Grand Unification

The Standard Model very successfully accounts for the known electroweak phenomena, and with a simple extension to include right-handed gauge singlet neutrinos it can also accomodate neutrino masses as well. However, being based on the non-simple gauge group $SU_c(3)\times SU_L(2)\times U_Y(1)$, it has three independent coupling constants g_3, g_2 and g_1. Had the gauge group been a simple group, there would be a single coupling constant, clearly a more complete unification of the fundamental forces.

It is remarkable how close this comes to working, in the Georgi-Glashow model (1974), which is based on the gauge group $G = SU(5)$. This group can be made to break spontaneously down to $SU(3)\times SU(2)\times U(1)$ by use of a Higgs field Φ transforming in the adjoint ($\underline{24}$) representation of $SU(5)$. One can construct a potential for a renormalizable theory (i.e. maximum dimension 4) from the three available $SU(5)$ invariants $V = -m_1^2\, tr(\Phi^2) + a\, tr(\Phi^4) + b\,[tr(\Phi^2)]^2$. Note that the vacuum value of Φ can always be made real and diagonal $\langle\Phi\rangle = \mathrm{diag}(l_1 l_2 l_3 l_4 l_5)$ with $\sum_i^5 l_i = 0$ by an appropriate basis choice. Choosing the coefficients m^2, a and b appropriately, one can arrange to have a vacuum value $\langle\Phi\rangle = v_1\, \mathrm{diag}(2,2,2,-3,-3)$, around which the excitations of the Φ field all have non-negative $(\text{mass})^2$ values.

The mass term for the $SU(5)$ gauge field A_μ^I ($I=1,\dots,24$) is obtained from the kinetic term for the Higgs field $-\frac{1}{2}tr(D_\mu\Phi D^\mu\Phi)$ [$D_\mu\Phi = \partial_\mu\Phi - ig_5[A_\mu\cdot T, \Phi]$] for $\Phi = \langle\Phi\rangle$, i.e. from $\frac{g_5^2}{2} tr([A_\mu\cdot T, \langle\Phi\rangle][A^\mu\cdot T, \langle\Phi\rangle])$. So the A_μ fields remaining massless, which belong to the unbroken subgroup H, belong to the $SU(5)$ generators that commute with $\langle\Phi\rangle$.

Write the SU(5) ~~generators~~ hermitian generators in the defining 5×5 fundamental representation as

$$(T^I)_A{}^B = \begin{pmatrix} a_1 & a_4 & a_5 & b_1 & b_2 \\ a_4^* & a_2 & a_6 & b_3 & b_4 \\ a_5^* & a_6^* & a_3 & b_5 & b_6 \\ b_1^* & b_3^* & b_5^* & c_1 & c_3 \\ b_2^* & b_4^* & b_6^* & c_3^* & c_2 \end{pmatrix} \qquad \begin{array}{l} I = 1, \dots, 24 \\ A,B = 1, \dots, 5 \end{array}$$

The T^I generate infinitesimal transformations of Φ by $\delta\Phi = i\epsilon_I[T^I, \Phi]$ which preserve hermiticity of Φ for real ϵ_I. Given the $\langle\Phi\rangle = v_1 \,\mathrm{diag}(2,2,2,-3,-3)$ vacuum value of Φ, the "a" 3×3 and "c" 2×2 diagonal blocks clearly commute with $\langle\Phi\rangle$ and so remain unbroken. These correspond to the $SU_c(3)$ and $SU_L(2)$ SM factor groups. There is one more unbroken generator, which is proportional to $\langle\Phi\rangle$ itself. In its conventional SM guise it is $Y = \mathrm{diag}(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2})$, where the normalization is chosen so as to give the correct hypercharge assignments for quarks and leptons.

One needs to be careful with the normalization of Y however. For any simple compact Lie group, there is a conventional choice of generators T^I so that in each reducible or irreducible representation R they satisfy the normalization condition $\mathrm{tr}(T^I T^J) = C_R \delta^{IJ}$. C_R is the Dynkin index of the representation R. For the fundamental representation F of $SU(N)$, $C_F = \frac{1}{2}$ by convention. [This then determines C_R for other representations. For example, for the adjoint A rep., $C_A = N$ for $SU(N)$; this is related to the Cartan-Killing metric $g^{IJ} = -\frac{1}{N} f^{IK}{}_L f^{JL}{}_K = \delta^{IJ}$ for compact semi-simple groups.]

For the SU(3) and SU(2) subgroups corresponding to the "a" and "c" blocks, the standard normalization is achieved by taking $T_{(3)}^{I=\vartheta} = \frac{1}{2}\lambda^\vartheta$, $\vartheta = 1, \dots, 8$, for which $\mathrm{tr}(\lambda^\vartheta \lambda^\varphi) = 2\delta^{\vartheta\varphi}$ and $T_{(2)}^{I=8+i} = \frac{1}{2}\sigma^i$ for which also $\mathrm{tr}(\sigma^i\sigma^j) = 2\delta^{ij}$. When it comes to Y, however, the traditional normalization does not agree with the $C_F = \frac{1}{2}$ convention for SU(5) fundamental representation generators. Instead, one has $tr(Y^2) = \frac{3}{9} + \frac{2}{4} = \frac{5}{6}$. So in order to incorporate the Y generator into a conventionally normalized SU(5) set, one needs to define $\hat{Y} = \sqrt{\frac{3}{5}}\, Y$, i.e. $\hat{Y} = \mathrm{diag}\left(-\frac{1}{\sqrt{15}}, -\frac{1}{\sqrt{15}}, -\frac{1}{\sqrt{15}}, \sqrt{\frac{3}{20}}, \sqrt{\frac{3}{20}}\right)$. This rescaling has an impact on the hypercharge coupling constant as well: the SM covariant derivative $D_\mu^{SM} = \partial_\mu - i g_3 G_\mu \cdot T_{(3)} - i g_2 W_\mu \cdot T_{(2)} - i g_1 B_\mu Y$ needs to become $D_\mu^{SM} = \partial_\mu - i g_3 G_\mu \cdot T_{(3)} - i g_2 W_\mu \cdot T_{(2)} - i \hat{g}_1 B_\mu \hat{Y}$ with $\hat{g}_1 = \sqrt{\frac{5}{3}}\, g_1$. Accordingly, at energy scales above the scale of SU(5) spontaneous breaking, one will have

$$g_5 = g_3 = g_2 = \hat{g}_1 = \sqrt{\tfrac{5}{3}}\, g_1 \quad \text{where } g_5 \text{ is the}$$

SU(5) coupling constant.

Of course, there are more SU(5) gauge fields than just the $8+3+1 = 12$ standard-model gauge fields for the $SU(3)\times SU(2)\times U(1)$ SM subgroup. SU(5) has 12 extra generators, corresponding to the off-diagonal "6" block. With respect to the SM $(SU(3), SU(2))_Y$ subgroup, the $X_{\mu \tilde{a} a}$ ($\tilde{a}=1,2,3$; $a=1,2$) corresponding gauge fields transform as $(3,2)_{-5/6}$ (or as $(3,2)_{-\sqrt{\frac{5}{12}}}$ with respect to $(SU(3), SU(2))_{\hat{Y}}$). Effects of such fields have not yet been seen, and so a successful grand unified theory needs to arrange for symmetry breaking from SU(5) down to $SU(3)\times SU(2)\times U(1)$ at a sufficiently high mass scale M_{GUT} in order to make the effects of the $X_{\mu \tilde{a} a}$ unobservable to date. This would be analogous to the $SU_L(2)\times U_Y(1) \to U_{EM}(1)$ breaking in the Standard Model, which would have to occur at lower ($\sim 100\ \mathrm{GeV}/c^2$) mass scales. It is precisely the occurrence of new, unseen phenomena that poses the key challenge to any proposal of larger unification: confirmation such as the discovery of the $W_\mu^\pm$ and Z_μ leads to triumph, but contradiction can kill the proposal.

The quarks and leptons of the Standard Model need to be assembled into representations of SU(5) as well, and the remarkable thing about the Georgi-Glashow model is how precisely this works, with no predictions of "exotics" that would have to be explained away. The fundamental $\underline{5}$ representation of SU(5) decomposes as $\underline{5} \to (3,1) \oplus (1,2)$ under $SU_c(3) \times SU_L(2)$. Label the $\underline{5}$ representation components as $(f_1, f_2, f_3, h_1, h_2)$ (written as a row). The traceless structure of the SU(5) $\hat{Y}$ generator requires that the corresponding hypercharges satisfy $3y_f + 2y_h = 0$. What can these spinor fields be? Standard Model fermions transforming as $(3,1)$ under $SU_c(3) \times SU_L(2)$ can only be found among the RH quarks: either u_{Rm} or d_{Rm}. The $(1,2)$ partner fermions under SU(5) must have the same RH spinor structure, since Lorentz and internal symmetries remain in a direct product. So one needs to find RH spinor fields in the $(3,1)$ and $(1,2)$ SM reps with $y_h = -\frac{3}{2} y_f$. Looking at the available SM species, the combination that works is to let the $f_{\breve{a}}$ triplet be the $d_{Rm\breve{a}}$ quarks, which have hypercharge $y_f = -\frac{1}{3}$, and to let the h_a doublet be $(\tilde{L}_{mc})_a = \epsilon_{ab}(L_{mc})^b$, which have $y_h = +\frac{1}{2}$. Recall that the L_m leptons transform as $(1,2)_{-1/2}$, so their charge conjugates L_{mc} are RH spinors transforming in the $(1,2^*)_{+1/2}$. However, in SU(2) one can lower an index with ϵ_{ab}, so $(\tilde{L}_{mc})_a = \epsilon_{ab}(L_{mc})^b$ are RH spinors transforming in the $(1,2)_{+1/2}$ as required. Note that the (y_f, y_h) charges fit precisely the traditional $Y = \mathrm{diag}(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2})$.

generation labels m diagonal to SU(5) throughout

We still need to find an SU(5) covariant way to incorporate the RH e_{Rm} in the $(1,1)_{-1}$, the u_{Rm} in the $(3,1)_{2/3}$ and the LH Q_{Lm} in the $(3,2)_{1/6}$. Consider the Q_{Lm} first. Since they transform in the $(SU(3), SU(2))$ rep. $(3,2)$, with both SU(3) and SU(2) indices, they cannot be lodged in a single-index SU(5) representation. Try instead an antisymmetric two-index

SU(5) tensor representation $\chi_{[AB]}$. This is a $5\cdot 4/2 = 10$ complex dimensional representation of SU(5): $A \to (\tilde{a}, a)$ $\tilde{a} = 1,2,3$, $a = 1,2$

$$\chi_{ABm} = \begin{pmatrix} \epsilon_{\tilde{a}\tilde{b}\tilde{\jmath}} (u_{Rm})_c^{\tilde{\jmath}} & -Q_{Lm\tilde{a}b} \\ Q_{Lm\tilde{a}b} & \epsilon_{ab}(e_{Rm})_c \end{pmatrix}$$

every component of χ_{ABm} is a LH spinor

See how the hypercharge assignments work out for this SU(5) representation. For a transformation matrix $M_A{}^B \in SU(5)$, χ_{AB} transforms as $\chi_{AB} \longrightarrow M_A{}^E M_B{}^F \chi_{EF}$, so for a transformation $M_A{}^B = \exp(i\hat{\rho}\hat{Y})_A{}^B = \exp(i\rho Y)$ with a $\rho = \sqrt{\frac{3}{5}}\hat{\rho}$ rescaled parameter and the traditional $Y = \mathrm{diag}(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2})$ generator, the hypercharges corresponding to the A and B indices simply <u>add</u>, so for the components of χ_{ABm} we find traditional Y hypercharge assignments

$$\begin{pmatrix} -\frac{1}{3}-\frac{1}{3} = -\frac{2}{3} & -\frac{1}{3}+\frac{1}{2} = \frac{1}{6} \\ -\frac{1}{3}+\frac{1}{2} = \frac{1}{6} & \frac{1}{2}+\frac{1}{2} = 1 \end{pmatrix}$$

which are the correct Y hypercharge assignments for the indicated SM LH fermions.

(Recall that u_{Rm} has $y = 2/3$ so $(u_{Rm})_c$ has $y = -\frac{2}{3}$ and e_{Rm} has $y = -1$ so $(e_{Rm})_c$ has $y = +1$.)

Thus, assembling the roster of SM spinor fields written all as LH spinors, and so charge conjugating the SU(5) 5 rep. $\psi_{mA} = (d_{Rm\tilde{a}}, \epsilon_{ab}(L_{mc})^b)$ to give an SU(5) $\bar{5}$ representation $(\psi_{mc})^A = ((d_{Rm})_c^{\tilde{a}}, \epsilon^{ab}L_{mb})$, all the LH Standard-Model fermions fit precisely into SU(5) representations $\bar{5} \oplus 10$ for each $m = 1,2,3$ generation.

Of course, neutrino masses can be incorporated via the seesaw mechanism by including gauge singlet right-handed neutrinos in each generation. Charge conjugating them for purpose of making a purely left-handed spinor roster, the SU(5) content then becomes $\bar{5} \oplus 10 \oplus 1$.

With just the adjoint $\underline{24}$ Higgs Φ taking the VeV $\langle\Phi\rangle = v_1 \, \mathrm{diag}(2,2,2,-3,-3)$, one directly arrives at a system with massless $SU(3)\times SU(2)\times U(1)$ gauge fields plus massive $(3,2)$ vector fields X_μ^{Aa} with masses of order $g_5 v_1$ after eating the Φ components corresponding to the broken "b" generators of $SU(5)$. This, of course, can't be the full story because one hasn't worked in the symmetry breaking structure of the Standard Model itself. In order to do this, one needs to include more $SU(5)$ Higgs fields. This can be done by including an $SU(5)$ $\underline{5}$ Higgs H_A. Respecting renormalizability, the general $SU(5)$-invariant potential containing terms of dimensions ≤ 4 is

$$V(\Phi,H) = V(\Phi) + V(H) + \lambda_1(\mathrm{tr}\,\Phi^2)(H^+H) + \lambda_2(H^+\Phi^2 H)$$

with $V(\Phi) = -m_1^2 \mathrm{tr}(\Phi^2) + a\,\mathrm{tr}(\Phi^4) + b[\mathrm{tr}(\Phi^2)]^2$

$$V(H) = -m_2^2(H^+H) + \lambda(H^+H)^2$$

Strictly speaking, one should start from $V(\Phi,H)$ and analyse the full symmetry breaking pattern down to just $SU(3)\times U(1)_{em}$. It is easier to present the symmetry-breaking structure as a two-stage process, however:

$$SU(5) \xrightarrow{\Phi} SU(3)\times SU(2)\times U(1) \xrightarrow{H} SU(3)\times U(1)_{em}$$

In the first stage, for $a>0$ and $b > -\frac{7a}{30}$, one has $\langle\Phi\rangle = v_1\,\mathrm{diag}(2,2,2,-3,-3)$ with $v_1^2 = m_1^2/(14a+60b)$. After this first stage of symmetry breaking, the X_μ^{Aa} $(3,2)$ vectors develop a mass $\sqrt{25/2}\, g_5 v_1$ after eating the "b" block Φ Higgs. Meanwhile, the $SU(5)$ H Higgs breaks into a $(3,1)$ colour triplet h_t and an $SU(2)_L$ $(1,2)$ doublet h_d. These acquire mass terms

$$m_t^2 = -m_2^2 + (30\lambda_1 + 4\lambda_2)v_1^2$$
$$m_d^2 = -m_2^2 + (30\lambda_1 + 9\lambda_2)v_1^2$$

In order for the second, $SU(3)\times SU(2)\times U(1) \longrightarrow SU(3)\times U(1)_{em}$, stage of symmetry breaking to occur at lower energies $v_2 \simeq 246\,\mathrm{GeV}/c^2$, one needs to have $|m_d^2| \ll v_1^2$. If that is the case, h_d will

need $m_d^2<0$

survive down to energies of order v_2 while the "superheavy" particles such as the X_μ^{Aa} with masses of order v_1 decouple. The relevant physics of the surviving "light" particles will then be determined by an $SU(3)\times SU(2)\times U(1)$ invariant h_d potential

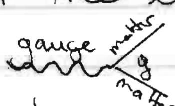
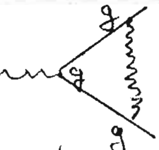
$$V_{eff}(h_d) = -|m_d^2|\,h_d^\dagger h_d + \lambda(h_d^\dagger h_d)^2$$

which is just the ordinary Higgs potential for the Standard Model. Then h_d develops a VeV $v_2 = (|m_d^2|/2\lambda)^{1/2}$, which should be about $246\ GeV/c^2$.

A proper treatment of the combined system $V(\Phi, H)$ would yield a $\langle\Phi\rangle$ VeV v_1 which is slightly shifted, together with other small corrections of order v_2/v_1. The X_μ^{A1} and X_μ^{A2} massive vectors also develop a mass difference of order v_2.

At the $v_1 \sim M_{GUT}$ scale, $SU(5)$ invariance imposes the relation between coupling constants $g_5 = g_3 = g_2 = \sqrt{\tfrac{5}{3}}\,g_1$, which gives $\sin^2\theta_w = g_1^2/(g_2^2+g_1^2) = \frac{3}{8} = 0.38$ as compared to the experimental value $\sin^2\theta_w = 0.231$. Even worse agreement is found for the strong $(\text{coupling})^2$ g_3^2, which is much larger than g_2^2 or g_1^2. What has not yet been taken into account in this discussion is the <u>evolution of couplings with scale</u>, which we now need to consider. At energy scales E much less than M_{GUT}, there will be radiative corrections proportional to $\ln(M_{GUT}/E)$.

<u>The Running of Coupling Constants</u>

At tree level in quantum field theory, one has vertices like  with coupling constant g. Radiative corrections to such interactions involve loop diagrams with the same external lines, like . These generally are <u>divergent</u> and require <u>renormalization</u> of the coupling constant g. In the process of renormalization, one needs to introduce a <u>regulator</u> to make such ultraviolet divergent diagrams finite:

so they can meaningfully be handled. The most straightforward regulator is simply a <u>cutoff</u>, i.e. a maximum scale Λ for $\int^{\Lambda} d^4k$ momentum-space integrals. There are other, more Lorentz-covariant regularization techniques, such as dimensional regularization $\int d^4k \to \int d^{4-\epsilon}k$, or zeta-function regularization, or Pauli-Villars regularization. All of them, in one way or another, involve the introduction of a reference <u>subtraction scale</u> μ.

In the process of renormalization, a renormalized coupling g_{Ren} is expressed in terms of an unrenormalized coupling g_0 and the regulator (eg. the cutoff Λ), referred to the subtraction scale μ. For example, in electrodynamics one has

$e_{Ren} = Z_3^{1/2} e_0$

$$e^2_{Ren}(\mu) = e_0^2\left(1 - \frac{e^2_{Ren}}{12\pi^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right)\right),$$ from corrections at the 1-loop order.

$Z_3 = 1 - \frac{e^2_{Ren}}{12\pi^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right)$

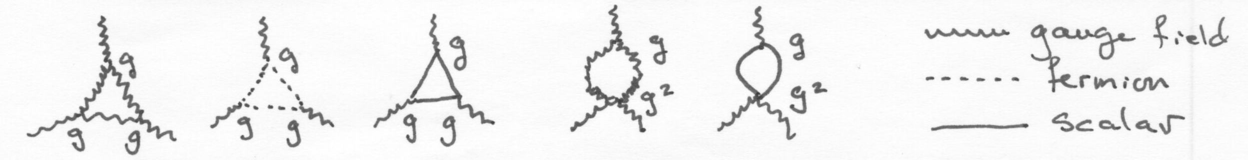
When amplitudes are rewritten in terms of e_{Ren} and other renormalized quantities like the electron mass, the Λ dependence of the divergent diagrams cancels against the Λ dependence introduced by rewriting expressions originally involving e_0 in terms of e_{Ren}. Subsequently, one may take the limit $\Lambda \to \infty$ and obtain non-divergent results, treating e_{Ren} as a finite coupling constant (which then needs to be determined by experiment or relation to other couplings). Although the renormalization procedure removes the ultraviolet divergences, it does leave a footprint of the subtraction procedure through dependence on the reference subtraction scale μ.

Considering once again the example of electrodynamics, if one compares the values of $e^2_{Ren}(\mu)$ and $e^2_{Ren}(\mu_0)$, one finds (again including just the leading 1-loop corrections)

$$\frac{1}{e^2_{Ren}(\mu)} = \frac{1}{e^2_{Ren}(\mu_0)} + \frac{1}{12\pi^2}\ln\left(\frac{\mu_0^2}{\mu^2}\right)).$$

In consequence of renormalization, $SU(N)$ gauge coupling constants g_N evolve with scale μ according to the

1-loop Diagram types



renormalization group equation $\mu\frac{\partial g_R}{\partial\mu} = \beta(g_R)$.

Writing $\beta = -\frac{b g^3}{4\pi}$, one finds the following contributions from generic spin-zero, spin ½ and spin 1 fields going around the one-loop diagram for SU(N) gauge symmetry with the standard normalization:

$$b_N = \frac{1}{12\pi}\left(11 C_A - 2 n_f^i C_{Ri} - n_\phi^i C_{Ri}\right)$$

treating all fermions as left-handed after charge-conjugating RH fermions

Include both n_L^i and n_R^i for L and $(R)_c$ fermions if they both have R^i rep.

where n_f^i is the number of fermions in the i^{th} representation, n_ϕ^i is the number of complex scalars in the i^{th} representation and C_{R^i} is the Dynkin index of the representation R^i. Recall that for SU(N), $C_A = N$ for the adjoint representation and $C_F = \frac{1}{2}$ for the fundamental representation and its conjugate. For $U_Y(1)$, the results are different owing to the difference in normalization of the traditional Y generator. To get b_Y, compare to electrodynamics, for which $-\frac{2}{e^3(\mu)}\mu\frac{\partial e(\mu)}{\partial\mu} = -\frac{2}{24\pi^2}\cdot 2$ where the final factor of 2 on the RHS arises from counting L+R projections of the electron field, each charge 1.

$\beta_{QED} = \frac{e^3}{12\pi^2}$

Analogously, for $U_Y(1)$ one has $\beta_Y = \frac{g_1^3}{24\pi^2}\,tr_L(Y^2) = -\frac{b_Y g_1^3}{4\pi}$ for LH fermions, so $b_Y = -\frac{1}{6\pi}\left(tr_{\text{LH fermi}}(Y^2) + \frac{1}{2}tr_{\text{scalars}}(Y^2)\right)$, including also the scalar contribution.

Leaving aside contributions from the Higgs scalars, which are small compared to the contributions from the massless vectors and spinors, in the Standard Model one has

for SU(3): $n_F = 2$, $n_{\bar F} = 1+1 = 2$

for SU(2): $n_F = 3+1$, $n_{\bar F} = 0$

$tr(Y^2) = \underbrace{\frac{5}{6}}_{\bar 5} + \underbrace{\frac{15}{6}}_{10}$ or $= \frac{1}{4}\cdot 2 + 1\cdot 1 + \frac{1}{36}\cdot 6 + \frac{4}{9}\cdot 3 + \frac{1}{9}\cdot 3 = \frac{10}{3}$ per generation

$$\mu\frac{\partial}{\partial\mu}g_3(\mu) = -\frac{g_3^3(\mu)}{4\pi^2}\left(\frac{11}{4} - \frac{n_g}{3}\right)$$

$$\mu\frac{\partial}{\partial\mu}g_2(\mu) = \frac{-g_2^3(\mu)}{4\pi^2}\left(\frac{11}{6} - \frac{n_g}{3}\right)$$ where n_g is the number of generations

$$\mu\frac{\partial}{\partial\mu}g_1(\mu) = \frac{-g_1^3(\mu)}{4\pi^2}\left(-\frac{5n_g}{9}\right)$$ (since $tr_L(Y^2) = \frac{10}{3}n_g$)

At the M_{GUT} scale where SU(5) should be restored, one would have $g_3^2(M_{GUT}) = g_2^2(M_{GUT}) = \frac{5}{3}g_1^2(M_{GUT})$. Starting from such SU(5)-compatible couplings and integrating the renormalization group equations down to scale μ, one has $\frac{1}{g_N^2(\mu)} = \frac{1}{g_N^2(M_{GUT})} - \frac{1}{2\pi} b_N \ln\left(\frac{M_{GUT}}{\mu}\right)$, so

$$\frac{1}{g_3^2(\mu)} = \frac{1}{g_3^2(M_{GUT})} - \frac{1}{8\pi^2}\left(11 - \frac{4n_g}{3}\right)\ln\left(\frac{M_{GUT}}{\mu}\right)$$

$$\frac{1}{g_2^2(\mu)} = \frac{1}{g_2^2(M_{GUT})} - \frac{1}{8\pi^2}\left(\frac{22}{3} - \frac{4n_g}{3}\right)\ln\left(\frac{M_{GUT}}{\mu}\right)$$

$$\frac{1}{g_1^2(\mu)} = \frac{1}{g_1^2(M_{GUT})} - \frac{1}{8\pi^2}\left(-\frac{20n_g}{9}\right)\ln\left(\frac{M_{GUT}}{\mu}\right)$$

Subtract the second of these from the first:

$$\frac{1}{g_3^2(\mu)} - \frac{1}{g_2^2(\mu)} = -\frac{11}{24\pi^2}\ln\left(\frac{M_{GUT}}{\mu}\right).$$

Subtract 3/5 of the third of these from the second:

$$\frac{1}{g_2^2(\mu)} - \frac{3}{5g_1^2(\mu)} = \frac{-11}{12\pi^2}\ln\left(\frac{M_{GUT}}{\mu}\right)$$

Then use $g_1 = \frac{e}{\cos\Theta_W}$, $g_2 = \frac{e}{\sin\Theta_W}$ together with

$\frac{1}{g_3^2(\mu)} - \frac{1}{g_2^2(\mu)} = \frac{1}{2}\left(\frac{1}{g_2^2(\mu)} - \frac{3}{5g_1^2(\mu)}\right)$ to deduce

$$\sin^2\Theta_W = \frac{g_1^2}{g_2^2+g_1^2} = \frac{1}{6} + \frac{5e^2(\mu)}{9g_3^2(\mu)}$$

and

$$\ln\left(\frac{M_{GUT}}{\mu}\right) = \frac{4\pi^2}{11e^2(\mu)}\left(1 - \frac{8e^2(\mu)}{3g_3^2(\mu)}\right).$$

The appropriate scale μ at which to evaluate these relations is $\mu \approx M_Z$, which is the typical energy of processes used to measure $\sin^2\Theta_W$. Extrapolation of g_3 from lower energy data gives $g_3^2(M_Z)/4\pi = 0.118 \pm .006$ and $e^2(M_Z)/4\pi = 1/128$. These then give $\sin^2\Theta_W \overset{SU(5)}{=} 0.203$ and $M_{GUT} \approx 1.1 \times 10^{15}$ GeV.

The SU(5) based prediction of $\sin^2\Theta_W$ is quite close to the measured value $\sin^2\Theta_W \overset{\text{measured}}{=} 0.231$. However, in the years since 1974, the accuracy of both measurements and calculations has improved to the point where it is now clear that there is not a precise agreement. A more serious problem arises from proton decay, however.

Spin(10) Grand Unification

Including an $SU(5)$ singlet to accommodate R/H neutrinos to generate neutrino masses via a seesaw mechanism, the $SU(5)$ fermion representations are $\bar{5} \oplus 10 \oplus 1$. A natural question is whether this representation content can fit nicely into a single representation of a larger grand unification group. Even before the Georgi-Glashow $SU(5)$ model, Howard Georgi realized that the 16 dimensional spinor representation of Spin(10) decomposes under $SU(5) \times U(1)$ as

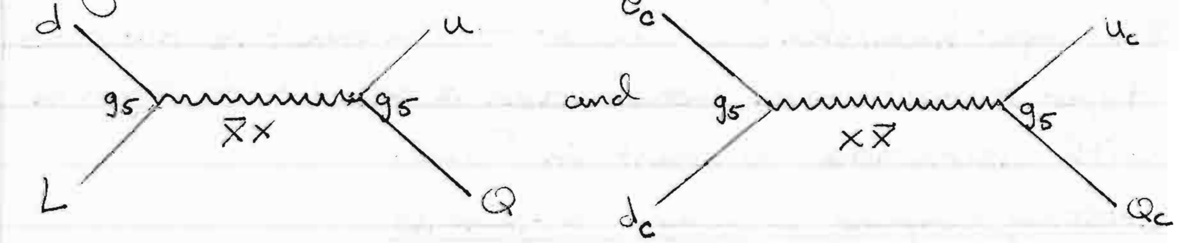
$16 \longrightarrow 10_1 \oplus \bar{5}_{-3} \oplus 1_5$, giving a way to obtain the correct $SU(5)$ representations for the roster of left-handed SM fermions of a given $m=1,2$ or 3 generation. Having all the SM fermions fit into an irreducible representation is clearly attractive. For the vector fields, the adjoint of Spin(10) $\Leftrightarrow$ adjoint of $SO(10)$ decomposes as $45 \longrightarrow 24_0 \oplus 10_{-4} \oplus 10_4 \oplus 1_0$.

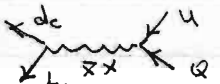
A key task for Spin(10) grand unification is to ensure that the extra vectors lying outside the SM $SU(3) \times SU(2) \times U(1)$ all become sufficiently massive to prevent any unseen physical effects being in conflict with observation.

Proton decay

The problem of physical predictions in conflict with observation comes to the fore in the $SU(5)$ Georgi-Glashow model. The adjoint $\underline{24}$ of $SU(5)$ contains the "6" block of generators, which after $SU(5) \longrightarrow SU(3) \times SU(2) \times U(1)$ symmetry breaking give rise to massive $X_{\mu\, \bar{a} b}$ vector fields. As we saw in the Standard Model itself, integrating out these vector fields will generate dimension 6 terms in the effective action (in particular, $(\text{fermion})^4$ terms) with an effective coupling constant $\frac{g_5^2}{M_X^2}$, where M_X is the mass of the $X_\mu^{\bar{a} b}$ vector field.

The $X_{\mu \tilde{a} b}$ vector bosons carry both SU(3) colour and $SU_L(2)$ indices and have traditionally-normalized Y hypercharge $-5/6$: $(3,2)_{-5/6}$ rep. Consequently, there can be interactions like $X_\mu \overline{u_{Rc}} \gamma^\mu Q_L \tilde{\epsilon} \epsilon + \text{h.c.}$, i.e. $X_{\mu \tilde{a} a} u_{R \dot{\alpha} \tilde{b}} \tilde{\sigma}^{\mu \dot{\alpha} \beta} Q_{L \beta \tilde{d} b} \epsilon^{\tilde{a} \tilde{b} \tilde{d}} \epsilon^{ab}$ $(Y: -\frac{5}{6} + \frac{2}{3} + \frac{1}{6} = 0)$ and $\overline{X}_\mu \overline{d_{Rc}} \gamma^\mu L + \text{h.c.}$, i.e. $\overline{X}_\mu{}^{\tilde{a} a} d_{R \dot{\alpha} \tilde{a}} \tilde{\sigma}^{\mu \dot{\alpha} \beta} L_{\beta a}$ $(Y: 5/6 - 1/3 - 1/2 = 0)$. Using these interactions together with $X_{\mu \tilde{a} a} \overline{X}_\nu{}^{\tilde{b} b}$ propagators which have leading structure $\eta_{\mu\nu} \delta_{\tilde{a}}{}^{\tilde{b}} \delta_a{}^b / M_X^2$, one can have diagrams like



which give rise to processes like $uu \to d_c e^+$ (view the left diagram as 

). This process conserves electric charge (Σm: $2/3 + 2/3 \to 1/3 + 1$). It violates baryon number (B: $1/3 + 1/3 \to -1/3 + 0$) and violates lepton number (L: $0 + 0 \to 0 - 1$), but it conserves $B-L$ ($B-L$: $2/3 - 0 \to -1/3 + 1$). The $uu \to d_c e^+$ process can give rise to proton decay: $p \leftrightarrow uud \to d d_c e^+ \leftrightarrow \pi^0 e^+$ with the π^0 subsequently decaying into a pair of γ-rays, as we have seen: $\pi^0 \to \gamma\gamma$.

Detailed analysis shows that the proton lifetime in the minimal SU(5) Georgi-Glashow model can't be more than 10^{31} years. In Spin(10) models, the lifetime can be pushed up, but by arranging specific cancellations for the most dangerous processes. Proton decay is accordingly a key vulnerability for grand unified models.

On the observational front, the best current limits come from the Super Kamiokande lab, located 1000 meters underground in the Mozumi Mine near the city of Hida in Gifu Prefecture, Japan. The detector consists of 50 000 tons of ultra-pure water watched by over 13 000 photomultiplier tubes.

The Super Kamiokande detector contains $\sim 7.5 \times 10^{33}$ protons. If one of the many protons in the water tank were to decay, the decay products would be detectable. However, the detector has found no evidence for proton decay in data spanning over 17 years. From this data, a bound on the proton lifetime has been obtained: $\tau_{proton} > 5.9 \times 10^{33}$ years. This kills the minimal $SU(5)$ Georgi-Glashow model.

There is a proposal to build a detector 10 times larger than the Super Kamiokande detector — the project is known as Hyper Kamiokande. Construction is expected to begin in 2018, with observations to start in 2025.

<u>Crossing coupling constant trajectories</u>

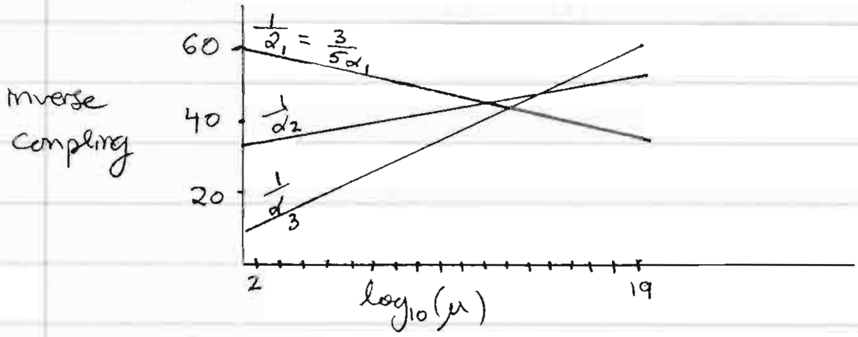
Another way to view the prospects for unification of forces beyond the Standard Model is from the bottom up. Instead of assuming a particular unified model based on some gauge group achieving unification of couplings at an energy scale like 10^{15} GeV, consider just the known evolution of $SU(3) \times SU(2)_L \times U(1)_Y$ couplings in the Standard Model. For the coupling constants g_3, g_2 and $\hat{g}_1 = \sqrt{\frac{5}{3}} g_1$, let $\alpha_N = \frac{g_N^2}{4\pi}$, $\hat{\alpha}_1 = \frac{\hat{g}_1^2}{4\pi}$.

One has the renormalization-group evolutions

$$\frac{1}{\alpha_N(\mu)} = \frac{1}{\alpha_N(\mu_0)} + b_N \ln\left(\frac{\mu^2}{\mu_0^2}\right) \quad \text{for } \hat{\alpha}_1, \alpha_2 \text{ and } \alpha_3$$

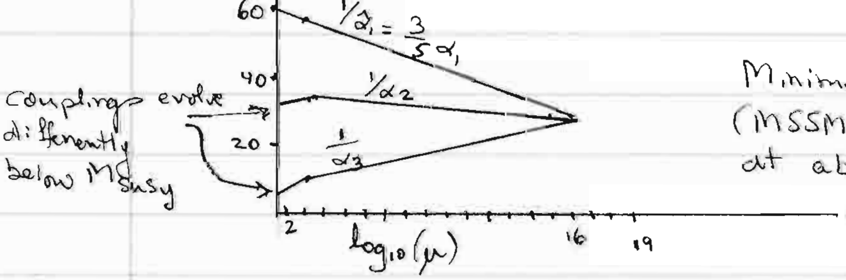
as μ increases away from the $\mu_0 \simeq M_Z$ electroweak scale. In a log plot, these coupling-constant trajectories are just straight lines. If there were a unified theory based on some simple gauge group (eg. $SU(5)$ or $Spin(10)$), there would have to be some scale μ where $\hat{\alpha}_1$, α_2 and α_3 <u>cross at a point</u>, just as happens in $SU(5)$ or $Spin(10)$ models. The problem that emerges then from current data and calculational expertise is that the Standard Model coupling trajectories <u>fail to cross at a point</u>. They almost cross at a point, but miss, leaving a small triangle.

Plotting the $\frac{1}{\alpha_3}$, $\frac{1}{\alpha_2}$ and $\frac{1}{\alpha_1}$ trajectories versus $\log(\mu)$ gives



The implication of this failure to cross at a point is that something else has to happen in between the $M_Z \sim 90$ GeV and the anticipated 10^{15} GeV grand unification scale. Moreover, one would like whatever new feature emerges to address as much as possible the naturalness problem of having to finely tune the theory so as to obtain the vast hierarchy of scales. In the SU(5) Georgi-Glashow model, the naturalness problem consisted in the need to have $v_2 \sim 246$ GeV/c^2 for SM symmetry breaking while v_1, which governs the $SU(5) \to SU(3) \times SU(2) \times U(1)$ symmetry breaking, corresponds to $M_{GUT} \sim 10^{15}$ GeV/c^2.

The most intensely studied additional feature of the unification picture is Supersymmetry, which introduces a "superpartner" for every current SM species. So one proposes the existence of selectrons, sneutrinos, gauginos, higgsinos, ... These change the renormalization coefficients b_1, b_2, b_3 in a very useful way, providing the scale of Supersymmetry breaking is not too large. One hopes for $M_{SUSY} \sim 1$ TeV; in that case the coupling constant trajectories naturally do cross at a point



Minimal Supersymmetric Standard Model (MSSM) couplings do cross at a point at about 10^{16} GeV/c^2.

Moreover, Supersymmetric SU(5) and Spin(10) models manage to have sufficiently long proton lifetimes to be consistent with current data.

Standard Model b_n Coefficients including Scalar Contributions

$b_3 = \frac{1}{4\pi}\left(11 - \frac{4n_g}{3}\right)$ $\quad b_2 = \frac{1}{4\pi}\left(\frac{22}{3} - \frac{4n_g}{3} - \frac{n_H}{6}\right)$ $\quad \hat{b}_1 = \frac{3b_Y}{5} = \frac{1}{4\pi}\left(-\frac{4n_g}{3} - \frac{n_H}{10}\right)$

n_g = No. of Generations $\quad n_H$ = No. of Higgs doublets

A quantitative way to express the failure of the g_3, g_2 and $\hat{g}_1$ coupling constants to unite at some unification scale $\mu = Q$ is to suppose $\frac{1}{\alpha_3(Q)} = \frac{1}{\alpha_2(Q)} = \frac{1}{\hat{\alpha}_1(Q)}$ at scale Q and then use the relations $\frac{1}{\alpha_N(Q)} = \frac{1}{\alpha_N(\mu_0)} + b_N \ln\left(\frac{Q^2}{\mu_0^2}\right)$ to eliminate the unknown

$$\ln\left(\frac{Q^2}{\mu_0^2}\right) = \left(\frac{1}{\hat{\alpha}_1(\mu_0)} - \frac{1}{\alpha_2(\mu_0)}\right) \Big/ (b_2 - \hat{b}_1)$$

to obtain a relation at the lower scale $\mu_0 \simeq M_Z$ that would have to be obtained: $\alpha_3^{-1}(\mu_0) = (1+B)\,\alpha_2^{-1}(\mu_0) - B\hat{\alpha}_1^{-1}$ where $B = \frac{b_3 - b_2}{b_2 - \hat{b}_1}$.

From the Standard Model theory with n_H Higgs doublets, one would have $B_{th} = \left(1 + \frac{n_H}{22}\right) \Big/ 2\left(1 - \frac{n_H}{110}\right) \simeq \frac{1}{2} + \frac{3}{110} n_H$.

Experiment, on the other hand, gives the ($\overline{MS}$ renormalized) values at $\mu_0 = M_Z$: $\quad \alpha_3^{-1} = 8.47 \pm .22 \quad \alpha_2^{-1} = 29.60 \pm .04$ and $\hat{\alpha}_1^{-1} = 58.98 \pm .08$ giving $B_{exp} = 0.719 \underset{\text{expt. error}}{\pm .008} \underset{\text{omission of higher order couplings}}{\pm .03}$

Clearly, B_{th} misses B_{exp} by a wide margin, for any plausible number of Higgs doublets n_H.

Supersymmetry

Supersymmetry enlarges Poincaré spacetime symmetry algebra to a graded Lie algebra containing both commutators and anticommutators:

$$[M_{\mu\nu}, M_{\rho\lambda}] = -i\left(\eta_{\nu\rho}M_{\mu\lambda} - \eta_{\mu\rho}M_{\nu\lambda} - \eta_{\nu\lambda}M_{\mu\rho} + \eta_{\mu\lambda}M_{\nu\rho}\right)$$

$$[M_{\nu\rho}, P_\mu] = i(\eta_{\mu\nu}P_\rho - \underline{\eta_{\mu\rho}P_\nu}) \qquad [P_\mu, P_\nu] = 0$$

$\underset{\sim}{\alpha}$: 4-cpt Dirac index

$$\{Q_{\underset{\sim}{\alpha}}, Q_{\underset{\sim}{\beta}}\} = 2(\gamma^\mu C)_{\underset{\sim}{\alpha}\underset{\sim}{\beta}} P_\mu \qquad C\text{: charge conjugation matrix}$$

$$[M_{\mu\nu}, Q_{\underset{\sim}{\alpha}}] = \frac{i}{2}(\gamma_{\mu\nu}Q)_{\underset{\sim}{\alpha}} \qquad \gamma_{\mu\nu} = \tfrac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$$

Lorentz spin ½ representation

$$[P_\mu, Q_{\underset{\sim}{\alpha}}] = 0$$

The key QQ anticommutator may be re-expressed as

$$\{Q_{\underset{\sim}{\alpha}}, \bar{Q}_{\underset{\sim}{\dot\beta}}\} = -2\gamma^\mu_{\underset{\sim}{\alpha}\underset{\sim}{\dot\beta}} P_\mu \qquad \text{where} \qquad Q = C(\bar{Q})^T = Q_c$$

a Majorana spinor

In order to understand the representations of this enlarged spacetime algebra on particle states, it is convenient to go to 2-component notation $Q_{\underset{\sim}{\alpha}} = \begin{pmatrix} Q_\alpha \\ \bar{Q}^{\dot\beta} \end{pmatrix}$ Majorana spinor. Then

2-cpt indices α, β, $\dot\alpha, \dot\beta = 1,2$

$$\{Q_\alpha, Q_\beta\} = 0 = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} \qquad \text{where} \quad \bar{Q}_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar{Q}^{\dot\beta}$$

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2P_{\alpha\dot\beta} \qquad P_{\alpha\dot\beta} = P^\mu(\sigma_\mu)_{\alpha\dot\beta}.$$

Representations on massive particle states will have the same mass M for all states in an irrep because $[P_\mu, Q_{\underset{\sim}{\alpha}}] = 0$. Go to the rest frame where $p^\mu = (M,0,0,0)$. Then the Lorentz little group is just spatial SO(3) and one has anticommutators

$$\{Q_\alpha, \bar{Q}_{\dot\beta}\} = M\delta_{\alpha\dot\beta} \quad ; \quad \{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0.$$

Accordingly, the rest-frame supersymmetry generators form a Clifford algebra. The basic irrep of this algebra is constructed by treating the Q_α as annihilation operators acting on a Clifford vacuum state Ξ, i.e. $Q_\alpha \Xi = 0$ for both $\alpha = 1, 2$, then building up a set of 4 states using $\bar{Q}_{\dot\alpha}$ as creation operators:

$$\underset{1\text{ state}}{\Xi}, \ \underset{2\text{ states}}{\bar{Q}_{\dot\alpha}\Xi}, \ \underset{1\text{ state}}{\bar{Q}_{\dot\alpha}\bar{Q}_{\dot\beta}\Xi} = -\tfrac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar{Q}_{\dot\gamma}\bar{Q}^{\dot\gamma}.$$

[Note: for extended supersymmetry, one has $Q_\alpha^{\ i}$, $\bar{Q}_{\dot\beta j}$ $\quad i=1,\dots,N$

N=1: Simple D=4 supersymmetry

and $\{Q_\alpha^{\ i}, Q_\beta^{\ j}\} = 0 = \{\bar{Q}_{\dot\alpha i}, \bar{Q}_{\dot\beta j}\}$; $\{Q_\alpha^{\ i}, \bar{Q}_{\dot\beta j}\} = 2\,\delta^i_j\, P_{\alpha\dot\beta}$. The massive basic Clifford algebra irrep then has 2^{2N} states $\Xi, \bar{Q}_{\dot\alpha i}\Xi, \dots, \bar{Q}_{\dot\alpha_1 1}\bar{Q}_{\dot\alpha_2 2}\cdots\bar{Q}_{\dot\alpha_{2N} N}\Xi$. Although supersymmetric theories exist for all $N\le 8$, $N=1$ is most relevant to particle physics.]

One may also build massive particle supermultiplets starting from a Clifford vacuum carrying non-zero spin, i.e. for massive representations use an $SO(3)$ spin j little group representation. The massive supermultiplet then is again formed by applying $\bar{Q}_{\dot\alpha}$ and then decomposing the (spin ½)⊗(spin j) reps into irreps. In this way one obtains a supersymmetry representation Ξ (spin j), $\bar{Q}_{\dot\alpha}\Xi$ (spin$(j-\frac12)$ ⊕ spin$(j+\frac12)$), $\bar{Q}_{\dot\gamma}\bar{Q}^{\dot\gamma}\Xi$ (spin j), i.e. a rep with spins $(j-\frac12, j, j, j+\frac12)$.

For massless supersymmetric particle irreps, one chooses a standard 4-momentum $p^\mu = (k,0,0,k)$, giving a supersymmetry stability subalgebra $\{Q_\alpha, \bar{Q}_{\dot\beta}\} = k(\mathbb{1}+\sigma_3)_{\alpha\dot\beta}$; $\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0$. Now the Q_2 and $\bar{Q}_{\dot 2}$ generators loose their annihilation/creation character, and the basic massless irrep is built using just Q_1 as annihilation operator and $\bar{Q}_{\dot 1}$ as creation operator. In the massless representation, states are classified by helicity instead of spin. The massless Lorentz little group is E_2, whose compact subgroup is just $SO(2)\cong U(1)$, reps of definite helicity. The normalized $U(1)$ little group generator is $\hat{h} = \epsilon^{ijk}p^i M^{jk}/(p^\rho p_\rho)^{1/2}$

E_2: Euclidean group of the $\mathbb{R}^2$ plane

so in the spin ½ representation where $(M^{\mu\nu})^{\dot\alpha}{}_{\dot\beta} = \frac{i}{4}(\tilde\sigma^\mu\sigma^\nu - \tilde\sigma^\nu\sigma^\mu)^{\dot\alpha}{}_{\dot\beta}$ one has $\hat{h} = \frac12\sigma^3$. The creation operator $\bar{Q} = \begin{pmatrix}\bar{Q}_{\dot 1}\\ 0\end{pmatrix}$ thus has the commutation relation $[\hat{h}, \bar{Q}] = \frac12\bar{Q}$, so $\bar{Q}$ raises helicity by ½. The basic massless supersymmetry irrep for simple N=1 supersymmetry is then just Ξ, $\bar{Q}_{\dot 1}\Xi$ and no more, since $\bar{Q}_{\dot 1}^{\,2}=0$. This basic massless supermultiplet thus has just helicities 0 and $+\frac12$. In a field theory, however, one must have a PCT conjugate state for every helicity. Accordingly one must in a field theory include the PCT conjugate supermultiplet $(-\frac12, 0)$ as well.

As for the massive particle supermultiplets, one can start from a Clifford vacuum state Ξ carrying a non-trivial Lorentz little group representation, i.e. non-zero helicity. The basic massless supermultiplet then has helicities $(h, h+\frac{1}{2})$. Adding the PCT conjugate supermultiplet, one then has helicities $(-h-\frac{1}{2}, -h) \oplus (h, h+\frac{1}{2})$. [Among supersymmetric field theories with extended supersymmetry, there are two cases where adding a PCT conjugate multiplet is not necessary: the intrinsically PCT self-conjugate $N=4$ Maxwell or Super Yang-Mills theories, and $N=8$ supergravity.]

For supersymmetric extensions of the Standard Model, the most relevant supermultiplets are

the Wess-Zumino multiplet with helicities $(0, \frac{1}{2}) \oplus (-\frac{1}{2}, 0)$ massless + massive

and supersymmetric gauge multiplets with helicities $(\frac{1}{2}, 1) \oplus (-1, -\frac{1}{2})$ massless.

Actions for supersymmetric field theories encounter the special feature of auxiliary fields, which are needed to complete supersymmetric field multiplets without regard to details of their field equations. Gauge field multiplets are generalizations of the Maxwell field supermultiplet (A_μ, λ, D), transforming as

$$\delta A_\mu(x) = i\bar{\epsilon}\gamma_\mu\lambda(x) \qquad \epsilon, \lambda \text{ Majorana spinors}$$
$$\delta\lambda(x) = -\tfrac{1}{2}F_{\mu\nu}\gamma^{\mu\nu}\epsilon + D(x)\gamma_5\epsilon \qquad \gamma^{\mu\nu} = \tfrac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)$$
$$\delta D(x) = i\bar{\epsilon}\gamma_5\not{\partial}\lambda(x), \qquad F_{\mu\nu} = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$

where ϵ is a constant anticommuting Majorana spinor infinitesimal parameter. The corresponding Maxwell supermultiplet action is $I_{\text{Smax}} = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{i}{2}\bar{\lambda}\not{\partial}\lambda + \frac{1}{2}D^2\right)$.

Varying A_μ, λ and D yields the field equations

$$\partial_\mu F^{\mu\nu} = 0 \;, \quad \not{\partial}\lambda = 0 \;; \quad D = 0 \;.$$

The $D(x)$ field is set to zero by its field equation, so it has no dynamical physical effect for an isolated I_{Smax} system. However, its presence is needed for off-mass-shell closure of the simple supersymmetry algebra.

Matter field multiplets employ the Wess-Zumino multiplet

(A, B, ψ, F, G), where again ψ can be written as a Majorana spinor field, with supersymmetry transformations

$$\delta A(x) = i\bar{\epsilon}\psi(x) \qquad \delta B(x) = i\bar{\epsilon}\gamma_5\psi(x)$$
$$\delta\psi(x) = \partial_\mu(A(x) - \gamma_5 B(x))\gamma^\mu\epsilon + (F(x) + \gamma_5 G(x))\epsilon$$
$$\delta F(x) = i\bar{\epsilon}\not{\partial}\psi(x) \qquad \delta G(x) = i\bar{\epsilon}\gamma_5\not{\partial}\psi(x).$$

This linear realization of simple supersymmetry allows for the construction of a variety of supersymmetric Lagrangian densities (densities because they transform by total derivatives, so giving invariant terms in the action):

$$\mathcal{L}_{kinetic} = -\tfrac{1}{2}\partial_\mu A\,\partial^\mu A - \tfrac{1}{2}\partial_\mu B\,\partial^\mu B - \tfrac{i}{2}\bar{\psi}\not{\partial}\psi + \tfrac{1}{2}F^2 + \tfrac{1}{2}G^2$$
$$\mathcal{L}_{mass} = m\left(FA + GB - \tfrac{i}{2}\bar{\psi}\psi\right)$$
$$\mathcal{L}_{\phi^3} = g\left(FA^2 - FB^2 + 2GAB - i\bar{\psi}(A - \gamma_5 B)\psi\right).$$

The field equations following from the WZ action $\int \mathcal{L}_{kinetic}\,d^4x + \int \mathcal{L}_{mass}\,d^4x + \int \mathcal{L}_{\phi^3}\,d^4x$ yield algebraic equations for $F(x)$ and $G(x)$:

$$F + mA + g(A^2 + B^2) = 0\ ; \quad G + mB + 2gAB = 0.$$

Now we come to an important point for the construction of supersymmetric extensions of the Standard Model, which is crucially based on chiral fields. Majorana spinors such as $\lambda(x)$ or $\psi(x)$ in the Maxwell or Wess-Zumino supermultiplets can straightforwardly be repackaged as chiral spinors and then endowed with appropriate complex representations of the $SU(3)\times SU(2)\times U(1)$ gauge group. An efficient way to encode this is in terms of chiral superfields $\phi(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$ where θ_α and $\bar{\theta}_{\dot{\alpha}}$ are superspace coordinates and the chiral superfield ϕ is subjected to the constraint $\bar{D}_{\dot{\alpha}}\phi = 0$, where D_α and $\bar{D}_{\dot{\alpha}}$ are superspace covariant derivatives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu$$
$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\ ,$$ satisfying the algebra

$\{D_\alpha, \bar{D}_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$

$\{D_\alpha, D_\beta\} = \{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\} = 0 \qquad [D_\alpha, \partial_\mu] = [\bar{D}_{\dot\alpha}, \partial_\mu] = 0.$

A key property of the covariant derivatives is their anticommutation with Q and $\bar{Q}$:

$\{D_\alpha, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot\beta}\} = \{\bar{D}_{\dot\alpha}, Q_\beta\} = \{\bar{D}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0.$

Since the chiral superfield constraint $\bar{D}_{\dot\alpha}\Phi = 0$ is linear, chirality is preserved by multiplication of chiral superfields, $\bar{D}_{\dot\alpha}(\Phi^n) = 0$. The conjugate of a chiral superfield, $\bar\Phi$, is antichiral: $D_\alpha\bar\Phi = 0$. The kinetic term in the action for a Wess-Zumino supermultiplet is given by the superspace integral of $\bar\Phi\Phi$: $I_{kinetic} = \frac{1}{16}\int d^4x\, d^2\theta\, d^2\bar\theta\, \bar\Phi\Phi$, where the Berezin rules for integration over anticommuting variables apply: $\int d\theta\,\theta = 1 = \int d\bar\theta\,\bar\theta$; $\int d\theta = 0 = \int d\bar\theta$

integration = differentiation for anticommuting variables

Accordingly in evaluating a superspace integral $\int d^4x\, d^2\theta\, d^2\bar\theta\, \mathcal{L}_{kinetic}$ one may replace each $\int d\theta_\alpha$ integral by $\frac{\partial}{\partial\theta^\alpha}$ and then, dropping also a total $\frac{\partial}{\partial x^\mu}$ derivative replace that by D_α. Similarly, each $\int d\bar\theta_{\dot\alpha}$ may be replaced by $\bar{D}_{\dot\alpha}$. After applying $D^2\bar{D}^2$ to $\mathcal{L}$ the resulting expression contains still expressions involving θ and $\bar\theta$, but these are total derivatives with respect to x^μ, so may be disregarded. Consequently, the action term $\int d^4x\, d^2\theta\, d^2\bar\theta\, \mathcal{L}_{kinetic}$ may be replaced by $\int d^4x\, [D^2\bar{D}^2\mathcal{L}_{kinetic}]|_{\theta=\bar\theta=0}$. Since $\bar{D}_{\dot\alpha}\Phi = 0$, this may then be written $\int d^4x\, [D^2(\Phi\bar{D}^2\bar\Phi)]|_{\theta=\bar\theta=0}$ and then further applying D^2 and using the $D, \bar{D}$ algebra one ends up with

$$I_{kinetic} = \frac{1}{16}\int d^4x\left(ff^* - 8i\chi^\alpha\partial_{\alpha\dot\alpha}\bar\chi^{\dot\alpha} + 16\, a\Box a^*\right)$$

where
$$a(x) = (A(x) + iB(x))/\sqrt{2} = \Phi|_{\theta=\bar\theta=0}$$
$$\chi_\alpha(x) = \sqrt{2}\,\psi(x) = (D_\alpha\Phi)|_{\theta=\bar\theta=0}$$
$$f(x) = 2\sqrt{2}(F(x) - iG(x)) = (D^2\Phi)|_{\theta=\bar\theta=0} \qquad D^2 = D^\alpha D_\alpha$$

For the mass and interaction terms of a chiral Wess-Zumino superfield, one needs a different form of superspace integral: $I_{chiral} = \int d^4x\, d^2\theta\, \mathcal{L}_{chiral}$ where $\bar{D}_{\dot\alpha}\mathcal{L}_{chiral} = 0$.

Invariance of I_{chiral} under supersymmetry transformations is obtained thanks to the constraint $\bar{D}_{\dot{\alpha}} \mathcal{L}_{chiral} = 0$ even though the full $N=1$ superspace is not integrated over. Thanks to the linearity of the chiral superspace constraint, Φ, Φ^2 and Φ^3 are all chiral superfields. Accordingly a holomorphic function $W(\Phi)$ (function of Φ alone, not $\bar{\Phi}$) produces an invariant $\int d^4x d^2\theta\, W(\Phi)$. $W(\Phi)$ is called a <u>superpotential</u>. For the W-Z mass and interaction terms, one has $W(\Phi) = \frac{m}{4}\Phi^2 + \frac{g}{6\sqrt{2}}\Phi^3$. Evaluation of $\int d^4x\, d^2\theta\, W(\Phi)$ produces the component-field $\int \mathcal{L}_{mass} d^4x$, $\int \mathcal{L}_{\phi^3} d^4x$ given earlier. Note that the auxiliary $f = 2\sqrt{2}(F - iG)$ appears both in these chiral superspace integrals and also in $I_{kinetic}$ (note also that for a real action one needs to take $\int d^4x\, d^2\theta\, W(\Phi) + \int d^4x\, d^2\bar{\theta}\, \bar{W}(\bar{\Phi})$). Elimination of the auxiliary fields via their algebraic equations of motion yields the potential $V(a_a, a_a^*) = 4\sum_a \left|\frac{\partial W}{\partial \Phi_a}\right|^2$ for a set of WZ superfields Φ_a.

Here is the key difference that supersymmetry brings to extensions of the Standard Model. If one has a single Higgs doublet field, one cannot build a supersymmetric extension of the potential $\lambda(a_a a^{\dagger a} - \frac{\mu^2}{2\lambda})^2$ because $\Phi_a \bar{\Phi}^a$ is not chiral, but this would be the only $SU(2)_L \times U(1)_Y$ invariant for a single chiral doublet superfield. [Of course, one can use $\bar{\Phi}^a \Phi_a$ in a full-superspace kinetic term.]
The only way to resolve this problem is to have <u>two Higgs doublet</u> chiral superfields: H_u transforming under $SU(3)\times SU(2)\times U(1)$ as $(1,2)_{1/2}$ and H_d transforming as $(1,2)_{-1/2}$.

The Yukawa interactions of the Standard Model would also run into the same problem, since they involve both Φ_a and $\tilde{\Phi}_a = \epsilon_{ab}\Phi^{*b}$. So one needs both H_u and H_d to write the Yukawa interactions supersymmetrically.

The Minimal Supersymmetric Standard Model (MSSM)

Aside from the Higgs sector, the MSSM is constructed by embedding each matter and gauge field in its own supermultiplet/superfield. This may seem highly inefficient: why can't one search for candidate known particle/superpartner pairings among the known SM spectrum? The problem is the assignments of $SU(3)\times SU(2)\times U(1)$ representations. The only conceivable pairing of a known fermion and boson would be one of the L_m left-handed leptons with $\tilde{\Phi}_a = \epsilon_{ab}\Phi^{*b}$, both of which transform as $(1,2)_{-1/2}$. This would lead to unwanted phenomenology, however, with certain cross sections that should be suppressed becoming too large.

Accordingly, every known SM physical field acquires a superpartner, and the two Higgs doublets do so as well. Spin ½ superpartners of SM bosons are given names ending in "ino", while scalar superpartners of SM fermions are given names with the prefix "s". Thus, one posits photinos, gluinos, etc, and squarks, selectrons, etc. Superpartners are denoted with the same letter as their Standard Model partner, overscored with a tilde: $\tilde{W}^i$ (wino), $\tilde{G}^a$ (gluino), $\tilde{Q}$ (squark $SU(2)$ doublet), $\tilde{u}$ (squark $SU(2)$ singlet), etc.

Interactions in the MSSM are governed by the same requirements of $SU(3)\times SU(2)\times U(1)$ gauge invariance and renormalizability as in the Standard Model, plus, initially, those of supersymmetry. Supersymmetry is guaranteed by deriving all interactions from superspace actions. Noting that $\int d\theta_\alpha \leftrightarrow D_\alpha$ and $\{D_\alpha, \bar{D}_{\dot\alpha}\} = 2i\,\partial_{\alpha\dot\alpha}$, one sees that a $\int d\theta_\alpha$ integration carries dimension ½ in mass dimension units. So $\int d^2\theta d^2\bar\theta\, \bar\Phi\Phi d^4x$ is of strictly renormalizable dimension $4-4=0$ if Φ has dimension 1, as is appropriate for its lowest ϕ component as a physical scalar field (auxiliary fields F and G then have dim 2)

Chiral superspace integrals like $\int d^4x\, d^2\theta\, W(\Phi)$ will correspond to renormalizable interactions provided $\dim(W) \leq 3$, so Φ^3 terms in the superpotential W are the maximum power monomials for a renormalizable theory. Gauge invariant kinetic terms for all gauge fields, matter fields and their superpartners are also strictly renormalizable. The structure of all kinetic terms is determined completely by the requirements of gauge invariance and supersymmetry.

The structure of the superpotential W is open to a wider choice than in the Standard Model, however. Supersymmetry will eventually have to be (softly) broken, and we will come to that, but even for fully supersymmetric interactions, there is a wide choice of possible terms, owing to the presence of all the superpartners. Simply requiring the Φ^3 type Yukawa couplings to generate the needed SM mass terms for known particles leads to the superpotential $W_{MSSM} = f_{mn} H_D L_m e_n + h_{mn} H_D Q_m d_n + k_{mn} H_u Q_m u_n + \mu H_u H_D$ where the L_m superfields transform as $(1,2)_{-1/2}$ and contain doublets of selectrons and sneutrinos plus the L_m left-handed leptons; e_n contain singlet selectrons plus $(e_{Rn})^c$ charge conjugates of the right-handed electrons, transforming as $(1,1)_1$. Similarly, the Q_m $(3,2)_{1/6}$ superfields contain squark doublets plus the Q_{Lm} left-handed quarks, while u_n contain the singlet up squarks plus the charge conjugated $(u_{Rn})^c$ singlet up quarks, transforming as $(\bar{3},1)_{-2/3}$ and d_n contain the singlet down squarks plus the charge conjugated $(d_{Rn})^c$ singlet down quarks, transforming as $(\bar{3},1)_{1/3}$. Note how the $U(1)_Y$ hypercharge assignments of $+\frac{1}{2}$ to H_u and $-\frac{1}{2}$ to H_D left chiral superfields are necessary for gauge invariance of W_{MSSM}.

The only new type of term in W_{MSSM} is the "mu term" $\mu H_u H_D$, of mass dimension 2. It contributes to the mass matrix for the Higgs fields, together with supersymmetry breaking terms, as is necessary to give H_u and H_d non-zero vevs. The mu term is also needed to give the Higgsinos $\tilde{H}_u$ and $\tilde{H}_D$ nonzero masses.

The superpotential W_{MSSM} is not the most general gauge-invariant renormalizable (i.e. max trilinear) superpotential built from the assumed MSSM matter superfields. Other terms are possible, which, however, violate baryon number or lepton number. The baryon and lepton number assignments to supermultiplets are the same for all supermultiplet members, agreeing with the assignment for the known-particle member. Examples of such B or L violating terms are

H_u, H_D are assigned $B=L=0$

$B(d) = -\frac{1}{3} = B(u)$

$L(e) = -1$

$$W_{BL} = \underbrace{C_m \epsilon_{ij} L^i_m H^j_u}_{L=1\ B=0} + \underbrace{C_{mnp} \epsilon_{ij} Q^i_{Am} L^j_n d^A_p}_{L=1\ B=0} + \underbrace{\tilde{C}_{mnp} \epsilon_{ij} L^i_m L^j_n e_p}_{L=1\ B=0} + \underbrace{C'_{mnp} \epsilon_{ABC} d^A_m d^B_n u^C_p}_{L=0\ B=-1}$$

where C_m has dim 1 and $C_{mnp}, \tilde{C}_{mnp}, C'_{mnp}$ are dimensionless. The presence of such B and L violating terms at the tree level is highly undesirable, as they can lead to catastrophic predictions such as rapid proton decay. This is in contrast to the Standard Model, where all gauge-invariant renormalizable interactions automatically conserve both B and L at the tree level.

Excluding terms such as W_{BL} from the superpotential of the MSSM is thus a key challenge. One may do this by assuming the existence of another symmetry. A common way to achieve this is to assume conservation of R-parity. The supersymmetry algebra $\{Q_\alpha, Q_\beta\} = 0 = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\}$; $\{Q_\alpha, \bar{Q}_{\dot\beta}\} = 2P_{\alpha\dot\beta}$ has outer automorphisms of Lorentz symmetry (i.e. Lorentz covariance) and also a chiral $U(1)$: $Q_\alpha \to e^{-i\delta} Q_\alpha$; $\bar{Q}_{\dot\alpha} \to e^{i\delta}$; $P_{\alpha\dot\beta} \to P_{\alpha\dot\beta}$. Such outer automorphisms are not required symmetries of supersymmetric field theories, but one may <u>choose</u> to build them in. This is certainly the common choice for Lorentz invariance.

The chiral $U(1)$ is called R-symmetry. One may build it into a field theory model by assigning R-symmetry phases $e^{ik\delta}$ to the various matter superfields so that the superpotential W has a homogeneous structure (i.e. every term in W acquires the same overall phase. This overall phase of W can then be cancelled by transforming the chiral superspace integration measure $\int d^2\theta$, so $\int d^2\theta W$ is invariant. R-symmetry, unlike the SM gauge symmetries, transforms the various members of a supermultiplet with different phases, since for $Q_\alpha \to e^{i\delta} Q_\alpha$ one must have $\theta_\alpha \to e^{-i\delta}\theta_\alpha$ in order to give a given superfield $\phi(x,\theta,\bar\theta)$ a homogeneous overall phase. An example of an R-symmetric model is $\int d^4x d^2\theta d^2\bar\theta \bar\Phi\Phi + \int d^4x d^2\theta W + \int d^4x d^2\bar\theta \bar W$ with $W(\phi) = \frac{g}{6}\phi^3$, a single monomial. R-symmetry is then obtained for $\theta_\alpha \to e^{-i\delta}\theta_\alpha$ (so $d\theta_\alpha \to e^{-i\delta} d\theta_\alpha$) and $\phi \to e^{2i\delta/3}\phi$.

Models with such continuous $U(1)$ R-symmetry turn out not to have promising phenomenology. However, one may also look for models with just a discrete subgroup of the continuous $U(1)$ R-symmetry. When this is a $\mathbb{Z}_2$ subgroup, it is called R-parity. The R-parity transformation of a superfield ϕ is $\phi(x,\theta,\bar\theta) \to \phi'(x,\theta,\bar\theta) = \eta\phi(x,-\theta,-\bar\theta)$ so in terms of components one has

$\eta = \pm 1 \in \mathbb{Z}_2$ $a(x) \to \eta a(x)$, $\chi_\alpha(x) \to -\eta\chi_\alpha(x)$, $f(x) \to \eta f(x)$. One can choose the $\pi_R(\Phi) = \eta$ R-parities such that $\pi_R = +1$ for all the conventional SM fields: gauge fields, quarks, leptons and both Higgs. Glennys Farrar and Stephen Weinberg showed (Phys. Rev. D27 (1983) 2732) that R-parity is equivalent to $\pi_R = (-1)^{3(B-L)}(-1)^F$, where B is baryon number, L is lepton number and F is fermion number. Since $(-1)^F = (-1)^{2S}$ where S is the spin of a particle, one may also write $\pi_R = (-1)^{3(B-L)+2S}$. Since $F=0$ for any term in a Lorentz-invariant Lagrangian, R-parity conservation forbids any interactions for which $3\Delta B \neq \Delta L$ mod 2.

Given the MSSM field content, requiring R-parity conservation prohibits writing down B- or L-violating renormalizable interactions. However, in an effective field theory treatment obtained after integrating out physical effects above some cutoff Λ, one can include non-renormalizable interactions with overall dimensions $d > 4$, but suppressed by $\Lambda^{-(d-4)}$ powers of the cutoff. Accordingly, R-parity conservation is potentially consistent with the existence of rare processes that don't conserve B or L, but do conserve (3B − L) mod 2 (i.e. since 2B is even, conserve (B−L) mod 2).

The definition of R-parity sets $\Pi_R = +1$ for all ordinary Standard Model particles, while $\Pi_R = -1$ for all of the superpartners. A consequence of this is the requirement that superpartners must be produced in pairs in interactions with ordinary matter, since all ordinary-matter initial states have $\Pi_R = +1$. Moreover, the lightest superpartner (called LSP) must be absolutely stable in an R-parity conserving theory. LSPs should be plentifully produced in interactions involving superpartners, and once produced they cannot decay. If the LSP is electrically neutral and a colour SU(3) singlet, then it will escape from a detector. A clear signal of such an event would thus be missing energy and momentum. Accordingly, the search for missing-energy signals is a key feature of the search for evidence of supersymmetry. The LSP is also a prime candidate for dark matter in cosmology.

R-parity violating models can also be considered, but then there will be no stable LSP. A key problem for such models is proton decay. Generic W_{BL} couplings can give a proton lifetime ~ 10^{-2} seconds, extendable subject to specific careful construction. For R-parity conserving MSSM models, proton lifetime is increased to 10^{32} years, nearly consistent with current observational data ($\tau_p \gtrsim 10^{33}$ or 10^{34} years).

Supersymmetry Breaking

Current physics shows no sign of supersymmetry, so it must be spontaneously broken. The key to spontaneous breaking of supersymmetry is to arrange for some auxiliary F or D field to take a non-zero vacuum expectation value. There are mechanisms for this: the O'Raifertaigh mechanism gives an F auxiliary field of a chiral superfield a VeV, while the Fayet-Iliopoulos mechanism gives a D auxiliary field of a vector supermultiplet a VeV. However, neither the O'Raifertaigh nor the Fayet-Iliopoulos mechanisms by themselves give desirable phenomenology.

Coupling to supergravity, however, gives different possibilities. The most versatile off-shell formulation of supergravity (with minimal auxiliary fields) has also a scalar & pseudoscalar (M, N) pair of dimension two auxiliary fields. One can arrange for M to have a VeV (Polony mechanism). Couplings between the supergravity fields and matter superfields can then impart supersymmetry breaking to the matter sector even when the spacetime metric is taken to be flat, $g_{\mu\nu} = \eta_{\mu\nu}$ and the $\psi_{\mu\alpha}$ gravitino is set to zero. One may describe the resulting breaking using a nondynamical "spurion" superfield without a kinetic term, but with an auxiliary field given a constant non-vanishing value. The result is called "soft supersymmetry breaking" because it does not disturb the most important ultraviolet divergence cancellations resulting from supersymmetry. In particular, the soft breaking terms maintain the cancellation of quadratic (e.g. Λ^2) divergences in scalar mass terms. Considering just the superrenormalizable terms (dimensions ≤ 3), and in particular the mass terms one has structures

Supergravity coupling incl. $(M+iN)W$

$$\mathcal{L}_{soft} = -\sum_j m_j^2 |\phi_j|^2 - \sum_k \underbrace{m_k \bar{\lambda}^k \lambda^k}_{\text{gauginos}} + \underbrace{B_\mu H_u H_D + A\, W(\phi)}_{\text{these lowest order parts of the superpotential come in to the potential after soft SUSY breaking}}$$

Note that no terms like $\phi^*\phi\phi$ or $\psi\psi$ chiral multiplet mass terms are generated. These are not wanted because they could regenerate unwanted quadratic mass term divergences. Higher order soft terms (dims > 4) can also be generated, but their effects should be suppressed by inverse powers of heavy masses.

The transmission of supersymmetry breaking from supergravity to the matter fields via $\mathcal{L}_{soft}$ terms has been discussed in two basic scenarios: supergravity-mediated supersymmetry breaking and gauge-mediated supersymmetry breaking. Both involve SUSY breaking initially at some higher scale M_S, which is then transmitted to the MSSM fields by

M auxiliary of dim 2

messenger fields. For some auxiliary field VeV $\langle M\rangle \sim M_S^2$ causing the supersymmetry breaking, subsequent transmission of the supersymmetry breaking to MSSM fields by messengers of mass M_{mess} will give soft supersymmetry breakings of scale $m_{soft} \sim M_S^2/M_{mess}$. For supergravity-mediated supersymmetry breaking, $M_{mess} \sim M_{Planck}$ $(\sim 10^{19}\,\text{GeV})$, so one needs $M_S \sim 10^{11}$ GeV in order to have $m_{soft} \sim 1$ TeV, as needed at the electroweak scale.

In gauge-mediated SUSY breaking, it is assumed that the messengers carry $SU(3)\times SU(2)\times U(1)$ representations, with $M_S \sim 10^2 - 10^3$ TeV in order to have $m_{soft} \sim 1$ TeV. Consider gauge-mediated SUSY breaking now and how it may impact electroweak $SU(2)\times U(1)$ symmetry breaking. In the tree approximation, the total scalar potential takes the form $V = V_D + V_\mu + V_{soft}$, where V_D comes from the gauge multiplet D^2 terms and V_μ are the supersymmetric mass terms deriving from the mu term $\mu H_u \cdot H_D$. Taken together with V_{soft}, one has

$$V = \frac{g_2^2}{2}\,|H_u \cdot H_D|^2 + \frac{(g_2^2+g_1^2)}{8}\left[H_u^* \cdot H_u - H_D^* \cdot H_D\right]^2$$
$$+ (\underset{\sim}{m_1^2} + |\mu|^2)(H_u^* \cdot H_u) + (\underset{\sim}{m_2^2} + |\mu|^2)(H_D^* \cdot H_D) - \underset{\sim}{B_\mu}\,\mathrm{Re}(H_u \cdot H_D)$$

where the ~ underlined terms are soft supersymmetry breaking terms. The B_μ coefficient can be arranged to be real + positive by an overall phase adjustment.

In order to achieve electroweak symmetry, one needs to have a negative eigenvalue in the $(\text{mass})^2$ matrix. However, there is also a danger of instability arising from this in directions where the quartic potential terms vanish. Specifically, one needs to ensure stability in directions where the Higgs fields take vacuum values $\langle H_u \rangle = \begin{pmatrix} 0 \\ \phi \end{pmatrix}$, $\langle H_D \rangle = \begin{pmatrix} \phi \\ 0 \end{pmatrix}$, for which the quartic terms vanish. Such vacuum

$v_u^2 + v_D^2 = v^2$ ↔ SM VeV for a single scalar alignment in the more general form $\langle H_u \rangle = \begin{pmatrix} 0 \\ v_u \end{pmatrix}$, $\langle H_D \rangle = \begin{pmatrix} v_D \\ 0 \end{pmatrix}$ so that $(\tau_3 + \frac{1}{2})\langle H_u \rangle = (\tau_3 - \frac{1}{2})\langle H_D \rangle = 0$ as needed in the MSSM in order to preserve an unbroken $U(1)_{EM}$ so that the photon is massless. Such alignment is driven by the $[H_u^* \cdot H_u - H_D^* \cdot H_D]^2$ term in V, which is minimized by having $\langle H_u \rangle$ and $\langle H_D \rangle$ align with opposite τ_3 eigenvalues. However, this then opens the door to a potential instability from the negative $(\text{mass})^2$ eigenvalues.

Stability is achieved by requiring $2|\mu|^2 + m_1^2 + m_2^2 - B_\mu \geq 0$. At the same time, one needs to have a negative eigenvalue

$\det(M^2) < 0 \Rightarrow \exists$ neg. eigenvalue for the $(\text{mass})^2$ matrix $\begin{pmatrix} m_1^2 + |\mu|^2 & -B_\mu/2 \\ -B_\mu/2 & m_2^2 + |\mu|^2 \end{pmatrix}$. This requirement gives the restriction $4(m_1^2 + |\mu|^2)(m_2^2 + |\mu|^2) < B_\mu^2$. These two restrictions on the tree-level potential are not mathematically inconsistent, but they are awkward for tree-level electroweak symmetry breaking. Gauge-mediated supersymmetry breaking naturally gives the same soft-breaking mass terms to fields carrying the same $SU(3)\times SU(2)\times U(1)$ representations. Up to the sign of the Y hypercharge, this is the case for H_u and H_D, so gauge-mediated supersymmetry breaking naturally gives $m_1^2 = m_2^2$ at tree level. Then the two restrictions on potential V coefficients become $0 < B\mu \leq 2(|\mu|^2 + m_1^2)$ and $(B\mu)^2 > 4(|\mu|^2 + m_1^2)^2$ which are inconsistent.

The resolution of this problem involves radiative corrections and the high mass scale of the top quark. One has $\langle H_u \rangle = \binom{0}{v_u}$ and $\langle H_D \rangle = \binom{v_D}{0}$ with v_u adjusted to be real and the vacuum conditions then requires v_D to be real as well. Define $\tan\beta = \frac{v_u}{v_D}$. One finds the $(\text{mass})^2$ of the Z_μ vector boson to be

$v_u^2 + v_D^2 = v^2 = (246\,\text{GeV})^2$

$M_Z^2 = \frac{1}{2}(g_2^2 - g_1^2)(v_u^2 + v_D^2)$; define also $M_A^2 = 2|\mu|^2 + m_1^2 + m_2^2$; $M_A^2 \geq 0$. One then finds $B_\mu = m_A^2 \sin 2\beta$,

$m_1^2 + |\mu|^2 = \frac{1}{2} m_A^2 - \frac{1}{2}(m_A^2 + M_Z^2)\cos 2\beta$, $m_2^2 + |\mu|^2 = \frac{1}{2} m_A^2 + \frac{1}{2}(m_A^2 + M_Z^2)\cos 2\beta$

For $B_\mu > 0$, this limits β to the range $0 \leq \beta \leq \pi/2$

In the 3rd generation, the ratio of top and bottom quark masses $\left(\frac{175\,\text{GeV}}{4.5\,\text{GeV}} \sim 39\right)$ is governed by $\tan\beta$. So, unless this large ratio is due to a large ratio of Yukawa couplings, which would be unnatural, one should have $\tan\beta$ large, so $\beta \sim \pi/2$. But then $m_1^2 + |\mu|^2 \sim M_A^2 + \frac{1}{2} M_Z^2$ while $m_2^2 + |\mu|^2 \sim -\frac{1}{2} M_Z^2$, which are hardly equal, so these values are in apparent contradiction with the mechanism of gauge-mediated supersymmetry breaking.

Best fits: $\tan\beta \sim 5-40$

The resolution of this problem comes from the renormalization group running of the $(\text{mass})^2$ values for H_u and the squark scalars (supersymmetric partners of the quarks), and in particular the stops. The effect of strong interaction couplings $\sim g_3^2$ is to push $M_{\tilde{t}_L}^2$ and $M_{\tilde{t}_R}^2$ ($(\text{mass})^2$ for L, R stops) up at low energies (viz. down at high energies) while the effect of λ_t (top quark Yukawa coupling) is to push $M_{H_u}^2$, $M_{\tilde{t}_L}^2$ and $M_{\tilde{t}_R}^2$ down at low energies. Consequently, in order for $M_{\tilde{t}_L}^2$ and $M_{\tilde{t}_R}^2$ to remain large at low energies, $M^2_{H_u}$ <u>must go down through zero and become negative.</u> Consequently, starting at high energies above the top scale with all Higgs $(\text{mass})^2$ parameters initially positive, renormalization-group evolution down to low energies <u>naturally</u> gives a negative $(\text{mass})^2$ to H_u, causing electroweak symmetry breaking.

Starting with 8 real component fields of the two Higgs doublets, three are eaten by the Higgs effect, leaving five physical scalars. These correspond to two charged ($H^\pm$) and three neutral scalar particles (h, H and A, with A a pseudoscalar). The pattern of $(\text{masses})^2$ emerging from analysis of the Higgs potential gives $m_H^2 = \frac{1}{2}(M_A^2 + M_Z^2 + \Delta)$, $m_h^2 = \frac{1}{2}(M_A^2 + M_Z^2 - \Delta)$ with $\Delta = [(M_A^2 - M_Z^2)^2 + 4\, m_A^2 M_Z^2 \sin^2(2\beta)]^{1/2}$, so the h scalar mass is smaller than the lighter of the A or Z masses. For $\tan\beta$ large, so $\beta \sim \pi/2$, one has $\sin(2\beta) \sim 0$ and m_h is then near either m_A or M_Z, whichever is lighter. Current bounds indicate $m_A > M_Z$, so one has a (tree-level) prediction that m_h is close to the Z_μ boson mass. Radiative corrections tend to raise m_h, and the result is in reasonable agreement with $M_Z \sim 91\ \text{GeV}/c^2$ and the now known $m_H \sim 126\ \text{GeV}/c^2$.

Finally, return to the question whether the $\frac{1}{\alpha_3}$, $\frac{1}{\alpha_2}$ and $\frac{1}{\hat{\alpha}_1}$ trajectories come to a point at some scale Q, using the renormalization group coefficients appropriate to the MSSM. Including the contributions of the gauginos, squarks and sleptons and the higgsinos as well as the Higgs scalars for n_g generations and n_H Higgs doublet supermultiplets, one has now

$$b_3 = \frac{1}{4\pi}(9 - 2n_g)\ ; \quad b_2 = \frac{1}{4\pi}\left(6 - 2n_g - \frac{n_H}{2}\right)\ ; \quad \hat{b}_1 = \frac{1}{4\pi}\left(-2n_g - \frac{3n_H}{10}\right).$$

Inserting these into $B = \frac{b_3 - b_2}{b_2 - \hat{b}_1}$ for $n_H = 2$, $n_g = 3$, one has $B^{th}_{MSSM} = \frac{5}{7} = 0.714$. This now agrees very well with the experimental value $B^{expt} = 0.719 \pm .008 \pm .03$. Thus, starting from the MSSM at low energies, one anticipates gauge coupling unification, and the arrival of yet more new physics around $10^{16}\ \text{GeV}/c^2$.

Some projected proton lifetimes for various models:

Model	years
Minimal SU(5) (Georgi-Glashow)	$10^{30}-10^{31}$
Susy SU(5) (MSSM)	$\sim 10^{34}$
Minimal non Susy SO(10)	$\lesssim 10^{35}$ maximum
Susy SO(10)	$10^{32}-10^{35}$
Flipped SU(5) MSSM	$10^{35}-10^{36}$

Flipped SU(5) has gauge group $(SU(5)\times U(1))/Z_5$ with fermions in $\bar{5}_{-3}$ for leptons L and u^c quarks

10_1 for Q quark doublet, d^c and RH neutrino N

1_5 for charged leptons e^c

Higgs are just in the 10_1

SM $U_Y(1)$ is in the $U(1)$ factor of the GUT group together with a mixed diag $(\frac{1}{15}\,\frac{1}{15}\,\frac{1}{15}\,-\frac{1}{10}\,-\frac{1}{10})$ from the SU(5)