Problem Sheet 6: Scalar Yukawa Theory

Comments on these questions are always welcome. For instance if you spot any typos or feel the wording is unclear, drop me an email at `T.Evans` at the usual Imperial address.

Note: problems marked with a $*$ are the most important to do and are core parts of the course. Those without any mark are recommended. It is likely that the exam will draw heavily on material covered in these two types of question. Problems marked with a ! are harder and/or longer. Problems marked with a $\sharp$ are optional. For the exam it will be assumed that material covered in these optional $\sharp$ questions has not been seen before and such optional material is unlikely to be used in an exam.

The scalar Yukawa theory for real scalar field ϕ of mass m and complex scalar field ψ with mass M has a cubic interaction

$$H_{\text{int}} = g \int d^3\boldsymbol{x}\; \psi^\dagger(x)\psi(x)\phi(x) \tag{1}$$

where the coupling constant g is real and a measure of the interaction strength. In the interaction picture, the field operators take the form

$$\phi(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\boldsymbol{p}}}}\left(a_{\boldsymbol{p}}e^{-ipx} + a^\dagger_{\boldsymbol{p}}e^{+ipx}\right), \qquad p_0 = \omega_{\boldsymbol{p}} = +\left|\sqrt{\boldsymbol{p}^2+m^2}\right| \geq 0\,. \tag{2}$$

$$\psi(x) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{1}{\sqrt{2\Omega_{\boldsymbol{p}}}}\left(b_{\boldsymbol{p}}e^{-ipx} + c^\dagger_{\boldsymbol{p}}e^{ipx}\right), \qquad p_0 = \Omega_{\boldsymbol{p}} = +\left|\sqrt{\boldsymbol{p}^2+M^2}\right| \geq 0\,. \tag{3}$$

where the annihilation and creation operators obey their usual commutation relations

$$\left[a_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = (2\pi)^3\delta^3(\boldsymbol{p}-\boldsymbol{q})\,, \qquad [a_{\boldsymbol{p}}, a_{\boldsymbol{q}}] = \left[a^\dagger_{\boldsymbol{p}}, a^\dagger_{\boldsymbol{q}}\right] = 0\,. \tag{4}$$

The b and c annihilation and creation operators obey similar commutation relations while the different types of annihilation and creation operator always commute e.g. $\left[a_{\boldsymbol{p}}, b^\dagger_{\boldsymbol{q}}\right] = 0$.

1. Vacuum Diagrams in Scalar Yukawa Theory

A **vacuum diagram** is a diagram which is not connected to any external leg. An external leg of a diagram is a propagator carrying at least one of the arguments of the Green function, coordinates $\{y_i\}$ or $\{z_f\}$ or momenta $\{p_i\}$ or $\{q_f\}$ in our conventions.

(i) Write down the Feynman diagrams describing the normalisation of the free vacuum $Z = \langle 0|S|\,0\rangle$ to $O(g^2)$ (i.e. all terms up to and including $(O(g^2))$.

(ii) Write down an expression for Z to $O(g^2)$ in terms of g, m, M in coordinate space.

(iii) Write this expression in terms of propagators in momentum space, $\Delta(p)$. Is Z one at order g^2 (i.e. all terms up to and including $O(g^2)$)? Try to find out how this integral diverges.

2. Vacuum Expectation Value for ϕ in Scalar Yukawa Theory

Diagrams with a **tadpole** contribution are contributions to the vacuum expectation value of a field, $\langle\hat{\phi}\rangle$. A tadpole is a part of a diagram which can be disconnected from all external legs if you cut just one line.

(i) Write down the Feynman diagrams describing the vev (vacuum expectation value) $v = \langle 0|\mathrm{T}\phi(x)S|\,0\rangle$ of field ϕ to $O(g^3)$.

(ii) Write down an expression for v to $O(g)$ in terms of g, m, M in coordinate space.

(iii) Write this expression in terms of propagator in momentum space, $\Delta(p)$. Is this zero? Try to find out how this integral diverges.

∗3. Symmetry factors in Scalar Yukawa Theory

This is best tackled in position/coordinate space not in terms of momenta.

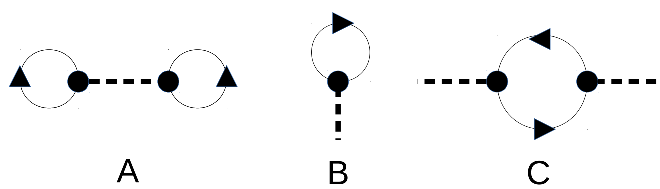

Figure 1: Diagrams representing contributions to three different quantities in scalar Yukawa theory of (1).

(i) Each of the diagrams in figure 1 contribute to different types of quantity: contribution to the vacuum, vacuum expectation value of the field, or a self-energy[1]. Which diagram contributes to which quantity?

(ii) There is one other diagram which contributes to the same quantity as diagram A at $O(g^2)$. Draw this diagram (call it A2).

(iii) Find the symmetry factors for the diagrams in figure 1 and for the other type A diagram, A2, found above.

(iv) *Optional*: Show that diagram (A) in figure 1 has an overall factor of space-time volume VT while diagram B is independent of position.

∗4. ψ^2 Scattering in Scalar Yukawa Theory

Consider the case of $\psi\psi \to \psi\psi$ scattering with incoming ψ particles of three-momenta $\boldsymbol{p}_1$ and $\boldsymbol{p}_2$ while the outgoing ψ particles have three-momenta $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$. We will define the matrix element $\mathcal{M}$ in terms of the free vacuum state[2] $|\,0\rangle$

$$\mathcal{M} = \langle f|S|\,i\rangle = \sum_{n=0}^{\infty} \mathcal{M}_n \,, \qquad \mathcal{M}_n \sim O(g^n) \tag{5}$$

$$= A\langle 0|\hat{b}(q_1)\hat{b}(q_2)S\hat{b}^\dagger(p_1)\hat{b}^\dagger(p_2)|\,0\rangle \tag{6}$$

where $A = (16\Omega(p_1)\Omega(p_2)\Omega(q_1)\Omega(q_2))^{1/2}$ for the normalisation of operators and states used here. Also $a_{\boldsymbol{p}} = a(\boldsymbol{p})$, $|\,0\rangle$ is the free vacuum state and all quantities are given in the Interaction picture (so no subscript is used in this question to indicate this picture).

[1]See handout or later questions for definitions of these various types of diagram.

[2]This is the state vacuum annihilated by the free creation operators used in the definition of the interaction picture fields. This is distinct from the vacuum in the fully interacting theory which we denote by $|\,\Omega\rangle$.

(i) Write down the Lagrangian density for this theory.

(ii) Derive an expression for the $O(g^0)$ term $\mathcal{M}_0$ (Wick's theorem is not needed here so this can be done directly if preferred).

(iii) Prove that $\mathcal{M}_n = 0$ if n is odd using Wick's theorem. (*Hint* no detailed calculations are needed here).

(iv) What is the relationship between the matrix element $\mathcal{M}$ of (5) and the corresponding Green function, $G(z_1, z_2, y_1, y_2)$, for this ψ^2 Scattering in Scalar Yukawa Theory, where

$$G(z_1, z_2, y_1, y_2) = \langle 0|\mathrm{T}\psi(z_1)\psi(z_2)\psi^\dagger(y_1)\psi^\dagger(y_2)S|0\rangle \tag{7}$$

Note we will define G_n as the order g^n term in a perturbative expansion for the Green function.

(v) Draw the two Feynman diagrams for G which contribute to $\mathcal{M}_0$.

(vi) Interpret the result that $\mathcal{M}_n = 0$ if n is odd in terms of Feynman diagrams for G.

(vii) Draw the Feynman diagrams which contribute to $\psi\psi \to \psi\psi$ scattering matrix element G_2 at $O(g^2)$. For each of these diagrams specify (a) the symmetry factor and (b) the number of loop momenta.

(viii) Write down an expression for $G_2(z_1, z_2, y_1, y_2)$ in terms of g, propagators and delta functions in coordinate space.

(ix) There are four distinct types of diagram:

(a) **vacuum diagrams**. These capture the virtual fluctuations in the vacuum of a fully interacting theory. These are diagrams (or subdiagrams, parts of diagrams) which are not connected to any external leg.

(b) **self-energy** contributions are parts of diagrams which represent quantum corrections to a propagator. These are diagrams[3] which can be dropped on top of one type of line to give a new legal diagram.

(c) Diagrams with a **tadpole** contribution are contributions to the vacuum expectation value of a field. A tadpole is a part of a diagram which can be disconnected from all external legs if you cut just one line.

(d) The remaining diagrams which are the only ones describing non-trivial scattering in the quantum theory at this order[4].

Identify which diagram is of which type.

(x) How do the diagrams or the expressions change if write Feynman diagrams for the Green function in momentum space? The Green function $G(q_1, q_2, p_1, p_2)$ in terms of energy-momentum is just the full four-dimensional Fourier transform in each coordinate, i.e.

$$G(z_1, z_2, y_1, y_2) = \int đ^4q_1 đ^4q_2 đ^4p_1 đ^4p_2 \, \exp(-ip_1y_1 - ip_2y_2 + iq_1z_1 + iq_2z_2) G(q_1, q_2, p_1, p_2) \tag{8}$$

[3] Formally they are two-point 1PI one particle irreducible diagram.

[4] Formally, these are four external legs (contractions involving at least one initial/final state field) multiplied by a four-point 1PI (one particle irreducible) function $\Gamma(x_1, x_2, x_3, x_4)$.

5. $\phi \to \psi\bar{\psi}$ Decay in Scalar Yukawa Theory

Consider the decay of a ϕ particle of mass m into a ψ-$\bar{\psi}$ pair (each of mass M) in the scalar Yukawa theory with interactions defined by (1).

(i) What is the relevant matrix element $\mathcal{M}$?

(ii) What is the relationship between the matrix element $\mathcal{M}$ and the corresponding Green function G (which you will have to define) for this $\phi \to \psi\bar{\psi}$ decay process in Scalar Yukawa Theory?

(iii) Draw the Feynman diagrams for the Green function in coordinate space which correspond to contributions to $\mathcal{M}$ up to and including $O(g^3)$.

For each of these diagrams specify (a) the symmetry factor and (b) the number of loop momenta.

(iv) Identify the type of each diagram: vacuum contribution, vev contribution (tadpole), propagator correction (has a self-energy insertion) or something which contributes to the core interaction behaviour (here three external legs multiplied by a three-point 1PI (one particle irreducible) function $\Gamma(x_1, x_2, x_3)$).