Solutions 7: Interacting Quantum Field Theory: $\lambda\phi^4$

Matrix Elements vs. Green Functions.
Physical quantities in QFT are derived from matrix elements $\mathcal{M}$ which represent probability amplitudes. Since particles are characterised by having a certain three momentum and a mass, and hence a specified energy, we specify such physical calculations in terms of such "on-shell" values i.e. four-momenta where $p^2 = m^2$. For instance the notation in momentum space for a $\psi\psi \to \psi\psi$ scattering in scalar Yukawa theory uses four-momenta labels p_1 and p_2 flowing into the diagram for initial states, with q_1 and q_2 flowing out of the diagram for the final states, all of which are on-shell.
However most manipulations in QFT, and in particular in this problem sheet, work with *Green functions* not matrix elements. Green functions are defined for arbitrary values of four momenta including unphysical off-shell values where $p^2 \neq m^2$. So much of the information encoded in a Green function has no obvious physical meaning. Of course to extract the corresponding physical matrix element from the Greens function we would have to apply such physical constraints in order to get the physics of scattering of real physical particles. Alternatively our Green function may be part of an analysis of a much bigger diagram so it represents contributions from virtual particles to some more complicated physical process.

∗1. The full propagator in $\lambda\phi^4$ theory

Consider a theory of a real scalar field ϕ

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{1}$$

(i) A theory with a $g\phi^3/(3!)$ interaction term will not have an energy which is bounded below. The energy can be made as large and negative as you like by taking the size of ϕ to be large (either positive or negative depending on the sign of g). Such a theory will not be stable classically and it is not well defined in QFT.

(ii) (a) The ϕ propagator in momentum space is $\Delta(k) = i/(k^2 - m^2 + i\epsilon)$. An internal line, one connecting two vertices, represents $\int đ^4k_l\,\Delta(k_l)$ with four-momentum k_l flowing along the line. It does not matter which direction you assign to the four-momenta (to which end of the line is it flowing) but you must be consistent. It means your k_l will be flowing into the vertex at one end and out of a vertex at the other end.

(b) An internal vertex with four legs represents $-i\lambda\tilde{\delta}^4(\sum_{l\in V} p_l)$ where p_l are the four-momenta flowing into the vertex. The delta function represents conservation of energy and momentum (a property encoded within Lorentz symmetry) in any process.

(c) *For matrix elements*, each initial or final state particle is associated with one external line. The four-momenta of the line is that of the associated physical particle. No other factor is included so external lines carry no factor of $\int đ^4k_l\,\Delta(k_l)$.
For Green functions, each initial or final state particle is associated with one external line. Again the four-momenta of the line is that of the associated physical particle but now we also add the appropriate propagator for this leg $\Delta(p_{\text{ext}})$. The reason is the external legs correspond to contractions ending on one of the fields explicitly given in the definition of a Green function and which do not come from the expansion of the S matrix. For example in configuration space that would be the $\phi(x)$ or $\phi(y)$ factors in the two-point Green function $\langle 0|\mathrm{T}\phi(x)\phi(y)S|\,0\rangle$. In

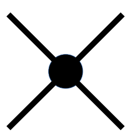
momentum space we just take the Fourier transform, $\int d^4x \exp\{-ip_{\rm ext}x\}$ for each coordinate, and the legs just turn into the appropriate $\Delta(p_{\rm ext})$.

(d) The diagram is divided by the symmetry factor, $\mathcal{S}$. This is the number of permutations of internal lines which leaves the diagram unchanged.

External Legs. The external legs are a confusing aspect as the rules for the change depending on the nature of the object being calculated. We showed in the lectures that converting initial/final states of matrix elements into fields acting on the vacuum, i.e. a Green function, involved a factor coming from a Fourier transform like factor of $\int d^3y \exp\{-ip_{\rm ext}y\}2\omega(p_{\rm ext})$. This is why converting from a Green function to a matrix element involves some extra manipulation which equates to removal of the propagator on external legs. Most analysis is done in terms of diagrams for Green functions.

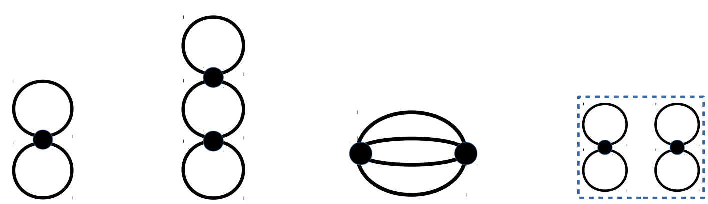
Parameter Values. It is important to note that a Green function is defined with arbitrary times, arbitrary energy variables. That is for a Green function we no longer distinguish between initial and final time parameters on fields, or set energies to be on-shell ($p_{\rm ext}^{\mu=0} = +\omega_p$ etc). Of course when we want to extract information about a real physical matrix element then these parameters will have to take on appropriate physical values.

In terms of diagrams we have

 $-i\lambda(2\pi)^4\delta^4(\sum_{l\in V} k_l)$ $\int \frac{d^4k}{(2\pi)^4}\,\Delta(k)$ (2)

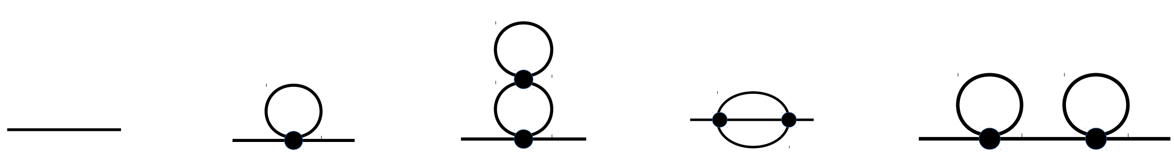
(iii) It is easiest if we classify the vacuum diagrams separately first. Then we can look at the connected diagrams for the full propagator. Finally we can combine any of the connected diagrams with one or more vacuum diagrams to create a diagram with more than one component up to $O(\lambda^2)$.

(a) The vacuum diagrams are



$O(\lambda^1)$ (3a) $O(\lambda^2)$ (3b) $O(\lambda^2)$ (3c) $O(\lambda^2)$ (3d) (3)

(b) The connected two-point diagrams, ones which contribute to the full propagator (connected two-point Green function) are



$O(\lambda^0)$ (4a) $O(\lambda^1)$ (4b) $O(\lambda^2)$ (4c) $O(\lambda^2)$ (4d) $O(\lambda^2)$ (4e) (4)

(c) The remaining diagrams consist of combining one or two vacuum diagrams in (3) with one of the connected diagrams in (4) provided we stay up to λ^2. Thus (4a) can be combined with any one of those in (3) to give total diagrams of two disconnected parts. We can also combine (4a) with (3d) (i.e. two copies of (3a)) to give a diagram with three disconnected parts. The last option is to combine (4b) with one (3a) to give a last $O(\lambda^2)$ diagram, one with two components.

(iv) The symmetry factors $\mathcal{S}$ and the number of loop momenta L are as follows

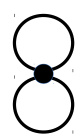
Diagram	$\mathcal{S}$	L
(4a)	1	0
(4b)	2	1
(4c)	4	2
(4d)	6	2
(4e)	4	2

(5)

I used Wick's theorem to check the symmetry factor of (46e) as it is used below and I wanted to be sure.

Diagram 4a

For a theory of a single real scalar field ϕ and an interaction Lagrangian density term $\mathcal{L}_{\text{int}} = -(\lambda/4!)\phi^4$, the symmetry factor for the only vacuum diagram at $O(\lambda)$ is indeed the following two-loop diagram



(6)

The $Z = \langle 0|S|\,0\rangle = 1 + \lambda_1 Z_1 + \ldots$ is given by sum over all vacuum diagrams (including those with disconnected components) and the exponential of the sum of all single component these In coordinate space we have

$$Z = \langle 0|\mathrm{T}\exp\{-i(\lambda/4!)\int d^4x\,\phi^4(x)\}|\,0\rangle \tag{7}$$

$$= 1 - i\frac{\lambda}{4!}\int d^4x\,\langle 0|\mathrm{T}\phi(x)\phi(x)\phi(x)\phi(x)|\,0\rangle + O(\lambda^2) \tag{8}$$

Now this four-field time-ordered product was considered in the question "Wick's theorem for four bosonic fields" (PS5, Q4 in 2015). There we showed using Wick's theorem that if we use the notation $\phi_i = \phi(x_i)$ then

$$\langle \mathrm{T}\,(\phi_1\phi_2\phi_3\phi_4)\rangle = +\wick{\c1{\phi_1}\c1{\phi_2}\c2{\phi_3}\c2{\phi_4}} + \wick{\c1{\phi_1}\c2{\phi_2}\c1{\phi_3}\c2{\phi_4}} + \wick{\c1{\phi_1}\c2{\phi_2}\c2{\phi_3}\c1{\phi_4}} \tag{9}$$

$$= \Delta_{12}\Delta_{34} + \Delta_{13}\Delta_{24} + \Delta_{14}\Delta_{23} \tag{10}$$

For our Z_1 calculation, all the coordinates are actually the same. Pretending that the four fields all have different coordinates and then setting them equal at the end helps us (well, helps me) count all the permutations. Basically here there are three different ways to choose two different pairs from four identical objects.

Since all the fields have the same coordinate for Z_1 term, the contractions in this expression (10) become $\Delta_{ij} = \Delta(x_i - x_j) = \Delta(x - x) = \Delta(0)$ where you must remember $\Delta(0)$ is in coordinate space. This means we have that

$$Z_1 \quad = \quad -i\frac{\lambda}{4!}\int d^4x\, 3(\Delta(0))^2 = -i\frac{\lambda}{8}\Omega(\Delta(0))^2 \tag{11}$$

That is we have a symmetry factor of $\mathcal{S} = 8$. The $\Omega = VT$ is the space-time volume factor. This is a two-loop diagram since in momentum space each of the $\Delta(0)$ is represented by a integration over an independent loop momentum

$$\Delta(0) = \lim_{x\to 0}\int \bar{d}^4k\, e^{-ikx}\frac{i}{k^2 - m^2 + i\epsilon} \,. \tag{12}$$

The integral, when regulated to control the infinities, can be done using contour integration tricks and this example is discussed in more detail in the context of the self-energy expression (24) below.

(v) The diagrams which contain a vacuum diagram and therefore do not contribute to the **full propagator** $G_c(x,y)$ or its Fourier transforms are clear from the discussion above. Only the diagrams with one connected piece contribute to the full propagator, that calculated with the full physical vacuum $|\,\Omega\rangle$. That is we have that

$$G_c(x,y) = \Pi(x-y) \quad = \quad \langle\Omega|\mathrm{T}\phi(x)\phi(y)S|\,\Omega\rangle \tag{13}$$

$$G_c(p_1,p_2) \quad = \quad \int d^4x_1\int d^4x_2\; e^{-ip_1x_1}e^{-ip_2x_2}G_c(x_1,x_2) \tag{14}$$

$$\Pi(p) \quad = \quad \int d^4x\; e^{-ipx}\Pi(x)\,. \tag{15}$$

$$\Rightarrow \quad G_c(p_1,p_2) \quad = \quad \bar{\delta}^4(p_1+p_2)\,\Pi(p_1) \tag{16}$$

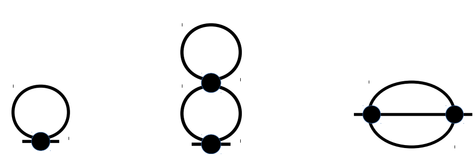
Note that Lorentz symmetry allows us to define a function $\Pi(x)$ through $\Pi(x-y) = G_c(x,y)$. This function, and its Fourier transform, is often referred to as the Full propagator. So the notation used here means that the 'full propagator' when written as $\Pi(p)$ is in terms of one four-momentum and so does not include the delta function representing the overall conservation, as (16) shows. Put another way, the Feynamn rules as given in this course give $G_c(p_1,p_2)$ but it is easy to get the $\Pi(p)$ function from that.

(vi) The full propagator may be written as

$$\Pi(p) = \Delta(p)\sum_{n=0}^{\infty}\left(\Sigma(p).\Delta(p)\right)^n \,, \tag{17}$$

where $\Delta(p) = i(p^2 - m^2 + i\epsilon)^{-1}$ is just the free propagator. The function $\Sigma(p)$ is called the **self-energy**. It is the two-point **1PI** (**one-particle irreducible**) function and it is described by the sum of 1PI diagrams with two amputated legs (two external legs but the propagator usually associated with them is not present). A 1PI diagram is one in which the external legs can not be separated by cutting *one* line.

The diagrams which contribute to $\Sigma(p)$ to $O(\lambda^2)$ are



(18)

$O(\lambda^1)$ (18a) $\quad O(\lambda^2)$ (18b) $\quad O(\lambda^2)$ (18c)

Note I have tried to denote the fact that external legs are truncated by drawing them as little 'stubs'. Each diagram should have two such stubs.

(vii) In our case Σ_1, the lowest order contribution to Σ at $O(\lambda^1)$ is simply the 'tadpole diagram' (18a).

To show that the formula (17) for the full propagator $\Pi(p)$ is consistent at $O(\lambda^2)$ with the contribution made by Σ_1 terms in the diagrammatic expansion we need to look at diagrams (4a), (4b) and (4e). These three diagrams represent a the terms in series containing zero, one and two factors of $\Sigma_1\Delta$ respectively. Note that from the table in (46) we see that the diagram (4e) has the correct symmetry factor, 4, twice the symmetry factor of 2 which is included in our definition of the Σ_1 function from diagram (18a). This ensures that we can express the diagram (4e) in the form $\Delta\Sigma_1\Delta\Sigma_1\Delta$ as required.

(viii) Using the binomial expansion we have that

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots = \frac{1}{1-x}\,, \qquad |x| < 1\,. \tag{19}$$

Treating Σ as being an $O(\lambda)$ object which can be treated as being "small" in perturbation theory, we have that

$$\Pi(p) = \frac{1}{(\Delta(p))^{-1} - \Sigma}\,. \tag{20}$$

Note that is really a notation which represents a particular infinite sum. It is not at all clear if the value of Σ falls in the radius of convergence, there are problems with high order terms in Σ and so far even our Σ_1 is infinite as no renormalisation has been performed.

(ix) When we say we have a particle of mass p in our experimental initial state or found in our detectors, we are making a statement that for widely separated particles (effectively free particles with no effective interactions) we can represent these particles as waves satisfying the Klein-Gordon equation. In momentum space, this corresponds to Green functions of the classical Klein-Gordon equation which are of the form $1/(p^2 - m^2)$ in momentum space. The Green functions of the classical Klein-Gordon equation are linked with the vacuum expectation values of the time-ordered products of two fields in QFT. This means we want the two-point Green function evaluated in the physical vacuum to be representing the propagation of our physical particles. As propagators blow up **on-mass shell**, i.e. at $p^2 = m^2$, it means we want our propagators to have a pole at $p^2 = m^2_{\text{phys}}$. This is how we would measure the mass of a particle given a measurement of a Green function or the related matrix element.

This means that

$$[\Pi(p)]^{-1} = -i(p^2 - m^2 + i\epsilon) = -i(p^2 - m^2) \qquad \text{as } \epsilon \to 0 \tag{21}$$

$$0 = [\Pi(p^2 = m^2_{\text{phys}})]^{-1} = -i(m^2_{\text{phys}} - m^2) - \Sigma(p^2 = m^2_{\text{phys}}) \tag{22}$$

$$\Rightarrow \qquad m^2_{\text{phys}} = m^2 - i\Sigma(p^2 = m^2_{\text{phys}}) \tag{23}$$

(x) We see that

$$\Sigma_1(p) = -ig \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} \tag{24}$$

Clearly the momentum of the external legs, p, does not contribute here so Σ_1 is independent of p and is only a function of the ϕ mass in the propagator. To estimate the divergence of the integral, observe that the small k regime is well behaved as the integrand is im^{-2} and constant. You might worry about the pole but you can show that this integral is equal to (see Wick rotation discussion below)

$$\Sigma_1(p) \quad = \quad -ig \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dk_4}{2\pi} \frac{1}{(k_4)^2 + \boldsymbol{k}^2 + m^2} \tag{25}$$

where in the complex plane we have $k_0 = ik_4$. In this form there is no pole so the region of integration around the mass shell $k^2 = m^2$ can not be a significant contribution to the integral.

So the dominant contribution must come from the large $|k|$ region, the UV part of the integration. Just on dimensional grounds you can see that the integral is of the form[1] $\int^{\Lambda} dK\, K/(K + m^2)$ (here I changed variables to $K = (k_4)^2 + \boldsymbol{k}^2$ and note you can now do this integral) so that Σ diverges quadratically, $\Sigma \sim \Lambda^2$. In fact there are also logarithmic divergences so exploiting Lorentz properties and dimensional arguments you might guess that $\Sigma \sim g(\Lambda^2 + m^2 \ln(\Lambda^2/m^2) +$ (finite parts).

Changing from a real energy variable to one which is pure imaginary is known as a **Wick Rotation**. To see this add these two terms, (24) and (25) together in terms of integral over k_0 in the complex

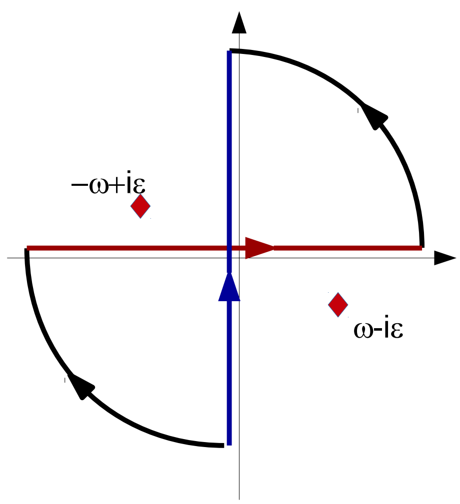


Figure 1: The curves used for Wick rotation. Here the real line represent real physical k_0 values and curve used in (24). The imaginary axis represents the integration in the variable $k_4 = -ik_0$ used in the final version of the integral (25). By working in the Euclidean energy variable we have that $k^2 = (k_0)^2 - \boldsymbol{k}^2 = -(k_4)^2 - \boldsymbol{k}^2$ which puts all four components (k_1, k_2, k_3, k_4) on an equal footing i.e. formally a Euclidean metric. The trick is to link the two integration paths into a single closed loop by reversing the direction for the k_4 part of the integration and adding two quarter circle circles at infinity as shown. The loop does not enclose any pole so the total integral is zero.

plane, one along the real axis, one along the imaginary axis. Add integrations over quarter circles at $|k_0| = \infty$, $0 \leq \arg(k_0) \leq \pi/2$ and $\pi \leq \arg(k_0) \leq 3\pi/2$ which you need to convince yourself have zero value. Arrange directions of integration so that these form a closed loop and observe that the

[1] To do this integral you realise that we have an integral which depends on on the length $\sqrt{K}$ of the Euclidean four-vector $(k_4, \boldsymbol{k})$ where $K = (k_4)^2 + \boldsymbol{k}^2 > 0$. You see that we need to integrate over spheres in four dimensions and you can find the factor of π you need for this elsewhere. Here you need only get the dimensional parts correct. Note that after this you have an integral you can actually do.

poles of the propagator are not inside the loops so the total must be zero. That is we have that

$$0 = \int_C dz \, \frac{1}{z^2 + \omega^2} \tag{26}$$

where C is the curve shown in figure 1.

Analyse the value of $\Sigma_1(p)$, the $O(\lambda^1)$ contribution to the self-energy. Do not calculate this in detail but argue that if we limit the size of the three- or four-momenta in the integration to be $O(\Lambda)$ or less, then $\Sigma_1 = c\Lambda^2$. Also deduce that Σ_1 is independent of the external momenta p. What does this tell us about the Lagrangian mass parameter m as $\Lambda \to \infty$?

*2. Scattering in $\lambda\phi^4$ theory

NOTE: in the diagrams in this question I have added labels p_1, p_2, q_1 and q_2. This mimics the notation used for initial and final state momenta used in the lectures of *matrix elements*. However here we are dealing with *Green functions* not matrix elements, where there is no limitation on the values of the four-momenta. So it is best to think in terms of $p_3 = -q_1$ and $p_4 = -q_2$ so that all four p_i four-momenta variables are flowing *into* the diagrams and there is no limit on the zero components $p_i^{\mu=0}$. In particular there is no need to put the momenta on mass-shell, no need to fix $p_i^{\mu=0} = \pm\omega_p$ as we would have if dealing with matrix elements. Of course to extract the matrix element from the Greens function we would have to apply such physical constraints in order to get the physics of scattering of real physical particles. The labelling of the diagrams is therefore annoying and will be changed in the future.
Consider the $\lambda\phi^4$ theory of a real scalar field ϕ described by the Lagrangian density (1).

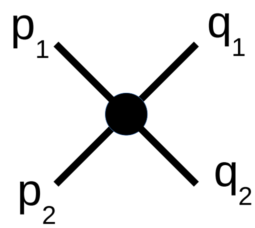
(i) See answer above.

(ii) The four-point Green function defined with respect to the free vacuum is

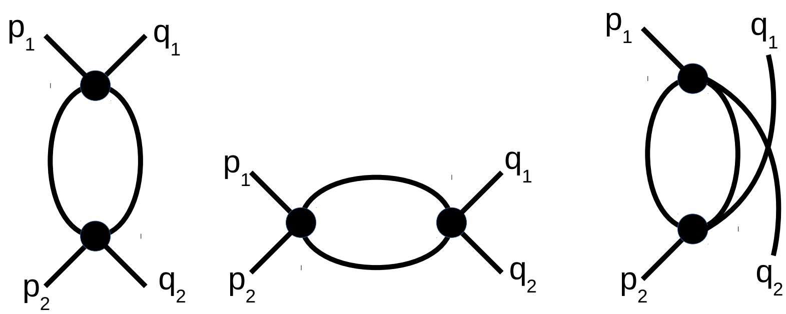
$$G_0(x_1, x_2, x_3, x_4) = \langle 0|\mathrm{T}[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)S]|\, 0\rangle \,. \tag{27}$$

is related to the matrix element $\mathcal{M}_0 = \langle q_1, q_2|S|\, p_1, p_2\rangle$ describing $\phi\phi \to \phi\phi$ scattering by truncation of the legs. More precisely we showed in the lectures that converting initial/final states into fields acting on the vacuum involved a factor coming from a Fourier transform like factor of $\int d^3y \, \exp\{-ip_{\text{ext}}y\} 2\omega(p_{\text{ext}})$.

1PI four-point diagrams with propagators added as legs. The diagrams with non-trivial contributions to scattering are those with a single propagator on each leg attached to either a single vertex or a loop of two ϕ-propagators. Note that there are three distinct $O(\lambda^2)$ diagrams even though their shape is the same. This is because the label on the legs is important. The reordering of the legs around the single vertex is accounted for in the Feynman rules.



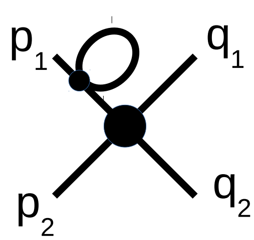
(28.1)



(28)

(28.2)

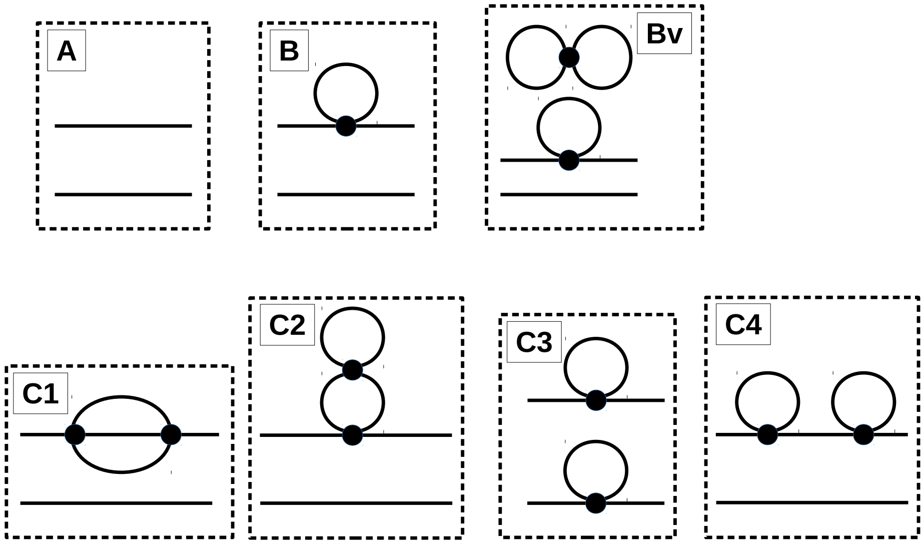
Since we are dealing with Green functions not matrix elements, in (28) we have $q_1 = -p_3$ and $q_2 = -p_4$, all p_i four-momenta are flowing into the diagrams and there is no limit on the zero components $p_i^{\mu=0}$. The $O(\lambda)$ diagram in (28) can replace the free propagator on any one leg with any $O(\lambda)$ contribution to the pull propagator to give an $O(\lambda^2)$ diagram overall. We see from (4) that there is only one $O(\lambda)$ contribution to the full propagator (i.e. to the self-energy) and one example is shown in (29)



$(q_1 = -p_3$ and $q_2 = -p_4)$ (29)

Again this diagram in (29) is a contribution to a Green function not a matrix element and we have $q_1 = -p_3$ and $q_2 = -p_4$ with all p_i four-momenta flowing into the diagram and with no limit on the zero components $p_i^{\mu=0}$. This means there are three more of these types of diagram (29) with the tadpole added to different external legs. Such diagrams are obviously really just telling us about the ϕ propagator and give no new information about the scattering process.

No Scattering diagrams Now we can can have diagrams which contribute only to pure propagation processes which have no contribution to scattering of physical particles. Such diagrams have two disconnected parts, each one a contribution to the full propagator as shown in (30). We find the following:-



(30)

In principle the diagrams B, Bv, C1, C2 and C4 in (30) come in six variants as we can choose the two four-momenta associated with the the free propagator line in six different ways. For any physical $\phi\phi \to \phi\phi$ scattering, two of these combinations would link what we have been thinking of as the energy/momentum of initial states, say p_1 and p_2, leading to a factor of $\Pi(p_1)(2\pi)^4\delta^4(p_1+p_2)$ or $\Pi(p_3)(2\pi)^4\delta^4(p_3+p_4)$. For a matrix element this would not be allowed kinematically if p_1 and p_2 were initial state momenta and $p_3 = -q_1$ and $p_4 = -q_2$ were final state momenta. In that case the zero components would satisfy $p_1 + p_2 \geq 2\omega_p >$ and $p_3 + p_4 \leq -2\omega_p < 0$, i.e. neither ever zero, and this diagram will not contribute to that matrix element. Such an assignment is, however, an allowed diagram for a Green function which is defined as a function of the unconstrained external four-momenta and these not put these on mass-shell. The reason for this is that we might be interested in the behaviour of the Green function off-mass shell because it is used to describe virtual processes as a part of some larger physical interaction problem.

Diagrams A and C3 have a symmetry between the two propagators so in this case there are only three distinct ways of assigning the four-momenta p_i.

Vacuum subdiagrams. The next step is to realise we can also add in disconnected vacuum diagrams to any of the diagrams above. For instance diagram B in (30) is one example as we can add the only $O(\lambda)$ vacuum diagram of (3a) to produce a three component diagram, Bv in (30). To diagram A in (30) we can add any of the three vacuum diagrams in (3) to give one more two-component diagrams of $O(\lambda)$ and two more of $O(\lambda^2)$. Finally we can add two copies of the $O(\lambda^1)$ vacuum diagram (3a) to diagram A of (30) to produce a four component $O(\lambda^2)$ contribution to $G_0(x_1, x_2, x_3, x_4)$ of (27). For all of these cases, there are three different assignments of the momenta.

Odd numbers of legs. Finally any diagram with an odd number of legs is zero. We could imagine diagrams of two components,one a three leg component, one a single leg component. An n-point Green function will have diagrams of v vertices representing the vacuum expectation value of the time-ordered product of $(n+4v)$ fields. Wick's theorem will only have terms which are just products of contractions if $(n+4v)$ is even. If n is odd then $(n+4v)$ is odd so every term in a Wick's theorem expansion of the time ordered product will have a normal ordered product present which we know is zero for a suitable choice of filed split. Thus n must be even to get a non-zero result and we need not consider any diagram with a connected component with an odd number of legs.

(iii) From the description above it is clear that any diagram component not connected to one of the external lines, so with a disconnected subdiagram (or two) from one of the vacuum diagrams of (3), are diagrams which do *not* contribute to this four-point Green function when defined with respect to the full physical vacuum $|\,\Omega\rangle$, i.e. when calculating $G(p_1,p_2,p_3,p_4)$ which is the appropriate Fourier transform of

$$G(x_1,x_2,x_3,x_4) \;=\; \langle\Omega|\mathrm{T}[\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4))S]|\,\Omega\rangle\,, \tag{31}$$

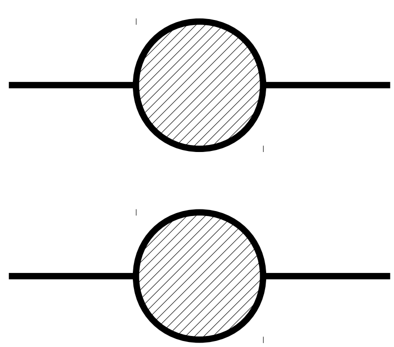
$$G(p_1,p_2,p_3,p_4) \;=\; \left(\prod_{a=1}^{4}\int d^4x_a\,\exp\{-ip_a\cdot x_a\}\right)G(x_1,x_2,x_3,x_4)\,. \tag{32}$$

(iv) Connected diagrams are diagrams where every vertex and line can trace a path along the lines of the diagram to *every* other vertex and line. The only connected diagrams here are shown in (28) and (29) have one connected component. Any based on (30) or with additional vacuum subdiagrams are all examples of disconnected diagrams.

(v) For the disconnected diagrams for G (so no vacuum subdiagrams), so those in (30), we can clearly see that there are two two-point connected components. In terms of diagrams we can see that these two component contributions to the four-point function G of (32) are of the form $\Pi(p_a)\Pi(p_b)$ when we compare (30) with the diagrams in (4) for Π, the full propagator of (16). The labels (a,b,c,d) are any permutation of $(1,2,3,4)$ and we define each p_a to be a four-momenta flowing into the diagram (or, equally good, they all flow out). Clearly each component must conserve energy and momentum separately (the Feynman rules enforce this) so we must get two appropriate delta functions, one for each component. So for each permutation we get something like $\Pi(p_a)\Pi(p_b)\bar{\delta}^4(p_a+p_c)\bar{\delta}^4(p_b+p_d)$. As we've pointed out above there are then three distinct ways to assign the momenta to the legs in this case ($\Pi(p)=\Pi(-p)$ as Π is a function of p^2 by Lorentz symmetry) so overall we will find

$$\Pi(p_1)\Pi(p_2)\bar{\delta}^4(p_1+p_3)\bar{\delta}^4(p_2+p_4)+\Pi(p_1)\Pi(p_2)\bar{\delta}^4(p_1+p_4)\bar{\delta}^4(p_2+p_3)+\Pi(p_1)\Pi(p_3)\bar{\delta}^4(p_1+p_2)\bar{\delta}^4(p_3+p_4)\,. \tag{33}$$

Diagrammatically this is represented as



(34)

where a circle with parallel line shading is used to show connected diagrams with a number of untruncated legs.

In a scattering process where p_1 and p_2 ($p_3=-q_1$ and $p_4=-q_2$) then $p_1^{\mu=0}=\omega(\boldsymbol{p}_1)>0$ and $p_2^{\mu=0}=\omega(\boldsymbol{p}_2)>0$ so $\bar{\delta}(p_1^{\mu=0}+p_2^{\mu=0})$ is always zero.

As explained above, a 1PI diagram is one which can not be cut into two separate parts by cutting *one* line. In particular this means there are no external lines on a 1PI diagram. See figure 2 for some examples.

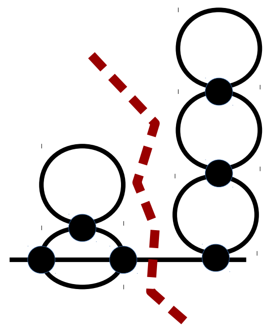
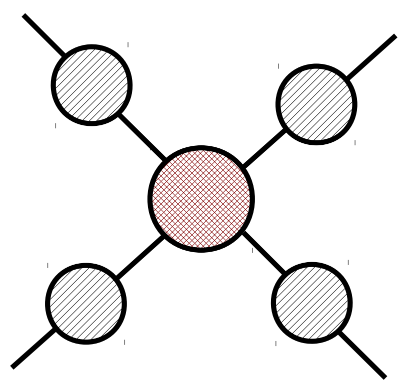


Figure 2: Example of a 1PI diagram. Here the external legs have been amputated/truncated so they are not present to be cut. However the line in the middle if cut, as indicated by the dashed line, would leave this diagram in two pieces. This is not a 1PI diagram. The two subdiagrams on either side of the dashed line, if the connecting propagator has been removed (the one intersecting the dashed line), are both in fact 1PI diagrams in this case.

(vi) Diagrammatically the diagrams in (28) are of the generic form



(35)

where circles with cross hatching represents 1PI diagram contributions, and we have shown that we can "pull out" full propagator legs, that is parts of the diagrams corresponding to contributions to connected two-point functions. Such connected diagrams are contributions in the expansion of the expression

$$\bar{\delta}^4(p_1+p_2+p_3+p_4)\Pi(p_1)\Pi(p_2)\Pi(p_3)\Pi(p_4).(-i\Gamma^{(4)}(p_1,p_2,p_3)) \tag{36}$$

to $O(\lambda^2)$. Here $-i\Gamma^{(4)}(p_1,p_2,q_1)$ is given by a sum of four-point 1PI diagrams to $O(\lambda^2)$with the explicit overall energy/momentum conserving delta function ($\bar{\delta}^4(p_1+p_2+p_3+p_4)$) written explicitly in this case.

Thus $-i\Gamma^{(4)}(p_1,p_2,p_3)$ to $O(\lambda^2)$ is just the sum of the diagrams in (28). Note in particular that to lowest order, $\Gamma^{(4)}=\lambda$ so this function $\Gamma^{(4)}$ given by four-point 1PI diagrams represents the quantum corrections to the coupling constant, the effect of virtual fluctuations on the strength of this interaction. Overall we have

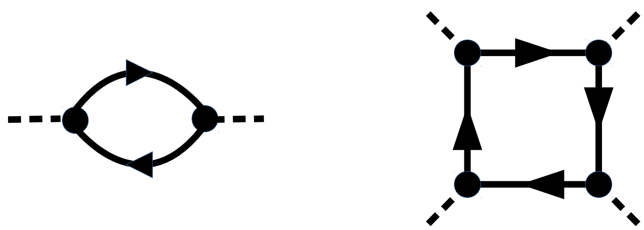
$$\begin{aligned}
-i\Gamma^{(4)}(p_1,p_2,p_3) &= -i\Gamma_1^{(4)}(p_1,p_2,p_3) - i\Gamma_2^{(4)}(p_1,p_2,p_3) && (37)\\
-i\Gamma_1^{(4)}(p_1,p_2,p_3) = -ig & && (38)\\
-i\Gamma_2^{(4)}(p_1,p_2,p_3) &= \frac{(-ig)^2}{2}\left(B(p_1+p_2)+B(p_1+p_3)+B(p_1+p_4)\right) && (39)\\
B(p_1,p_2) &= \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}\frac{i}{(k+p_1+p_2)^2-m^2+i\epsilon} && (40)
\end{aligned}$$

Dimensional analysis suggests that $\Gamma_2^{(4)}$ diverges as $\ln(\Lambda)$ at high energy scales Λ.

(vii) The Lagrangian density for scalar Yukawa theory is given by

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 + (\partial_\mu\psi^\dagger)(\partial^\mu\psi) - M^2\psi^\dagger\psi - g\psi^\dagger(x)\psi(x)\phi(x) \qquad (41)$$

This is the left hand diagram in (42)



(42)

In the limit where we are working at energy scales E_{CM} well below the ψ mass M then the only way we get something interesting is if $M \gg E_{\mathrm{CM}} \sim m$. In this situation the ψ particles are barely excited and we can think of M as a large parameter. So first consider the ϕ propagator. The lowest order self-energy correction for the ϕ propagator will look like

$$\Sigma_\phi(p) = (-ig)^2\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-M^2+i\epsilon}\,\frac{i}{(p+k)^2-M^2+i\epsilon} \sim \frac{g^2}{M^4} \qquad (43)$$

where we will assume the UV infinities have been dealt with (the integral is formally infinite). Thus the physical ϕ mass squared is really $m^2_{\mathrm{phys}} = m^2 + c(g^2/M^4)$ where c is some constant.

In the same way if we consider a four-point ϕ diagram we will we can generate terms of $\Gamma_\phi^{(4)} \sim (g^4/M^8)$, for instance

$$\begin{aligned}
\Gamma_\phi^{(4)}(p) &= (-ig)^4\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-M^2+i\epsilon}\,\frac{i}{(p_1+k)^2-M^2+i\epsilon}\\
&\qquad \frac{i}{(p_1+p_2+k)^2-M^2+i\epsilon}\,\frac{i}{(p_1+p_2+p_3+k)^2-M^2+i\epsilon} && (44)\\
&\sim \frac{g^4}{M^8}\,. && (45)
\end{aligned}$$

from the right-hand diagram in (42). We saw above that the lowest order term in a ϕ^4 interaction was $\Gamma^{(4)} = \lambda$ so in some sense the quantum corrections, here coming from heavy unseen undiscovered ψ particles with masses much higher than the experimental scale E_{CM}, can mimic the effect of a simple pure ϕ^4 self-interaction. Of course by doing the calculations carefully you will find that the pure ϕ^4 self-interaction and the heavy $\phi\psi^\dagger\psi$ theory differ in their detailed predictions though you can guess these will only be really noticeable close to M. Thus, by probing energies approaching $M \sim E_{\text{CM}}$ we might be able to guess a new ψ particle is there. Ultimately you would confirm your suspicions most clearly if you run the experiment at higher energies of $E_{\text{CM}} \geq 2M^2$ as then a ϕ can decay into a ψ-$\bar{\psi}$ (anti-ψ) pair and we can observe these new particles directly.

♯3. Exponential form for Z

(i) The diagrams in the expansion for Z are vacuum diagrams. The relevant diagrams to $O(\lambda^2)$ have already been given in (3a), (3b), (3c) and (3d) above. The symmetry factors $\mathcal{S}$ and the number of loop momenta L are as follows

Diagram	$\mathcal{S}$	L
(3a)	8	2
(3b)	16	3
(3c)	48	2
(3d)	128	4

(46)

The vacuum diagram (3d) illustrates that some of the contributions to Z come from diagrams which consist of disconnected pieces — 'components' as they are called in graph theory. Each of these components also appears in the expansion with a single 'figure-of-eight' diagram (3a) being the unique $O(\lambda^1)$ contribution to Z. That is if V is a **fully connected**[2] vacuum diagram so it appears in the expansion of Z, then there is always a higher order contribution proportional to V^2 which is represented as a single diagram consisting of two connected subdiagrams.

(ii) Focus on the single $O(\lambda^1)$ vacuum diagram, V_1, the 'figure-of-eight' diagram (3a). Let us write this as $V_1 = \lambda F_1/\mathcal{S}_1$ where $\mathcal{S} = 8$ is the symmetry factor. What we observe is that the second order diagram containing two figure-of-eight components (3d) can be written as $V_1 = (\lambda F_1)^2/(2\times\mathcal{S}_1\times\mathcal{S}_1)$ since the symmetry factor for this diagram is 128. Thus, in tyerms of these figure-of-eight diagrams only, we have that

$$Z = 1 + \lambda\frac{F_1}{\mathcal{S}_1} + \lambda^2\frac{(F_1)^2}{2(\mathcal{S}_1)^2} + O(\lambda^3) + (\text{other diagrams}) \tag{47}$$

$$= 1 + \lambda\frac{F_1}{\mathcal{S}_1} + \frac{\lambda^2}{2}\left(\frac{F_1}{\mathcal{S}_1}\right)^2 + O(\lambda^3) + (\text{other diagrams}) \tag{48}$$

$$= \exp\left(\frac{F_1}{\mathcal{S}_1}\right) + O(\lambda^3) + (\text{other diagrams}) \tag{49}$$

That is we have that in the expansion for Z, the terms which correspond to diagrams made up only of single component V_1 diagrams, that is terms proportional to $(V_1)^n$ contribute to Z in such a way that we may write

$$Z = \exp\{V_1\} + \dots \tag{50}$$

To see that the n-th order term in this expansion has all teh correct factors we need to convince ourselves that the vacuum diagram with n components, each a figure-of-eight diagram (3a), we need to have that such a diagram has a symmetry factor of $(\mathcal{S}_1)^n n!$. The $n!$ is simply left over from the expansion of the exponential in the S matrix. As we do exactly the same set of contractions

[2] In a **fully connected** diagram, there is a path from every vertex (internal or external) to *every* other vertex in the diagram.

at each vertex (3 different ways to do that which with the 4! from the interaction leave the $\mathcal{S}_1 = 8$ factor), permuting the vertices leaves us with the same set of contractions. So it will contribute exactly once to Wicks theorem which only every counts each distinct set of contractions. The $n!$ is never cancelled in this case. Hence the general result.