The Standard Model and Beyond

Outline

- The Standard Model and its generations
  - Review of the $SU(3)\times SU(2)\times U(1)$ representation content
  - Custodial $SU(2)$ and consequences of the simple structure with a single Higgs doublet
  - Mixing of generations and CKM structure
- Neutrino masses
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- Approximate symmetries
  - $U_L(3)\times U_R(3)$ and QCD
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- Grand Unification : $SU(5)$, $SO(10)$
- The Running of Coupling Constants + renormalization group eqn.
- Problems with unification beyond the SM
- Supersymmetry and the Minimal Supersymmetric Standard Model

Some books:

Michael Dine - Supersymmetry and String Theory Beyond the Standard Model

Cliff Burgess + Guy Moore - The Standard Model: A Primer

Matthew Schwarz - Quantum Field Theory and the Standard Model

Steven Weinberg - The Quantum Theory of Fields, Vol. 2

Gordon Kane - Modern Elementary Particle Physics

E.S. Abers and B.W. Lee - Gauge Theories
Physics Reports 9C, No. 1, November 1973

Notation: mostly-plus, as for Weinberg

The Standard Model

Prior to consideration of neutrino masses, the Standard Model Lagrangian is determined by $SU(3) \times SU(2)_L \times U(1)_Y$ Yang-Mills gauge invariance, the requirements of renormalizability ($\Rightarrow$ only Lagrangian terms of dimensions ≤ 4 in $D=4$ spacetime) and the following choice of group representations $(SU(3), SU(2))_{Y\ \text{charge}}$:

$(SU(3), SU(2))_{U(1)}$

- $SU(3) \times SU(2) \times U(1)$ gauge fields: $G_\mu^g\ (8,1)_0$ ($g=1,\dots,8$), $W_\mu^i\ (1,3)_0$ ($i=1,2,3$), $B_\mu\ (1,1)_0$
- Leptons:
  - cap: Left $(1,2)_{-1/2}$ $\quad L_m = \begin{pmatrix} \nu_m \\ e_{Lm} \end{pmatrix}$ $\quad m=1,2,3$ generations
  - l.c.: right $(1,1)_{-1}$ $\quad e_{Rm}$ $\quad m=1$ "electron" $m=2$ "muon" $m=3$ "Tau"
- Quarks:
  - cap: Left $(3,2)_{1/6}$ $\quad Q_{LmA} = \begin{pmatrix} u'_{LmA} \\ d'_{LmA} \end{pmatrix}$ $\quad A=1,2,3$ colour $SU(3)$ index
  - l.c.: right $(3,1)_{2/3}$ $\quad u'_{RmA}$
  - l.c.: $(3,1)_{-1/3}$ $\quad d'_{RmA}$
- Higgs doublet $(1,2)_{1/2}$ $\quad \phi_a = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$ $\quad$ Simplest model has a single Higgs doublet

With these choices, the most general gauge invariant and renormalizable Lagrangian is

$$\mathcal{L}_{SM} = \mathcal{L}_{gauge} + \mathcal{L}_{spinor} + \mathcal{L}_{Higgs} + \mathcal{L}_{Yukawa} + \mathcal{L}_{theta}$$

$\eta_{\mu\nu} = \begin{pmatrix} - & & & \\ & + & & \\ & & + & \\ & & & + \end{pmatrix}$ "mostly plus"

$$\mathcal{L}_{gauge} = -\tfrac{1}{4} G^g_{\mu\nu} G^{g\mu\nu} - \tfrac{1}{4} W^i_{\mu\nu} W^{i\mu\nu} - \tfrac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

where

$$G^g_{\mu\nu} = \partial_\mu G^g_\nu - \partial_\nu G^g_\mu + g_3 f^{g\sigma\tau} G^\sigma_\mu G^\tau_\nu \qquad f^{g\sigma\tau}: SU(3) \text{ structure constants } (g=1,\dots,8)$$

$$W^i_{\mu\nu} = \partial_\mu W^i_\nu - \partial_\nu W^i_\mu + g_2 \epsilon^{ijk} W^j_\mu W^k_\nu \qquad \epsilon^{ijk}: SU(2) \text{ structure constants } (i=1,2,3)$$

$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$\mathcal{L}_{spinor} = -i\overline{L}_m \not{D} L_m - i\overline{e}_{Rm} \not{D} e_{Rm} - i\overline{Q}_{Lm} \not{D} Q_{Lm} - i\overline{u}'_{Rm} \not{D} u'_{Rm} - i\overline{d}'_{Rm} \not{D} d'_{Rm}$$

$D_\mu = \partial_\mu - i g W_\mu \cdot T$

where the covariant derivatives are

$$D_\mu L_m = \partial_\mu L_m + \left(-\tfrac{i}{2} g_2 W_\mu \cdot \sigma + \tfrac{i}{2} g_1 B_\mu\right) L_m \,;\quad D_\mu e_{Rm} = \partial_\mu e_{Rm} + i g_1 B_\mu e_{Rm}$$

$$D_\mu Q_{Lm} = \partial_\mu Q_{Lm} + \left(-\tfrac{i}{2} g_3 G_\mu \cdot \lambda - \tfrac{i}{2} g_2 W_\mu \cdot \sigma - \tfrac{i}{6} g_1 B_\mu\right) Q_{Lm}$$

$$D_\mu u'_{Rm} = \partial_\mu u'_{Rm} + \left(-\tfrac{i}{2} g_3 G_\mu \cdot \lambda - \tfrac{2i}{3} g_1 B_\mu\right) u'_{Rm} \qquad \tfrac{1}{2}\lambda^g: SU(3) \text{ generators for } \underline{3} \text{ rep.}$$

$$D_\mu d'_{Rm} = \partial_\mu d'_{Rm} + \left(-\tfrac{i}{2} g_3 G_\mu \cdot \lambda + \tfrac{i}{3} g_1 B_\mu\right) d'_{Rm} \qquad \tfrac{1}{2}\sigma^i: SU(2) \text{ generators for } \underline{2} \text{ rep.}$$

$$\mathcal{L}_{Higgs} = -D_\mu \phi^\dagger D^\mu \phi - \lambda\left(\phi^\dagger\phi - \frac{\mu^2}{2\lambda}\right)^2 \qquad \phi^\dagger = (\phi^{+*}, \phi^{0*})$$

$$D_\mu \phi = \partial_\mu \phi + \left(\frac{-i}{2} g_2 W_\mu \cdot \sigma - \frac{i}{2} g_1 B_\mu\right)\phi$$

- Note that for the direct-product gauge group $SU(3) \times SU(2)_L \times U(1)_Y$, the three simple factor groups can each have their own separate coupling constants: g_3, g_2, g_1

$$\mathcal{L}_{Yukawa} = -i\left(f_{mn} \overline{L}_m e_{Rn} \phi + h_{mn} \overline{Q_{Lm}} d'_{Rn} \phi + k_{mn} \overline{Q_{Lm}} u'_{Rn} \tilde{\phi}\right) + h.c.$$

- Note how all the gauge-group contractions make invariants:

$\overline{L_m} e_{Rn} \phi = \overline{L_m}^a e_{Rn} \phi_a$, hypercharge $1/2 - 1 + 1/2 = 0$

$\overline{Q_{Lm}} d'_{Rn} \phi = \overline{Q_{Lm}}^{Aa} d'_{RnA} \phi_a$, hypercharge $-\frac{1}{6} - \frac{1}{3} + \frac{1}{2} = 0$

$\overline{Q_{Lm}} u'_{Rn} \tilde{\phi} = \overline{Q_{Lm}}^{Ab} u'_{RnA} \tilde{\phi}_b$, hypercharge $-\frac{1}{6} + 2/3 - 1/2 = 0$

where $\tilde{\phi}_a = \epsilon_{ab} \phi^{*b} = \begin{pmatrix} \phi^{0*} \\ -\phi^{+*} \end{pmatrix}$ transforms in the $(1,2)_{-1/2}$

— complex conjugation of complex representations like ϕ_a, d'_{RA} produce conjugate representations denoted with raised indices: $(\phi_a)^* = \phi^{*a}$, $(d'_{RA})_c = (d'_R)_c{}^A$ where the Lorentz charge conjugate is $(\lambda)_c = C(\bar{\lambda})^T$, $\bar{\lambda} = \lambda^\dagger \gamma^0$, $C =$ charge conjugation matrix

- The prime notation on $u'_{Lm}, d'_{Lm}, u'_{Rm}, d'_{Rm}$ anticipate redefinitions in the family (generation) labels "m" which will be made to diagonalize the quark mass matrix. These redefinitions will also have an important effect on the quark couplings to the charged $W_\mu^\pm$ gauge fields.

The remaining part $\mathcal{L}_{theta}$ of the Standard Model Lagrangian has no effect at the classical level, but nonetheless has an important set of rôles to play in the quantum theory.

$$\mathcal{L}_{theta} = -\frac{g_3^2\theta_3}{32\pi^2} G^{\vartheta}_{\mu\nu}\tilde{G}^{\vartheta\mu\nu} - \frac{g_2^2\theta_2}{32\pi^2} W^i_{\mu\nu}\tilde{W}^{i\mu\nu} - \frac{g_1^2\theta_1}{32\pi^2} B_{\mu\nu}\tilde{B}^{\mu\nu}$$

in which the field strengths with tildes $\tilde{G}^{\vartheta}_{\mu\nu}$, $\tilde{W}^i_{\mu\nu}$, $\tilde{B}^{\mu\nu}$ are <u>Hodge duals</u> of the original field strengths:

$$\tilde{G}^{\vartheta}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}G^{\vartheta\alpha\beta},\ \tilde{W}^i_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}W^{i\alpha\beta},\ \tilde{B}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\alpha\beta}B^{\alpha\beta}$$

• Note that the Levi-Civita tensor $\epsilon_{\mu\nu\alpha\beta}$ is a numerically invariant tensor only under $SO(3,1)$, but not under $O(3,1)$. Under $O(3,1)$ transformations $\Lambda_\mu{}^\nu$ with $\det(\Lambda) = -1$, $\epsilon_{\mu\nu\alpha\beta}$ flips by a factor of (-1). Hence $\mathcal{L}_{theta}$ is <u>odd</u> under transformations such as parity, $(x^1,x^2,x^3,t) \to (-x^1,-x^2,-x^3,t)$.

— Noninvariance under parity is also a standout feature of the Standard Model in the fermionic sector because of the key rôle of chiral fields:

$$P_+L_m = L_m,\quad P_-e'_{Rm} = e_{Rm};\quad P_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma_5),\ \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

$$P_+Q_{Lm} = Q_{Lm},\quad P_-u'_{Rm} = u'_{Rm},\quad P_-d'_{Rm} = d'_{Rm}.$$

• The reason why $\mathcal{L}_{theta}$ does not affect the theory at the classical level is that each of the three terms is a total divergence. One quick way to see this is to consider the variation of such a term and use the Yang-Mills version of the Palatini identity:

Palatini id.

$$\delta G^{\vartheta}_{\mu\nu} = D_\mu\delta G^{\vartheta}_\nu - D_\nu\delta G^{\vartheta}_\mu,$$ in which D_μ is the covariant derivative in the adjoint representation. Then

δG^{ϑ}_ν is tensorial

$$\delta(G^{\vartheta}_{\mu\nu}\tilde{G}^{\vartheta\mu\nu}) = 2(\delta G^{\vartheta}_{\mu\nu}\tilde{G}^{\vartheta\mu\nu}) = 4(D_\mu\delta G^{\vartheta}_\nu\tilde{G}^{\vartheta\mu\nu})$$
$$= 4\partial_\mu(\delta G^{\vartheta}_\nu\tilde{G}^{\vartheta\mu\nu}) - 4(\delta G^{\vartheta}_\nu D_\mu\tilde{G}^{\vartheta\mu\nu})$$

but $D_\mu\tilde{G}^{\vartheta\mu\nu} \equiv 0$, so $\delta(G^{\vartheta}_{\mu\nu}\tilde{G}^{\vartheta\mu\nu}) = 4\partial_\mu(\delta G^{\vartheta}_\nu\tilde{G}^{\vartheta\mu\nu})$.

The total divergence property also holds for the unvaried terms. One has, e.g.,

$$G^{a}_{\mu\nu}\tilde{G}^{a\mu\nu} = \partial_\mu K^\mu \; ; \; K^\mu = 2\epsilon^{\mu\alpha\beta\gamma}\left(G^a_\alpha \partial_\beta G^a_\gamma + \tfrac{1}{3} f^{a b c} G^a_\alpha G^b_\beta G^c_\gamma\right),$$ where K^μ is the Chern-Simons current.

Bosonic masses and the mixing angle

Under $SU(2)_L \times U(1)_Y$, the Higgs doublet transforms as

$$\phi_a \to \exp\left(\tfrac{i}{2}\zeta\cdot\sigma\right)_a{}^b \phi_b \qquad SU(2)_L$$

$$\phi_a \to \exp\left(\tfrac{i}{2}\rho\right)\phi_a \qquad U(1)_Y$$

For the potential $V = \lambda\left(\phi^\dagger\phi - \frac{\mu^2}{2\lambda}\right)^2$, the vacuum expectation value ϕ_0 may be taken to be

$$\phi_0 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}v \end{pmatrix} \quad \text{with} \quad v = \frac{\mu}{\sqrt{\lambda}}, \quad v \text{ real.}$$

The stability subgroup H for this broken symmetry system is identified by making infinitesimal transformations around ϕ_0:

$$\delta\phi_0 = \frac{i}{2}\left(\zeta\cdot\sigma + \rho\right)\phi_0 = \frac{i}{2}\begin{pmatrix} \zeta_3+\rho & \zeta_1 - i\zeta_2 \\ \zeta_1 + i\zeta_2 & -\zeta_3+\rho \end{pmatrix}\begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$$

Requiring $\delta_H\phi_0 = 0$, one finds $\zeta_1 = \zeta_2 = 0$ and $\zeta_3 = \rho$, so the unbroken stability subgroup $H = U(1)_{em}$ is generated by $T^3 + Y$.

Expand ϕ about ϕ_0 using an exponential parametrization:

$$\phi = \exp\left(\frac{i\sqrt{2}}{v}\xi_i(x) k^i\right)\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{pmatrix}$$

where the k^i are the <u>broken</u> generator combinations $\frac{1}{2}\sigma^1$, $\frac{1}{2}\sigma^2$ and $\frac{1}{2}\sigma^3 - Y$ and the $\xi_i(x)$ are the would-be Goldstone bosons. Eliminate the $\xi_i(x)$ by going to <u>unitary gauge</u> where $\xi_i(x) = 0$, in which case

$$\phi_{unitary} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{pmatrix}$$

In order to work out the boson field masses, we need the covariant derivative $D_\mu\Phi$ written in unitary gauge:

$$D_\mu\Phi = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ \partial_\mu H\end{pmatrix} - \frac{i}{2\sqrt{2}}\begin{pmatrix} g_2 W_\mu^3 + g_1 B_\mu & g_2 W_\mu^1 - i g_2 W_\mu^2 \\ g_2 W_\mu^1 + i g_2 W_\mu^2 & -g_2 W_\mu^3 + g_1 B_\mu \end{pmatrix}\begin{pmatrix}0\\ v+H\end{pmatrix}$$

Use this to expand the kinetic term for Φ:

$$-D_\mu\Phi^\dagger D^\mu\Phi = -\frac{1}{2}\partial_\mu H\partial^\mu H - \frac{1}{8}(v+H)^2 g_2^2 (W_\mu^1 - iW_\mu^2)(W^{1\mu} + iW^{2\mu})$$
$$-\frac{1}{8}(v+H)^2(-g_2 W^{3\mu} + g_1 B^\mu)(-g_2 W_\mu^3 + g_1 B_\mu)$$

while the potential term $-V$ becomes

$$-V = -\frac{\lambda}{4}\left[(v+H)^2 - \mu^2/\lambda\right]^2 = -\left(\lambda v^2 H^2 + \lambda v H^3 + \frac{\lambda}{4}H^4\right).$$

We can now read off the bosonic field masses. Compare the expansion of $-V$ out to second order with

H a real field

a standard mass term $-\frac{1}{2}m_H^2 H^2$ to find $\underline{m_H^2 = 2\lambda v^2 = 2\mu^2}$.

For the spin-one particles, one has from the gauge couplings in $\mathcal{L}_{Higgs}$

$$-\frac{1}{8}g_2^2 v^2(W_\mu^1 - iW_\mu^2)(W^{1\mu} + iW^{2\mu}) - \frac{1}{8}v^2(-g_2 W_\mu^3 + g_1 B_\mu)(-g_2 W^{3\mu} + g_1 B^\mu)$$

so the fields W_μ^1 and W_μ^2 appear nicely in the diagonal combination $-\frac{1}{8}g_2^2 v^2 (W_\mu^1 W^{1\mu} + W_\mu^2 W^{2\mu})$.

Comparing to a standard spin-one mass term $-\frac{1}{2}M_1^2 W_\mu^1 W^{1\mu} - \frac{1}{2}M_2^2 W_\mu^2 W^{2\mu}$, one identifies $M_1^2 = M_2^2 = g_2^2 v^2/4$. It is no accident that W_μ^1 and W_μ^2 have the same mass, because they transform into each other under the unbroken stability subgroup $H = U(1)_{em}$ which is generated by $T^3 + Y$. Setting $\xi_1 = \xi_2 = 0$ and $\xi_3 = \delta\rho$, find $\delta\begin{pmatrix}W_\mu^1\\ W_\mu^2\end{pmatrix} = \delta\rho\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\begin{pmatrix}W_\mu^1\\ W_\mu^2\end{pmatrix}$. So W_μ^1 and W_μ^2 are charged under $U(1)_{em}$. Combine them into a conjugated pair of complex vector fields $W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2)$. These transform as $W_\mu^\pm \longrightarrow e^{\pm i\rho}W_\mu^\pm$ under $U(1)_{em}$. Compare the mass term to $-M_W^2 W_\mu^\pm W^{\mp\mu}$ for a complex vector to obtain $\underline{M_W = M_1 = M_2 = \frac{g_2 v}{2}}$

The remaining vector fields W_μ^3 and B_μ appear only in the combination $-g_1 B_\mu + g_2 W_\mu^3$ in the mass term. So define a normalized combination

$$Z_\mu = \frac{-g_1 B_\mu + g_2 W_\mu^3}{\sqrt{g_1^2+g_2^2}}$$

which can also be rewritten $Z_\mu = W_\mu^3 \cos\Theta_W - B_\mu \sin\Theta_W$ where $\cos\Theta_W = \frac{g_2}{\sqrt{g_1^2+g_2^2}}$ $\sin\Theta_W = \frac{g_1}{\sqrt{g_1^2+g_2^2}}$ determine the <u>weak mixing angle</u> (or Weinberg angle) Θ_W.

In terms of Z_μ, the kinetic term has the standard form $-\frac{1}{4}(\partial_\mu Z_\nu - \partial_\nu Z_\mu)(\partial^\mu Z^\nu - \partial^\nu Z^\mu)$ while the mass term $-\frac{1}{2}M_Z^2 Z_\mu Z^\mu$ is $-\frac{1}{8}v^2(g_1^2+g_2^2)Z_\mu Z^\mu$, so one identifies $M_Z = \frac{1}{2}v\sqrt{g_1^2+g_2^2}$.

The remaining vector field combination, orthogonal to Z_μ, is $A_\mu = W_\mu^3 \sin\Theta_W + B_\mu \cos\Theta_W = \frac{g_1 W_\mu^3 + g_2 B_\mu}{\sqrt{g_1^2+g_2^2}}$.

N.B. from the coefficient of A_μ in D_μ, identify the charge $e = g_1\cos\Theta_W = g_2 \sin\Theta_W$

The A_μ field combination has <u>no mass term</u> and corresponds to the unbroken $U(1)_{em}$ gauge symmetry. The vector field A_μ is accordingly identified as the <u>photon field</u>. $D_\mu e_{Rm} = \partial_\mu e_{Rm} + ig_1(A_\mu \cos\Theta_W - Z_\mu \sin\Theta_W)e_{Rm}$

$D_\mu^{(em)} e_{Rm} = \partial_\mu e_{Rm} - iq A_\mu e_{Rm}$ $q(e_R) = -e$ so $e = g_1 \cos\Theta_W$

<u>Current values</u>

$m_H = 125.09 \pm 0.21 \text{(stat)} \pm 0.11 \text{(syst.)}\ \mathrm{GeV}/c^2$

$M_W = 80.385 \pm 0.015\ \mathrm{GeV}/c^2$

$M_Z = 91.1876 \pm 0.0021\ \mathrm{GeV}/c^2$

$\sin^2\Theta_W = 0.23120 \pm 0.00015$ at $91.2\ \mathrm{GeV}/c$ momentum transfer (so $\Theta_W \sim 30°$). Θ_W, however, "runs" as a function of momentum transfer — a key prediction of electroweak theory at the quantum level.

General structure of non-abelian symmetry breaking:

T.W.B. Kibble, Symmetry breaking in non Abelian gauge theories Phys. Rev. 155 (1967) 1554 - 1561

Custodial SU(2) symmetry

The Standard Model is formulated in the simplest way that fits with elementary particle physics data. In particular, it presumes just one Higgs doublet for the symmetry-breaking mechanism. There are many ways in which this could be generalized (e.g. by including more Higgs scalars), and it is

$M_W = M_Z \cos\theta_W$ special to SM

appropriate to keep track of which features are required by the symmetry-breaking pattern $SU(2)_L \times U(1) \to U(1)_{em}$ and which features may be accidental.

Custodial SU(2) symmetry is such an accidental feature. It is a rigid symmetry of the non-gauge theory obtained upon setting $g_2 = g_1 = 0$ under which the Goldstone bosons transform as an SU(2) (equivalently SO(3)) triplet. This also implies for the Standard Model gauge theory that the vector mass matrix sub-block for W^1_μ, W^2_μ and W^3_μ leaves unbroken a rigid SU(2) (equiv. SO(3)) under which the $W^{1,2,3}_\mu$ transform as a triplet.

Not the same as $SU(2)_L$!

Consider the non-abelian Higgs effect in general for a system with original local gauge symmetry G which breaks down to $H \subset G$. This system has Yang-Mills gauge fields A^i_μ, $i = 1, \ldots, \dim(G)$ and a set of n real scalars ϕ_a, $a = 1, \ldots, n$. Even for symmetry breaking involving complex Higgs fields, one may consider them in real form by separating real and imaginary parts. (So for the Standard Model, the complex Higgs doublet may be considered as a real quartet $\phi_{1,2,3,4}$.)

The Yang-Mills + Higgs sector then has an action

$$I_{YMH} = \int d^4x \left(-\frac{1}{4} F^i_{\mu\nu} F^{i\mu\nu} - \frac{1}{2} \left[(\partial_\mu - i g_T A_\mu \cdot T)\phi \right]_a \left[(\partial^\mu - i g_T A^\mu \cdot T)\phi \right]_a - V(\phi) \right)$$

where the g_T coupling constants are the same for all T^i generators within any simple subgroup of G, but may differ between different subgroup factors, eg. between the $SU(3) \times SU(2)_L \times U(1)_Y$ factors.

The potential $V(\Phi)$ in a renormalizable theory will be a polynomial with up to quartic terms in Φ_a. Let it be minimized for $\Phi_a = \bar{\Phi}_a$, constant in x^μ and invariant under the stability subgroup $H \subset G$ (aka the "little group").

Divide the $\dim(G)$ generators T^i into $t^{\hat{i}}$, $\hat{i} = 1, \ldots, \dim(H)$ little group generators annihilating the Higgs vacuum, $t^{\hat{i}} \bar{\Phi} = 0$, plus $(\dim(G) - \dim(H))$ broken generators $k^{\check{i}}$ that do not annihilate the Higgs vacuum, $k^{\check{i}} \bar{\Phi} \neq 0$. The $k^{\check{i}} \bar{\Phi}$ are independent and span a $(\dim(G) - \dim(H))$ dimensional subspace of the original n dimensional representation target space of the Φ_a fields. In the absence of gauge coupling, there would be $(\dim(G) - \dim(H))$ massless Goldstone boson scalar fields corresponding to the $k^{\check{i}}$ broken generators.

difference from earlier expansion there Φ_a are real

Parametrize Φ by $\Phi = \exp\left(\frac{i}{v} \xi_{\check{i}}(x) k^{\check{i}}\right)\left(\bar{\Phi} + \eta(x)\right)$ where $\check{i}$ repeated is Einstein summed and $v^2 = \bar{\Phi}_a \bar{\Phi}_a$. The Higgs fields $\eta_{\hat{a}}(x)$ belong to the $n - (\dim(G) - \dim(H))$ complement to the $k^{\check{i}} \bar{\Phi}$ subspace of the original n dimensional representation space. In general, the $\eta_{\hat{a}}(x)$ Higgs fields will be massive (barring accidental vanishing) with a $(\text{mass})^2$ matrix $\left(\partial^2 V / \partial\phi_{\hat{a}} \partial\phi_{\hat{b}}\right)\big|_{\phi = \bar{\Phi}}$.

One eliminates the would-be Goldstone fields by going to unitary gauge where $\xi_{\check{i}}(x) = 0$. In unitary gauge, the action depends only on the $\eta_{\hat{a}}(x)$ Higgs fields and the vector fields A^i_μ, of which $(\dim(G) - \dim(H))$ are now massive. The term in the Lagrangian responsible for the vector field masses is $\frac{1}{2}(g_T T^i \bar{\Phi}, g_T T^j \bar{\Phi}) A^i_\mu A^{j\mu}$ where (v, w) is the ordinary inner product in the real n-dimensional scalar representation space. Hence, the vector-field $(\text{mass})^2$ matrix is

$(v,w) = v_A w_A$ real inner prod.

T^i_{AB} hermitian: imaginary + antisymmetric

$$(M^2)^{ij} = -g_i g_j (T^i \bar{\Phi}, T^j \bar{\Phi}) = g_i g_j (\bar{\Phi}, T^i T^j \bar{\Phi})$$

in which i and j are not summed; g_i is g_T for all T^i in a simple G factor.

The second form of $(M^2)^{ij}$ follows since the T^i are Hermitian for a compact gauge group G, i.e. antisymmetric and purely imaginary. The matrix $(M^2)^{ij}$ is <u>real and symmetric</u>. Restricting $(M^2)^{ij}$ to the broken $k^i\bar{\phi} \neq 0$, one finds a positive definite $(\text{mass})^2$ submatrix for $((\dim G) - \dim(H))$ vector fields corresponding to the broken symmetries.

Now consider $SU(2)_L \times U(1)_Y$ symmetry breaking to $U(1)_{EM}$. In the Standard Model with a single Higgs doublet ϕ_a, reformulate this complex field system in terms of four real fields as above. Write $\phi_{(2c)} = \frac{1}{\sqrt{2}}\begin{pmatrix}\phi_3 + i\phi_4 \\ \phi_1 + i\phi_2\end{pmatrix}$ and consider the gauge-field couplings in $D_\mu \phi$:

$$-\frac{ig_2}{2\sqrt{2}} W_\mu^1 \begin{pmatrix}\phi_1 + i\phi_2 \\ \phi_3 + i\phi_4\end{pmatrix} - \frac{ig_2}{2\sqrt{2}} W_\mu^2 \begin{pmatrix}\phi_2 - i\phi_1 \\ -\phi_4 + i\phi_3\end{pmatrix}$$

$$-\frac{ig_2 W_\mu^3}{2\sqrt{2}} \begin{pmatrix}\phi_3 + i\phi_4 \\ -\phi_1 - i\phi_2\end{pmatrix} - \frac{ig_1}{2\sqrt{2}} B_\mu \begin{pmatrix}\phi_3 + i\phi_4 \\ \phi_1 + i\phi_2\end{pmatrix} = -ig_i A_\mu^i T^i_{(2c)} \phi_{(2c)} \qquad A_\mu^{i=1,2,3} = W_\mu^i \quad A_\mu^4 = B_\mu$$

rewrite this now in terms of the real quartet $\phi_{(4R)} = \begin{pmatrix}\phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4\end{pmatrix}$. Then for the four contributions to $-ig_i A_\mu^i T^i_{(4R)} \phi_{(4R)}$ in $D_\mu \phi_{(4R)}$, one has

$$\frac{g_2 W_\mu^1}{2}\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}\phi_{(4R)}, \quad \frac{g_2 W_\mu^2}{2}\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix}\phi_{(4R)}$$

$$\frac{g_2 W_\mu^3}{2}\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}\phi_{(4R)}, \quad \frac{g_1 B_\mu}{2}\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}\phi_{(4R)}$$

Then, recalling that one writes the covariant derivative gauge field terms as $-ig_i A_\mu^i T^i \phi$ with T^i hermitian, one needs to insert $1 = (-i)i$ to identify

$$T^1_{(4R)} = \frac{i}{2}\begin{pmatrix}0&0&0&1\\0&0&-1&0\\0&1&0&0\\-1&0&0&0\end{pmatrix}, \quad T^2_{(4R)} = \frac{i}{2}\begin{pmatrix}0&0&1&0\\0&0&0&1\\-1&0&0&0\\0&-1&0&0\end{pmatrix}, \quad T^3_{(4R)} = \frac{i}{2}\begin{pmatrix}0&-1&0&0\\1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}.$$

$$T^4_{(4R)} = Y_{(4R)} = \frac{i}{2}\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}.$$ All purely imaginary and antisymmetric.

One can now obtain the vector field $(\text{mass})^2$ matrix from $-D_\mu \phi^\dagger_{(2C)} D^\mu \phi_{(2C)} = -\frac{1}{2} D_\mu \phi_{(4R)} D^\mu \phi_{(4R)}$ by setting $\phi_{(4R)}$ to its vacuum value $\bar{\Phi}_{(4R)} = \begin{pmatrix} v \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

One obtains
$$(M^2)^{ij}_{sm} = \frac{1}{4}\begin{pmatrix} g_2^2 v^2 & 0 & 0 & 0 \\ 0 & g_2^2 v^2 & 0 & 0 \\ 0 & 0 & g_2^2 v^2 & -g_2 g_1 v^2 \\ 0 & 0 & -g_2 g_1 v^2 & (g_1)^2 v^2 \end{pmatrix}$$

N.B. $g_1 = g_2 = g_3 = g_2$
i.e. $SU(2)_L$ coupling
and $g_4 = g_1$
i.e. $U(1)_Y$ coupling

noting that
$$(T^1)^2 = (T^2)^2 = (T^3)^2 = \tfrac{1}{4}\mathbb{1} = (T^4)^2 \quad \text{and} \quad T^3T^4 = T^4T^3 = \tfrac{1}{4}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now ask whether $(M^2)^{ij}_{sm}$ has a more special structure than one might expect just from a requirement that $SU(2)_L \times U(1)_Y$ break spontaneously to $U(1)_{EM}$ generated by $T^3 + Y$. As we have seen, requiring T^3+Y to be unbroken implies $(M^2)^{11} = (M^2)^{22}$. This can also be seen directly using $(T^3+Y)\bar{\Phi} = 0$, so one requires $0 = g_i g_j (\bar{\Phi}, [(T^3+Y), T^i T^j]\bar{\Phi})$ (recall: i and j are not summed here). Using $[T^1, T^2] = iT^3$, etc. and $[Y, T^i] = 0$, one gets for $i=1$, $j=2$
$$g_2^2(\bar{\Phi}, [T^3, T^1T^2]\bar{\Phi}) = ig_2^2\left((\bar{\Phi}, T^2T^2\bar{\Phi}) - (\bar{\Phi}, T^1T^1\bar{\Phi})\right) = 0$$

So $(M^2)^{11} = (M^2)^{22}$. Similarly, taking $i=j=1$ one finds $ig_2^2(\bar{\Phi}, T^1T^2\bar{\Phi}) = ig_2^2(\bar{\Phi}, T^2T^1\bar{\Phi}) = 0$ so $(M^2)^{12} = (M^2)^{21} = 0$. From $i=1, j=3$ one finds $ig_2^2(\bar{\Phi}, T^2T^3\bar{\Phi}) = 0$ so $(M^2)^{23} = 0$ while from $i=2, j=3$ one finds $(M^2)^{13} = 0$. From $g_2 g_1 (\bar{\Phi}, [T^3, T^i T^4]\bar{\Phi}) = 0$ for $i = 2, 1$ one obtains $(M^2)^{14} = (M^2)^{24} = 0$. So the upper right-hand block of $(M^2)^{ij}$ must vanish, and similarly for the lower left-hand block.

The remaining lower right-hand block is symmetric, but is not otherwise constrained by the above argument, since $[T^3, T^3] = [T^3, T^4] = 0$. However, this block is constrained by the requirement that $(M^2)^{ij}$ have a zero eigenvalue for the unbroken $U(1)_{EM}$. This requires the lower right-hand block's determinant to vanish.

Taken together with the other block structure, one accordingly finds that $(M^2)^{ij}$ can thus be written

$$(M^2)^{ij}_{gen} = \frac{1}{4}\begin{pmatrix} g_2^2 v^2 & 0 & 0 & 0 \\ 0 & g_2^2 v^2 & 0 & 0 \\ 0 & 0 & g_2^2 u^2 & -g_2 g_1 u^2 \\ 0 & 0 & -g_2 g_1 u^2 & (g_1)^2 u^2 \end{pmatrix}$$

which is similar to the structure of $(M^2)^{ij}_{SM}$ except that in $(M^2)^{ij}_{gen}$ one doesn't necessarily have $u = v$. The zero eigenvector of the lower right-hand block is, up to normalization, $\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$, or, normalizing, $\begin{pmatrix} \sin\Theta_W \\ \cos\Theta_W \end{pmatrix}$ with $\sin\Theta_W = \frac{g_1}{\sqrt{g_1^2+g_2^2}}$, $\cos\Theta_W = \frac{g_2}{\sqrt{g_1^2+g_2^2}}$ as before.

The non-zero eigenvalue of the lower right-hand block is given by $M_Z^2 = u^2 \operatorname{tr}\begin{pmatrix} g_2^2 & -g_2 g_1 \\ -g_2 g_1 & (g_1)^2 \end{pmatrix} = u^2(g_2^2 + (g_1)^2) = (M^2)^{33}(1+\tan^2\Theta_W)$ since $\tan\Theta_W = \frac{g_1}{g_2}$. So $M_Z^2 = (M^2)^{33}\sec^2\Theta_W$, ie. $(M^2)^{33} = \cos^2\Theta_W M_Z^2$

Letting $(M^2)^{11} = (M^2)^{22} = M_W^2$ and $(M^2)^{33} = M_3^2$, the special property of the Standard Model vector mass pattern is $M_3 = M_W$, or $u = v$. This specific structure is a consequence of having used just one $SU(2)_L$ scalar field doublet to break $SU(2)_L \times U(1)_Y$ to $U(1)_{EM}$. The Standard Model potential $V(\phi^\dagger\phi)$ can be written in terms of $\phi_{(4R)}$ as $V(\phi_1^2+\phi_2^2+\phi_3^2+\phi_4^2)$. As required by the $SU(2)_L \times U(1)_Y$ gauge symmetry, this is locally invariant under this 4-parameter gauge group. But it is also invariant under an $SO(4) \cong SO(3)\times SO(3)$ rigid symmetry.

Symmetry breaking with a vacuum $\bar{\phi}_{(4R)} = \begin{pmatrix} v \\ 0 \\ 0 \\ 0 \end{pmatrix}$ breaks this $SO(4)$ rigid symmetry down to $SO(3)_{cust} \cong SU(2)_{cust}/Z_2$. This unbroken symmetry of the Higgs sector is the custodial $SU(2)_{cust}$

The Standard Model relation $M_W = M_Z \cos\theta_W$ is thus a consequence of the $SU(2)_{cust}$ electroweak custodial symmetry, following from the simple Higgs spontaneous symmetry breaking mechanism with a single $SU(2)_L$ scalar doublet Φ_a. More complicated models of the Higgs mechanism can give deviations from this, even at the tree level. Even in the simple Standard Model, custodial symmetry is <u>broken</u> by couplings of Φ_a to B_μ and the Yukawa couplings. Accordingly, radiative corrections can be expected to alter the M_Z to M_W mass relation. It is useful to define a parameter describing such deviations from the mass formula: $\rho = \dfrac{M_W^2}{M_Z^2 \cos^2\theta_W}$. (Current experimental value: $\rho = 1.0004$)

Denoting the tree-level value of ρ as ρ_0, one has $\rho_0 = 1$ in the Standard Model.

Consider now couplings to currents $J_{\mu 1,2}$, $J_{\mu 3}$ and $J_{\mu Y}$:

$$-g_2(W_{1\mu} J_1^\mu + W_{2\mu} J_2^\mu) - g_2 W_{3\mu} J_3^\mu - g_1 B_\mu J_Y^\mu .$$

Identify the electromagnetic charge $e = g_1 c = g_2 s$ (with shorthand $c = \cos\theta_W$, $s = \sin\theta_W$).

Then the $W_{3\mu}$ and B_μ couplings can be rewritten

$$-g_2 W_{3\mu} J_3^\mu - g_1 B_\mu J_Y^\mu = -\frac{e}{s}(A_\mu s + Z_\mu c) J_3^\mu - \frac{e}{c}(A_\mu c - Z_\mu s) J_Y^\mu$$
$$= -e A_\mu (J_3^\mu + J_Y^\mu) - Z_\mu\left(\frac{ec}{s} J_3^\mu - \frac{es}{c} J_Y^\mu\right)$$

then writing $J^\mu_{EM} = J_3^\mu + J_Y^\mu$, one has the couplings

$$-e A_\mu J^\mu_{EM} - Z_\mu\left(\frac{e}{sc} J_3^\mu - \frac{es}{c} J^\mu_{EM}\right)$$

as is appropriate for electromagnetic coupling to A_μ.

After symmetry breaking, W_1^μ and W_2^μ become massive with mass M_W, and Z_μ with mass M_Z. Ignore higher (cubic, quartic) self couplings and consider integrating out these massive fields to obtain current-current effective theory interactions.

For a real vector field V_μ with mass M coupled to a conserved ($\partial_\mu J^\mu$) current J_μ with charge coupling q, one has the action

$$\int d^4x \left(-\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} - \tfrac{M^2}{2} V_\mu V^\mu - q V_\mu J^\mu\right) \quad , \quad F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$$

giving field equations $\partial_\mu F^{\mu\nu} - M^2 V^\nu = q J^\nu$, which in momentum space becomes

$$\left(k_\nu k_\mu - (k^2 + M^2)\eta_{\nu\mu}\right) V^\mu(k) = q J_\nu$$

Solve this:

$$V^\mu = -\frac{q\left(J^\mu + \frac{k^\mu k^\nu J_\nu}{M^2}\right)}{k^2 + M^2} = \frac{-q\left(\eta^{\mu\nu} + \frac{k^\mu k^\nu}{M^2}\right) J_\nu}{k^2 + M^2}$$

For low momenta $|k^\nu| \ll M$, the factor $\frac{1}{k^2+M^2} \approx \frac{1}{M^2}$ and for a conserved current J_ν, $k^\nu J_\nu = 0$, so the propagator $-\left(\eta^{\mu\nu} + k^\mu k^\nu/M^2\right)/(k^2+M^2)$ becomes just $-\eta^{\mu\nu}/M^2$ and accordingly one has $V^\mu \simeq -q J^\mu/M^2$.

At low momentum transfers, the terms in the effective Lagrangian arising from W and Z vector field exchanges are

$$-\frac{e}{s}\left(W_{1\mu} J_1^\mu + W_{2\mu} J_2^\mu\right) - Z_\mu\left(\frac{e}{sc} J_3^\mu - \frac{es}{c} J^\mu_{EM}\right)$$

$$\cong \left(\frac{e}{s}\right)^2 (J_{1\mu} + i J_{2\mu}) \frac{\eta^{\mu\nu}}{M_W^2} (J_{1\nu} - i J_{2\nu}) + \left(\frac{e}{sc}\right)^2 (J_{3\mu} - s^2 J_{EM\mu}) \frac{\eta^{\mu\nu}}{M_Z^2} (J_{3\nu} - s^2 J_{EM\nu})$$

The conventional definition of the Fermi constant is $G_F = \sqrt{2} e^2/(8 s^2 M_W^2)$, so the current-current interactions arising from W and Z exchange can be written

$[G_F] = m^{-2}$ appropriate for dim 6 operator

$$\frac{8}{\sqrt{2}} G_F \left[\,|J_{1\mu} + i J_{2\mu}|^2 + \rho\,(J_{3\mu} - s^2 J_{EM\mu})^2\right]$$

So the custodial symmetry implying $\rho_0 = 1$ is a symmetry relating the strength of the weak (Z) part of the neutral-current interactions to the strength of the charged-current ($W^\pm$) interactions. The currents $J_{\pm\mu}$ and the neutral current $(J_{3\mu} - s^2 J_{EM\mu})$ are quadratic in fermions, so this gives the structure of the current-current 4-fermi effective interactions: For the leptons, one has a chiral charged current

V-A structure

$$l_\rho(x) = \underbrace{\tfrac{1}{2}\bar\Psi_e \gamma_\rho (1+\gamma_5)\Psi_{\nu_e}}_{\text{electron}} + \underbrace{\tfrac{1}{2}\bar\Psi_\mu \gamma_\rho (1+\gamma_5)\Psi_{\nu_\mu}}_{\text{muon}} + \underbrace{\tfrac{1}{2}\bar\Psi_\tau \gamma_\rho (1+\gamma_5)\Psi_{\nu_\tau}}_{\text{Tau}}.$$

A bit of history: in December 1956 - January 1957, George Sudarshan, a student of Robert Marshak at the University of Rochester, was studying non-parity-violating beta decays and found the known experiments on angular correlations to be inconsistent. By the time of the Rochester Series conference in spring 1957, he had come to the conclusion that the interaction had to have a chiral V-A structure (vector minus axial vector). He asked to present this at the conference, but was not allowed, since he was only a graduate student. P.T. Matthews (a founder, together with Salam of the Imperial Theory Group) was asked to report on the V-A theory briefly, but forgot to do so.

Marshak was going to be at the RAND Corporation in Los Angeles and offered Sudarshan and another student one-month salaries during the summer to work in LA. However, as an alien, Sudarshan could not enter RAND, so they met outside in restaurants. Gell-Mann was also at RAND at that time and Marshak invited him to such a lunch. Sudarshan gave a full presentation on the V-A theory. As above, the data at that time were internally inconsistent; he also pointed out which experiments were likely to be mistaken - 4 of them in particular.

Marshak asked Sudarshan to write up the work, which he did. Instead of publishing it immediately, however, Marshak decided to wait until a Sept. 1957 conference in Padua and Venice. A later paper was sent to the Physical Review, but in the meantime, Feynman and Gell-Mann published a paper in the Physical Review asserting the V-A structure of the weak interactions. They thanked Sudarshan for "important discussions". Most often, reference is made only to the Feynman and Gell-Mann paper. Feynman, however, acknowledged Sudarshan's discovery of V-A in later writings.

Yukawa and gauge field couplings of fermions

In addition to the masses of $W_\mu^\pm$ and Z_μ, spontaneous symmetry breaking also generates masses for the leptons and quarks via the Yukawa interactions. In the unitary gauge, one has $\phi = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v+H(x)) \end{pmatrix}$, $\tilde{\phi} = \begin{pmatrix} \frac{1}{\sqrt{2}}(v+H(x)) \\ 0 \end{pmatrix}$ (recall $\tilde{\phi}_a = \epsilon_{ab}\phi^{*b}$), so upon inserting these into the SM Yukawa terms one has

$$\mathcal{J}^{\text{unitary}}_{\text{yukawa}} = -\frac{i}{\sqrt{2}}(v+H)\left(f_{mn}\overline{e_{Lm}}e_{Rn} + h_{mn}\overline{d'_{Lm}}d'_{Rn} + k_{mn}\overline{u'_{Lm}}u'_{Rn}\right) + \text{h.c.}$$

Fermion masses arise from the $\frac{v}{\sqrt{2}}$ terms, while the other terms involving $H(x)$ are Higgs-(lepton)^2 and Higgs-(quark)^2 couplings.

The mass terms for the fermions involve mixtures of the three $m = 1,2,3$ generations for general f_{mn}, k_{mn} and h_{mn}. However, one can diagonalize the structure of the mass terms by making non-symmetry rigid $\mathcal{U}(3)$ unitary field redefinition transformations independently for all the leptons and quark flavours: $e_{Lm} \to \mathcal{U}^{e_L}_{mn} e_{Ln}$, $e_{Rm} \to \mathcal{U}^{e_R}_{mn} e_{Rn}$, $u'_{Lm} \to u_{Lm} = \mathcal{U}^{u_L}_{mn} u'_{Ln}$, $u'_{Rm} \to u_{Rm} = \mathcal{U}^{u_R}_{mn} u'_{Rn}$
$d'_{Lm} \to d_{Lm} = \mathcal{U}^{d_L}_{mn} d'_{Ln}$, $d'_{Rm} \to d_{Rm} = \mathcal{U}^{d_R}_{mn} d'_{Rn}$.
It is at this point that we define the unprimed quarks. The Dirac conjugates of these fields transform contragrediently: $\overline{e_{Lm}} \to \overline{e_{Ln}}\,\mathcal{U}^{e_L\dagger}_{nm}$, etc. Note that all kinetic terms are consequently left unchanged by these transformations:
$-i\,\overline{e_{Lm}}\,\not{\partial} e_{Lm} \to -i\,\overline{e_{Lm}}\,\mathcal{U}^{e_L\dagger}_{mn}\mathcal{U}^{e_L}_{np}\not{\partial} e_{Lp} = -i\,\overline{e_{Lm}}\,\not{\partial} e_{Lm}$, etc.

A note on nomenclature: for the quarks, there are six flavours, comprising the up and down quarks of each of the three generations, or families. At this point, after $SU(2)_L \times \mathcal{U}(1)_Y$ symmetry breaking, the up and down quarks of each generation are treated separately.

The effects of these field-redefinition transformations on the Yukawa coupling matrices are biunitary transformations
$f_{mn} \to U^{e_L \dagger}_{ms} f_{st} U^{e_R}_{tn}$, $h_{mn} \to U^{d_L \dagger}_{ms} h_{st} U^{d_R}_{tn}$, $k_{mn} \to U^{u_L \dagger}_{ms} k_{st} U^{u_R}_{tn}$.
Note that although prior to symmetry breaking the Yukawa interactions $f_{mn} \bar{L}_m e_{Rn} \Phi$, $h_{mn} \bar{Q}_{Lm} d_{Rn} \Phi$, $k_{mn} \bar{Q}_{Lm} u_{Rn} \tilde{\Phi}$ all involved $SU(2)_L$ doublets (capitalized fields), the unitary gauge conditions for Φ and $\tilde{\Phi}$ effect that only one field out of each doublet participates in these biunitary transformations of f, h and k.

With the unitary transformation matrices acting on the first and second indices of f_{mn}, h_{mn} and k_{mn} independently, the biunitary transformations can be used to make f_{mn}, h_{mn} and k_{mn} diagonal and with positive real values:

$$f_{mn} = \mathrm{diag}(f_1, f_2, f_3),\ h_{mn} = \mathrm{diag}(h_1, h_2, h_3),\ k_{mn} = \mathrm{diag}(k_1, k_2, k_3)$$

where the f_m, h_m and k_m are all real and positive.
The mass terms are then

$$\frac{-iv}{\sqrt{2}}\left(f_{\underline{m}} \overline{e_{Lm}} e_{Rm} + h_{\underline{m}} \overline{d_{Lm}} d_{Rm} + k_{\underline{m}} \overline{u_{Lm}} u_{Rm} \right)$$

where the underlined $\underline{m}$ indices take the same values as their unadorned counterparts, but do not trigger Einstein summation.
Then the lepton and quark masses are

$$m_n^{(e)} = \frac{v}{\sqrt{2}} f_n\ ,\ m_n^{(d)} = \frac{v}{\sqrt{2}} h_n\ ,\ m_n^{(u)} = \frac{v}{\sqrt{2}} k_n\ .$$

Note that since right-handed neutrinos have not been introduced into the original Standard Model, no mass terms arise, so far, for the neutrinos.

The absence of right-handed neutrinos, and consequently of neutrino mass terms has another consequence. One may freely transform the left-handed neutrino fields ν_m with the same $U^{e_L}(3)$ transformation as for the left-handed e_{Lm}. Then the whole $SU(2)_L$ doublet L_m transforms into $U^{e_L}_{mn} L_n$, and consequently the gauge-coupled kinetic term $-i \bar{L}_m \not{D} L_m$ is left unchanged under this $U^{e_L}(3)$ redefinition. Similarly, the full gauge-coupled right-handed kinetic term $-i \overline{e_{Rm}} \not{D} e_{Rm}$ is unchanged under $U^{e_R}(3)$.

CKM structure

Now consider what happens to the quark field gauge couplings. In order to diagonalize the quark mass matrix, one needs to make independent $U(3)$ transformations $U^{u_L}_{mn}$ and $U^{d_L}_{mn}$, so there is no uniform transformation of the left-handed quark doublet Q_{Lm}. The pure derivative kinetic terms $-i\overline{u'_{Lm}}\not{\partial}u'_{Lm}$ and $-i\overline{d'_{Lm}}\not{\partial}d'_{Lm}$ are naturally diagonal in the u'_{Lm} and d'_{Lm} quarks, and consequently are themselves left unchanged. The same is the case for the other terms that are naturally diagonal in u'_{Lm} and d'_{Lm}. Since the T^3 and Y generators do not mix the top and bottom components of $SU(2)_L$ doublets, this also applies to the quark gauge couplings to W^3_ρ and B_ρ (i.e. to A_ρ and Z_ρ). So the neutral gauge couplings are unchanged.

The gauge-coupling terms involving the charged gauge fields $W^\pm_\rho$ are not all invariant, however. The charged-current interaction terms for the quarks are
$\mathcal{L}_{cc} = \frac{-g_2}{\sqrt{2}}\left[W^+_\rho \overline{u'_{Lm}}\gamma^\rho d'_{Lm} + W^-_\rho \overline{d'_{Lm}}\gamma^\rho u'_{Lm}\right]$, and these transform into

$$\mathcal{L}_{cc} = \frac{-g_2}{\sqrt{2}}\left[W^+_\rho V_{mn}\overline{u_{Lm}}\gamma^\rho d_{Ln} + W^-_\rho V^\dagger_{mn}\overline{d_{Lm}}\gamma^\rho u_{Ln}\right]$$

where $V_{mn} = \left(U^{u_L}(U^{d_L})^\dagger\right)_{mn}$ is the Cabbibo-Kobayashi-Maskawa (CKM) matrix. Note that $V^\dagger V = \mathbb{1}$, so V_{mn} is itself a unitary matrix. [Note that the couplings of $W^\pm_\rho$ to the leptons, i.e. $\frac{-g_2}{\sqrt{2}}\left[W^+_\rho \overline{\nu_{Lm}}\gamma^\rho e_{Lm} + W^-_\rho \overline{e_{Lm}}\gamma^\rho \nu_{Lm}\right]$ are invariant under $U^{u_L}(3)$, as noted above, once one transforms ν_{Lm} with the same $U^{u_L}_{mn}$ matrix as e_{Lm}.]

A priori, the CKM matrix, which is a $U(3)$ matrix, would depend on $3^2 = 9$ real parameters. However, not all components of V_{mn} are of equal significance: some can be eliminated by putting V_{mn} into one or another special form using further non-symmetry transformations.

In order to simplify V_{mn}, one can make pure phase transformations of the 6 quark flavours, acting in the same way on L and R chiralities, ie. like $u_{Lm} \to e^{i\alpha_m} u_{Lm}$, $u_{Rm} \to e^{i\alpha_m} u_{Rm}$, etc. Such pure phase transformations leave the pure derivative $\not\partial$ kinetic terms and also mass terms such as $k_{mm} \bar{u}_{Lm} u_{Rm}$ invariant. Note, however, that only non-symmetry transformations can be used in this way. There is one phase symmetry of the whole SM Lagrangian which has no effect on V_{mn}: overall transformations of all six quark flavours by the same phase. Consequently, the number of parameters that can be adjusted to put V_{mn} into a special form is $6-1=5$. The remaining, physically significant, number of components of V_{mn} is $9-5=4$.

There are a number of different conventional ways to parametrize the CKM matrix. It is appropriate first to count the number of parameters V_{mn} would have if it were purely real. In that case, it would be an orthogonal $O(3)$ matrix, determind by three parameters. In addition to these, there is one parameter determining the essentially complex nature of V_{mn}, a phase. The three $O(3)$ parameters can be taken to be three angles θ_{12}, θ_{23} and θ_{13} corresponding to rotations in the [mn] generation/family planes. The phase is often denoted δ. In the parametization advocated by the Particle Data Group, one has
$$V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$$

with $c_{mn} = \cos\theta_{mn}$
$s_{mn} = \sin\theta_{mn}$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} .$$

Numerical values for the angles and phase are $\theta_{12} = 13.02° \pm 0.04°$, $\theta_{23} = 2.36° \pm 0.08°$, $\theta_{13} = 0.20° \pm 0.02°$ and $\delta = 69° \pm 5°$ [Particle Data Group 2012]. The CKM matrix V_{mn} expresses the transformation between the original basis of fields in which generations (families) do not mix in the gauge boson interactions, known as flavour basis and the basis in which all the fermionic mass terms are diagonal, known as mass basis.

All the rotation angles are relatively small: one has $\theta_{13} \ll \theta_{23} \ll \theta_{12} \ll 1$ when expressed in radians. So charged-current interactions that link fermions of different generations are highly suppressed $\longleftrightarrow$ V_{mn} is close to being a unit matrix. The dominant amount of flavour mixing is given by θ_{12} known as the Cabibbo angle. Letting $\lambda = s_{12} = \sin\theta_{12} \simeq .23$, another common parametrization of V_{mn} is the Wolfenstein parametization, which more clearly indicates the size of each matrix element. Through third order in λ it is

$$V = \begin{pmatrix} 1-\frac{1}{2}\lambda^2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\frac{1}{2}\lambda^2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}$$

where A and $\rho^2+\eta^2$ are $\mathcal{O}(1)$.

The above discussion is for the currently understood case of $N=3$ generations. It is also instructive to consider what happens for general N generations of fermions. The CKM matrix is an $N\times N$ unitary matrix, so depends a priori on N^2 real parameters. If it were real, it would be an $O(N)$ matrix depending on $\frac{N(N-1)}{2}$ parameters. The difference between these two numbers, $N^2 - \frac{1}{2}N(N-1) = \frac{1}{2}N(N+1)$ is the number of "phase parameters" determining the complex structure of V_{mn}. However, one can, as in the $N=3$ case, make non-symmetry phase transformations of the $2N$ quark flavours. Among these, however is one combination that is a symmetry (overall fermi field phase transf.).

Thus, from the $\frac{1}{2}N(N+1)$ a priori phase parameters in V_{mn}, one may use $(2N-1)$ parameters to fix a special form for V_{mn} without disturbing the diagonal mass structure. The number of physically meaningful phases is then

$$P = \left[N^2 - \tfrac{1}{2}N(N-1)\right] - (2N-1) = \tfrac{1}{2}(N-1)(N-2).$$

If there were only $N=2$ generations, which seemed to be the case before the discovery of the tau family, then one would have $P=0$. In that case, the CKM matrix would be an $O(2)$ matrix

$$V_{mn} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix} \qquad \theta_c = \theta_{12} \quad \text{Cabbibo angle}$$

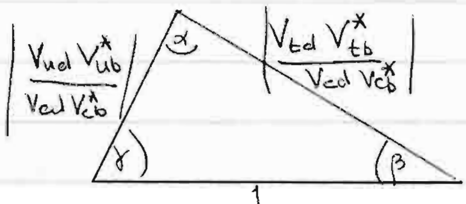
As one can see from the Wolfenstein parmetization of the $N=3$ case, the leading 2×2 submatrix of the $N=3$ CKM matrix V_{mn} contains just this $N=2$ submatrix. Experimental values for the θ_{23} and θ_{13} angles give only very small corrections to this structure.

Another question that can be addressed is that of a <u>possible fourth generation</u>. The overall CKM matrix is unitary by construction. However, if there were a fourth generation, the restriction of the CKM matrix to the three-generation subsector would not be unitary. So testing the $N=3$ V_{mn} for unitarity is a way to look indirectly for physics beyond the Standard Model. Unitarity requires $\sum_m' V_{mp}V^*_{mq} = \delta_{pq}$ for any p and q; for

$11(13)^* + 21(23)^* + 31(33)^*$ example $V_{ud}V^*_{ub} + V_{cd}V^*_{cb} + V_{td}V^*_{tb} = 0$. The best measured

$(V^T V^*)_{13} = 0$ of these three quantities is $V_{cd}V^*_{cb}$; dividing by it leads to

$$\frac{V_{ud}V^*_{ub}}{V_{cd}V^*_{cb}} + \frac{V_{td}V^*_{tb}}{V_{cd}V^*_{cb}} + 1 = 0.$$ This expresses closure of the <u>unitarity triangle</u>:



If V_{mn} were real, the unitarity triangle would collapse to a line. For the non-rescaled triangle, the <u>Jarlskog invariant</u> is defined as

$$J = 2(\text{area}) = \left|\mathrm{Im}\left(V_{ud}V_{tb}V^*_{td}V^*_{ub}\right)\right| = (2.96 \pm 0.20)\times 10^{-5}.$$

Neutrino masses

When the Standard Model was originally formulated, there was no evidence for neutrino masses, so the SM was constructed with massless neutrinos. A key puzzle regarding neutrinos emerged, however, from the neutrino-counting experiment of Ray Davis in the Homestake Gold Mine, Lead, South Dakota (1970 to 1994). John Bahcall had calculated the expected rate of neutrino capture for neutrinos emitted by nuclear fusion in the Sun. The Homestake experiment consisted of 100,000 gallons of dry-cleaning fluid (perchloroethylene), which is rich in chlorine, 1478 meters underground. The neutrino capture reaction is $\nu_e + {}^{37}Cl \rightarrow {}^{37}Ar + e^-$. Every few weeks, Davis bubbled helium through the tank to collect the argon. Argon 37 is radioactive, with a half-life of 35 days. In this way, a few tens of ^{37}Ar atoms could be detected, and the count of neutrinos captured made.

The puzzle posed by the Homestake experiment was that the detected rate of neutrino capture was only about 1/3 of that predicted by Bahcall. The experiment could, however, be calibrated by bringing radioactive, neutrino-producing sources down near the detector. Another experimental finding was the discovery by Masatoshi Koshiba at the Super Kamiokande experiment (Kamioka, Japan) of oscillations between the three generations of neutrinos in a well controlled beam from the KEK accelerator in Tsukuba. The Kamioka experiments also found a deficit in the flux of ν_e neutrinos from the Sun. The conversions $\nu_e \rightarrow \nu_\mu$ & ν_τ can explain this deficit, since the Homestake detector was sensitive only to ν_e electron neutrinos and the Kamioka detector preferentially detected ν_e.

The idea of neutrino oscillations had been put forward by Bruno Pontecorvo in 1957, not long after he emerged from sequestration by the KGB following his defection to Russia in 1950.

Pontecorvo's original proposal was for neutrino-antineutrino transitions. This has not been observed, but a 1968 paper by Pontecorvo and V.N. Gribov ("Neutrino astronomy and lepton charge", Physics Letters B28 (1969), 493) elaborated the key idea that neutrino mixing is a natural consequence of massive neutrinos. Since 2001, results from the Sudbury Neutrino Observatory provide clear evidence of neutrino flavour change, with only about 35% of solar neutrinos remaining as ν_e when they reach the earth.

These observations give evidence for neutrino masses, in particular for differences in the $(\text{mass})^2$ values for ν_e, ν_μ and ν_τ, but they don't give any single value. The Δm^2 differences are now known to be on the order of $10^{-4}\,\text{eV}^2$, with indications

Δm^2_{32} that one of the Δm^2 differences is 0.0027. Consequently, at least one neutrino species must have a mass that is at least the square root of this, i.e. 0.05 eV. Bounds on the sum of the masses for the known three neutrinos are obtained from cosmology, particularly galaxy surveys and analysis of cosmic microwave background data. These generally indicate that the sum of the masses is less than about 0.3 eV.

The upshot of the various experimental and observational results is that at least some neutrino species have masses on the order of $10^{-2}\,\text{eV}$. This is vastly different from the masses of other SM leptons, e.g. the electron mass of 0.511 MeV. So a big challenge is to find a mechanism that makes such a tiny scale occur naturally.

One way to obtain small masses for the known neutrino species is to consider the possibility of "sterile" right-handed singlets with respect to all SM gauge symmetries (i.e. $(1,1)_0$), not currently observed. Denoting such right-handed spinor fields ν_{Rm}, there are two new types of term that can occur in the Lagrangian. The first of these is a type of Yukawa coupling: $-i\, p_{mn} \bar{L}_m \nu_{Rn} \tilde{\Phi} + h.c.$ Check hypercharge invariance: $+1/2 - 1/2 = 0$. Taken alone, and assuming the p_{mn} coefficients to be of the same order of magnitude as the SM f_{mn}, h_{mn} and k_{mn} coefficients, these would produce neutrino mass terms after symmetry breaking that are far too large, perhaps comparable to the electron, muon or tau masses ($m_e \sim .5$ MeV, $m_\mu \sim 100$ MeV, $m_\tau \sim 1.8$ GeV).

Achieving small neutrino masses can be arranged in a somewhat paradoxical way by including the second type of new term in the Lagrangian, with an even larger mass value. For a sterile neutrino that is an $SU(3) \times SU(2)_L \times U(1)_Y$ singlet, one can construct bilinear Majorana mass terms:

m Einstein insensitive

$$-i M_{\underline{m}} \left(\overline{\nu_{Rm}} (\nu_{Rm})_c + \overline{(\nu_{Rm})_c}\, \nu_{Rm} \right), \text{ where } (\nu_{Rm})_c = C (\overline{\nu_{Rm}})^T$$

are the charge-conjugated fields. Such spinor bilinear mass terms are ruled out for all the other SM spinor fields because they would be incompatible with gauge invariance. Only for $(1,1)_0$ gauge singlets are they allowed. Instead of being small, the $M_{\underline{m}}$ masses could be very large, perhaps on the order of $10^{10} - 10^{11}$ GeV.

To see how this counterintuitive system can work, first review some spinor field formalism. The correct "Lorentz" group needed to characterize spinor fields is $SL(2, \mathbb{C})$, which bears a similar relationship to $SO(3,1)$ as the relationship of $SU(2)$ to $SO(3)$, i.e. $SO(3,1) \cong SL(2,\mathbb{C}) / Z_2$ (i.e. $SL(2,\mathbb{C})$ is the "double cover" of $SO(3,1)$). This relationship is made explicit in 2-component notation, using the Weyl representation for the Dirac γ-matrices:

2x2 blocks of 2x2 submatrices

$$\gamma_\mu = \begin{pmatrix} 0 & i(\sigma_\mu)_{\alpha\dot{\beta}} \\ i(\tilde{\sigma}_\mu)^{\dot{\gamma}\delta} & 0 \end{pmatrix} \qquad \begin{array}{l} \alpha, \delta = 1,2 \text{ undotted} \\ \dot{\beta}, \dot{\gamma} = 1,2 \text{ dotted} \end{array}$$

where the σ_μ, $\tilde{\sigma}_\mu$ are the Van der Waerden matrices:

$$(\sigma_\mu)_{\alpha\dot{\beta}} = (\underset{\mu=0}{\mathbb{1}}, \underset{\mu=i=1,2,3}{\sigma_i}), \quad (\tilde{\sigma}_\mu)^{\dot{\alpha}\beta} = (\mathbb{1}, -\sigma_i)$$

and $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the Pauli matrices.

In the 2-component notation, standard 4-component Dirac spinors are composed of two 2-component $SL(2,\mathbb{C})$ spinors: $\Psi_{Dirac} = \begin{pmatrix} \lambda_\alpha \\ \bar{\chi}^{\dot{\beta}} \end{pmatrix}$ $\alpha = 1,2$ $\dot{\beta} = 1,2$ Note index positions!

The fundamental λ_α and $\bar{\chi}^{\dot{\beta}}$ spinors are complex, giving rise to $2\times 4 = 8$ real components, just like any complex 4-component Dirac spinor.

The undotted-index λ_α and the dotted-index $\bar{\chi}^{\dot{\beta}}$ $SL(2,\mathbb{C})$ spinors are independent, and one or the other may consistently be set to zero in Ψ_{Dirac}: the 4-component Dirac spinor representation is <u>reducible</u>. Setting $\bar{\chi}^{\dot{\beta}} = 0$, one obtains a <u>left-handed chiral</u> spinor

$$\Psi_L = \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix}$$

while setting ~~setting~~ $\lambda_\alpha = 0$, one obtains a <u>right-handed</u> chiral spinor

$$\Psi_R = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot{\beta}} \end{pmatrix}.$$

In the 4-component notation, one obtains ~~right~~ left chiral spinors by projection with $P_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma_5)$ on a general Dirac spinor, where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \delta_\alpha{}^\beta & 0 \\ 0 & -\delta^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix}$ in Weyl rep. Note: $(P_\pm)^2 = P_\pm$

Taken together with their contragredient forms, there are four basic $SL(2,\mathbb{C})$ representations. For $A_\alpha{}^\beta \in SL(2,\mathbb{C})$, one has the hermitian conjugate $A^{\dagger\dot{\beta}}{}_{\dot{\alpha}}$, giving transformation types

$A^\dagger = A^{*T}$

$$\lambda_\alpha \to A_\alpha{}^\beta \lambda_\beta \quad ; \text{ conjugated to } \quad \bar{\lambda}_{\dot{\alpha}} \to \bar{\lambda}_{\dot{\beta}} A^{\dagger\dot{\beta}}{}_{\dot{\alpha}}$$

and contragredient transformations

$$\chi^\alpha \to \chi^\beta (A^{-1})_\beta{}^\alpha \quad ; \text{ conjugated to } \quad \bar{\chi}^{\dot{\alpha}} \to (A^{-1})^{\dagger\dot{\alpha}}{}_{\dot{\beta}} \bar{\chi}^{\dot{\beta}}.$$

The relationship between $SL(2,\mathbb{C})$ and $SO(3,1)$ is now expressed precisely by the existence of a <u>homomorphism</u> $\mu: SL(2,\mathbb{C}) \longrightarrow SO(3,1)$ given by $\mu: A_\alpha{}^\beta \longrightarrow L^\mu{}_\nu$ where $L^\mu{}_\nu(A) = -\frac{1}{2}\mathrm{Tr}(\tilde{\sigma}^\mu A \sigma_\nu A^\dagger)$. Note the rule that 2-component index contractions are only made between upper and lower indices. The index structures of $A, A^\dagger, \tilde{\sigma}^\mu = \eta^{\mu\nu}\tilde{\sigma}_\nu$ and σ_ν are just as needed to make this covariant construction. This homomorphism map is clearly 2 to 1 since both the $SL(2,\mathbb{C})$ matrices $A_\alpha{}^\beta$ and $(-A_\alpha{}^\beta)$ (both with determinant 1) map to the same $L^\mu{}_\nu \in SO(3,1)$. So $SO(3,1) \cong SL(2,\mathbb{C})/Z_2$.

Now return to the right-handed neutrinos. In 2-component notation, they are $\nu_{Rm} = \begin{pmatrix} 0 \\ \bar{\nu}_{(R)m}{}^{\dot\beta} \end{pmatrix}$. Another way to repackage covariantly such a spinor is as a 4-component <u>Majorana spinor</u> $\nu_{maj\,m} = \begin{pmatrix} \nu_{(R)m\,\alpha} \\ \bar{\nu}_{(R)m}{}^{\dot\beta} \end{pmatrix}$, where $\nu_{(R)m}{}^{\alpha} = (\bar{\nu}_{(R)m}{}^{\dot\alpha})^*$ (complex conjugate)

and $\nu_{(R)m\,\alpha} = \epsilon_{\alpha\beta}\nu_{(R)m}{}^\beta$ in which $\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is a numerically invariant $SL(2,\mathbb{C})$ bispinor. [One also has $\epsilon_{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\epsilon^{\dot\alpha\dot\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ numerically invariant bispinors. Note: $\epsilon^{\alpha\gamma}\epsilon_{\gamma\beta} = \delta^\alpha{}_\beta$, $\epsilon_{\dot\alpha\dot\gamma}\epsilon^{\dot\gamma\dot\beta} = \delta_{\dot\alpha}{}^{\dot\beta}$ and $\lambda^\alpha = \epsilon^{\alpha\beta}\lambda_\beta$, $\bar{\chi}^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar{\chi}_{\dot\beta}$ allow for $SL(2,\mathbb{C})$ covariant raising and lowering (with $\epsilon_{\alpha\beta}, \epsilon_{\dot\alpha\dot\beta}$) of 2-component indices.]

In the Weyl representation, the 4x4 <u>charge conjugation matrix</u> is $C = \begin{pmatrix} i\epsilon_{\alpha\beta} & 0 \\ 0 & i\epsilon^{\dot\alpha\dot\beta} \end{pmatrix}$, so for $\psi_{Dirac} = \begin{pmatrix} \lambda_\alpha \\ \bar{\chi}^{\dot\beta} \end{pmatrix}$ one has the charge conjugate $\psi_c = C(\bar{\psi})^T = \begin{pmatrix} \chi_\alpha \\ \bar{\lambda}^{\dot\beta} \end{pmatrix}$.
A Majorana spinor may thus be constructed from a right-handed spinor as $\psi_{Maj} = \psi_R + (\psi_R)_c = \begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot\beta} \end{pmatrix}$ since the Majorana constraint $\psi = (\psi)_c$ implies $\lambda_\alpha = \chi_\alpha$. [Note: $\psi_L + (\psi_L)_c$ also makes a Majorana spinor, $\begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}^{\dot\beta} \end{pmatrix}$.]

The Majorana mass term for the right-handed neutrinos when written in 2-component notation is

$$-i M_{\underline{m}}\left(\overline{\nu_{Rm}}(\nu_{Rm})_c + \overline{(\nu_{Rm})_c}\,\nu_{Rm}\right) = -M_{\underline{m}}\left(\nu_{(R)m}{}^{\alpha}\nu_{(R)m\,\alpha} + \bar{\nu}_{(R)m\dot\alpha}\bar{\nu}_{(R)m}{}^{\dot\alpha}\right).$$

This Majorana mass term can also be written in terms of the Majorana spinor as $-i M_m \overline{\nu_{Maj\,m}}\, \nu_{Maj\,m}$.

Now consider what happens after $SU(2)_L \times U(1)_Y$ symmetry breaking and making the unitary gauge choice in which $\tilde{\phi} = \begin{pmatrix} \frac{1}{\sqrt{2}}(v+H) \\ 0 \end{pmatrix}$. The Yukawa terms $-i\rho_{mn}\bar{L}_m \nu_{Rn}\tilde{\phi}$ then yield $L \leftrightarrow R$ off-diagonal mass terms $-\frac{iv}{\sqrt{2}}\rho_{mn}\overline{\nu_{Lm}}\,\nu_{Rn} = \frac{-v}{\sqrt{2}}\rho_{mn}\overline{\nu_{(L)m\dot{\alpha}}}\;\overline{\nu_{(R)n}}^{\dot{\alpha}}$ and similarly for the hermitian conjugate. Adding these to their conjugates, one has $-\frac{v}{\sqrt{2}}\left(\rho_{mn}\overline{\nu_{(L)m\dot{\alpha}}}\;\overline{\nu_{(R)n}}^{\dot{\alpha}} + \rho^*_{mn}\nu_{(L)m}{}^{\alpha}\nu_{(R)n\alpha}\right)$.

To understand more clearly what happens in such a system with coefficients ρ_{mn} and M_m, consider first a single generation, e.g. $m=1$. Phase rotation of ν_{L1} can be used to make $\rho_{11} = \rho$ real and positive. For simplicity, repackage now both ν_{L1} and ν_{R1} spinors into Majorana spinors (always covariant for ν_R, harmless after symmetry breaking and unitary gauge choice for ν_L): let

$$n = \begin{pmatrix} \nu_{(L)1\alpha} \\ \overline{\nu_{(L)1}}^{\dot{\beta}} \end{pmatrix} \qquad N = \begin{pmatrix} \nu_{(R)1\alpha} \\ \overline{\nu_{(R)1}}^{\dot{\beta}} \end{pmatrix}$$

and combine the mass terms: $-i\left(M\bar{N}N + \frac{v\rho}{2\sqrt{2}}(\bar{n}N + \bar{N}n)\right)$, giving a mass matrix

$$\begin{array}{cc} n \\ N \end{array}\begin{pmatrix} 0 & \frac{v\rho}{2\sqrt{2}} \\ \frac{v\rho}{2\sqrt{2}} & M \end{pmatrix}$$
$$\begin{array}{cc} n & N \end{array}$$

The vanishing matrix element in the nn position reflects the ordinary impossibility of having a diagonal mass term for ν_L spinors in the original Standard Model, as imposed by the starting $SU(2)_L \times U(1)_Y$ gauge invariance prior to symmetry breaking.

One finds the eigenvalues of the nondiagonal neutrino mass matrix in the standard way by solving:

$$-\lambda(M-\lambda) - v^2 \rho^2/8 = 0\,, \text{ giving } \lambda_\pm = \frac{M}{2}\left(1 \pm \sqrt{1-\frac{\rho^2 v^2}{2M^2}}\right).$$

Thus, if M is very large compared to $\frac{\rho v}{\sqrt{2}}$, i.e. for $\rho^2 v^2/2M^2 \ll 1$, one finds one very large eigenvalue $\lambda_+ \simeq M$ corresponding to a (currently unknown) very heavy neutrino species and also a very small eigenvalue $\lambda_- \simeq \frac{\rho^2 v^2}{8M} \ll \frac{\rho v}{2}$. This mechanism thus naturally produces a very light mass value for the known types of neutrino, at the cost of proposing some very heavy unknown neutrino species.

The above mechanism for generating a very small neutrino mass by introducing a large mass for an as-yet unknown sterile neutrino species is called the seesaw mechanism. It has the virtue of preserving renormalizability, since all the extensions to the minimal Standard Model involve operators of dimensions $d \le 4$. The fact that the Majorana mass values M_m for the right-handed neutrino singlets ν_{Rm} may be very large (perhaps 10^{10} GeV or so) invites another kind of description in terms of an effective theory. If one wants to describe the effects of mass for the known light neutrinos, one can integrate out the heavy ν_{Rm} fields. We saw earlier how integrating out the W_μ and Z_μ fields produces current-current interactions of dimension 6 in the effective 4-fermion theory. One may do something similar to integrate out the ν_{Rm} fermion fields. Instead of constructing the ν_{Rm} propagators and extracting the leading local parts of the interactions with light fields produced by ν_{Rm} exchange, it is simplest to consider just the Majorana mass terms and Yukawa terms and eliminate the ν_{Rm} using their lowest-order algebraic equations.

For simplicity of presentation, combine the ν_{Rm} and their charge conjugates $\nu_{Rmc} = C(\bar{\nu}_{Rm})^T$ into Majorana spinors $N_m = \nu_{Rm} + \nu_{Rmc}$, or, in 2-component form, $\begin{pmatrix} \nu_{(R)m\alpha} \\ \bar{\nu}_{(R)m}^{\dot\beta} \end{pmatrix}$.

The Majorana mass and Yukawa interactions involving N_m are then

$$-iM_{\underline{m}}\bar{N}_m N_m - i\rho_{mn}\bar{N}_n (L_{mc})^a \tilde{\Phi}_a - i\rho^*_{mn}\bar{N}_n L_{ma}\epsilon^{ab}\Phi_b .$$

Note that since L_{mc} is right-handed ($P_- L_{mc} = L_{mc}$), the first Yukawa term involves only the right-handed projection of $\bar{N}_n$ (i.e. $\overline{\nu_{Rnc}}$) and the second Yukawa term involves only the left-handed projection of $\bar{N}_n$ (i.e. $\bar{\nu}_{Rn}$). Taking just the above terms and deriving the algebraic equation for N_m, one has $N_m = -\frac{1}{2M_{\underline{m}}}\left[(\tilde{L}_{nc})^a \rho_{nm}\Phi_a + L_{na}\rho^*_{nm}\epsilon^{ab}\Phi_b\right]$. Using this algebraic result together with its Dirac conjugate for $\bar{N}_m$, one obtains the dimension $d=5$ operators

coefficient $\frac{1}{M_{\underline{m}}}$ of mass dim -1

$$\frac{i}{4M_{\underline{m}}}\left[(\overline{L_n})^a (L_{kc})^d \tilde{\Phi}_a \tilde{\Phi}_d \rho_{nm}\rho_{km} + (\overline{L_{nc}})_a L_{kd}\epsilon^{ab}\Phi_b\epsilon^{de}\Phi_e \rho^*_{nm}\rho^*_{km}\right]$$

involving two dimension 3/2 spinor fields and two dimension 1 Higgs fields.

This dimension 5 operator is non-renormalizable, similarly to the dimension 6 four-fermion operator obtained by integrating out the W_μ and Z_μ fields. One does not insist upon renormalizability in the context of an effective theory, since it is only taken to be valid up to some scale where other physical effects can take place. In this case, the validity of the above dimension 5 operator could not extend beyond the $M_{\underline{m}}$ scales of the ν_{Rm} Majorana masses. Consideration of an effective theory always presumes the existence of a satisfactory <u>ultraviolet completion</u>. For the neutrino masses, the ultraviolet completion could be a seesaw mechanism, as above, or it could be some other type of much higher energy physics. Carrying out the integration-out of the ν_{Rm} to higher order involves expanding fermionic propagators $[\not{p} - iM]^{-1} = \frac{\not{p} + iM}{p^2+M^2} = \frac{i}{M} + \frac{\not{p}}{M^2} + \cdots$, the second term of which generates dimension 6 terms in the effective theory, and eventually higher. Phenomenological studies often involve dims 5+6.

The PMNS matrix

When neutrino masses are included, one faces a similar issue upon diagonalizing their mass matrix, to that encountered with the quarks: a mixing matrix V^{PMNS}_{mn} occurs ($m,n=1,2,3$) distinguishing the diagonal mass basis from the flavour basis in which the $W^{\pm}_{\mu}$ couplings are m,n diagonal. This is the PMNS matrix (Pontecorvo, Maki, Nakagawa & Sakata), the neutrino analogue of the CKM matrix for the quarks.

Consider the mass terms for the light neutrinos after integrating out the heavy species N_m and going to the unitary gauge where $\phi_a = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$, $\tilde{\Phi}_a = \begin{pmatrix} v/\sqrt{2} \\ 0 \end{pmatrix}$:

$$\frac{v^2 i}{8 M_{\underline{m}}}\left(\overline{\nu_{Ln}}\, \nu_{Lkc}\, \rho_{nm}\rho_{km} + \overline{\nu_{Lnc}}\, \nu_{Lk}\, \rho^*_{nm}\rho^*_{km}\right)$$

One sees that the resulting mass matrix $\left(\frac{-v^2}{8M_{\underline{m}}}\right)\rho_{nm}\rho_{km} = M_{nk}$ is a complex, symmetric 3×3 matrix. Note already a difference with respect to the initial quark mass matrix prior to diagonalization, where the f_{mn}, h_{mn} and k_{mn} matrices were not in general symmetric. The symmetry of the M_{nk} mass matrix derives from the relation $-i\bar{\Psi}\chi_c = -i\bar{\chi}\Psi_c$ in the case $\Psi = \nu_{Ln}$, $\chi = \nu_{Lk}$, so $M_{nk}\overline{\nu_{Ln}}\nu_{Lkc} = M_{kn}\overline{\nu_{Ln}}\nu_{Lkc}$, and similarly for the conjugate term $-i M^*_{nk}\overline{\nu_{Lnc}}\nu_{Lk}$.

$\lambda^\alpha \chi_\alpha = \chi^\alpha \lambda_\alpha$

Under $\overset{\nu}{U}(3)$ transformations of the left-handed neutrinos $\nu_{nL} \to \overset{\nu}{U}_{np}\nu_{pL}$, the matrix M_{nk} transforms into $U^\dagger_{np}M_{ps}(U^\dagger)^T_{sk}$. By the process of Takagi Factorization, any complex symmetric matrix M can be decomposed as $M = UDU^T$ where D is a real nonnegative diagonal matrix and U is unitary. Hence one can find $\overset{\nu}{U}$ such that $\overset{\nu}{U}^\dagger M(\overset{\nu}{U}^\dagger)^T = D$ is real, diagonal and nonnegative. Accordingly, the neutrino mass matrix M_{nk} may be diagonalized by $\nu_{Ln} \to \overset{\nu}{U}_{np}\nu_{Lp}$ for some $\overset{\nu}{U}(3)$ matrix $\overset{\nu}{U}$.

With the diagonalization of the neutrino mass matrix comes a change in the charge-current couplings of leptons to $W^\pm_\rho$. As in the case of the quark gauge couplings, couplings to A_ρ and Z_ρ do not mix top and bottom components of $SU(2)_L$ doublets, so those remain undisturbed.

The situation is now different for lepton couplings to the charged $W_\rho^\pm$. Recall that in the minimal Standard Model without neutrino masses, the couplings of $W_\rho^\pm$ to the leptons

$$\mathcal{L}_{cc}^{leptons} = \frac{ig_2}{\sqrt{2}} \left[W_\rho^+ \overline{\nu_{Lm}} \gamma^\rho e_{Lm} + W_\rho^- \overline{e_{Lm}} \gamma^\rho \nu_{Lm} \right]$$

remained unchanged under the $\mathcal{U}^{e_L}(3)$ transformations needed to diagonalize the electron-muon-tau mass matrix, because one could without other consequence transform the ν_{Lm} neutrinos with the same matrix $\mathcal{U}^{e_L}(3)$, causing the leptonic charged current transformations to cancel. After including neutrino masses and the need to diagonalize them, however, the charged current couplings transform to

$$\mathcal{L}_{cc}^{leptons} = \frac{ig_2}{\sqrt{2}} \left[W_\rho^+ V_{mn}^{\nu lep} \overline{\nu_{Lm}} \gamma^\rho e_{Ln} + W_\rho^- V_{mn}^{\nu lep \dagger} \overline{e_{Lm}} \gamma^\rho \nu_{Ln} \right]$$

where $V_{mn}^{\nu lep} = \mathcal{U}_{mp}^{\nu \dagger} \mathcal{U}_{pn}^{e_L}$ is the PMNS mixing matrix. As for the CKM mixing matrix for the quarks, $V_{mn}^{\nu lep}$ is unitary.

Now count the number of physically important parameters needed to determine $V_{mn}^{\nu lep}$ for a system with N generations. The $N \times N$ unitary matrix $V^{\nu lep}$ depends a priori on N^2 real parameters. However, as in the CKM case, one may use <u>non-symmetry</u> phase transformations to put $V^{\nu lep}$ into some standard form. Note, however, that the $\overline{\nu_{Ln}} \nu_{Lmc}$ structure of the neutrino mass terms does <u>not</u> allow for phase transformations of the ν_{Ln} leaving the neutrino mass terms invariant. There are, however, phase changes of the e_{Lm} together with identical phase changes of the e_{Rn} that leave the $\overline{e_{Lm}} e_{Rm}$ mass terms invariant. So there are just N phases that can be used to put the PMNS matrix into a standard form. Thus, for $N = 3$ generations, there are $N^2 - N = 9 - 3 = 6$ physically important parameters. Of these, 3 correspond to orthogonal $O(3)$ transformation parameters, similarly to the CKM case for the quarks. In addition, there are 3 phases determining the complex structure of $V_{mn}^{\nu lep}$.

Note that in the count of physically relevant parameters for the PMNS matrix, an overall phase transformation of all fermions has not been excluded from the number of adjustable phases. This is because the $M_{nk}\overline{\nu_{Ln}}\nu_{Lk}^{c}$ neutrino mass term structure does not admit such an overall phase symmetry. The count of N adjustable phases from the e_{Lm}, e_{Rm} fields remains undiminished, unlike the situation for the CKM matrix.

A standard parametrization of the lepton charged current mixing matrix $V^{\nu lep}_{mn}$ is $V^{\nu lep} = \mathcal{U}^{PMNS}(\vartheta_{12}, \vartheta_{23}, \vartheta_{13}, \delta)\times \mathrm{diag}(e^{i\alpha_1}, e^{i\alpha_2}, 1)$ where the PMNS matrix $\mathcal{U}^{PMNS}$ has the same structure as the CKM matrix for the quark couplings:

$$\mathcal{U}^{PMNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}\begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix}\begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -c_{23}s_{12} - s_{23}c_{12}s_{13}e^{i\delta} & c_{23}c_{12} - s_{23}s_{12}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{23}s_{12} - c_{23}c_{12}s_{13}e^{i\delta} & -s_{23}c_{12} - c_{23}s_{12}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$

where $c_{mn} = \cos\vartheta_{mn}$ and $s_{mn} = \sin\vartheta_{mn}$.

The additional two phases in $V^{\nu lep}$ occurring in the $\mathrm{diag}(e^{i\alpha_1}, e^{i\alpha_2}, 1)$ factor arise because of the Majorana mass structure of $M_{nk}\overline{\nu_{Ln}}\nu_{Lk}^{c}$, which prevents these from being adjusted to zero. [Note, however, that one phase, $e^{i\alpha_3}$, has conventionally been rotated away by making a common phase rotation of the charged leptons e_{Lm}, e_{Rm}] The two phases α_1 and α_2 can only be observed by considering processes that are sensitive to the difference between Majorana and Dirac mass terms. In particular, neutrino oscillations are not sensitive to this difference.

Another aspect of the extra α_1, α_2 phases is the violation of <u>lepton number</u> by the Majorana mass term structure.

Leptons: $L=1$
lepton antiparticles $L=-1$

Lepton number L is a Noether charge corresponding to overall $U(1)$ phase rotations of all leptons: $L(e^-_{Lm}) = L(e^-_{Rm}) = L(\nu_{Lm}) = 1$ and $L(e^+_{Rm} = e^-_{Lmc}) = L(e^+_{Lm} = e^-_{Rmc}) = L(\nu_{Lmc}) = -1$. Non-leptonic particles are assigned lepton number zero. Lepton number conservation is an approximate rigid symmetry consequence. It is violated by anomalies at the quantum level, but so far there is no experimental evidence for lepton number non-conservation. The $e^{i\alpha_1}$ and $e^{i\alpha_2}$ phases are not observable in processes that conserve total lepton number L, such as neutrino oscillations.

Neutrino Oscillations

Most of the information known about neutrino masses and mixings comes from the phenomenon of neutrino oscillations. Neutrinos are usually created by charged-current interactions, produced in association with a charged lepton. Accordingly, they start off in a flavour eigenstate ν_{Lm}. For example, muon decay $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$ (Note: lepton number conservation) produces muon neutrinos and electron antineutrinos. Neutrinos are typically detected by observing the charged lepton in another charged-current reaction. In this way one may observe the final flavour state and find whether it differs from the initial state. The production and capture processes involve interaction flavour eigenstates. For a generic neutrino mass matrix, however, the interaction eigenstates don't coincide with mass eigenstates.

Consider for simplicity just a two generation system $m=1,2$ with the mixing matrix $\begin{pmatrix} c & s \\ -s & c \end{pmatrix}\begin{pmatrix} e^{i\alpha_1} & 0 \\ 0 & e^{i\alpha_2} \end{pmatrix} = \begin{pmatrix} ce^{i\alpha_1} & se^{i\alpha_2} \\ -se^{i\alpha_1} & ce^{i\alpha_2} \end{pmatrix}$.

$c = \cos\theta \quad s = \sin\theta \quad \theta = \theta_{12}$

The flavor basis is related to the mass basis for neutrinos by

$$|\nu_e\rangle = e^{i\alpha_1}\cos\theta\,|\nu_1\rangle + e^{i\alpha_2}\sin\theta\,|\nu_2\rangle$$
$$|\nu_\mu\rangle = -e^{i\alpha_1}\sin\theta\,|\nu_1\rangle + e^{i\alpha_2}\cos\theta\,|\nu_2\rangle$$

$|\nu_e\rangle, |\nu_\mu\rangle$: flavour eigenstates
$|\nu_1\rangle, |\nu_2\rangle$: mass eigenstates

The mass eigenstates have the standard quantum mechanical evolution in time:

$$|\nu_n, t\rangle = e^{iHt}|\nu_n, 0\rangle = e^{it\sqrt{(\vec{p}_n)^2+m_n^2}}|\nu_n, 0\rangle \qquad n=1,2$$

Consequently, an initially electron-generation neutrino state $|\nu_e,0\rangle$ with spatial momentum $|\vec{p}|=p$ evolves after time T to the state

$$|\nu_e, T\rangle = e^{i\alpha_1}\cos\vartheta\, e^{iT\sqrt{p^2+m_1^2}}|\nu_1\rangle + e^{i\alpha_2}\sin\vartheta\, e^{iT\sqrt{p^2+m_2^2}}|\nu_2\rangle .$$

The overlap amplitude of this state with the $|\nu_e\rangle = |\nu_e, 0\rangle$ interaction flavour eigenstate is

α_1, α_2 phases cancel out

$$\langle\nu_e,0|\nu_e,T\rangle = \left(e^{-i\alpha_1}\cos\vartheta\langle\nu_1| + e^{-i\alpha_2}\sin\vartheta\langle\nu_2|\right)\left(e^{i\alpha_1}\cos\vartheta\, e^{iT\sqrt{p^2+m_1^2}}|\nu_1\rangle + e^{i\alpha_2}\sin\vartheta\, e^{iT\sqrt{p^2+m_2^2}}|\nu_2\rangle\right)$$
$$= \cos^2\vartheta\, e^{iT\sqrt{p^2+m_1^2}} + \sin^2\vartheta\, e^{iT\sqrt{p^2+m_2^2}}$$

Hence the probability of finding an electron neutrino in the final state is

$$P(\nu_e\to\nu_e) = |\langle\nu_e,0|\nu_e,T\rangle|^2 = \left|\cos^2\vartheta\, e^{iT\sqrt{p^2+m_1^2}} + \sin^2\vartheta\, e^{iT\sqrt{p^2+m_2^2}}\right|^2$$
$$= \cos^4\vartheta + \sin^4\vartheta + 2\cos^2\vartheta\sin^2\vartheta\cos T\left(\sqrt{p^2+m_1^2}-\sqrt{p^2+m_2^2}\right)$$
$$= 1 - \sin^2 2\vartheta\sin^2\tfrac{T}{2}\left(\sqrt{p^2+m_1^2}-\sqrt{p^2+m_2^2}\right)$$

Now make the approximation that the neutrino mass values m_1, m_2 are much smaller than p, and that $p = E$, up to corrections of order m^2/E^2. Then $\sqrt{p^2+m_1^2} - \sqrt{p^2+m_2^2} \approx (m_1^2-m_2^2)/2E$. Express the resulting probability in terms of the distance travelled L instead of the time T, noting $L \simeq T$ for highly relativistic particles (with $c=1$). Thus

$$P(\nu_e\to\nu_e) = 1 - \sin^2 2\vartheta\,\sin^2(\Delta m^2 L/4E)$$

and since probabilities must add up to 1, one also has

$$P(\nu_e\to\nu_\mu) = \sin^2 2\vartheta\,\sin^2(\Delta m^2 L/4E)$$

where $\Delta m^2 = |m_1^2 - m_2^2|$.

Some points to note about this neutrino oscillation result:

- The Majorana phases $e^{i\alpha_1}, e^{i\alpha_2}$ have cancelled out. One cannot distinguish between Dirac and Majorana mass term structures on the basis of neutrino oscillation experiments.
- The oscillation probability involves only Δm^2 differences between squares of masses, not the masses directly.

A more general treatment gives the probability of a neutrino being produced as a flavour eigenstate ν_a with energy E and then detected in flavour eigenstate ν_b, a distance L away:

$$P_{\nu_a \to \nu_b}(E,L) = |\langle \nu_b(L,t) | \nu_a(0,0)\rangle|^2$$
$$= \sum_{ij} e^{-i(m_i^2 - m_j^2)L/2E}\, \mathcal{U}_{bi}\mathcal{U}^*_{bj}\mathcal{U}_{aj}\mathcal{U}^*_{ai}$$

in which the Majorana phases $e^{i\alpha_j}$ have cancelled out, so only the PMNS mixing matrix $\mathcal{U} = \mathcal{U}^{PMNS}$ appears in the oscillation probability. The sums over i and j arise from inserting a complete basis of mass eigenstates in the probability amplitude.

Defining the "oscillation length" $\lambda(E) = \frac{2E}{\Delta m^2}$, or $\lambda(E) = 500\,m \left(\frac{E}{1\,GeV}\right)\left(\frac{1eV^2}{\Delta m^2}\right)$, and noting that the observed neutrino $(mass)^2$ splittings are bounded by $\Delta m^2 \lesssim 5\times10^{-3} eV^2$ with typical energies $E \lesssim 10\,GeV$, oscillation lengths typically need to be in ranges of 100s of km to get useful information about Δm^2 values. Note that for lengths $L \ll \lambda(E)$, the probability $P_{\nu_a\to\nu_b}$ tends to $P_{\nu_a\to\nu_b}(E,0) = \delta_{ab}$. Accordingly, neutrino oscillation experiments involve very long-baseline setups, eg. 295 km for the T2K experiment with a muon neutrino beam from J-PARC in Tokai to a detector in Kamioka. The T2K experiment has a range of muon neutrino energies around 600 MeV, chosen to get the highest $P_{\nu_\mu\to\nu_e}$ over a distance of 295 km.

Two distinct types of neutrino oscillations have been clearly observed. The first type concerns the disappearance of ν_e neutrinos coming from the Sun as well as the disappearance of $\bar{\nu}_e$ neutrinos coming from reactors. Considered in the two-flavour case (a good approximation, even with $N=3$ flavours), such experiments give $30° < \theta_\odot < 38°$, $\Delta m^2_\odot = |m_2^2 - m_1^2| = 7.53 \pm 0.18 \times 10^{-5} eV^2$ (99% confidence level) $\sin^2 2\theta_\odot = 0.846 \pm 0.021$ $\quad \theta_\odot = \theta_{12}$

The second type of observed neutrino oscillation involves the disappearance of ν_μ and $\bar{\nu}_\mu$ neutrinos from cosmic-ray showers in the upper atmosphere or of ν_μ neutrinos from long-baseline beam experiments. Again, interpretation in terms of a two-flavour system proves to be a good approximation.

$\Delta m^2_{31} \approx \Delta m^2_{32}$ Results give $38° < \vartheta_{atm} < 52°$, $\Delta m^2_{atm} = |m_3^2 - m_2^2| = 2.44 \pm 0.06 \times 10^{-3} eV^2$ (90% CL) $\sin^2 2\vartheta_{atm} > 0.92$ $\vartheta_{atm} = \vartheta_{23}$. Since $\sin^2 2\vartheta_{atm}$ is consistent with unity, such oscillations are close to maximal.

From these experimental results, one sees that the splitting between the two neutrino species involved in the solar-neutrino oscillations is much smaller than the splitting between either of them from the third neutrino species. Common practice is to label the neutrino-mass eigenstates so the small splitting is between ν_1 and ν_2. Since $\Delta m^2_{12} = \Delta m^2_{\odot}$ determines the solar-neutrino oscillations, which involve the disappearance of ν_e neutrinos, either ν_1 or ν_2 must have a significant overlap with the ν_e flavour basis neutrino. Conventionally, one chooses ν_1 to be the mass eigenstate with this significant overlap.

Note that since the neutrino oscillations involve $|\Delta m^2_{ij}|$ differences, they do not determine the values of the individual masses. This leaves open two possibilities for the neutrino mass hierarchy. If the nearly degenerate (ν_1, ν_2) pair is less massive than ν_3, the mass pattern is called the "normal" hierarchy. Alternatively, if the (ν_1, ν_2) pair is more massive than ν_3, the hierarchy is called the "inverted" hierarchy.

The results of experiment may be summarized in an approximate PMNS matrix obtained by setting $\vartheta_{13} = 0$, maximal $\vartheta_{23} = \pi/4$ (so $s_{13} = 0$, $c_{13} = 1$, $s_{23} = c_{23} = 1/\sqrt{2}$):

$$U^{PMNS} \approx \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12}/\sqrt{2} & c_{12}/\sqrt{2} & 1/\sqrt{2} \\ s_{12}/\sqrt{2} & -c_{12}/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}.$$

CP invariance and breakdown

Owing to the chiral nature of the Standard Model, i.e. the key rôle γ_5 plays in the definition of the fermion field representations, parity is clearly not a symmetry. Nor is charge conjugation. However, the combined CP discrete transformation is a symmetry of the gauge and leptonic sector provided neutrino masses are not turned on. We will come to what happens with quarks and neutrino masses in due course (the stories are similar, and involve CKM and PMNS matrices.) ... First, we need to establish how parity and charge conjugation act on spinor fields.

We have seen that the proper Lorentz group $SO(3,1)$ is related to the $Spin(3,1)$ group by $Spin(3,1) \simeq SL(2,\mathbb{C})$ and $SO(3,1) \simeq SL(2,\mathbb{C})/\mathbb{Z}_2$, ie $Spin(3,1)$ is the "double cover" of $SO(3,1)$. When one wants to extend $SO(3,1)$ to $O(3,1)$ by the inclusion of transformations $L^\mu{}_\nu$ with $\det(L) = -1$, defining the corresponding extension of $Spin(3,1)$ requires some care. The corresponding group is called <u>$Pin(3,1)$</u> — a mathematician's joke, attributed to J.P. Serre (according to Atiyah, Bott & Shapiro), although when asked about the name, would not confirm his rôle in it.

Although the "bosonic" Lorentz groups $O(1,3)$ and $O(3,1)$ are isomorphic – making the choice of "mostly plus" or "mostly minus" formalisms simply one of preference – this is not the case for $Pin(1,3)$ and $Pin(3,1)$, which are not isomorphic. [A reference: M. Berg, C. DeWitt-Morette, S. Gwo & E. Kramer, "The Pin Groups in Physics: C, P and T", Rev. Math. Phys. 13 (2001) 953-1034, math-ph/0012006]. The relation between Pin and Spin groups is $Pin(m,n) \simeq Spin(m,n) \rtimes \mathbb{Z}_2$, where $\rtimes$ denotes a semidirect product.

The key difference between $Pin(3,1)$ and $Spin(3,1)$ is the existence of group elements built from odd powers of gamma matrices. The generators of $Spin(3,1)$ are represented by gamma-matrix commutators: $S_{\mu\nu} = \frac{-i}{4}[\gamma_\mu, \gamma_\nu]$, which satisfy the algebra $[S_{\mu\nu}, S_{\rho\sigma}] = i(\eta_{\mu\rho}S_{\nu\sigma} - \eta_{\nu\rho}S_{\mu\sigma} - \eta_{\mu\sigma}S_{\nu\rho} + \eta_{\nu\sigma}S_{\mu\rho})$ just the same as the generators of the Lorentz vector representation $(J_{\mu\nu})^\rho{}_\sigma = -i(\delta_\mu{}^\rho\eta_{\nu\sigma} - \delta_\nu{}^\rho\eta_{\mu\sigma})$. [Check infinitesimal transformation $\delta V^\rho = \frac{i}{2}(\omega^{\mu\nu}J_{\mu\nu})^\rho{}_\sigma V^\sigma = \omega^\rho{}_\sigma V^\sigma$] Group element representations $\exp(\frac{i}{2}\omega^{\mu\nu}S_{\mu\nu})$ clearly involve only <u>even</u> powers of the γ matrices.

Now consider how <u>parity</u> inversions may be realized when acting on spinor fields. When acting on a scalar field $\phi(x^\mu)$, one has $\phi(x) \xrightarrow{P} \eta_P\phi(x_P)$, where for space inversion $x^\mu_P = (x^0, -x^i)$ $i=1,2,3$ and the phase η_P is the <u>intrinsic parity</u> of the field ϕ: $\eta_P = 1$ for scalars, $\eta_P = -1$ for pseudoscalars. When it comes to spinor fields, however, it is not just a matter of changing the field's argument $x^\mu \to x^\mu_P$ and supplying a phase. One needs to ensure that the Dirac equation remains valid after a parity transformation. For a massive spinor field $\psi(x)$, one has $(\gamma^\mu\partial_\mu + m)\psi(x) = 0$. Let the parity operation be $\psi(x) \to \psi_P(x) = \eta_P\Lambda_P\psi(x_P)$ where $x^\mu_P = (x^0, -x^i)$ and Λ_P is some unitary matrix. Since $\partial_i\psi(x_P) = -\frac{\partial}{\partial x_P^i}\psi(x_P)$, one has $(\gamma^0\partial_0 - \gamma^i\partial/\partial x^i + m)\psi(x_P) = 0$, so the flip of sign in the $\gamma^i\partial_i$ term must be reversed by the Λ_P operator. Requiring $\{\Lambda_P, \gamma^i\} = 0$ and choosing Λ_P hermitian leads to $\Lambda_P = \pm i\gamma^0$. Pick $\Lambda_P = i\gamma^0$. Then $\psi_P(x) = \eta_P i\gamma^0\psi(x_P)$ and the Dirac equation is preserved: $(\gamma^\mu\partial_\mu + m)\psi_P(x) = 0$. The phase η_P is called the intrinsic parity of ψ, and needs to be chosen carefully in the context of a given theory to achieve overall invariance, including interactions if possible.

The Dirac conjugate spinor $\bar{\Psi} = \Psi^\dagger \gamma^0$ consequently transforms under parity as $\bar{\Psi}(x) \xrightarrow{P} \bar{\Psi}_p(x) = \eta_p^* \bar{\Psi}(x_p) i\gamma^0$, and since η_p is a phase, $|\eta_p|^2 = 1$, one obtains the following transformations of spinor bilinears under parity:

$$\bar{\Psi}(x)\Psi(x) \rightarrow \bar{\Psi}(x_p)\Psi(x_p) \quad \text{scalar}$$
$$\bar{\Psi}(x)\gamma_5\Psi(x) \rightarrow -\bar{\Psi}(x_p)\gamma_5\Psi(x_p) \quad \text{pseudoscalar}$$
$$\bar{\Psi}(x)\gamma^0\Psi(x) \rightarrow \bar{\Psi}(x_p)\gamma^0\Psi(x_p) \quad \text{charge density}$$
$$\bar{\Psi}(x)\gamma^i\Psi(x) \rightarrow -\bar{\Psi}(x_p)\gamma^i\Psi(x_p) \quad \text{current density}.$$

N.B. limit flips: $\int_{-\infty}^{\infty} d^3x\, L(x_p) = \int_{-\infty}^{\infty} d^3x_p\, L(x_p) = \int_{-\infty}^{\infty} d^3x\, L(x)$

Thus, one finds that a standard kinetic term for a non-chiral Dirac spinor $\frac{-i}{2}\bar{\Psi}\gamma^\mu\partial_\mu\Psi \rightarrow \frac{-i}{2}\bar{\Psi}_p\gamma^\mu\partial_\mu\Psi_p$ is parity invariant, as is a mass term $\frac{-i}{2}m\bar{\Psi}\Psi \rightarrow \frac{-i}{2}m\bar{\Psi}_p\Psi_p$.

For <u>chiral</u> spinor fields, however, parity is broken: $P_+\Psi(x) = \frac{1}{2}(1+\gamma_5)\Psi(x) \rightarrow \frac{1}{2}(1-\gamma_5)\Psi_p(x)$, i.e. parity switches left and right projections $P_+ \leftrightarrow P_-$, and $-i\bar{\Psi}_L(x)\gamma^\mu\partial_\mu\Psi_L(x) = \frac{-i}{2}\bar{\Psi}(x)\gamma^\mu\partial_\mu(1+\gamma_5)\Psi(x) \rightarrow \frac{-i}{2}\bar{\Psi}_p(x)\gamma^\mu\partial_\mu(1-\gamma_5)\Psi_p(x)$.

In the Standard Model, with its chiral representations and couplings, parity is accordingly not a good symmetry.

Next consider <u>charge conjugation</u>. We have already introduced the notion of the charge conjugate $\Psi_c(x) = C(\bar{\Psi}(x))^T$, where in the Weyl representation of the gamma matrices, numerically one has $C = -\gamma^0\gamma^2$, becoming in 2-component notation $C = \begin{pmatrix} i\epsilon_{\alpha\beta} & 0 \\ 0 & i\epsilon^{\dot{\alpha}\dot{\beta}} \end{pmatrix}$. [Note: relative to Weinberg, $C_{here} = i C_{Weinberg}$. This is because for Weinberg $\bar{\Psi}_{Weinberg} = \Psi^\dagger\beta$ where $\beta = i\gamma^0$.] Consider now the Dirac equation for a charged spinor field: $(\gamma^\mu\partial_\mu + ieA_\mu\gamma^\mu + m)\Psi = 0$.

Take the adjoint: $\partial_\mu\Psi^\dagger\gamma^{\mu\dagger} - ie\Psi^\dagger\gamma^{\mu\dagger}A_\mu + m\Psi^\dagger = 0$

then use $\gamma^{\mu\dagger}\gamma^0 = -\gamma^0\gamma^\mu$ to get $-\partial_\mu\bar{\Psi}\gamma^\mu + ie\bar{\Psi}\gamma^\mu A_\mu + m\bar{\Psi} = 0$

then transpose: $-\gamma^{\mu T}(\partial_\mu\bar{\Psi})^T + ieA_\mu\gamma^{\mu T}(\bar{\Psi})^T + m(\bar{\Psi})^T = 0$

and note $C\gamma^{\mu T} = -\gamma^\mu C$ for $C = -\gamma^0\gamma^2$, so for $\Psi_c = C(\bar{\Psi})^T$ one finds $(\gamma^\mu\partial_\mu - ieA_\mu\gamma^\mu + m)\Psi_c = 0$, i.e. the Dirac equation goes into itself but with $e \rightarrow -e$.

In a system with a variety of spinor fields, it may be necessary to include also charge conjugation phases, so the charge conjugation transformation can more generally be written $\psi \to \eta_c C(\overline{\psi})^T$. Now consider the effect of a charge-conjugation transformation on the Lagrangian for a Dirac spinor field $-i\overline{\psi}(\gamma^\mu(\partial_\mu + ieA_\mu) + m)\psi$. As we have seen, $\overline{\psi}\chi = \overline{\chi}_c\psi_c$, so the transform $-i\overline{\psi}_c(\gamma^\mu(\partial_\mu + ieA_\mu) + m)\psi_c$, noting that the phases $\eta_c^*\eta_c = 1$ cancel out, may be rewritten $-i\left[\overline{(\not{\partial}\psi)}_c\psi + ie\overline{(\not{A}\psi)}_c\psi + m\overline{\psi}\psi\right]$, noting that $\psi_{cc} = \psi$.
Then, using $\gamma^{\mu\dagger}\gamma^0 = -\gamma^0\gamma^\mu$ and $C(\gamma^\mu)^T = -\gamma^\mu C$, one finds $(\gamma^\mu\psi_c)_c = C[-\overline{\psi}_c\gamma^\mu]^T = \gamma^\mu\psi_{cc} = \gamma^\mu\psi$, so also $(\not{\partial}\psi_c)_c = \not{\partial}\psi$. Then $\overline{\gamma^\mu\psi} = -\overline{\psi}\gamma^\mu$
so $-i\overline{\psi}_c[\gamma^\mu(\partial_\mu + ieA_\mu) + m]\psi_c = -i[-\partial_\mu\overline{\psi}\gamma^\mu\psi - ie\overline{\psi}\not{A}\psi + m\overline{\psi}\psi]$
and integrating ∂_μ by parts in the action makes this into
$-i\overline{\psi}(\gamma^\mu(\partial_\mu - ieA_\mu) + m)\psi$, corresponding exactly to the effect of the charge-conjugation transformation on the field equation. Invariance under such a transformation consequently would require transforming $A_\mu \to -A_\mu$, treating e as a fixed coupling constant. This works in QED, which consequently has a charge-conjugation symmetry (as well as parity). In more complicated models, one just has to search for spinor field η_c phases and vector sign flips to see if invariance of the action can be achieved.

In order to investigate the degree to which charge-conjugation symmetry works in the Standard Model, we need to know its effect on the various types of terms in the Lagrangian.

$C = -\gamma^0\gamma^2$ It may be useful to make a summary of relations: $C^\dagger = C$, $C^T = -C$, $C^2 = 1$
$\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$ (or $\gamma^{\mu\dagger}\gamma^0 = -\gamma^0\gamma^\mu$); $C\gamma^{\mu T}C = -\gamma^\mu$; $C\gamma_5^T C = \gamma_5$
$\psi_c = C(\overline{\psi})^T = -\gamma^0 C\psi^*$, $\overline{\psi}_c = -\psi^T C$; $\overline{(\gamma^\mu\psi)} = -\overline{\psi}\gamma^\mu$, $\overline{(\gamma_5\psi)} = -\overline{\psi}\gamma^5$

From these follow a number of transform relations for fermion bilinears which are needed to find the effect of charge conjugation on the various types of terms in the SM action:

(leaving out phases)

$$\bar\chi\psi \xrightarrow{c} \bar\chi_c\psi_c = \bar\psi\chi \qquad \bar\chi\gamma^\mu\psi \xrightarrow{c} \bar\chi_c\gamma^\mu\psi_c = -\bar\psi\gamma^\mu\chi$$

$$\bar\chi\gamma_5\psi \xrightarrow{c} \bar\chi_c\gamma_5\psi_c = \bar\psi\gamma_5\chi \qquad \bar\chi[\gamma^\mu,\gamma^\nu]\psi \to \bar\chi_c[\gamma^\mu,\gamma^\nu]\psi_c = -\bar\psi[\gamma^\mu,\gamma^\nu]\chi$$

$$\bar\chi\gamma_5\gamma_\mu\psi \xrightarrow{c} \bar\chi_c\gamma_5\gamma_\mu\psi_c = -\bar\psi\gamma_\mu\gamma_5\chi = \bar\psi\gamma_5\gamma_\mu\chi$$

The above relations are valid for arbitrary Dirac (general) spinor fields, which anticommute. In performing charge conjugation transformations, only fields transform – quantities like γ^μ in bilinears are not directly transformed, although it may be useful to consider charge conjugates of products of γs acting on spinor fields in rearranging the result of a transformation. We have already done this above in discussing the transformation of the gauge coupled QED fermion kinetic terms. Another example could be $\bar\psi\gamma_5\chi \xrightarrow{c} \bar\psi_c\gamma_5\chi_c$ (transformation just of the fields) followed by rearrangement: $\bar\psi_c\gamma_5\chi_c = \overline{(\gamma_5\chi_c)_c}\psi_{cc} = \overline{(\gamma_5\chi_c)_c}\psi$ using $\bar\chi\psi = \bar\psi_c\chi_c$, then $(\gamma_5\chi_c)_c = -\gamma_5\chi_{cc} = -\gamma_5\chi$, so one has $-\overline{(\gamma_5\chi)}\psi = -\chi^\dagger\gamma_5\gamma^0\psi = \bar\chi\gamma_5\psi$, as above.

Now consider the effect of a charge-conjugation transformation on the pure derivative part of a kinetic term for spinor fields carrying a representation of some group G, e.g. $SU(3)$. For a field ψ_A, where A is an index corresponding to a complex representation of the group G, the charge-conjugation transformation sends ψ_A to $\eta_c\psi_c{}^A$, with a raised index for the conjugate G representation. For now, let ψ_A be a general Dirac spinor field. The charge-conjugation transformation then transforms the kinetic term for ψ_A as $-\frac{i}{2}\bar\psi^A\gamma^\mu\partial_\mu\psi_A \xrightarrow{c} -\frac{i}{2}\bar\psi_{cA}\gamma^\mu\partial_\mu\psi_c{}^A$, noting that the η_c phases cancel between the transformations of ψ_A and $\bar\psi^A$. Then, using the above rules, the kinetic term can be reorganised to $+\frac{i}{2}\partial_\mu\bar\psi_{cc}{}^A\gamma^\mu\psi_{ccA} = \frac{i}{2}\partial_\mu\bar\psi^A\gamma^\mu\psi_A$. When such a Lagrangian term is integrated over in the action, one may integrate the ∂_μ derivative by parts giving back $-\frac{i}{2}\bar\psi^A\gamma^\mu\partial_\mu\psi_A$, so such pure-derivative kinetic terms are C-invariant.

Next, consider what happens when there is nonabelian gauge coupling to the spinor fields. We have seen in the QED case that the gauge field may itself need to transform under charge conjugation in order to achieve invariance. The key requirement is to preserve covariance. Consider what happens for some general putatively covariant derivative $D_{\mu A}{}^{B}$ acting on a spinor field ψ_A carrying a representation of the gauge group G denoted by the index A. [For example, for the group $SU(3)$, still consider ψ_A to be a general Dirac spinor field.] The charge-conjugate transformation sends $\psi_A \xrightarrow{C} \psi_c{}^A$ (again ignoring phases η_c). The covariant derivative of ψ_A is $(D_\mu \psi)_A = \partial_\mu \psi_A - ig A_\mu^i T^i_A{}^B \psi_B$ (assuming a simple group G for which one has a single coupling constant g; for non-simple groups, one can have separate g_i for each simple factor).

Applying a charge-conjugation transformation now to ψ_A (making only the group G index manifest; ψ_A has also spinor indices), one obtains $\partial_\mu \psi_c{}^A - ig A_\mu^i T^i_A{}^B \psi_c{}^B$, which seems very noncovariant. However, we need to see what can be achieved by careful choice of A'^i_μ. For a unitary representation $U_A{}^B = \exp(i\theta^i T^i_A{}^B)$ of G, the generators $T^i_A{}^B$ are hermitian, so $(T^{i*})^A{}_B = T^i_B{}^A$. The desired, covariant, form of the C-transformed $(D'_\mu \psi_c)^A$ is

$\psi_c{}^A \to \psi_c{}^B U^{-1}_B{}^A = \psi_c{}^B U^\dagger{}_B{}^A$

$(D'_\mu \psi_c)^A = \partial_\mu \psi_c{}^A + ig \psi_c{}^B A'^i_\mu T^i_B{}^A$. So we see that what is required of the gauge field is

$$A'^i_\mu T^i_A{}^B = -A^i_\mu T^i_B{}^A, \text{ i.e.}$$

for $T^i_A{}^B = T^i_B{}^A$ symmetric, one must have $A'^i_\mu = -A^i_\mu$ and for $T^i_A{}^B = -T^i_B{}^A$ antisymmetric, one has $A'^i_\mu = A^i_\mu$, C-invariant. These transformations of the A^i_μ are just what is needed to make the nonabelian field strength $F^i_{\mu\nu}$ transform coherently under charge conjugation with A^i_μ: $F^i_{\mu\nu} T^i = \frac{i}{g}[D_\mu, D_\nu]$, so $F^i_{\mu\nu}$ flips sign if A^i_μ does, and stays invariant if A^i_μ does.

Check whether this transformation of the gauge fields A_μ^i under C agrees with what one finds in QED, where $G = U(1)$. The $U(1)$ unitary representation is just $U = e^{i\theta}$, so one may take $T = -1$. This is a rather trivial symmetric 1×1 matrix. Accordingly, the gauge field A_μ in $D_\mu\psi = (\partial_\mu + ieA_\mu)\psi$ must transform $A_\mu \to -A_\mu$, as we found above. For $SU(2)$, with generators $\frac{1}{2}\sigma^i$ with $\sigma^1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma^2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\sigma^3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$ the A_μ^1 and A_μ^3 gauge fields need to flip sign, while the A_μ^2 gauge field remains invariant under C.

Now consider what charge conjugation does to chiral spinor fields when coupled to gauge fields. For simplicity, consider the chiral coupling of a spinor field ψ to a $U(1)$ gauge field like B_μ : $-i\bar\psi\partial\!\!\!/\psi - \frac{q}{2}B_\mu\bar\psi\gamma^\mu(1+\gamma_5)\psi$, where in this example q is the charge of the left-handed part of ψ, while the right-handed part of ψ remains uncharged, i.e. $D_\mu\psi_L = \partial_\mu\psi_L - iqB_\mu\psi_L$, $D_\mu\psi_R = \partial_\mu\psi_R$. Under charge conjugation, the pure derivative kinetic term $-i\bar\psi\partial\!\!\!/\psi = -i(\bar\psi_L\partial\!\!\!/\psi_L + \bar\psi_R\partial\!\!\!/\psi_R)$ is C-invariant, but for the chiral gauge coupling $-\frac{q}{2}B_\mu\bar\psi\gamma^\mu(1+\gamma_5)\psi \to -\frac{q}{2}B_\mu\bar\psi_c\gamma^\mu(1+\gamma_5)\psi_c$ considering just the spinor field transformation $\psi \to \psi_c\eta_c$. Rearrange using the earlier rules: $\bar\psi_c\gamma^\mu\psi_c = -\bar\psi\gamma^\mu\psi$, $\bar\psi_c\gamma^\mu\gamma_5\psi_c = +\bar\psi\gamma^\mu\gamma_5\psi$, so the gauge coupling transforms to $+\frac{q}{2}B_\mu\bar\psi\gamma^\mu(1-\gamma_5)\psi$, i.e. charge conjugation has transformed a left-handed chiral coupling into a right-handed chiral coupling. Note that, unlike the nonchiral QED coupling case, there is no transformation $B_\mu \to B_\mu'$ that can achieve invariance. The expected $B_\mu \to -B_\mu$ works for the γ^μ term, but not for the $\gamma^\mu\gamma_5$ term in the coupling.

It makes things clearer to consider the effect of charge conjugation in 2-component notation.

For a Dirac field, $\Psi = \begin{pmatrix} \lambda_\alpha \\ \bar{\chi}^{\dot\beta} \end{pmatrix}$, one has $\Psi_c = \begin{pmatrix} \chi_\alpha \\ \bar{\lambda}^{\dot\beta} \end{pmatrix}$, so the charge-conjugation transformation is $\begin{pmatrix} \lambda_\alpha \\ \bar{\chi}^{\dot\beta} \end{pmatrix} \to \eta_c \begin{pmatrix} \chi_\alpha \\ \bar{\lambda}^{\dot\beta} \end{pmatrix}$, i.e. $\lambda_\alpha \to \eta_c \chi_\alpha$, $\chi_\alpha \to \eta_c^* \lambda_\alpha$

In a chiral gauge coupling, C-invariance can not be achieved because the couplings of λ_α and χ_α are not symmetrical. [In the above example, χ_α does not couple at all.]

An even more incompatible situation for C-invariance occurs if one chirality is simply absent for some spinor field type. This is the case for the original Standard model without right-handed neutrinos.

Consider the effect now of a C transformation on the charged-current couplings in the Standard Model:

$$\frac{-g_2}{2\sqrt{2}}\left[V_{mn} W_\mu^+ \bar{u}_m \gamma^\mu (1+\gamma_5) d_n + (V^\dagger)_{mn} W_\mu^- \bar{d}_m \gamma^\mu (1+\gamma_5) u_n\right]$$

under charge conjugation, this becomes

$$\frac{+g_2}{2\sqrt{2}}\Big\{(\eta^c_{um})^*(\eta^c_{dn}) V_{mn} W_\mu^- \bar{u}_{cm}\gamma^\mu(1+\gamma_5) d_{cn}$$
$$+ (\eta^c_{un})(\eta^c_{dm})^* V^\dagger_{mn} W_\mu^+ \bar{d}_{cm}\gamma^\mu(1+\gamma_5) u_{cn}\Big\}$$

where we have used the anticipated $W_\mu^\pm \xrightarrow{C} -W_\mu^\mp$ based on the symmetric / antisymmetric nature of the SU(2) generators. Then $\bar{u}_{cm}\gamma^\mu(1+\gamma_5)d_{cn} = -\bar{d}_n\gamma^\mu(1-\gamma_5)u_m$ as we have seen, so the coupling cannot be charge-conjugation invariant: there has been a $P_+ \to P_-$ switch of projectors.

Now consider the effect of a parity transformation on this coupling. Under parity, the spatial components of gauge fields will have to switch sign in order to make couplings like $A_i \bar\Psi \gamma^i \Psi \to (-A_i)(-\bar\Psi\gamma^i\Psi) = A_i \bar\Psi \gamma^i \Psi$ invariant. The time components don't flip signs, however, just as for the charge density $\bar\Psi\gamma^0\Psi$. So $A_\mu \bar\Psi \gamma^\mu \Psi$ terms remain parity invariant. For $\gamma_5\gamma^\mu$ coupling terms, however, there is a sign flip: $A_\mu \bar\psi \gamma_5 \gamma^\mu \psi \xrightarrow{P} -A_\mu \bar\psi \gamma_5 \gamma^\mu \psi$. Consequently, $A_\mu \bar\psi \gamma^\mu (1+\gamma_5) \psi \xrightarrow{P} A_\mu \bar\psi \gamma^\mu (1-\gamma_5)\psi$, also producing a $P_+ \to P_-$ switch of projectors.

Consequently, combining C and P transformations, it is possible to arrange for invariance of kinetic and gauge coupling terms in the Standard model as long as the mixing between generations doesn't cause a problem, eg. if V_{mn} were just δ_{mn}.

Under the combined CP transformation for the charge current coupling, one obtains

$$\frac{-g_2}{2\sqrt{2}} \Big\{ (\eta^{c}_{um})^* (\eta^{c}_{dn}) (\eta^{P}_{um})^* (\eta^{P}_{dn}) \, V_{mn} W^-_\mu \bar d_n \gamma^\mu (1+\gamma_5) u_m + (\eta^{c}_{un}) (\eta^{c}_{dm})^* (\eta^{P}_{un}) (\eta^{P}_{dm})^* V^\dagger_{mn} W^+_\mu \bar u_n \gamma^\mu (1+\gamma_5) d_m \Big\}$$

If the phases can be chosen to make $(\eta^{c}_{um})^* (\eta^{c}_{dn}) (\eta^{P}_{um})^* (\eta^{P}_{dn}) = 1$ and if the CKM matrix V_{mn} can be chosen to be real, then the above term becomes the complex conjugate of the original term before the CP transformation. Note the inversion of the (n,m) indices in the fermion bilinears, which can be corrected by rewriting the CP transformed charged current coupling

$$\frac{-g_2}{2\sqrt{2}} \Big\{ (\eta^{c}_{un}) (\eta^{c}_{dm})^* (\eta^{P}_{un}) (\eta^{P}_{dm})^* V^*_{nm} W^+_\mu \bar u_n \gamma^\mu (1+\gamma_5) d_m + (\eta^{c}_{um})^* (\eta^{c}_{dn}) (\eta^{P}_{um})^* (\eta^{P}_{dn}) V^{\dagger *}_{nm} W^-_\mu \bar d_n \gamma^\mu (1+\gamma_5) u_m \Big\}$$

which agrees with the starting coupling terms provided $(\eta^{c}_{un})(\eta^{P}_{un})(\eta^{c}_{dm})^*(\eta^{P}_{dm})^* V^*_{nm} = V_{nm}$. Clearly, this can be achieved if $(\eta^{c}_{un})(\eta^{P}_{un}) = 1 = (\eta^{c}_{dm})(\eta^{P}_{dm})$ for all m, n and if $V_{mn} = V^*_{mn}$.

An easier way to see that complex coefficients in the SM Lagrangian imply CP violation is to consider the implications of CPT invariance. A basic theorem on the relation between spin and statistics (appeared first implicitly in the work of Schwinger, then more explicitly in work by Lüders & Pauli; see Streater and Wightman, "PCT, spin and statistics, and all that") shows that CPT violation implies breaking of Lorentz symmetry. On pinors, a CPT transformation corresponds to an orientation preserving Lorentz transformation. Consequently, in a Lorentz invariant field theory any term in the Lagrangian must be CPT invariant, whether it has real or complex coefficients, and T invariance implies CP invariance. The T transformation, however, is antiunitary, so it sends $i \to -i$ in addition to whatever it does to fields. Thus, if a given Lagrangian term is T invariant for real coefficients, it must be T violating for imaginary coefficients. Then by CPT invariance, one concludes that imaginary coefficients imply CP violation. In the standard Particle Data Group presentation of the CKM matrix V_{mn}, this means that a non-zero value of the phase δ implies CP violation.

Discussing CP violation in terms of a particular phase in one version of the CKM matrix is not optimal, because different choices of basis or phases of V_{mn} can alter the discussion without altering the physics. It is better to have a basis-independent measure of CP violation. For example, in the flavour basis where the $W_\mu^\pm$ interactions are flavour diagonal, $V_{mn} = \delta_{mn}$, but the quark mass matrix is then nondiagonal and complex, and CP is still violated. Conversely, if the quark mass matrix were diagonal and V_{mn} were complex, there could still be an absence of CP violation if there were some remaining phase change that could eliminate the complex character of V_{mn}. If one of the quarks is massless, this is the case

Recall how the original Yukawa coupling matrices are related to the diagonal quark mass matrices:

$y^{(u)}_{mn} = k_{mn} = U^{u_L\dagger}_{ms} D^{(u)}_{st} U^{u_R}_{tn}$ for uquarks, $y^{(d)}_{mn} = h_{mn} = U^{d_L\dagger}_{ms} D^{(d)}_{st} U^{d_R}_{tn}$ for dquarks

where $D^{(u)} = \frac{v}{\sqrt{2}}\,\mathrm{diag}(k_1,k_2,k_3)$ and $D^{(d)} = \frac{v}{\sqrt{2}}\,\mathrm{diag}(h_1,h_2,h_3)$ are the diagonal mass matrices in mass basis. The CKM matrix is $V_{mn} = \left(U^{u_L}(U^{d_L})^\dagger\right)_{mn}$. Thus, if $U^{u_L} = U^{d_L}$, then $V = \mathbb{1}$ with no flavour mixing or CP violation.

Instead of rotating $u_L' = U^{u_L\dagger}u_L$, $u_R' = U^{u_R\dagger}u_R$, $d_L' = U^{d_L\dagger}d_L$, $d_R' = U^{d_R\dagger}d_R$ to make the mass matrix diagonal, one could first leave $u'_{L_m} = u_{L_m}$ unchanged, but take $u'_{Rm} = (U^{u_R\dagger}U^{u_L})_{mn}u_{Rn}$ to make $y^{(u)}_{mn} = (U^{u_L\dagger}D^{(u)}U^{u_L})_{mn}$ which is hermitian, and similarly take $d'_{Lm} = d_{Lm}$, $d'_{Rm} = (U^{d_R\dagger}U^{d_L})_{mn}d_{Rn}$ so $y^{(d)}_{mn} = (U^{d_L\dagger}D^{(d)}U^{d_L})_{mn}$ also hermitian. So consider $y^{(u)}$ and $y^{(d)}$ to be hermitian without loss of generality.

(ie using same U) If $y^{(u)}$ and $y^{(d)}$ could be simultaneously (keeping them hermitian) diagonalized, then one would have $V = \mathbb{1}$ and there would be no CP violation. A hermitian matrix is diagonalized by a unitary transformation $y \to U y U^\dagger$. In order for two hermitian matrices to be simultaneously diagonalizable (with ie. the same U), they must commute. So the amount of CP violation is encoded in the commutator

$$-iK = [y^{(u)}, y^{(d)}] = [U^{u_L\dagger}D^{(u)}U^{u_L}, U^{d_L\dagger}D^{(d)}U^{d_L}]$$
$$= U^{u_L\dagger}[D^{(u)}, V D^{(d)} V^\dagger]\,U^{u_L}$$

The matrix K is traceless and hermitian, because $y^{(u)}$ and $y^{(d)}$ are hermitian.

The natural basis-invariant quantity is its determinant: $\det K = -i\det([D^{(u)}, V D^{(u)} V^\dagger])$. It turns out that

$$\det K = -2J(m_t - m_c)(m_t - m_u)(m_c - m_u)(m_b - m_s)(m_b - m_d)(m_s - m_d)$$

where $J = \mathrm{Im}(V_{11}V_{22}V_{12}^*V_{21}^*) = -\mathrm{Im}(V_{11}V_{32}V_{21}^*V_{31}) = \mathrm{Im}(V_{22}V_{33}V_{23}^*V_{32}^*) =$ etc.

or, in general $\mathrm{Im}(V_{ij}V_{kl}V_{il}^*V_{kj}^*) = J\sum_{m,n}\epsilon_{ikm}\epsilon_{jln}$. J is the Jarlskog invariant.

The different products giving expressions for J are all equal owing to the unitarity of the CKM matrix V. In the standard Particle Data Group parametrization,

$$J = s_{12}s_{23}s_{31}c_{12}c_{23}c_{31}^2 \sin\delta$$

As we saw earlier, J is twice the area of the unitarity triangle. All the weak CP violation in the Standard Model is proportional to $\det K$, i.e. proportional to $\mathrm{Im}\det[y^{(u)}, y^{(d)}]$.

If V is real, we have seen that there is no CP violation, $J=0$ and so $\det K = 0$. Moreover, if any two up-type or two down-type quarks are degenerate in mass, then $\det K = 0$. If there are such degenerate masses, an extra phase rotation becomes possible to remove the CP-violating phase. And we have already seen that if there were only two generations, there would be no CP violation because one could then remove all phases from V. Accordingly, CP-violating effects must involve all three generations. The CKM elements coupling either of the first two generations to the third are small, so the amount of CP violation is suppressed by products of such small coefficients.

The first measurement of CP violation was through decays of neutral kaons. Their quark content is $K^0 = \bar{s}d$ and $\bar{K}^0 = \bar{d}s$, which are CP conjugates of each other, mixing generations 1 and 2. The CP eigenstates are

$$K_1 = \frac{K^0 + \bar{K}^0}{\sqrt{2}}, \quad K_2 = \frac{K^0 - \bar{K}^0}{\sqrt{2}}.$$

K_1 is CP even, while K_2 is CP odd. Were CP a perfect symmetry, only K_1 could decay to $\pi^+\pi^-$ or $\pi^0\pi^0$, which are CP even. K_2 could in that case only decay to $\pi^0\pi^0\pi^0$ or $\pi^+\pi^-\pi^0$. This makes K_2 (52 ns) much longer lived than K_1 (0.089 ns). However Christensen, Cronin, Fitch and Turlay found in 1964 that K_2 <u>can</u> decay to $\pi\pi$, about 0.2% of the time, indicating CP violation.

Non-gauged continuous approximate symmetries

In addition to the $SU(3) \times SU(2)_L \times U(1)_Y$ local gauge symmetry of the Standard Model, there can be additional rigid (aka "global") symmetries and also approximate symmetries that appear when certain SM parameters are considered to be small. In the bosonic sector, we have already seen an example of such an approximate symmetry in the Higgs sector, which develops an $SO(4) \simeq SU(2) \times SU(2)$ (up to $\mathbb{Z}_2$ double covering) when the $U(1)_Y$ gauge coupling is turned off. This is the origin of the $SU(2)$ custodial symmetry.

In the fermionic sector, the SM gauge group representations in each $m=1,2,3$ generation for $SU(3) \times SU(2)_L \times U(1)_Y$ are $(3,2)_{1/6} \oplus (\bar{3},1)_{-2/3} \oplus (\bar{3},1)_{1/3} \oplus (1,2)_{-1/2} \oplus (1,1)_1$ when all spinor fields are presented in their left-handed chiral versions. Thus, instead of listing right-handed e_{Rm} as $(1,1)_{-1}$, the left-handed charge conjugate field e_{Rcm} is listed as $(1,1)_1$, and similarly for $u_{Rc}^{(\bar{3},1)_{-2/3}}$ and $d_{Rc}^{(\bar{3},1)_{1/3}}$. The point of presenting the spinor fields this way is similar to that of presenting the Higgs ϕ_a as 4 real fields: to see if there are any degeneracies of representations that could give rise to additional, non-gauged, symmetries. However, all 5 of these irreducible representations for left-handed chiral fields are inequivalent. So there are no accidental symmetries relating fermions within a single generation. However, if the various f_{mn}, h_{mn}, k_{mn} Yukawa couplings are turned off, each of these 5 irreps of spinor fields develops an independent $U(3)$ rigid symmetry in generation space. So in that limit, there is an accidental $(U(3))^5$ rigid symmetry.

When the Yukawa interactions are turned on, and incorporating experimental information that none of the eigenvalues of f_{mn}, h_{mn} or k_{mn} vanish or are degenerate, the candidate rigid accidental symmetry group is reduced to a product of independent $U(1)$ phases for the various fermi species, subject to the requirement that they leave

f_{mn}, h_{mn} and k_{mn} invariant. This requirement imposes the relations $U_{QL} = U_{uRc}^* = U_{dRc}^*$ and $U_L = U_{eRc}^*$. These constraints reduce the accidental symmetry of the lepton sector to

$$U_L = U_{eRc}^* = \begin{pmatrix} e^{i\theta_e} & & \\ & e^{i\theta_\mu} & \\ & & e^{i\theta_\tau} \end{pmatrix}$$

For the quarks, a phase transformation symmetry must preserve the form of the charged-current interactions when written in the mass basis (ie. with diagonal mass matrices). A candidate rigid accidental symmetry must therefore commute with the CKM matrix. For a general unitary CKM matrix, the only such candidates must be proportional to the unit matrix in generation space. This leaves only a single $U(1)$:

$$U_{QL} = U_{uRc}^* = U_{dRc}^* = \begin{pmatrix} e^{i\theta_B/3} & & \\ & e^{i\theta_B/3} & \\ & & e^{i\theta_B/3} \end{pmatrix}$$

The factors of $1/3$ are chosen so that the charges (i.e. conserved quantities under above $U(1)_B$) of all quarks are $1/3$. Bound states of quarks need to be $SU(3)$ singlets, and thus contain a multiple of 3 quarks (excluding quark-antiquark pairs). Consequently, they have integer charges (0 or ±1) under this symmetry.

Note that with the above constraints, the left and right handed components of each flavour of leptons (6 flavours) or quarks (6 flavours) transform the same way, eg. $e_L^- \to e^{i\theta_e} e_L$, $e_R^- \to e^{i\theta_e} e_R$. Antiparticles have opposite transformations, hence opposite charges, eg. $e_{L,R}^+ \to e^{-i\theta_e} e_{L,R}^+$. Note that antiparticles are CPT transforms of particles. There are four such accidental symmetries, which to date appear to be experimentally conserved to a high degree

- electron number $L_e(e^-) = L_e(\nu_e) = +1$, $L_e(e^+) = L_e(\bar\nu_e) = -1$
- muon number $L_\mu(\mu^-) = L_\mu(\nu_\mu) = +1$, $L_\mu(\mu^+) = L_\mu(\bar\nu_\mu) = -1$
- Tau number $L_\tau(\tau^-) = L_\tau(\nu_\tau) = +1$, $L_\tau(\tau^+) = L_\tau(\bar\nu_\tau) = -1$

otherwise zero

- baryon number $B(q) = 1/3$ for all quarks, $B(\bar q) = -\frac{1}{3}$ for antiquarks; $B = 0$ otherwise.

The sum $L = L_e + L_\mu + L_\tau$ is known as lepton number

It is a striking feature of the Standard Model that its $SU(3) \times SU(2)_L \times U(1)_Y$ gauge symmetry and the four $U(1)$ accidental symmetries correspond to the conservation / approximate conservation of corresponding quantum numbers as observed experimentally. At least, that was the situation until neutrino masses and neutrino oscillations were discovered, which show that the individual lepton family numbers have very small violations. It is now known that rare muon decays like $\mu^- \to e^- + \gamma$ are possible.

The total lepton number $L = L_e + L_\mu + L_\tau$ is conserved to a very high degree, but it too is subject to very small amounts of violation. One type of L violation arises from Majorana neutrino mass terms: mass terms with structure $\overline{\nu_{Ln}} \nu_{Lkc} \rho_{nm} \rho_{km}$ have net lepton number -2, and the conjugates $\overline{\nu_{Lnc}} \nu_{Lk} \rho^*_{nm} \rho^*_{km}$ have net lepton number $+2$. We shall also see that lepton number is subject to anomalies.

$J_R \nu_{Rc}$ mass terms also have lepton number -2

Baryon number B is similarly conserved to a very high degree, but, like lepton number, is subject to anomalies. No experimental evidence for baryon number violation has been observed. A classic example of baryon number violation would be proton decay.

The difference $(B-L)$ is even more robust. Rigid $(B-L)$ symmetry, unlike baryon number alone or lepton number alone, is not broken by chiral anomalies. If neutrinos are given mass via the seesaw mechanism, $|\Delta(B-L)| = 2$ is possible, but the experimental limits are very stringent ($\tau > 10^{29}$ years). (Proton ($B=1, L=0$) decay to e^+ ($B=0, L=-1$) π^0 ($B=L=0$) conserves $B-L$.)

$p \to e^+ + \pi^0$
$\pi^0 \to \gamma\gamma$

Searches for violations of any of these rigid accidental symmetries of the Standard Model are important ways to probe for hints of physics beyond the Standard Model. Such symmetry violations could point toward Grand Unification, or some non-minimal versions of Supersymmetry.