all operators to be distinct taking any equal time limits at the end of the calculation as needed. This will make no difference to any physical quantity.

Show that for an arbitrary split of bosonic fields we have that

$$\contraction{}{\phi_1}{}{\phi_2}\phi_1\phi_2 = \Delta_{12} \;=\; \theta(t_1-t_2)\left[\phi_1^+,\phi_2^-\right] + \theta(t_2-t_1)\left(\left[\phi_2^+,\phi_1^+\right] + \left[\phi_2^+,\phi_1^-\right] + \left[\phi_2^-,\phi_1^-\right]\right) \tag{13}$$

Hence show that a contraction is symmetric only if we split the field such that

$$\left[\phi_2^+,\phi_1^+\right] + \left[\phi_2^-,\phi_1^-\right] = 0\,. \tag{14}$$

Assuming (19) is true, give the simplified form for the contraction $\contraction{}{\phi_1}{}{\phi_2}\phi_1\phi_2 = \Delta_{12}$.

(ii) For the remainder of this question consider the standard definition of normal ordering (denoted in this course with $:\ldots:$) where annihilation (creation) operators are put to the right (left) so that $\langle 0| : (\text{any fields}) : |0\rangle = 0$.

Show that for this choice of split, we have the following proeprties

(a) $[\phi_1^+,\phi_2^-] \propto \widehat{\mathbf{1}}$ i.e.s this a c-number which commutes with everything.

(b) Equation (19) is true.

(c) the two-point normal ordering is symmetric under interchange of the fields under the ordering, i.e.

$$:\phi_1\phi_2:=:\phi_2\phi_1:\,. \tag{15}$$

(iii) Consider a theory with a real scalar field ϕ of mass m and a complex scalar field ψ of mass M. The scalar Yukawa theory is one example but the details of any interaction are not relevant in this question. Start from the fields in the interaction picture

$$\hat\phi(x) \;=\; \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{p})}}\left(\hat a_{\boldsymbol{p}}e^{-ipx} + \hat a^\dagger_{\boldsymbol{p}}e^{+ipx}\right),\quad p_0=\omega(\boldsymbol{p}) = \left|\sqrt{\boldsymbol{p}^2+m^2}\right|, \tag{16}$$

$$\hat\psi(x) \;=\; \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\Omega(\boldsymbol{p})}}\left(\hat b_{\boldsymbol{p}}e^{-ipx} + \hat c^\dagger_{\boldsymbol{p}}e^{ipx}\right),\quad p_0=\Omega(\boldsymbol{p}) = \left|\sqrt{\boldsymbol{p}^2+M^2}\right|, \tag{17}$$

where the annihilation and creation operators obey the usual commutation relations

$$\left[\hat a_{\boldsymbol{p}},\hat a^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\,,\qquad [\hat a_{\boldsymbol{p}},\hat a_{\boldsymbol{q}}] = \left[\hat a^\dagger_{\boldsymbol{p}},\hat a^\dagger_{\boldsymbol{q}}\right] = 0\,. \tag{18}$$

Use the definition of a contraction (17) to show that the value for each of following contractions is as given (assuming standard normal ordering)

$$\contraction{}{\phi(x)}{}{\phi(y)}\phi(x)\phi(y) \;=\; \Delta_m(x-y) = \int\frac{d^4k}{(2\pi)^4}\,e^{-ik(x-y)}\frac{i}{k^2-m^2+i\epsilon} \tag{19}$$

$$\contraction{}{\phi(x)}{}{\psi(y)}\phi(x)\psi(y) \;=\; 0 \tag{20}$$

$$\contraction{}{\phi(x)}{}{\psi^\dagger(y)}\phi(x)\psi^\dagger(y) \;=\; 0 \tag{21}$$

$$\contraction{}{\psi(x)}{}{\psi(y)}\psi(x)\psi(y) \;=\; 0 \tag{22}$$

$$\contraction{}{\psi^\dagger(x)}{}{\psi^\dagger(y)}\psi^\dagger(x)\psi^\dagger(y) \;=\; 0 \tag{23}$$

$$\contraction{}{\psi(x)}{}{\psi^\dagger(y)}\psi(x)\psi^\dagger(y) \;=\; \Delta_M(x-y) = \int\frac{d^4k}{(2\pi)^4}\,e^{-ik(x-y)}\frac{i}{k^2-M^2+i\epsilon} \tag{24}$$

$$\contraction{}{\psi^\dagger(x)}{}{\psi(y)}\psi^\dagger(x)\psi(y) \;=\; \Delta_M(y-x) = \int\frac{d^4k}{(2\pi)^4}\,e^{+ik(x-y)}\frac{i}{k^2-M^2+i\epsilon} \tag{25}$$

∗3. Wick's theorem for four bosonic fields

(i) Using Wick's theorem, write down an expression for the time ordered product of four scalar[3] fields $T_{1234} = \mathrm{T}\left(\phi_1\phi_2\phi_3\phi_4\right)$, where $\phi_i = \phi_i(x_i)$. This should be given in terms of the normal ordered products of two and four fields, denoted $N(\text{fields})$, and in terms of the contractions $\Delta_{ij} = \mathrm{T}\left(\phi_i\phi_j\right) - \mathrm{N}\left(\phi_i\phi_j\right)$ (assumed to be c-numbers). Do this for generic (any) normal ordering.

(ii) Consider some general expectation value $G_{1234} = \langle \mathrm{T}\left(\phi_1\phi_2\phi_3\phi_4\right)\rangle$. Assuming that normal ordering has been defined such that the relevant expectation values of any normal product are zero, i.e. $\langle N(\text{fields})\rangle = 0$, find an expression for G_{1234} in terms of $\Delta_{ij} = \langle \mathrm{T}\left(\phi_i\phi_j\right)\rangle$.

(iii) For the typical definitions of normal ordering, all the terms in Wick's theorem are symmetric under interchange of fields. So without loss of generalisation you may choose $t_1 > t_2 > t_3 > t_4$. Assume that we have only one real scalar field $\phi(x) = \phi_i(x)$ for all i, but that the coordinates of the four terms are still distinct. We will consider the vacuum expectation value,

$$G_{1234} = \langle 0|\mathrm{T}\left(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\right)|0\rangle\,. \tag{26}$$

Substitute the form for the interaction picture field into this expression and verify directly that we obtain the result found in the previous part.

♯4. Normal Ordering for Thermal Expectation Values

This question illustrates one example where a different normal ordering is used.
In typical high energy experiments at colliders such as CERN, the inputs and outputs are one, two or a few physical particles added to the quantum vacuum. However there are many situations where we are interested in processes taking place in a background of many particles e.g. condensed matter problems, in the early universe or in heavy ion collisions at CERN (see the ALICE experiment). To deal with such situations with many particles we need to combine QFT with the ideas of statistical mechanics and thermodynamics, a topic known as **Thermal Field Theory** (or Finite Temperature Field Theory). In this question we will look at a the simplest type of Green functions used in Thermal Field Theory, two point functions for a free field.
The expectation values $\langle\ldots\rangle$ in this question are now thermal expectation values where

$$\langle\widehat{\mathcal{O}}\rangle = \frac{1}{Z}\mathrm{Tr}\{e^{-\beta\widehat{H}}\widehat{\mathcal{O}}\}\,, \qquad Z = \mathrm{Tr}\{e^{-\beta\widehat{H}}\}\,. \tag{27}$$

where $\beta = 1/(KT)$ is the inverse temperature[4]. Here $\mathrm{Tr}\{\ldots\}$ indicates a sum over all states in any basis, i.e.

$$\mathrm{Tr}\{\mathcal{O}\} \equiv \sum_n \langle n|\mathcal{O}|\,n\rangle\,. \tag{28}$$

By way of comparison 'normal' QFT , based on the vacuum expectation values, is for few particles and is obtained as the zero temperature limit of thermal field theory, $\beta \to \infty$.

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[3]For this question, bosonic fields behave exactly as scalar fields so the result is more general.

[4]Here T is the temperature and, guess what, we set $K = 1$ in relativistic thermal field theory to define our temperatures in natural units.

(i) Consider a single quantum harmonic oscillator with the usual annihilation and creation operators $\hat{a}^\dagger$ and $\hat{a}$ so the Hamiltonian may be written as $\widehat{H} = \omega \hat{a}^\dagger \hat{a}$ (ignoring constants which are not relevant here). The states are the usual normalised Fock space energy/number eigenstates

$$|\, n\rangle = \frac{1}{\sqrt{n!}} (\hat{a}^\dagger)^n |\, 0\rangle \,. \tag{29}$$

Show that $Z = \mathrm{Tr}\{e^{-\beta \widehat{H}}\} = (1 - e^{-\beta\omega})^{-1}$.

By taking an appropriate derivative of Z show that

$$\langle \hat{a}^\dagger \hat{a} \rangle = \frac{1}{Z} \mathrm{Tr}\{e^{-\beta \widehat{H}} \hat{a}^\dagger \hat{a}\} \tag{30}$$

is the Bose-Einstein distribution.

Using commutators or otherwise calculate

$$\langle \hat{a} \hat{a}^\dagger \rangle = \frac{1}{Z} \mathrm{Tr}\{e^{-\beta \widehat{H}} \hat{a} \hat{a}^\dagger\} \,. \tag{31}$$

Why are $\langle \hat{a}\hat{a} \rangle$ and $\langle \hat{a}^\dagger \hat{a}^\dagger \rangle$ both zero?

(ii) Normal ordering in normal QFT (zero temperature QFT) is indicated here by $: \ldots :$, and is defined such that annihilation (creation) operators are moved to the right (left) of creation (annihilation) operators[5].

Compare $\langle 0| : \hat{a}\hat{a}^\dagger : |\, 0\rangle$ and $\langle : \hat{a}\hat{a}^\dagger : \rangle$.

!♯5. Normal Ordering for Thermal Field Theory

The answers are for this question are in Thouless, Phys.Rev. 107 (1957) 4 for non-relativistic fields and Evans and Steer, Nucl.Phys.B, 474 (1996) 481–496 (online as `arXiv:hep-ph/9601268`[6]) for relativistic fields.

Thermal expectation values are defined as in (32). Note that we are working with a continuous momentum labelling the different states so we need to remember that the normalisation for these continuous momentum space $\boldsymbol{p}$ states is

$$\langle n_p |\, n_q \rangle = (2\pi)^3 \delta^3(\boldsymbol{p} - \boldsymbol{q}) \delta_{n_p, n_q} \,. \tag{32}$$

where $|\, n_p\rangle$ is the normalised state obtained by acting $\hat{a}^\dagger_{\boldsymbol{p}}$ n_p times on the vacuum state for the oscillator associated with mode $\boldsymbol{p}$..

(i) Consider a single real scalar field $\phi(x)$ in the interaction picture as given in (21). The Hamiltonian is then just $\widehat{H} = \int d^3k \hat{a}^\dagger_k \hat{a}_k$ where we can ignore any (infinite) constants.

Using (36) and (35) or otherwise, find the **thermal Wightman function**

$$D_\beta(x - y) = \langle \phi(x)\phi(y) \rangle = \frac{1}{Z} \mathrm{Tr}\{e^{-\beta \widehat{H}} \phi(x)\phi(y)\} \,, \tag{33}$$

[5] To be more precise if asked explicitly for a definition, as you might want to be in an exam, we should also note that within the set of annihilation operators, their relative order is maintained. Likewise for the creation operators.

[6] See `http://arxiv.org/abs/hep-ph/9601268`.

where the Tr is a generalisation to QFT of (33). Hint: you should find that the expression factorises into a product over different momenta $\boldsymbol{p}$, each term of a familiar form for a bosonic state of energy ω_p.

Compare this thermal Wightman function D_β with the form used in zero temperature QFT, i.e. $D(x-y) = \langle 0|\phi(x)\phi(y)|\,0\rangle$.

(ii) Suppose we split our field using a general *linear* split of the annihilation and creation parts. That is we define

$$\hat{\phi}^+(x) \;=\; \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{p})}}\left((1-f_p)\hat{a}_{\boldsymbol{p}}e^{-ipx} + g_p\hat{a}^\dagger_{\boldsymbol{p}}e^{+ipx}\right), \tag{34}$$

$$\hat{\phi}^-(x) \;=\; \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{p})}}\left(f_p\hat{a}_{\boldsymbol{p}}e^{-ipx} + (1-g_p)\hat{a}^\dagger_{\boldsymbol{p}}e^{+ipx}\right), \tag{35}$$

where $p_0 = \omega(\boldsymbol{p}) = \left|\sqrt{\boldsymbol{p}^2+m^2}\right|$ as before. Here $f_p =\equiv f(\boldsymbol{p})$ and $g_p =\equiv g(\boldsymbol{p})$ are two functions to be determined.

Now write down $\langle \mathrm{N}(\phi(x)\phi(y))\rangle$ in terms of f_p, g_p and where N is normal ordering which moves $\hat{\phi}^+$ to the right of all $\hat{\phi}^-$ but which maintains the relative order within the set of $\hat{\phi}^+$ and which also maintains the relative order within the set of $\hat{\phi}^-$.

(iii) Demand that $\langle \mathrm{N}(\phi(x)\phi(y))\rangle = 0$ and hence show that functions f and g satisfy

$$f_p g_p = -n(\omega_p)\,, \qquad (1-f_p)(1-g_p) = 1 + n(\omega_p)\,. \tag{36}$$

Hence show that there are two possible solutions

$$f_p = -n + s\sqrt{n(n+1)}\,, \qquad g_p = -n - s\sqrt{n(n+1)}\,, \qquad s = \pm 1\,. \tag{37}$$

(iv) Find the time-ordered two point Green function for a real scalar field in Thermal Field Theory (i.e. the replacement for our usual Feynman propagator in perturbation theory) by using the definition that $\Delta_\beta(x-y) = \langle \mathrm{T}(\phi(x)\phi(y))\rangle$. Compare with the zero temperature case.