# Problem Sheet 7: Interacting Quantum Field Theory: $\lambda\phi^4$

**Comments** on these questions are always welcome. For instance if you spot any typos or feel the wording is unclear, drop me an email at `T.Evans` at the usual Imperial address.

**Note**: problems marked with a $*$ are the most important to do and are core parts of the course. Those without any mark are recommended. It is likely that the exam will draw heavily on material covered in these two types of question. Problems marked with a ! are harder and/or longer. Problems marked with a $\sharp$ are optional. For the exam it will be assumed that material covered in these optional $\sharp$ questions has not been seen before and such optional material is unlikely to be used in an exam.

### $*$1. The full propagator in $\lambda\phi^4$ theory

Consider a theory of a real scalar field $\phi$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{1}$$

This is the simplest interacting relativistic QFT as it involves a single type of spinless particle (which is its own anti-particle) yet it contains an interaction term. It is also renormalisable in four space-time dimensions[1] though it turns out to be trivial in a precise mathematical sense. Nevertheless this is a standard example used to illustrate QFT. When part of a larger theory it can represent real physics.

(i) Thinking in terms of the energy (the interaction terms in the Hamiltonian might help here), why is theory with a $g\phi^3/(3!)$ interaction term not a viable option?

(ii) Write down the Feynman rules for *Green functions* in momentum space.

Why do we *not* include a propagator for external legs when calculating matrix elements but we do include propagators on external legs when representing Green functions with Feynman diagrams?

(iii) The full propagator is the two-point Greens function. If we denote this as $G_0(x,y) = \Pi_0(x-y)$ when it is defined using the vacuum of the free theory, $|0\rangle$, then we have that

$$\begin{aligned}
G_0(x,y) = \Pi_0(x-y) &= \langle 0|\mathrm{T}\phi(x)\phi(y)S|0\rangle && (2)\\
G_0(p_1,p_2) &= \int d^4x_1 \int d^4x_2\; e^{-ip_1x_1}e^{-ip_2x_2}G_0(x_1,x_2) && (3)\\
\Pi_0(p) &= \int d^4x\; e^{-ipx}\,\Pi_0(x)\,. && (4)\\
\Rightarrow \quad G_0(p_1,p_2) &= \delta^4(p_1+p_2)\,\Pi_0(p_1) && (5)
\end{aligned}$$

Remember that we are exploiting Lorentz symmetry as we know $G_0(x,y)$ can only be a function of $(x-y)$ so we set $G_0(x,y) = \Pi_0(x-y)$. In terms of momentum space notation, it means $\Pi_0(p)$ does not contain the overall energy-momentum conserving delta-function in its definition. The Feynman rules, as given in this course, give $G_0(p_1,p_2)$ rather than $\Pi_0(p)$ directly.

(a) Draw *all* the vacuum diagrams up to $O(\lambda^2)$.

(b) Draw all the connected diagrams which contribute to $\Pi_0$ up to $O(\lambda^2)$.

(c) Describe the remaining diagrams which contribute to $\Pi_0$ up to $O(\lambda^2)$.

[1] For relativistic scalar fields, any terms of $\phi^n$ for $n > 4$ are not renormalisaable in $d = 3+1$ dimensions. That means the UV (ultraviolet, i.e. high energy) infinities can not be systematically removed.

(iv) Specify the symmetry factor and the number of loop momenta for

(a) Draw *all* the vacuum diagrams up to $O(\lambda^2)$.

(b) Draw all the connected diagrams which contribute to $\Pi_0$ up to $O(\lambda^2)$.

(v) Which diagrams contain a vacuum diagram and therefore do not contribute to the **full propagator** $\Pi(p)$? That is the two-point Green function calculated with the full interacting vacuum $|\,\Omega\rangle$ $G_c(x,y)$ (as opposed to using the vacuum of the free non-interacting theory $|\,0\rangle$ which gives $G_0(x,y)$)

$$G_c(x,y) = \Pi(x-y) = \langle\Omega|\mathrm{T}\phi(x)\phi(y)S|\,\Omega\rangle \tag{6}$$

$$G_c(p_1,p_2) = \int d^4x_1 \int d^4x_2\; e^{-ip_1x_1}e^{-ip_2x_2}G_c(x_1,x_2) \tag{7}$$

$$\Pi(p) = \int d^4x\; e^{-ipx}\Pi(x)\,. \tag{8}$$

$$\Rightarrow \quad G_c(p_1,p_2) = \tilde{\delta}^4(p_1+p_2)\,\Pi(p_1) \tag{9}$$

As before, Lorentz symmetry allows us to define $\Pi(x)$ through $G_c(x-y) = \Pi(x-y)$. Again the notation used here means that the full propagator $\Pi(p)$ is in terms of one momentum and so does not include the delta function representing the overall conservation as (9) shows.

(vi) The full propagator may be written as

$$\Pi(p) = \Delta(p)\sum_{n=0}^{\infty}\left(\Sigma(p).\Delta(p)\right)^n\;, \tag{10}$$

where $\Delta(p) = i(p^2 - m^2 + i\epsilon)^{-1}$ is just the free propagator. The function $\Sigma(p)$ is called the **self-energy**. It is the two-point **1PI** (**one-particle irreducible**) function and it is described by the sum of 1PI diagrams with two amputated legs (two external legs but the propagator usually associated with them is not present). A 1PI diagram is one in which external legs are not separated if you cut a single line[2] In particular this means there are no contributions from external lines on a 1PI diagram (I draw such external lines on 1PI diagrams as little stubs not long lines). For instance a diagram contributing to the contribution to $\Pi(p)$ in the $\Delta(p)\Sigma(p)\Delta(p)$ term may be cut into two by cutting either of the lines representing the $\Delta(p)$ factors, which are external lines in the diagram. See figure 1 for a further example.

Find the diagrams which contribute to $\Sigma(p)$ to $O(\lambda^2)$.

(vii) Let $\Sigma_1$ be the lowest order contribution to $\Sigma$. This is $O(\lambda^1)$.

Show that the formula (10) for the full propagator $\Pi(p)$ is consistent at $O(\lambda^2)$ with the contribution made by $\Sigma_1$ terms in the diagrammatic expansion. You will have to evaluate all symmetry factors to make sure the coefficients are correct.

(viii) Show formally from (10) (e.g. treating $\Sigma$ as being an $O(\lambda)$ object which can be treated as being "small" in perturbation theory) that

$$\Pi(p) = \frac{1}{(\Delta(p))^{-1} - \Sigma}\,. \tag{11}$$

[2] In theories with no tadpoles, this is equivalent to saying that a 1PI diagram cannot be cut into two separate parts by cutting *one* line. In almost all theories, tadpole diagrams (or their sum at each order in the expansion) are zero.

(ix) The pole of the full propagator should be at the physical mass squared, $p^2 = m^2_{\rm phys}$. Why? Hence show that the physical mass is *not* the parameter $m$ in the Lagrangian but is given by the self-consistent equation

$$m^2_{\rm phys} = m^2 - i\Sigma(p^2 = m^2_{\rm phys})\,. \tag{12}$$

(x) Analyse the value of $\Sigma_1(p)$, the $O(\lambda^1)$ contribution to the self-energy. Do not calculate this in detail but argue that if we limit the size of the three- or four-momenta in the integration to be $O(\Lambda)$ or less, then $\Sigma_1 = c\Lambda^2$. Also deduce that $\Sigma_1$ is independent of the external momenta $p$. What does this tell us about the Lagrangian mass parameter $m$ as $\Lambda \to \infty$?

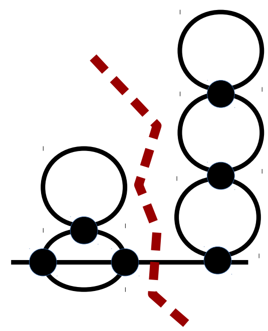

Figure 1: Example of a 1PI diagram. Here the external legs have been amputated/truncated so they are not present to be cut. However the line in the middle if cut, as indicated by the dashed line, would leave this diagram in two pieces. This is not a 1PI diagram. The two subdiagrams on either side of the dashed line, if the connecting propagator has been removed (the one intersecting the dashed line), are both in fact 1PI diagrams in this case.

## ∗2. Scattering in $\lambda\phi^4$ theory

Consider the $\lambda\phi^4$ theory of a real scalar field $\phi$ described by the Lagrangian density (1).

(i) Write down the Feynman rules for Green functions in momentum space.

What changes are needed to the rule depending on whether we care calculating the contribution to a Green function or to a matrix element?

(ii) Consider the four-point Green function

$$G_0(x_1, x_2, x_3, x_4) = \langle 0|\mathrm{T}[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)S]|\,0\rangle\,. \tag{13}$$

How is this related to the matrix element $\mathcal{M}_0 = \langle q_1, q_2|S|\,p_1, p_2\rangle$ describing $\phi\phi \to \phi\phi$ scattering as described by the matrix element? Just sketch this relationship and do not give a full algebraic expression.

Write down *all* the diagrams up to $O(\lambda^2)$ which contribute to the Fourier transform of this four-point Green function, $G_0(p_1, p_2, p_3, p_4)$. Note we may choose to set $p_1$ and $p_2$ ($q_1 = p_3$ and $q_2 = p_4$ ) to be the four-momenta of the two initial (final) states in the $\phi\phi \to \phi\phi$ scattering process, in which case they would be on-shell so that $p_i^2 = m^2$. However the Green function is defined for arbitrary momenta and we will not impose any conditions on the $p_i$.

(iii) Which of these diagrams contain a vacuum diagram? Therefore these diagrams do *not* contribute to this four-point Green function when defined with respect to the full physical vacuum $|\,\Omega\rangle$, i.e. $G_c(p_1, p_2, p_3, p_4)$, which is the Fourier transform of

$$G_c(x_1, x_2, x_3, x_4) = \langle\Omega|\mathrm{T}[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)S]|\,\Omega\rangle\,. \tag{14}$$

(iv) A component of a Feynman diagram is a set of nodes and edges where every vertex and line in the component is connected to every other node and edge in the same component. For example see figure 2. Classify your diagrams for $G_c$ (so no vacuum diagrams here) by the number of components in the diagram.

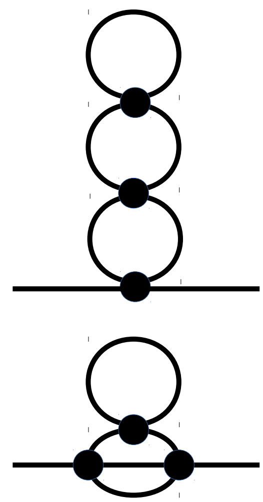

Figure 2: Example of a single Feynman diagram of two components which contributes to the four-point Green function at higher order. The single diagram is the whole lot, both parts, and is a contribution to the four-point Green function $G_c(x_1, x_2, x_3, x_4)$ (or the momentum equivalent) defined above. However this diagram represents quantum fluctuations for the propagation of two phi particles which don't interact with each other. Such terms will be dealt with by considering the two-point Green function. We would only focus on diagrams of one single connected component when wanting to look at the physics of $\phi\phi \to \phi\phi$ scattering.

(v) For the diagrams with more than one component, explain (no detailed calculation needed) why, working to $O(\lambda^2)$ approximation, these two-component diagrams for the *Green function* may be written in the form

$$\begin{aligned} &\Pi(p_1)\Pi(p_2)\tilde{\delta}^4(p_1+p_3)\tilde{\delta}^4(p_2+p_4) + \Pi(p_1)\Pi(p_2)\tilde{\delta}^4(p_1+p_4)\tilde{\delta}^4(p_2+p_3) \\ + \quad &\Pi(p_1)\Pi(p_3)\tilde{\delta}^4(p_1+p_2)\tilde{\delta}^4(p_3+p_4) \end{aligned} \tag{15}$$

where $p_i$ are four-momenta all defined to be flowing into the diagram (or equally good they all flow out). Here $\Pi(p)$ is the full propagator of (9) with the 1PI function $\Sigma$ calculated to $O(\lambda^2)$.

Why must this last term give no contribution to a scattering process where $p_1$ and $p_2$ ($p_3 = -q_1$ and $p_4 = -q_2$) are the four-momenta of initial (final) state particles?

(vi) Using a diagrammatic approach (don't write this out algebraically), show that the one component diagrams may be written approximately to $O(\lambda^2)$ as

$$\tilde{\delta}^4(p_1+p_2+p_3+p_4)\Pi(p_1)\Pi(p_2)\Pi(p_3)\Pi(p_4).(-i\Gamma^{(4)}(p_1,p_2,p_3)) \tag{16}$$

where $-i\Gamma^{(4)}(p_1, p_2, p_3)$ is given by a sum of four-point 1PI (one-particle irreducible) diagrams to $O(\lambda^2)$. Hence give a diagrammatic representation for $-i\Gamma^{(4)}(p_1, p_2, p_3)$ to $O(\lambda^2)$. 1PI has the meaning that if we cut any one line in a diagram then the external legs in that diagram will still be connected. In particular, here this means that we must amputate the propagators on the external legs, here leaving four external 'stubs' represented in a diagram as no leg or just a short stub. See figure 1 for some examples.

Write down an expression for the scattering amplitude for this process. Do not evaluate this in detail but estimate the nature of its UV divergence. That is how does the loop integration diverge at high energies.

(vii) How does this compare to the $M \to \infty$ limit of the scalar Yukawa theory? The Lagrangian density for scalar Yukawa theory is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 + (\partial_\mu \psi^\dagger)(\partial^\mu \psi) - M^2\psi^\dagger\psi - g\psi^\dagger(x)\psi(x)\phi(x) \tag{17}$$

Only a brief qualitative answer is needed and you should ignore the issue of UV infinities. Just indicate the lowest order type of diagrams in the propagation and scattering of $\phi$ particles in scalar Yukawa theory of (17) which can give a behaviour which is roughly like that of pure $\phi^4$ theory of (1). Hence specify, very roughly, the relationship between the two theories in terms of their parameters.

## ♯3. Exponential form for $Z$

The expression for $Z = \langle 0|S|0\rangle$ is given by the sum of all vacuum diagrams. In fact we find that $\ln(Z)$ is given by the sum of *connected vacuum diagrams* alone. Put another way, $Z = \exp\{C\}$ where $C$ is the sum of connected vacuum diagrams. We will illustrate this property for $Z$ by working with one example, the $\lambda\phi^4$ theory of (1).

(i) Give the diagrammatic expansion for $Z$ to $O(\lambda^2)$. You should find the symmetry factors and the number of loops in each case.

Use this to illustrate how some of the contributions to $Z$ come from diagrams which consist of multiple **components**, disconnected pieces. A component in graph theory[3] is a graph (Feynman diagram) which is a **connected** diagram, where you can find a path from every vertex to every other vertex by following the edges. Further each these components also appears in the expansion for $Z$ as a vacuum diagram in its own right. Put another way, if $V$ is a connected vacuum diagram which appears in the expansion for $Z$ then there is always a higher order contribution proportional to $V^2$ which is represented as a single diagram consisting of two components, each the connected subdiagrams $V$.

(ii) Focus on the single $O(\lambda^1)$ vacuum diagram, $V_1$. Show that, in the expansion for $Z$, the terms which correspond to diagrams made up only of disconnected $V_1$ diagrams, that is terms proportional to $(V_1)^n$ contribute to $Z$ in such a way that we may write

$$Z = \exp\{V_1\} + \ldots \tag{18}$$

See if you can justify this for arbitrary orders but only working in terms of the single connected diagram $V_1$.

[3] Feynman diagrams are 'graphs' in 'graph theory'.