Solutions 5: Bosonic Interacting Quantum Field Theory

1. The Interaction Picture Evolution Operator $U(t_2, t_1)$

The relationship between the Schrödinger and Interaction pictures is given by

$$|\psi, t\rangle_{\mathrm{I}} = \exp\{+iH_{0,\mathrm{S}}t\}|\psi, t\rangle_{\mathrm{S}}\,, \tag{1}$$

$$\mathcal{O}_{\mathrm{I}}(t) = \exp\{+iH_{0,\mathrm{S}}t\}\,\mathcal{O}_{\mathrm{S}}\exp\{-iH_{0,\mathrm{S}}t\}\,. \tag{2}$$

The evolution of interaction picture states is found through the interaction picture evolution operator $U(t_2, t_1)$ where

$$|\psi, t_2\rangle_{\mathrm{I}} = U(t_2, t_1)|\psi, t_1\rangle_{\mathrm{I}}\,. \tag{3}$$

(i) Show that the U operator satisfies the following conditions:

(a) This is clearly needed as $|\psi, t\rangle_{\mathrm{I}} = |\psi, t\rangle_{\mathrm{I}}$.

$$U(t, t) = 1\,. \tag{4}$$

(b) From the definition of U in (3) we have

$$|\psi, t_2\rangle_{\mathrm{I}} = U(t_2, t_1)|\psi, t_1\rangle_{\mathrm{I}}\,, \tag{5}$$

$$|\psi, t_1\rangle_{\mathrm{I}} = U(t_1, t_0)|\psi, t_0\rangle_{\mathrm{I}}\,, \tag{6}$$

$$|\psi, t_2\rangle_{\mathrm{I}} = U(t_2, t_0)|\psi, t_0\rangle_{\mathrm{I}}\,. \tag{7}$$

Substituting (5) into (6) and comparing to (7) gives us the result

$$U(t_2, t_1)U(t_1, t_0) = U(t_2, t_0)\,. \tag{8}$$

(c) We need the normalisation of states to be time independent so that ${}_{\mathrm{I}}\langle\psi, t_2|\,\psi, t_2\rangle_{\mathrm{I}} = {}_{\mathrm{I}}\langle\psi, t_1|\,\psi, t_1\rangle_{\mathrm{I}}$. Taking the hermitian conjugate of (3) we have that

$${}_{\mathrm{I}}\langle\psi, t_2| = {}_{\mathrm{I}}\langle\psi, t_1|\,.[U(t_2, t_1)]^{\dagger} \tag{9}$$

so that

$${}_{\mathrm{I}}\langle\psi, t_2|\,\psi, t_2\rangle_{\mathrm{I}} = {}_{\mathrm{I}}\langle\psi, t_1|[U(t_2, t_1)]^{\dagger}.U(t_2, t_1)|\psi, t_1\rangle_{\mathrm{I}} = {}_{\mathrm{I}}\langle\psi, t_1|\,\psi, t_1\rangle_{\mathrm{I}}\,. \tag{10}$$

This demands the unitary condition that

$$[U(t_2, t_1)]^{\dagger}\,.U(t_2, t_1) = 1\,. \tag{11}$$

(d) If we set $t_2 = t_0$ in (8) we get that $U(t_0, t_1)U(t_1, t_0) = U(t_0, t_0) = 1$ using the identity (4). Comparing this with (11) we arrive at

$$[U(t_2, t_1)]^{\dagger} = U(t_1, t_2)\,. \tag{12}$$

(ii) The Schrödinger equation defines all the time-evolution in the Schrödinger picture which lies in their states

$$i\frac{d}{dt}|\psi, t\rangle_{\mathrm{S}} = H_{\mathrm{S}}\,|\psi, t\rangle_{\mathrm{S}}\,. \tag{13}$$

This tells us that

$$|\psi,t\rangle_{\rm S} \;=\; \exp\{-iH_{\rm S}t\}\,|\psi,t=0\rangle_{\rm S}\,, \tag{14}$$

as we can check by substituting this into (13). Substituting (14) into (1) gives us

$$|\psi,t\rangle_{\rm I} \;=\; \exp\{+iH_{0,{\rm S}}t\}\exp\{-iH_{\rm S}t\}\,|\psi,t=0\rangle_{\rm S}\,, \tag{15}$$

From this we deduce that

$$|\psi,t\rangle_{\rm I} \;=\; \exp\{-iH_{{\rm int},{\rm S}}t\}\,|\psi,t=0\rangle_{\rm S} \quad \text{iff} \quad [H_{\rm S},H_{0,{\rm S}}]=0 \tag{16}$$

but this is not generally true. (EFS: What is happening if this is true?)

(iii) To make progress we have to do this in infinitesimal steps which we can do by producing a differential equations. Substitute in $|\psi,t\rangle_{\rm S}=\exp\{-iH_{0,{\rm S}}t\}|\psi,t\rangle_{\rm I}$ from (1) to the left hand side of (13) to find

$$i\frac{d}{dt}|\psi,t\rangle_{\rm S} \;=\; i\frac{d}{dt}\left(\exp\{-iH_{0,{\rm S}}t\}\,|\psi,t\rangle_{\rm I}\right) \tag{17}$$

$$\;=\; H_{0,{\rm S}}e^{-iH_{0,{\rm S}}t}\,|\psi,t\rangle_{\rm I}+ie^{-iH_{0,{\rm S}}t}\frac{d}{dt}\,|\psi,t\rangle_{\rm I} \tag{18}$$

The right hand side of (13) gives us $H_{\rm S}|\psi,t\rangle_{\rm S}=H_{\rm S}\exp\{-iH_{0,{\rm S}}t\}\,|\psi,t\rangle_{\rm I}$ noting the order carefully as $H_{\rm S}$ and $H_{0,{\rm S}}$ do not commute. Put these two sides of the Schrödinger equation together gives

$$ie^{-iH_{0,{\rm S}}t}\frac{d}{dt}\,|\psi,t\rangle_{\rm I}+H_{0,{\rm S}}e^{-iH_{0,{\rm S}}t}\,|\psi,t\rangle_{\rm I} \;=\; H_{\rm S}\exp\{-iH_{0,{\rm S}}t\}\,|\psi,t\rangle_{\rm I} \tag{19}$$

$$\Rightarrow \quad ie^{-iH_{0,{\rm S}}t}\frac{d}{dt}\,|\psi,t\rangle_{\rm I} \;=\; -H_{0,{\rm S}}\exp\{-iH_{0,{\rm S}}t\}\,|\psi,t\rangle_{\rm I}+H_{\rm S}\exp\{-iH_{0,{\rm S}}t\}\,|\psi,t\rangle_{\rm I} \tag{20}$$

$$\;=\; (H_{\rm S}-H_{0,{\rm S}})\exp\{-iH_{0,{\rm S}}t\}\,|\psi,t\rangle_{\rm I} \tag{21}$$

$$\;=\; H_{{\rm int},{\rm S}}\exp\{-iH_{0,{\rm S}}t\}\,|\psi,t\rangle_{\rm I} \tag{22}$$

$$\Rightarrow \quad \frac{d}{dt}\,|\psi,t\rangle_{\rm I} \;=\; -i\exp\{+iH_{0,{\rm S}}t\}H_{{\rm int},{\rm S}}\exp\{-iH_{0,{\rm S}}t\}\,|\psi,t\rangle_{\rm I} \tag{23}$$

$$\frac{d}{dt}\,|\psi,t\rangle_{\rm I} \;=\; -iH_{{\rm int},{\rm I}}(t)\,|\psi,t\rangle_{\rm I} \tag{24}$$

where we have used the definition of an interaction picture operator in (2) to produce the $H_{{\rm int},{\rm I}}(t)$ factor. Thus we have

$$i\frac{d}{dt}|\psi,t\rangle_{\rm I}=H_{{\rm int},{\rm I}}(t)\,|\psi,t\rangle_{\rm I}\,. \tag{25}$$

Substituting in the definition of U from (3) then gives us

$$i\frac{d}{dt}U(t,t_0)=H_{{\rm int},{\rm I}}(t)U(t,t_0)\,. \tag{26}$$

(iv) From (25) we have for infinitesimal ϵ (not necessarily positive just yet) that

$$\frac{i}{\epsilon}\left(|\psi,t+\epsilon\rangle_{\rm I}-|\psi,t\rangle_{\rm I}\right) \;\approx\; H_{{\rm int},{\rm I}}(t)\,|\psi,t\rangle_{\rm I} \tag{27}$$

$$\Rightarrow \quad i|\psi,t+\epsilon\rangle_{\rm I} \;\approx\; i|\psi,t\rangle_{\rm I}+\epsilon H_{{\rm int},{\rm I}}(t)\,|\psi,t\rangle_{\rm I} \tag{28}$$

$$\Rightarrow \quad |\psi,t+\epsilon\rangle_{\rm I} \;\approx\; |\psi,t\rangle_{\rm I}-i\epsilon H_{{\rm int},{\rm I}}(t)\,|\psi,t\rangle_{\rm I}=(1-i\epsilon H_{{\rm int},{\rm I}}(t))\,|\psi,t\rangle_{\rm I} \tag{29}$$

$$\Rightarrow \quad |\psi,t+\epsilon\rangle_{\rm I} \;\approx\; \exp\{-i\epsilon H_{{\rm int},{\rm I}}(t)\}\,|\psi,t\rangle_{\rm I}\,. \tag{30}$$

Note that $H_{\text{int,I}}(t)$ is an operator so it is vital that we are careful with the order. Now we iterate to find that

$$|\psi, t+2\epsilon\rangle_{\text{I}} \approx \exp\{-i\epsilon H_{\text{int,I}}(t+\epsilon)\}\exp\{-i\epsilon H_{\text{int,I}}(t)\}\;|\psi,t\rangle_{\text{I}} \tag{31}$$

and then

$$|\psi, t+N\epsilon\rangle_{\text{I}} \approx \exp\{-i\epsilon H_{\text{int,I}}(t+(N-1)\epsilon)\}\exp\{-i\epsilon H_{\text{int,I}}(t+(N-2)\epsilon)\} \tag{32}$$

$$\ldots\exp\{-i\epsilon H_{\text{int,I}}(t+\epsilon)\}\exp\{-i\epsilon H_{\text{int,I}}(t)\}\;|\psi,t\rangle_{\text{I}}\,. \tag{33}$$

Again note we are being very careful with operator ordering. This may be written in a short hand notation as a product

$$|\psi, t+N\epsilon\rangle_{\text{I}} \approx \text{T}\left(\prod_{n=0}^{(N-1)}\exp\{-i\epsilon H_{\text{int,I}}(t+n\epsilon)\}\right)\;|\psi,t\rangle_{\text{I}}\,. \tag{34}$$

as the time-ordering operator T encode the operator ordering we need **provided** that $\epsilon > 0$. This notation only works in this case which will be useful only if $t_2 > t_1$ in our $U(t_2,t_1)$ of (3).

Now we need the identity proved in the next part that the time ordering operator takes care of all the operator ordering issues so that we can write $\text{T}(e^A e^B) = \text{T}(e^{(A+B)})$. We find that

$$|\psi, t+N\epsilon\rangle_{\text{I}} \approx \text{T}\left(\exp\{-i\sum_{n=0}^{(N-1)}\epsilon H_{\text{int,I}}(t+n\epsilon)\}\right)\;|\psi,t\rangle_{\text{I}}\,. \tag{35}$$

Taking the limit $\epsilon \to 0^+$ with $N = (t_2 - t)/\epsilon$, we arrive at the key result

$$|\psi, t_2\rangle_{\text{I}} = \text{T}\left(\exp\{-i\int_t^{t_2} dt'\; H_{\text{int,I}}(t')\}\right)\;|\psi,t\rangle_{\text{I}}\,. \tag{36}$$

Looking at the definition of U in (3) we see that we have

$$U(t_2,t_1) = \text{T}\left(\exp\{-i\int_{t_1}^{t_2} dt'\; H_{\text{int,I}}(t')\}\right)\,. \tag{37}$$

(v) The Baker-Campbell-Hausdorf identity (BCH) is

$$\exp\{\widehat{A}\}\exp\{\widehat{B}\} = \exp\{\widehat{A}+\widehat{B}+\frac{1}{2}[\widehat{A},\widehat{B}]+\frac{1}{12}[\widehat{A},[\widehat{A},\widehat{B}]]-\frac{1}{12}[\widehat{B},[\widehat{A},\widehat{B}]]+\ldots\} \tag{38}$$

The additional terms in the $\ldots$ represent terms containing all possible combinations of $\widehat{A}$ and $\widehat{B}$ operators in all possible multiple commutators, multiplied by a known c-number.

Second order proof:

Consider two operators, A and B. The expansion of $e^A e^B$ to second order is

$$e^A e^B = (1+A+\frac{1}{2}A^2+\ldots)(1+B+\frac{1}{2}B^2+\ldots) \tag{39}$$

$$= 1+A+B+\frac{1}{2}A^2+AB+\frac{1}{2}B^2+\ldots \tag{40}$$

Then compare this to the expansion of e^{A+B}

$$\begin{aligned}\exp\{A+B\} &= 1+(A+B)+\frac{1}{2}(A+B)^2+\dots && (41)\\ &= 1+A+B+\frac{1}{2}(A^2+AB+BA+B^2)+\dots. && (42)\end{aligned}$$

We note that the order of the terms is important if, as for many operators, A and B do not commute. The difference between these expressions is the commutator which is the first non-trivial term in the BCH expression (38). That is

$$\begin{aligned}\exp\{A+B+\frac{1}{2}[A,B]\} &= 1+(A+B)+\frac{1}{2}(A+B)^2+\frac{1}{2}[A,B]+\dots && (43)\\ &= 1+A+B+\frac{1}{2}(A^2+AB+BA+B^2)+\frac{1}{2}[A,B]+\dots && (44)\\ &= 1+A+B+\frac{1}{2}A^2+AB+\frac{1}{2}B+\dots. && (45)\end{aligned}$$

agrees with (40) to second order as the BCH formula (38) says it should.

However we also note that a time-ordering operator changes (42) to

$$\begin{aligned}\mathrm{T}\left(\exp\{A+B\}\right) &= \mathrm{T}\left(1+A+B+\frac{1}{2}(A^2+AB+BA+B^2)+\dots\right) && (46)\\ &= 1+A+B+\frac{1}{2}A^2+\mathrm{T}(AB)+\frac{1}{2}B^2+\dots. && (47)\end{aligned}$$

as this overrides any operator ordering in the expression. Note that only the AB term has any ordering issue at this order. Likewise

$$\begin{aligned}\mathrm{T}\left(e^Ae^B\right) &= \mathrm{T}\left((1+A+\frac{1}{2}A^2+\dots)(1+B+\frac{1}{2}B^2+\dots)\right) && (48)\\ &= 1+A+B+\frac{1}{2}A^2+\mathrm{T}(AB)+\frac{1}{2}B^2+\dots && (49)\end{aligned}$$

So now there is complete agreement between (49) and (42) and we have that up to second order

$$\mathrm{T}\left(\exp\{A+B\}\right) = \mathrm{T}\left(e^Ae^B\right)=1+A+B+\frac{1}{2}A^2+\mathrm{T}(AB)+\frac{1}{2}B^2+\dots \qquad (50)$$

All orders proof:

Essentially the issue of operator ordering is encoded in the commutators in the BCH expression of (38). However we see that

$$\mathrm{T}\left([A,B]\right)=\mathrm{T}\left(AB\right)-\mathrm{T}\left(BA\right) \qquad (51)$$

but because the T specifies the ordering $\mathrm{T}\left(AB\right)=\mathrm{T}\left(BA\right)$ and so $\mathrm{T}\left([A,B]\right)=0$. This will guarantee the commutator corrections in the BCH expression give zero e.g. when we expand out the exponential in (38), so we arrive at the conclusion that

$$\mathrm{T}\left(\exp\{\widehat{A}\}\exp\{\widehat{B}\}\right)=\mathrm{T}\left(\exp\{\widehat{A}+\widehat{B}\}\right). \qquad (52)$$

(vi) As we have written out the derivation of the form of $U(t_2,t_1)$ in detail for the case $t_2>t_1$, we can just pick this up from where we used the time ordering. So we can start from the expression in (33) which is valid for any t_2 and t_1. It is the next line, (34), which we must change for the case $t_2 < t_1$. So suppose that $\epsilon < 0$ then we have in terms of an infinitesimal $\eta = -\epsilon > 0$ that we must write

$$|\psi, t - N\eta\rangle_{\rm I} \approx \overline{\rm T}\left(\prod_{n=0}^{(N-1)} \exp\{+i\eta H_{\rm int,I}(t-n\eta)\}\right)|\psi, t\rangle_{\rm I}\,. \tag{53}$$

Here $\overline{\rm T}$ is the anti-time-ordering operator, that is $\overline{\rm T}(AB) = \theta(t_a - t_b)BA + \theta(t_b - t_a)AB$ where t_a is the time associated with operator A and t_b is the time associated with operator B. So $\overline{\rm T}$ puts the operator in order of their time with operators at the *earliest* times on the left.

Another way to see this is to realise that $U(t_1, t_2)U(t_2, t_1) = 1$ from (8). So if $t_2 > t_1$ we have the largest times in the middle of the expression so we must have $U(t_1, t_2)$ ordered in the opposite way from $U(t_1, t_2)$ (and also with a minus sign difference in the exponential) in order for cancellation to occur.

2. Contractions

(i) For bosonic fields, a contraction is defined and denoted as

$$\overbracket{\phi_1\phi_2} = \Delta_{12} = {\rm T}[\phi_1\phi_2] - {\rm N}[\phi_1\phi_2] \tag{54}$$

where $\phi_1 = \phi_1(x_1)$ and $\phi_2 = \phi_2(x_2)$ are any two bosonic[1] fields. Here N[...] is general normal ordering where, for a given split of fields $\phi_i = \phi_i^+ + \phi_i^-$, ϕ_i^+ are moved to the right of all ϕ_i^-, switching terms as few times as possible.

$${\rm N}[\phi_1\phi_2] = {\rm N}[(\phi_1^+ + \phi_1^-)(\phi_2^+ + \phi_2^-)] = \left(\phi_1^+\phi_2^+ + \underbrace{\phi_2^-\phi_1^+}_{\text{(order changed)}} + \phi_1^-\phi_2^+ + \phi_1^-\phi_2^-\right). \tag{55}$$

Time ordering moves fields with latest times to the left.

$$\begin{aligned} {\rm T}[\phi_1\phi_2] = {\rm T}[(\phi_1^+ + \phi_1^-)(\phi_2^+ + \phi_2^-)] &= \theta(t_1 - t_2)\left(\phi_1^+\phi_2^+ + \phi_1^+\phi_2^- + \phi_1^-\phi_2^+ + \phi_1^-\phi_2^-\right) && (56)\\ &\quad +\theta(t_2 - t_1)\left(\phi_2^+\phi_1^+ + \phi_2^-\phi_1^+ + \phi_2^+\phi_1^- + \phi_2^-\phi_1^-\right). && (57)\end{aligned}$$

Substituting (55) and (57) into (54) gives

$$\overbracket{\phi_1\phi_2} = \Delta_{12} = \theta(t_1 - t_2)\left[\phi_1^+, \phi_2^-\right] + \theta(t_2 - t_1)\left(\left[\phi_2^+, \phi_1^+\right] + \left[\phi_2^+, \phi_1^-\right] + \left[\phi_2^-, \phi_1^-\right]\right). \tag{58}$$

The same argument works if we consider $\overbracket{\phi_2\phi_1}$ so we deduce that

$$\overbracket{\phi_2\phi_1} = \Delta_{21} = \theta(t_2 - t_1)\left[\phi_2^+, \phi_1^-\right] + \theta(t_1 - t_2)\left(\left[\phi_1^+, \phi_2^+\right] + \left[\phi_1^+, \phi_2^-\right] + \left[\phi_1^-, \phi_2^-\right]\right). \tag{59}$$

Comparing the two expressions (58) and (59) we see we have a symmetric contraction $\overbracket{\phi_1\phi_2} = \overbracket{\phi_2\phi_1}$ only if the term with a commutator of two plus parts cancels the commutator with two minus parts, i.e. for splits of the fields where

$$\left[\phi_2^+, \phi_1^+\right] + \left[\phi_2^-, \phi_1^-\right] = 0 \tag{60}$$

for all times.

[1] Fermionic fields have some extra signs in these definitions.

(ii) For the remainder of this question we will consider the standard definition of normal ordering (denoted with $: \ldots :$) where annihilation (creation) operators are put to the right (left) so that $\langle 0| : (\text{any fields}) : |0\rangle = 0$.

(a) we have to show that $[\phi_1^+, \phi_2^-] \propto \widehat{\mathbf{1}}$ i.e. this a c-number which commutes with everything. The commutator $[\phi_1^+, \phi_2^-]$ will be proportional to commutators of annihilation operators (coming from ϕ_1^+) and creation operators (these are just in ϕ_2^-) as that is how the standard normal ordering is defined. So the operator part of this commutator will always be proportional to $[\hat{a}_{i\boldsymbol{p}}, \hat{a}^\dagger_{ij\boldsymbol{q}}] = \delta_{ij}\delta^3(\boldsymbol{p}-\boldsymbol{q})$ i.e. a c-number. Hence whatever scalar (or indeed bosonic) fields we use here e.g. relativistic, non-relativistic, the commutator is not an operator, it is just some number, often defined in terms of integrals.

(b) We now need to show that equation (60) is true here, i.e. that $\left[\phi_2^+, \phi_1^+\right] + \left[\phi_2^-, \phi_1^-\right] = 0$. The same approach just used works here. Since now the + parts of fields are always pure annihilation operators they will always commute with each other, whether they are commutators of the + parts of the same or different fields. So we will have $\left[\phi_2^+, \phi_1^+\right] = 0$ for our chosen field split. Similar argument for the minus parts tells us $\left[\phi_2^-, \phi_1^-\right] = 0$. So we see each term in (60) is zero and the sum is therefore zero.

(c) We showed above that the difference of the normal ordered products of two scalar (or bosonic) fields is proportional to $\left[\phi_2^+, \phi_1^+\right] + \left[\phi_2^-, \phi_1^-\right] = 0$ of (60). We have just shown this is zero for our chosen split, the standard normal ordering, it follows that

$$: \phi_1(x)\phi_2 :=: \phi_2(x)\phi_1 : . \tag{61}$$

(iii) The fields we have are in the interaction picture and they are given by

$$\hat{\phi}(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{p})}}\left(\hat{a}_{\boldsymbol{p}}e^{-ipx} + \hat{a}^\dagger_{\boldsymbol{p}}e^{+ipx}\right), \quad p_0 = \omega(\boldsymbol{p}) = \left|\sqrt{\boldsymbol{p}^2+m^2}\right|, \tag{62}$$

$$\hat{\psi}(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\Omega(\boldsymbol{p})}}\left(\hat{b}_{\boldsymbol{p}}e^{-ipx} + \hat{c}^\dagger_{\boldsymbol{p}}e^{ipx}\right), \quad p_0 = \Omega(\boldsymbol{p}) = \left|\sqrt{\boldsymbol{p}^2+M^2}\right|, \tag{63}$$

where the annihilation and creation operators obey the usual commutation relations

$$\left[\hat{a}_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q}), \qquad [\hat{a}_{\boldsymbol{p}}, \hat{a}_{\boldsymbol{q}}] = \left[\hat{a}^\dagger_{\boldsymbol{p}}, \hat{a}^\dagger_{\boldsymbol{q}}\right] = 0. \tag{64}$$

For the standard split used in QFT we find that

$$\phi^+(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}a_{\boldsymbol{p}}\,e^{-ipx}, \qquad p_0 = \omega_{\boldsymbol{p}}, \tag{65}$$

$$\phi^-(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}a^\dagger_{\boldsymbol{p}}\,e^{+ipx}, \qquad p_0 = \omega_{\boldsymbol{p}}. \tag{66}$$

For the complex fields we have to be careful with the notation, distinguish the hermitian conjugate operator (denoted with a dagger †) from the plus symbol (+) used to indicate the annihilation operator parts. The split for the field ψ in terms of $\Omega_{\boldsymbol{p}}$ of (63) is

$$\psi^+(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\Omega_{\boldsymbol{p}}}}b_{\boldsymbol{p}}\,e^{-ipx}, \qquad p_0 = \Omega_{\boldsymbol{p}}, \tag{67}$$

$$\psi^-(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\Omega_{\boldsymbol{p}}}}c^\dagger_{\boldsymbol{p}}\,e^{+ipx}, \qquad p_0 = \Omega_{\boldsymbol{p}}. \tag{68}$$

The split for the hermitian conjugate field $\psi^\dagger$ is (now the notation is getting a little clumsy)

$$(\psi^\dagger)^+(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\Omega_{\boldsymbol{p}}}} c_{\boldsymbol{p}}\, e^{-ipx}, \qquad p_0 = \Omega_{\boldsymbol{p}}, \tag{69}$$

$$(\psi^\dagger)^-(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\Omega_{\boldsymbol{p}}}} b^\dagger_{\boldsymbol{p}}\, e^{+ipx}, \qquad p_0 = \Omega_{\boldsymbol{p}}. \tag{70}$$

Note that the hermitian conjugate of the 'positive' part of $\psi^\dagger$, that is $((\psi^\dagger)^+)^\dagger$, is not ψ^+.

Now from (60) we find the following results

(a)

$$\overbrace{\phi(x)\phi}(y) = \theta(x_0 - y_0)\left[\phi^+(x), \phi^-(y)\right] + \theta(y_0 - x_0)\left[\phi^+(y), \phi^-(x)\right] \tag{71}$$

Now use (65) and (66) to find

$$\left[\phi^+(x), \phi^-(y)\right] = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{d^3\boldsymbol{q}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}} \frac{1}{\sqrt{2\omega_{\boldsymbol{q}}}} e^{-ipx+iqy}\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] \tag{72}$$

$$= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{2\omega_{\boldsymbol{p}}} e^{-i\omega_p(x_0-y_0)+i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})}. \tag{73}$$

Now we need to use the standard tricks used for the energy integrations in the complex plane used for two-point functions. Here we need to see that

$$\int_{-\infty}^{+\infty} dz\, \frac{e^{-izt}}{z - \omega + i\epsilon} = -2\pi i \theta(t) e^{-i\omega t} \tag{74}$$

where ϵ is the usual infinitesimal but positive real parameter and ω is real and positive. To prove this you need to see that you can only close the contour with the upper semicircle at infinity (so with positive imaginary part) when $t < 0$ and then this part picks up no pole inside the contour and is zero. For $t > 0$ you have to close the contour in the lower half plane which then picks up the pole at $z = \omega - i\epsilon$ though you are going around the pole in the negative sense giving a factor of $-2\pi i$.

Using this on the $t = (x_0 - y_0)$ dependent terms in (73) we can rewrite our expression as an integral over $z = p_0$ as follows

$$\theta(x_0 - y_0)\left[\phi^+(x), \phi^-(y)\right] = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} dz\, \frac{e^{-izt}}{z - \omega + i\epsilon} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{2\omega_{\boldsymbol{p}}} e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} \tag{75}$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{1}{2\omega_{\boldsymbol{p}}} \frac{i}{p_0 - \omega + i\epsilon} e^{-ip(x-y)} \tag{76}$$

where we have also changed variables from $\boldsymbol{p}$ to $-\boldsymbol{p}$ exploiting the fact that $\omega_{\boldsymbol{p}}$ is independent of this change.

For the second term of (71) is identical except we have x and y switched round.

$$\overbrace{\phi(x)\phi}(y) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{2\omega_{\boldsymbol{p}}} \frac{i}{p_0 - \omega + i\epsilon} e^{-ip(x-y)} + \int \frac{d^4p}{(2\pi)^4} \frac{1}{2\omega_{\boldsymbol{p}}} \frac{i}{p_0 - \omega + i\epsilon} e^{-ip(y-x)} \tag{77}$$

If we change variables in the second term, switching p^μ to $-p^\mu$, then we find

$$\overbrace{\phi(x)\phi}(y) = \int \frac{d^4p}{(2\pi)^4} \frac{1}{2\omega_{\boldsymbol{p}}} \frac{i}{p_0 - \omega + i\epsilon} e^{-ip(x-y)} + \int \frac{d^4p}{(2\pi)^4} \frac{1}{2\omega_{\boldsymbol{p}}} \frac{i}{-p_0 - \omega + i\epsilon} e^{-ip(x-y)} \quad (78)$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{1}{2\omega_{\boldsymbol{p}}} \left(\frac{i}{p_0 - \omega + i\epsilon} + \frac{i}{-p_0 - \omega + i\epsilon} \right) e^{-ip(x-y)} \quad (79)$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{2\omega_{\boldsymbol{p}}} \left(\frac{-2\omega + 2i\epsilon}{(p_0 - \omega + i\epsilon)(-p_0 - \omega + i\epsilon)} \right) e^{-ip(x-y)} \quad (80)$$

In the numerator $\omega \geq m > 0$ is assumed so we can always drop the infinitesimal here. This leaves use with

$$\overbrace{\phi(x)\phi}(y) = \int \frac{d^4p}{(2\pi)^4} \frac{i}{2\omega_{\boldsymbol{p}}} \left(\frac{-2\omega}{-(p_0)^2 + (\omega - i\epsilon)^2} \right) e^{-ip(x-y)} \quad (81)$$

$$= \int \frac{d^4p}{(2\pi)^4} \left(\frac{i}{(p_0)^2 - \omega^2 - 2i\epsilon\omega - \epsilon^2} \right) e^{-ip(x-y)} \quad (82)$$

Now the ϵ^2 term is negligible while the $2\epsilon\omega = \epsilon'$ is infinitesimal, real and positive (not negligible at the pole of course). However we can relabel this ϵ' as ϵ (it has the usual properties) giving us

$$\overbrace{\phi(x)\phi}(y) = \int \frac{d^4p}{(2\pi)^4} \left(\frac{i}{(p_0)^2 - \omega^2 + i\epsilon} \right) e^{-ip(x-y)} = \Delta_m(x-y) \quad (83)$$

Note: I will use the same Δ_m notation for propagators in both position and momenta with the arguments indicating which is meant, i.e. it is clear which form is meant for $\Delta_m(x-y)$ and $\Delta_m(p-q)$.

(b) Here

$$0 = \overbrace{\phi(x)\psi}(y) \quad (84)$$

as ψ and ϕ fields have different types of annihilation and creation operators (i.e. they represent different types of particle). Thus all the commutators of any parts of these fields always commute.

(c) The previous argument applies here too

$$0 = \overbrace{\phi(x)\psi}^\dagger(y) \quad (85)$$

(d) Here we have from (60) that

$$\overbrace{\psi(x)\psi}(y) = \theta(t) \left[\psi^+(x), \psi^-(y)\right] + \theta(-t) \left[\psi^+(y), \psi^-(x)\right] \quad (86)$$

Again by inspection we see that from (67) and (68) that ψ^+ and ψ^- contain different types of annihilation and creation operators, b's in ψ^+ and c's in ψ^-.

(e) The argument for the previous parts works here too.

$$0 = \overbrace{\psi^\dagger\psi}^\dagger \quad (87)$$

(f) We start from

$$\overbracket{\psi(x)\psi}^\dagger(y) \quad = \quad \theta(x_0 - y_0) \left[\psi^+(x), (\psi^\dagger)^-(y)\right] + \theta(y_0 - x_0) \left[(\psi^\dagger)^+(y), \psi^-(x)\right] \tag{88}$$

Now use (67), and (70) we find that

$$\left[\psi^+(x), (\phi^\dagger)^-(y)\right] \quad = \quad \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{d^3\boldsymbol{q}}{(2\pi)^3} \frac{1}{\sqrt{2\Omega_{\boldsymbol{p}}}} \frac{1}{\sqrt{2\Omega_{\boldsymbol{q}}}} e^{-ipx+iqy} \left[b_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{q}}\right] \tag{89}$$

$$= \quad \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{2\Omega_{\boldsymbol{p}}} e^{-i\Omega_p(x_0-y_0)+i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} . \tag{90}$$

For the second term in (88), using (68) and (69) we get

$$\left[(\psi^\dagger)^+(y), \psi^-(x)\right] \quad = \quad \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{d^3\boldsymbol{q}}{(2\pi)^3} \frac{1}{\sqrt{2\Omega_{\boldsymbol{p}}}} \frac{1}{\sqrt{2\Omega_{\boldsymbol{q}}}} e^{-ipx+iqy} \left[c_{\boldsymbol{p}}, c^\dagger_{\boldsymbol{q}}\right] \tag{91}$$

$$= \quad \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{2\Omega_{\boldsymbol{p}}} e^{-i\Omega_p(x_0-y_0)+i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{y})} . \tag{92}$$

These two forms are identical to those found for the real scalar field contraction of (71) which gave us the two terms (73) and (77) except ω is replaced by Ω. So we can immediately deduce that

$$\overbracket{\psi(x)\psi}^\dagger(y) \quad = \quad \Delta_M(x-y) = \int \frac{d^4k}{(2\pi)^4} e^{-ik(x-y)} \frac{i}{k^2 - M^2 + i\epsilon} \tag{93}$$

Note: I will use the same Δ_M notation for propagators in both position and momenta with the arguments indicating which is meant, i.e. it is clear which form is meant for $\Delta_M(x-y)$ and $\Delta_M(p-q)$.

(g) We just have to note our result above in (61) to see that with the standard normal ordering definition the contractions are symmetric so that

$$\overbracket{\psi^\dagger(x)\psi}(y) = \overbracket{\psi(y)\psi}^\dagger(y) \quad = \quad \Delta_M(y-x) \tag{94}$$

which is exactly the same as (88) except the space-time arguments are reversed.

The contraction is symmetric in our standard choice of normal ordering for vacuum expectation values. This means the order of the field and its hermitian conjugate in the definition of the contraction is irrelevant provided we label the coordinates appropriately. Equation (94) is correctly labelled with the y and x swapped compared to (93) above.

However for complex (and real) scalar fields the propagator is in fact completely symmetric $\Delta(x-y) = \Delta(y-x)$. You can show this explicitly by switching each component of the four $k_\mu = -k'_\mu$ integration variables in turn (remembering to set the range of integration appropriately).

3. Wick's theorem for four bosonic fields

(i) Let $\phi_i = \phi_i(x_i)$, $\Delta_{12} = \wick{\c{\phi}_1 \c{\phi}_2}$, $N_{12} = N(\phi_1\phi_2)$, $N_{1234} = N(\phi_1\phi_2\phi_3\phi_4)$, etc., then for four scalar (or indeed bosonic) fields Wick's theorem states that

$$\begin{aligned}
T_{1234} = \mathrm{T}\left(\phi_1\phi_2\phi_3\phi_4\right) &= \mathrm{N}\left(\phi_1\phi_2\phi_3\phi_4\right) \\
&+\mathrm{N}\left(\wick{\c{\phi}_1\c{\phi}_2\phi_3\phi_4}\right) + \mathrm{N}\left(\wick{\c{\phi}_1\phi_2\c{\phi}_3\phi_4}\right) + \mathrm{N}\left(\wick{\c{\phi}_1\phi_2\phi_3\c{\phi}_4}\right) \\
&+\mathrm{N}\left(\wick{\phi_1\c{\phi}_2\c{\phi}_3\phi_4}\right) + \mathrm{N}\left(\wick{\phi_1\c{\phi}_2\phi_3\c{\phi}_4}\right) + \mathrm{N}\left(\wick{\phi_1\phi_2\c{\phi}_3\c{\phi}_4}\right) \\
&+\mathrm{N}\left(\wick{\c1{\phi}_1\c1{\phi}_2\c2{\phi}_3\c2{\phi}_4}\right) + \mathrm{N}\left(\wick{\c1{\phi}_1\c2{\phi}_2\c1{\phi}_3\c2{\phi}_4}\right) + \mathrm{N}\left(\wick{\c1{\phi}_1\c2{\phi}_2\c2{\phi}_3\c1{\phi}_4}\right) \qquad (95)\\
&= N_{1234} \\
&+\Delta_{12}N_{334} + \Delta_{13}N_{24} + \Delta_{14}N_{23} \\
&+\Delta_{23}N_{14} + \Delta_{24}N_{13} + \Delta_{34}N_{12} \\
&+\Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23} \qquad (96)
\end{aligned}$$

(ii) Whenever normal ordering has been defined such that expectation values of any normal product are zero, i.e. $\langle N(\text{fields})\rangle = 0$. Then when we take expectation values of a time ordered product, and if we use Wick's theorem any term containing a normal ordered operator will be zero so only terms which are products of contractions $\Delta_{ij} = \langle \mathrm{T}\left(\phi_i\phi_j\right)\rangle$ will survive. In this case we will just pick up the last term

$$G_{1234} = \langle \mathrm{T}\left(\phi_1\phi_2\phi_3\phi_4\right)\rangle = \Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23} \qquad (97)$$

(iii) For the usual vacuum expectation values we want to define our normal ordering in terms of the usual split into annihilation parts for ϕ^+ and creation parts in ϕ^- as given in (65) and (66). We can substitute this in to our expression for a vacuum expectation value of the time-ordered product of four fields with $t_1 > t_2 > t_3 > t_4$

$$\begin{aligned}
G_{1234} &= \langle 0|\mathrm{T}\left(\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\right)|\,0\rangle = \langle 0|\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|\,0\rangle \qquad (98)\\
&= \left(\prod_{i=1,2,3,4}\int \frac{d^3p_i}{(2\pi)^3}\frac{1}{\sqrt{2\omega_i}}\right) \\
&\quad \times\langle 0|(a_1e^{-ip_1x_1} + a_1^\dagger e^{+ip_1x_1})(a_2e^{-ip_2x_2} + a_2^\dagger e^{+ip_2x_2}) \\
&\qquad .(a_3e^{-ip_3x_3} + a_3^\dagger e^{+ip_3x_3})(a_4e^{-ip_4x_4} + a_4^\dagger e^{+ip_4x_4})|\,0\rangle \qquad (99)
\end{aligned}$$

where $a_i = a_{p_i}$. Now for a single operator $a_{\boldsymbol{p}}$ the only non-zero answers must come from terms with two annihilation operators a_j and two creation operators $a^\dagger$. Otherwise we will have in initial and final states with different numbers of quanta and hence the overlap will be zero.

So writing $A_j = a_je^{-ip_jx_j}$, and exploiting that $a_i|\,0\rangle = 0$ and $\langle 0|a_i^\dagger = 0$, then we have using the commutation relations (64) that

$$\begin{aligned}
&\langle 0|A_1(A_2+A_2^\dagger)(A_3+A_3^\dagger)A_4^\dagger|\,0\rangle = && (100)\\
&= \langle 0|A_1A_2A_3^\dagger A_4^\dagger|\,0\rangle \\
&\quad +\langle 0|A_1A_2^\dagger A_3A_4^\dagger|\,0\rangle && (101)\\
&= \langle 0|A_1\left(A_3^\dagger A_2+\bar\delta^3(p_2-p_3)e^{-ip_2(x_2-x_3)}\right)A_4^\dagger|\,0\rangle \\
&\quad +\langle 0|\left(A_2^\dagger A_1+\bar\delta^3(p_1-p_2)e^{-ip_1(x_1-x_2)}\right)A_3A_4^\dagger|\,0\rangle && (102)\\
&= \langle 0|\left(A_3^\dagger A_1+\bar\delta^3(p_1-p_3)e^{-ip_1(x_1-x_3)}\right)A_2A_4^\dagger|\,0\rangle+\bar\delta^3(p_2-p_3)e^{-ip_2(x_2-x_3)}\langle 0|A_1A_4^\dagger|\,0\rangle \\
&\quad +\bar\delta^3(p_1-p_2)e^{-ip_1(x_1-x_2)}\langle 0|A_3A_4^\dagger|\,0\rangle && (103)\\
&= \bar\delta^3(p_1-p_3)e^{-ip_1(x_1-x_3)}\bar\delta^3(p_2-p_4)e^{-ip_2(x_2-x_4)} \\
&\quad +\bar\delta^3(p_2-p_3)e^{-ip_2(x_2-x_3)}\bar\delta^3(p_1-p_4)e^{-ip_1(x_1-x_4)} \\
&\quad +\bar\delta^3(p_1-p_2)e^{-ip_1(x_1-x_2)}\bar\delta^3(p_3-p_4)e^{-ip_3(x_3-x_4)} && (104)
\end{aligned}$$

An inspection for the form of the propagator shows that for $t_1 > t_2$ we have that

$$\Delta_{12} = D(x_1-x_2) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2\omega}e^{-ip(x_1-x_2)}\,. \tag{105}$$

and likewise for other combinations. This makes it clear what we have in (104) are just the terms found above in (97), so indeed we have that

$$G_{1234} \;=\; \Delta_{12}\Delta_{34}+\Delta_{13}\Delta_{24}+\Delta_{14}\Delta_{23} \tag{106}$$

♯4. Normal Ordering for Thermal Expectation Values

The expectation values here, $\langle\ldots\rangle$, are now thermal expectation values where

$$\langle\widehat{\mathcal{O}}\rangle = \frac{1}{Z}\mathrm{Tr}\{e^{-\beta\widehat{H}}\widehat{\mathcal{O}}\}\,, \qquad Z=\mathrm{Tr}\{e^{-\beta\widehat{H}}\}\,, \tag{107}$$

and $\beta = 1/(KT)$ is the inverse temperature. Here $\mathrm{Tr}\{\ldots\}$ indicates a sum over all states in any basis, i.e. $\mathrm{Tr}\{\mathcal{O}\} \equiv \sum_n \langle n|\mathcal{O}|\,n\rangle$.

(i) We have a single quantum harmonic oscillator with the usual annihilation and creation operators $\hat a^\dagger$ and $\hat a$ and a Hamiltonian $\widehat{H} = \omega\hat a^\dagger\hat a$. The states are the usual normalised Fock space energy/number eigenstates

$$|\,n\rangle = \frac{1}{\sqrt{n!}}(\hat a^\dagger)^n|\,0\rangle\,. \tag{108}$$

This gives that

$$Z = \mathrm{Tr}\{e^{-\beta\widehat{H}}\} = \sum_{n=0}^{\infty}\langle n|\exp(-\beta\omega\hat{a}^{\dagger}\hat{a})|\,n\rangle \tag{109}$$

$$= \sum_{n=0}^{\infty}\langle n|\sum_{m=0}^{\infty}(-\beta\omega\hat{a}^{\dagger}\hat{a})^{m}|\,n\rangle \tag{110}$$

$$= \sum_{n=0}^{\infty}\langle n|\sum_{m=0}^{\infty}(-\beta\omega n)^{m}|\,n\rangle \tag{111}$$

$$= \sum_{n=0}^{\infty}\langle n|e^{-\beta\omega n}|\,n\rangle \tag{112}$$

$$= \sum_{n=0}^{\infty}\left(e^{-\beta\omega}\right)^{n} \tag{113}$$

$$= \frac{1}{1-e^{-\beta\omega}} \tag{114}$$

Note we used the binomial expansion of $(1-x)^{-1}$ and the normalisation of the states.

We can calculate $\langle\hat{a}^{\dagger}\hat{a}\rangle$ directly as above or use the usual trick with partition functions and recognise that

$$\langle\hat{a}^{\dagger}\hat{a}\rangle = \frac{1}{\omega}\langle\widehat{H}\rangle = \frac{1}{\omega}\frac{1}{Z}.-\frac{d}{d\beta}\mathrm{Tr}\{e^{-\beta\widehat{H}}\} = -\frac{1}{\omega}\frac{1}{Z}\frac{dZ}{d\beta} \tag{115}$$

$$= -\frac{1}{\omega}\frac{1}{Z}\frac{d}{d\beta}\left(\frac{1}{1-e^{-\beta\omega}}\right) \tag{116}$$

$$= -\frac{1}{\omega}\frac{1}{Z}\left(\frac{-\omega e^{-\beta\omega}}{(1-e^{-\beta\omega})^{2}}\right) \tag{117}$$

$$= \frac{e^{-\beta\omega}}{(1-e^{-\beta\omega})} \tag{118}$$

$$= \frac{1}{(e^{\beta\omega}-1)} = n(\omega) \tag{119}$$

This is the Bose-Einstein distribution as should have been expected

Using the commutator $[\hat{a},\hat{a}^{\dagger}]=1$ we have that

$$\langle\hat{a}\hat{a}^{\dagger}\rangle = \langle\hat{a}^{\dagger}\hat{a}+1\rangle = 1+\frac{1}{(e^{\beta\omega}-1)} = \frac{e^{\beta\omega}}{(e^{\beta\omega}-1)} = 1+n(\omega) = \frac{1}{(1-e^{-\beta\omega})} \tag{120}$$

We can see that $\langle\hat{a}\hat{a}\rangle$ and $\langle\hat{a}^{\dagger}\hat{a}^{\dagger}\rangle$ both zero because all the terms in the sum under the trace are of the form $\langle n|\hat{a}\hat{a}|\,n\rangle \propto \langle n|\,n-2\rangle = 0$ and $\langle n|\hat{a}^{\dagger}\hat{a}^{\dagger}|\,n\rangle \propto \langle n|\,n+2\rangle = 0$.

(ii) Here we will use the notation $: \ldots :$ to indicate the 'usual' 'traditional' normal ordering as used in QFT based around vacuum expectation values — zero temperature QFT. This normal ordering $: \ldots :$ is defined such that annihilation (creation) operators are moved to the right (left) of creation (annihilation) operators[2].

Clearly

$$\langle 0|:\hat{a}\hat{a}^{\dagger}:|\,0\rangle = \langle 0|\hat{a}^{\dagger}\hat{a}|\,0\rangle = 0 \tag{121}$$

[2]To be more precise, as you might want to be in an exam if asked explicitly for a definition, we should also note that within the set of annihilation operators, their relative order is maintained. Likewise for the creation operators.

but from above we have that

$$\langle : \hat{a}\hat{a}^\dagger : \rangle = \langle \hat{a}^\dagger \hat{a} \rangle = \frac{1}{(e^{\beta\omega}-1)} \neq 0\,. \tag{122}$$

So the traditional normal ordering : ... : while perfectly well defined in thermal field theory will turn out not to simplify the calculations. In thermal field theory it is best to work with a different normal ordering, see Thouless, Phys.Rev. 107 (1957) 4 or Evans and Steer, Nucl.Phys.B, 474 (1996) 481–496 `arXiv:hep-ph/9601268`[3].

♯5. Normal Ordering for Thermal Field Theory

(i) Consider a single real scalar field $\phi(x)$, defined as usual as

$$\hat{\phi}(x) \quad = \quad \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{p})}}\left(\hat{a}_{\boldsymbol{p}}e^{-ipx} + \hat{a}^\dagger_{\boldsymbol{p}}e^{+ipx}\right), \;\; p_0 = \omega(\boldsymbol{p}) = \left|\sqrt{\boldsymbol{p}^2+m^2}\right|, \tag{123}$$

where $\hat{a}_{\boldsymbol{p}}$ and $\hat{a}^\dagger_{\boldsymbol{p}}$ are the usual annihilation and creation operators obeying standard commutation relations. The Hamiltonian is then just $\widehat{H} = \int d^3k \hat{a}^\dagger_{\boldsymbol{k}}\hat{a}_k$. Note we can ignore any (infinite) constants because this represents a shift in the zero of energy. Equilibrium statistical mechanics does not depends on the zero of energy e.g. you can include rest mass energy or not and it does not make any difference to the statistical mechanics.

The **thermal Wightman function** is given as follows:-

$$D(x-y) \quad = \quad \langle\phi(x)\phi(y)\rangle = \frac{1}{Z}\mathrm{Tr}\{e^{-\beta\widehat{H}}\phi(x)\phi(y)\}\,. \tag{124}$$

$$= \quad \frac{1}{Z}\mathrm{Tr}\left\{e^{-\beta\widehat{H}}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{p})}}\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{q})}} (\hat{a}_{\boldsymbol{p}}e^{-ipx} + \hat{a}^\dagger_{\boldsymbol{p}}e^{+ipx})(\hat{a}_{\boldsymbol{q}}e^{-iqy} + \hat{a}^\dagger_{\boldsymbol{q}}e^{+iqy})\right\} \tag{125}$$

We can generalise the result that $\langle\hat{a}\hat{a}\rangle$ and $\langle\hat{a}^\dagger\hat{a}^\dagger\rangle$ are both zero to see that $\langle\hat{a}_{\boldsymbol{p}}\hat{a}_{\boldsymbol{q}}\rangle$ and $\langle\hat{a}_{\boldsymbol{p}}\hat{a}_{\boldsymbol{q}}\rangle$ are always zero as the bra and kets will have different numbers of $\boldsymbol{p}$ and $\boldsymbol{q}$ states even if $\boldsymbol{p}=\boldsymbol{q}$. For instance if $\boldsymbol{p}\neq\boldsymbol{q}$ then

$$\langle\hat{a}_{\boldsymbol{p}}\hat{a}_{\boldsymbol{q}}\rangle \quad = \quad \frac{1}{Z}\mathrm{Tr}\left\{e^{-\beta\widehat{H}}\hat{a}_{\boldsymbol{p}}\hat{a}_{\boldsymbol{q}}\right\} \tag{126}$$

$$= \quad \frac{1}{Z}\left(\sum_{n_p=0}^{\infty}\langle n_p|e^{-\beta\hat{a}_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}}}\hat{a}_{\boldsymbol{p}}|\,n_p\rangle\right)\left(\sum_{n_q=0}^{\infty}\langle n_q|e^{-\beta\hat{a}_{\boldsymbol{q}}\hat{a}_{\boldsymbol{q}}}\hat{a}_{\boldsymbol{q}}|\,n_q\rangle\right) \times\prod_{\boldsymbol{r}\neq\boldsymbol{p},\boldsymbol{q}}\left(\sum_{n_r=0}^{\infty}\langle n_r|e^{-\beta\hat{a}_{\boldsymbol{r}}\hat{a}_{\boldsymbol{r}}}|\,n_r\rangle\right) \tag{127}$$

$$\propto \quad \frac{1}{Z}\left(\sum_{n_p=0}^{\infty}\langle n_p|e^{-\beta\hat{a}_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}}}|\,n_p-1\rangle\right)\left(\sum_{n_q=0}^{\infty}\langle n_q|e^{-\beta\hat{a}_{\boldsymbol{q}}\hat{a}_{\boldsymbol{q}}}|\,n_q-1\rangle\right) \times\prod_{\boldsymbol{r}\neq\boldsymbol{p},\boldsymbol{q}}\left(\sum_{n_r=0}^{\infty}\langle n_r|e^{-\beta\hat{a}_{\boldsymbol{r}}\hat{a}_{\boldsymbol{r}}}|\,n_r\rangle\right)\,. \tag{128}$$

[3] See `http://arxiv.org/abs/hep-ph/9601268`.

The $\exp(-\beta\hat{a}_{\boldsymbol{p}}\hat{a}_{\boldsymbol{p}})$ terms do not alter particle number of the states so the terms with $\hat{a}_{\boldsymbol{p}}$ and $\hat{a}_{\boldsymbol{q}}$ always involve overlaps between $\langle n|\, n-1\rangle = 0$. If $\boldsymbol{p} = \boldsymbol{q}$ then we will have one term of $\langle n|\, n-2\rangle = 0$. The same applies to all the other cases for thermal expectation values of pairs of annihilation and creation operators when $\boldsymbol{p} \neq \boldsymbol{q}$.

In fact we can see the only cases where the thermal expectation values of a pair of annihilation and creation operators is non-zero is when then are a number operator, i.e. the two cases

$$\langle\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{q}}\rangle = n(\omega_p)(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\,, \tag{129}$$

$$\langle\hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{q}}\rangle = (1+n(\omega_q))\,(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q}) \tag{130}$$

where we have used (120) and (119) and written the answer in terns of the Bose-Einstein distribution defined as $n(\omega_p) = (e^{\beta\omega}-1)^{-1}$. Note the normalisation for these continuous momentum space $\boldsymbol{p}$ states and operators is now

$$\langle n_p|\, n_q\rangle = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\delta_{n_p,n_q}\,. \tag{131}$$

and so forth. It is sometimes easier to do this calculation in discrete momentum space and take the continuum limit at the end but we would have to change the notation and definitions used in this course.

So we can reduce this thermal Wightman function to

$$D_\beta(x-y) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{d^3\boldsymbol{q}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{p})2\omega(\boldsymbol{q})}}\langle\hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{p}}\rangle e^{-ipx+iqy} + \langle\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{q}}\rangle e^{+ipx-iqy} \tag{132}$$

$$= \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{d^3\boldsymbol{q}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{p})2\omega(\boldsymbol{q})}} \times\left(n(\omega_p)(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})e^{-ipx+iqy} + (1+n(\omega_q))(2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})e^{+ipx-iqy}\right) \tag{133}$$

$$D_\beta(x-y) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2\omega(\boldsymbol{p})}\left(n(\omega_p)e^{-ipx+iqy} + (1+n(\omega_q))e^{+ip(x-y)}\right) \tag{134}$$

Note that if we take $\beta\to\infty$ we get the usual Wightman function we encountered in zero temperature QFT, i.e.

$$D(x-y) = \langle 0|\phi(x)\phi(y)|\,0\rangle = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2\omega(\boldsymbol{p})}e^{+ip(x-y)}\,. \tag{135}$$

(ii) We split our field using a general *linear* split of the annihilation and creation parts as follows

$$\hat{\phi}^+(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{p})}}\left((1-f_p)\hat{a}_{\boldsymbol{p}}e^{-ipx} + g_p\hat{a}^\dagger_{\boldsymbol{p}}e^{+ipx}\right), \tag{136}$$

$$\hat{\phi}^-(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{p})}}\left(f_p\hat{a}_{\boldsymbol{p}}e^{-ipx} + (1-g_p)\hat{a}^\dagger_{\boldsymbol{p}}e^{+ipx}\right), \tag{137}$$

where $p_0 = \omega(\boldsymbol{p}) = \left|\sqrt{\boldsymbol{p}^2+m^2}\right|$ as usual. Here $f_p =\equiv f(\boldsymbol{p})$ and $g_p =\equiv g(\boldsymbol{p})$ are two functions to be determined.

We have from the definition of normal ordering that if $\phi_1 = \phi(x)$ and $\phi_2 = \phi(y)$ the as in (55) we have that

$$\mathrm{N}(\phi(x)\phi(y)) = \left(\phi_1^+\phi_2^+ + \phi_2^-\phi_1^+ + \phi_1^-\phi_2^+ + \phi_1^-\phi_2^-\right) \tag{138}$$

For instance the first term is

$$\begin{aligned}
\langle\phi_1^+\phi_2^+\rangle &= \left\langle\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{p})}}\left((1-f_p)\hat{a}_{\boldsymbol{p}}e^{-ipx}+g_p\hat{a}^\dagger_{\boldsymbol{p}}e^{+ipx}\right)\right.\\
&\qquad\left.\times\int\frac{d^3\boldsymbol{q}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{q})}}\left((1-f_q)\hat{a}_{\boldsymbol{q}}e^{-iqy}+g_q\hat{a}^\dagger_{\boldsymbol{q}}e^{+iqy}\right)\right\rangle \qquad (139)\\
&= \int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{d^3\boldsymbol{q}}{(2\pi)^3}\frac{1}{\sqrt{2\omega(\boldsymbol{p})2\omega(\boldsymbol{q})}}\\
&\qquad\times\Big((1-f_p)g_q\langle\hat{a}_{\boldsymbol{p}}\hat{a}^\dagger_{\boldsymbol{q}}\rangle e^{+iqy-ipx}+g_p(1-f_q)\langle\hat{a}^\dagger_{\boldsymbol{p}}\hat{a}_{\boldsymbol{q}}\rangle e^{-iqy+ipx}\Big) \qquad (140)\\
&= \int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2\omega(\boldsymbol{p})}\Big((1-f_p)g_p(1+n(\omega_p))e^{+ip(y-x)}+g_p(1-f_p)n(\omega_p)e^{-ip(y-x)}\Big) \qquad (141)
\end{aligned}$$

The remaining terms are

$$\begin{aligned}
\langle\phi_2^-\phi_1^+\rangle &= \int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2\omega(\boldsymbol{p})}\Big(f_pg_p(1+n(\omega_p))e^{-ip(y-x)}+(1-g_p)(1-f_p)n(\omega_p)e^{+ip(y-x)}\Big) \qquad (142)\\
\langle\phi_1^-\phi_2^+\rangle &= \int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2\omega(\boldsymbol{p})}\Big(f_pg_p(1+n(\omega_p))e^{+ip(y-x)}+(1-g_p)(1-f_p)n(\omega_p)e^{-ip(y-x)}\Big) \qquad (143)\\
\langle\phi_1^-\phi_2^-\rangle &= \int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2\omega(\boldsymbol{p})}\Big(f_p(1-g_p)(1+n(\omega_p))e^{+ip(y-x)}+g_p(1-f_p)n(\omega_p)e^{-ip(y-x)}\Big) \qquad (144)
\end{aligned}$$

(iii) We now demand that $\langle \mathrm{N}(\phi(x)\phi(y))\rangle = 0$. This gives us the following

$$\begin{aligned}
0 &= \int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2\omega(\boldsymbol{p})}\Big[\Big((1-f_p)g_p(1+n(\omega_p))e^{+ip(y-x)}+g_p(1-f_p)n(\omega_p)e^{-ip(y-x)}\Big)\\
&\qquad+\Big(f_pg_p(1+n(\omega_p))e^{-ip(y-x)}+(1-g_p)(1-f_p)n(\omega_p)e^{+ip(y-x)}\Big)\\
&\qquad+\Big(f_pg_p(1+n(\omega_p))e^{+ip(y-x)}+(1-g_p)(1-f_p)n(\omega_p)e^{-ip(y-x)}\Big)\\
&\qquad+\Big(f_p(1-g_p)(1+n(\omega_p))e^{+ip(y-x)}+g_p(1-f_p)n(\omega_p)e^{-ip(y-x)}\Big)\Big] \qquad (145)\\
&= \int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2\omega(\boldsymbol{p})}\\
&\qquad\Big[e^{+ip(y-x)}\left((1-f_p)g_p(1+n(\omega_p))+(1-g_p)(1-f_p)n(\omega_p)\right.\\
&\qquad\qquad\left.+f_pg_p(1+n(\omega_p))+f_p(1-g_p)(1+n(\omega_p))\right)\\
&\qquad+e^{-ip(y-x)}\left(g_p(1-f_p)n(\omega_p)+f_pg_p(1+n(\omega_p))\right.\\
&\qquad\qquad\left.+(1-g_p)(1-f_p)n(\omega_p)+g_p(1-f_p)n(\omega_p)\right)\Big] \qquad (146)\\
&= \int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2\omega(\boldsymbol{p})}\\
&\qquad\Big[e^{+ip(y-x)}\left((1-(1-f_p)(1-g_p))+n(\omega_p)\right)+e^{-ip(y-x)}\left(f_pg_p+n(\omega_p)\right)\Big] \qquad (147)
\end{aligned}$$

So we require that

$$f_pg_p = -n(\omega_p)\,, \qquad (1-f_p)(1-g_p) = 1+n(\omega_p)\,. \qquad (148)$$

These have two solutions

$$f_p = -n+s\sqrt{n(n+1)}\,, \qquad g_p = -n-s\sqrt{n(n+1)}\,, \qquad s=\pm 1\,. \qquad (149)$$

(iv) Perturbation theory works formally in thermal field theory exactly as before.

If we choose a field split such that the thermal expectation value of a pair of normal ordered fields is zero then from the definition of the contraction we still have that

$$\langle \mathrm{T}\phi(x)\phi(y)\rangle = \langle \overbracket{\phi(x)\phi(y)} \rangle \tag{150}$$

Exactly as before, this leads to (see (58))

$$\overbracket{\phi(x)\phi(y)} \quad = \quad \theta(t_1-t_2)\left[\phi_1^+,\phi_2^-\right] + \theta(t_2-t_1)\left(\left[\phi_2^+,\phi_1^+\right] + \left[\phi_2^+,\phi_1^-\right] + \left[\phi_2^-,\phi_1^-\right]\right) . \tag{151}$$

where $\phi_1 = \phi(x)$ and $\phi_2 = \phi(y)$ here.

Only when we substitute in the specific form for the split to be used in thermal theory do we find the new form for the propagator to be used in perturbation theory of thermal field theory.

We could calculate the contraction directly as that will give us our thermal propagator for the Feynman rules as the contraction has the same form as in (54) and (58). By choosing the split such that the thermal expectation value of normal ordered products is zero we will find the correct propagator to use in perturbation theory.

However it is easier just to find the expectation value of the time-ordered product of two interaction picture fields directly. This is simply given in term of the thermal Wightman function (134)

$$\begin{aligned}
\Delta_\beta(x-y) &= \langle \mathrm{T}\phi(x)\phi(y)\rangle && (152)\\
&= \theta(x_0-y_0)D_\beta(x-y) + \theta(y_0-x_0)D_\beta(y-x) && (153)\\
&= \Delta_0(x-y) + \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{2\omega(\boldsymbol{p})} n(\omega_p)\left(e^{-ipx+iqy} + e^{+ip(x-y)}\right) && (154)\\
&= \Delta_0(x-y) + \int \frac{d^4p}{(2\pi)^4} n(|p_0|)(2\pi)^4\delta^4(p^2-m^2)e^{-ip(x-y)} && (155)
\end{aligned}$$

where $\Delta_0(x-y)$ is our usual Feynman propagator i.e. for zero temperature field theory. Using our previous results for $\Delta_0(x-y)$ we have that

$$\begin{aligned}
\Delta_\beta(x-y) &= \int \frac{d^4p}{(2\pi)^4}\left(\frac{i}{p^2-m^2+i\epsilon} + n(|p_0|)(2\pi)^4\delta^4(p^2-m^2)\right)e^{-ip(x-y)} && (156)\\
\Delta_\beta(p) &= \frac{i}{p^2-m^2+i\epsilon} + n(|p_0|)(2\pi)^4\delta^4(p^2-m^2) . && (157)
\end{aligned}$$

So we see that the thermal correction to our usual propagator only comes from physical on-shell particles weighted by the Bose-Einstein factor. These correspond to the effects of propagating through a heat bath at inverse temperature β of *real physical* particles (i.e. no virtual energy fluctuations, their energies are always equal to ω_p).