Imperial College London

MSc EXAMINATION May 2017

This paper is also taken for the relevant Examination for the Associateship

THE STANDARD MODEL AND BEYOND

For Students in Quantum Fields and Fundamental Forces

Friday, May 12$^{\text{th}}$ 2017: 14:00 to 17:00

Answer TWO out of the following THREE questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the TWO answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in TWO answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

1. The Standard Model of elementary particle physics is based on the gauge symmetry group $\mathrm{SU(3)} \times \mathrm{SU(2)_L} \times \mathrm{U(1)_Y}$. It contains the Yang-Mills gauge fields and the Higgs field plus three generations of fermionic fields, each containing electron and neutrino type fields, left and right handed quarks. In Weinberg's conventions, the unbroken $U(1)_{\mathrm{em}}$ symmetry is generated by $Q = T_3 + Y$ where T_3 is the diagonal generator of $\mathrm{SU(2)_L}$. The u type quarks have electromagnetic charge $\frac{2}{3}$ and the d type quarks have charge $-\frac{1}{3}$.

(i) State the $\mathrm{SU(3)} \times \mathrm{SU(2)_L} \times \mathrm{U(1)_Y}$ gauge-group representations in which the left- and right-handed components of the various spinor fields transform in the Standard Model. Explain why a bare mass term for the electron is not allowed in the $\mathrm{SU(3)} \times \mathrm{SU(2)_L} \times \mathrm{U(1)_Y}$ gauge-invariant action. Explain how the incorporation of Yukawa couplings nonetheless allows mass terms to arise for the electron after shifts in fields occasioned by symmetry breaking. [12 marks]

(ii) Assuming that a hermitian matrix can be diagonalised by a unitary transformation, show that a general complex but nonsingular (*i.e.* having a nonvanishing determinant) matrix $\mathbf{M}$ can be made purely diagonal with positive real entries by a biunitary transformation of the form $\mathbf{M} \to \mathbf{V}^\dagger \mathbf{M} \mathbf{U}$ where $\mathbf{V}$ and $\mathbf{U}$ are unitary matrices. Do this as follows:

(a) Explain why $\mathbf{M}^\dagger \mathbf{M}$ is straightforward to diagonalise by some unitary transformation $\mathbf{U}$. Show that the resulting diagonal entries are real and positive. Call this diagonal matrix $\mathbf{D}^2$, *i.e.* let $\mathbf{U}^\dagger \mathbf{M}^\dagger \mathbf{M} \mathbf{U} = \mathbf{D}^2$.

(b) Define the matrix $\mathbf{D}$ to be the diagonal matrix whose entries are the positive square roots of the corresponding entries in $\mathbf{D}^2$. Then let $\mathbf{H} = \mathbf{U}\mathbf{D}\mathbf{U}^\dagger$. Show that $\tilde{\mathbf{U}} = \mathbf{M}\mathbf{H}^{-1}$ is unitary. Finally, show that $\mathbf{V}^\dagger \mathbf{M} \mathbf{U} = \mathbf{D}$ where $\mathbf{V} = \tilde{\mathbf{U}}\mathbf{U}$ is unitary. [16 marks]

(iii) The Yukawa couplings between the Higgs field ϕ and the Standard Model fermions are of the form $\mathcal{L}_{\mathrm{Yukawa}} = -\mathrm{i}(f_{mn}\overline{L_m} e_{Rn}\phi + h_{mn}\overline{Q_{Lm}} d'_{Rn}\phi + k_{mn}\overline{Q_{Lm}} u'_{Rn}\tilde{\phi}) + \text{hermitian conj.}$, where the vacuum values of ϕ and $\tilde{\phi}$ are $\phi^T_{\mathrm{vac}} = (0, \frac{1}{\sqrt{2}}v)$ and $\tilde{\phi}^T_{\mathrm{vac}} = (\frac{1}{\sqrt{2}}v, 0)$ (in which v is a constant and the T notation denotes a transpose), $L^T_m = (\nu_m, e_{Lm})$ are the left-handed lepton doublets and $Q^T_{Lm} = (u'_{Lm}, d'_{Lm})$ are the left-handed quark doublets in generations $m = 1, 2, 3$. The coefficients f_{mn}, h_{mn} and k_{mn} describe the mixing between generations.

(a) Starting from the original "flavour" basis where the gauge field couplings are diagonal in m, n generation labels, explain how biunitary transformations $U^{\mathrm{field-type}}_{mn}$ such as those of part (ii) may be used to diagonalise mass terms for the six fermion types $e_{{}^{\mathrm{L}}_{\mathrm{R}}m}$, $u'_{{}^{\mathrm{L}}_{\mathrm{R}}m}$ and $d'_{{}^{\mathrm{L}}_{\mathrm{R}}m}$ in the m, n indices without disturbing diagonal fermion kinetic terms (*i.e.* terms of the form $\overline{(\text{spinor})}\gamma^\mu \partial_\mu(\text{same spinor})$).

(b) Explain why quark couplings to the electromagnetically *neutral* A_μ and Z_μ vector fields are not altered by such biunitary basis transformations. Explain why Lagrangian terms coupling the *charged* vector fields $W^\pm_\mu$ to the quarks become mixed between the $m = 1, 2, 3$ generations by the Cabbibo-Kobayashi-Maskawa (CKM) matrix $V_{mn} = ((U^{u_L})^\dagger U^{d_L})_{mn}$ as a result of the biunitary basis transformations of part (a), since one cannot simultaneously diagonalise in the m indices both the fermion mass terms and the charged-vector-field couplings to the quarks.

(c) Explain why the CKM matrix for the three-generation system has just four physically relevant parameters. How many physically relevant CKM parameters would there be for an N-generation system? How many of these are phase parameters that go beyond what one would have in a purely real $\mathrm{SO}(N)$ matrix? [22 marks]

[Total 50 marks]

2. (i) How does one ensure that neutrinos are massless in the original Standard Model? How is the representation content of the model changed, and which types of mass terms are included when the Standard Model is extended to incorporate neutrino masses via the see-saw mechanism?

[10 marks]

(ii) The process of Takagi factorisation involves writing a complex symmetric matrix S in the form $\mathbf{S} = \mathbf{U}\tilde{\mathbf{D}}\mathbf{U}^{\mathrm{T}}$ where $\mathbf{U}$ is unitary and $\tilde{\mathbf{D}}$ diagonal. Show that this may be carried out as follows. For a complex symmetric matrix $\mathbf{S}$, consider the eigenvectors of $\mathbf{M} = \mathbf{S}\mathbf{S}^\dagger$ and first show that $\mathbf{M}$ may be diagonalised by a unitary matrix $\mathbf{U}$ constructed from the eigenvectors of $\mathbf{M}$, *i.e.* show that $\mathbf{U}^\dagger\mathbf{M}\mathbf{U} = \mathbf{D}$ where $\mathbf{D} = \mathrm{diag}(h_1, h_2, \ldots, h_N)$ is a diagonal matrix with real nonnegative elements. Then let $\mathbf{T} = \mathbf{U}^\dagger\mathbf{S}(\mathbf{U}^\dagger)^{\mathrm{T}}$ so $\mathbf{T}\mathbf{T}^\dagger = \mathbf{D}$. Show that $[\mathbf{D}, \mathbf{T}] = \mathbf{0}$ and consequently that $T_{i\ell}(h_i - h_\ell) = 0$, so that $\mathbf{T}$ must itself be diagonal provided the eigenvalues h_i are all distinct. Consequently, show that one may choose phases for the eigenvalues of $\mathbf{T}$ such that $\tilde{\mathbf{D}} = \mathbf{T} = \mathrm{diag}(\sqrt{h_1}, \sqrt{h_2}, \ldots, \sqrt{h_N})$.

[12 marks]

(iii) Explain the relevance of Takagi factorisation to the counting of physically relevant parameters in the PMNS matrix governing charged-current couplings of leptons to massive neutrinos. Explain why the number of physically relevant parameters in the PMNS matrix is six as opposed to four for the CKM matrix governing the analogous charged-current couplings of quarks. Why, in practice, can one only obtain experimental information about four of these parameters at present?

[12 marks]

(iv) In a system with N generations and PMNS unitary matrix U which diagonalises the neutrino mass matrix, the relation between flavour eigenstates $|\nu_\alpha\rangle$ and mass eigenstates $|i\rangle$ is $|\nu_\alpha\rangle = \sum_i U^*_{\alpha i}|i\rangle$. Consider ultrarelativistic neutrinos with momentum $p = |\vec{p}| >> m_i$ for any of the mass eigenvalues m_i, whose energies can accordingly be approximated by $E_i = \sqrt{p_i^2 + m_i^2} \simeq p_i + \frac{m_i^2}{2p_i} \approx E + \frac{m_i^2}{2E}$. For ultrarelativistic neutrinos and $c = 1$, one has a time of flight T to distance travelled L relation $T \approx L$.

Show that the probability for a neutrino originally of flavour α to be later observed with flavour β is then to leading order

$$P_{\alpha\to\beta} = |\langle\nu_\beta|\nu_\alpha(T)\rangle|^2 = |\sum_i U^*_{\alpha i}U_{\beta i}e^{-im_i^2 L/2E}|^2 \quad .$$

Show that this probability may be rewritten as

$$P_{\alpha\to\beta} = \delta_{\alpha\beta} - 4\sum_{i>j}\mathrm{Re}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin^2(\frac{\theta_{ij}}{2}) + 2\sum_{i>j}\mathrm{Im}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin(\theta_{ij})$$

where $\Delta m^2_{ij} = m_i^2 - m_j^2$ and $\theta_{ij} = \left(\frac{\Delta m^2_{ij}L}{2E}\right)$.

The CP asymmetry is $A^{\alpha\beta}_{\mathrm{CP}} = P_{\alpha\to\beta} - P_{\bar{\alpha}\to\bar{\beta}} = 4\sum_{i>j}\mathrm{Im}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin(\theta_{ij})$. In terms of the Jarlskog invariant J, determined in the $N = 3$ case by $\mathrm{Im}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}) = -J\sum_{\gamma,k}\epsilon_{\alpha\beta\gamma}\epsilon_{ijk}$, use the identity $\sin(a+b)\sin(a+c)\sin(c+b) = \frac{1}{4}\left(-\sin(2a+2b+2c) + \sin(2a) + \sin(2b) + \sin(2c)\right)$ to show that the CP asymmetry is given by

$$A^{\alpha\beta}_{\mathrm{CP}} = 16J\sum_\gamma \epsilon_{\alpha\beta\gamma}\sin(\frac{\theta_{31}}{2})\sin(\frac{\theta_{32}}{2})\sin(\frac{\theta_{21}}{2}) \ .$$

[16 marks]

[Total 50 marks]

3. A symmetric-space nonlinear realisation of a group G with linear realisation on a subgroup H is based on a Lie algebra

$$[V_i, V_j] = i f_{ij}{}^k V_k \qquad [V_i, A_\ell] = i f_{il}{}^m A_m \qquad [A_\ell, A_m] = i f_{\ell m}{}^k V_k$$

where the V_i are generators of the stability subgroup H and the A_ℓ are generators of the G/H coset representatives. Note that this symmetric-space symmetry algebra admits an automorphism $A_\ell \to -A_\ell$. For the nonlinear realisation, write $g(x) = e^{i\xi^k(x)A_k} e^{i\theta^i(x)V_i} = u(\xi^k(x))h(\theta^i(x))$ where the $\xi^k(x)$ are the nonlinearly transforming Goldstone fields. Upon transforming by an x^μ independent element g_0 of G, one has

$$g_0 g = u(\xi'^k(x))h(g_0, \xi^j(x)) \tag{3.1}$$

by repolarising into G/H and H factors.

(i) For $g_0 = e^{i\xi_0\cdot A}$ with ξ_0^k constant and $\xi_0 \cdot A = \xi_0^\ell A_\ell$, etc., show that $e^{2i\xi'\cdot A} = e^{i\xi_0\cdot A} e^{2i\xi\cdot A} e^{i\xi_0\cdot A}$.

[10 marks]

For chiral symmetry derived from N underlying quark species, $G = \mathrm{SU}(N)_\mathrm{L} \times \mathrm{SU}(N)_\mathrm{R}$ where H is the diagonal "vector" $\mathrm{SU}(N)_\mathrm{V}$ subgroup, and the individual $\mathrm{SU}(N)_\mathrm{L,R}$ factors act on the q_L left (γ_5 eigenvalue +1) and the q_R right (γ_5 eigenvalue -1) chiral components of the quarks independently. An arbitrary element g of G may be written $g = L(\alpha_\mathrm{L})R(\alpha_\mathrm{R}) = e^{i\alpha_\mathrm{L}\cdot T_\mathrm{L}} e^{i\alpha_\mathrm{R}\cdot T_\mathrm{R}} = e^{\frac{i}{2}(\alpha_\mathrm{L}^i + \alpha_\mathrm{R}^i)T_i + \frac{i}{2}(\alpha_\mathrm{L}^j - \alpha_\mathrm{R}^j)\gamma_5 T_j}$ corresponding to $V_i = \mathbb{1}_{4\times 4} T_i$ for the diagonal $\mathrm{SU}(N)_\mathrm{V}$ vector subgroup H generators and $A_k = \gamma_5 T_k$ for the G/H axial coset generators, in which the T_k are generators of $\mathrm{SU}(N)$. The g_0 transformation of equation (3.1) thus would have $\xi_0^i = \frac{1}{2}(\alpha_\mathrm{L}^i - \alpha_\mathrm{R}^i)$ and $\theta_0^j = \frac{1}{2}(\alpha_\mathrm{L}^j + \alpha_\mathrm{R}^j)$. The $\mathbb{1}_{4\times 4}$ and γ_5 matrices are inherited from the underlying spinorial quark structure; even though one is now dealing only with bosonic fields, these matrices play a key rôle in representing the $\mathrm{SU}(N)_\mathrm{L} \times \mathrm{SU}(N)_\mathrm{R}$ algebra.

For original left and right chiral-projection quark transformations $q \to q' = Lq_\mathrm{L} + Rq_\mathrm{R}$, now define modified chiral quark fields $q_\mathrm{L} = u\tilde{q}_\mathrm{L}$, $q_\mathrm{R} = u^\dagger \tilde{q}_\mathrm{R}$, where the form of the second definition is determined by the parity transform of the first, which sends $\gamma_5 \to -\gamma_5$. It may be helpful to use a representation in which $\gamma_5 = \mathrm{diag}(\mathbb{1}, -\mathbb{1})$ is diagonal (with $\mathbb{1}$ now a 2×2 unit submatrix). Keeping the +1 eigenvalue associated to the upper-left 2×2 subblock of this 4×4 matrix, the effect of a left $\leftrightarrow$ right parity transformation may then be viewed as interchanging the upper-left and lower-right 2×2 subblocks of γ_5 so $u_P = PuP^{-1}$ with $P = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$.

(ii) Show that the G/H coset representative structure $u = e^{i\xi^k(x)\gamma_5 T_k}$ implies $u(x) = \mathrm{diag}(w(x)\mathbb{1}, w^\dagger(x)\mathbb{1})$ where $w(x) = e^{i\xi(x)\cdot T}$. The transformation of u is determined by the requirement of preserving this G/H coset representative structure for $u' = e^{i\xi'^k(x)\gamma_5 T_k}$. [5 marks]

For a quark transformation $q \to q' = Lu\tilde{q}_\mathrm{L} + Ru^\dagger \tilde{q}_\mathrm{R}$ one accordingly has the nonlinear transformations $Lu = u'h$, $Ru^\dagger = u^{\dagger\prime}h$ and $\tilde{q}'_\mathrm{L} = h\tilde{q}_\mathrm{L}$, $\tilde{q}'_\mathrm{R} = h\tilde{q}_\mathrm{R}$, where $u' = u(\xi')$, $h = h(g_0, u(x))$ and the forms of the R transformation laws are determined by the parity transforms of the L transformation laws, *i.e.* using $R = L_P = PLP^{-1}$.

(iii) Show that for $\mathrm{SU}(N)_\mathrm{L}$ or $\mathrm{SU}(N)_\mathrm{R}$ transformations, *i.e.* $g_0 = L$ resp. $g_0 = R$, one thus has

$$u' = Luh^\dagger \qquad \text{resp.} \qquad u' = huR^\dagger \tag{3.2a}$$

$$u^{\dagger\prime} = hu^\dagger L^\dagger \qquad \text{resp.} \qquad u^{\dagger\prime} = Ru^\dagger h^\dagger \tag{3.2b}$$

where in each case $h = h(g_0, u(x))$. [5 marks]

[This question continues on the next page . . .]

(iv) In a case where the R transformation in (3.2a) is the parity transform of L, *i.e.* where $R = L_P = PLP^{-1}$, show that the two transformations of the form (3.2a) produce equivalent results for u' with the same $h \in H$. [5 marks]

(v) For a general constant $G = \mathrm{SU}(N)_{\mathrm{L}} \times \mathrm{SU}(N)_{\mathrm{R}}$ transformation $g_0 = X = LR = \mathrm{diag}(\ell\mathbb{1}, r\mathbb{1})$, show that $h(g_0, u(x))$ is determined by the relation $hw^\dagger\ell^\dagger = rw^\dagger h^\dagger$. Show that with h determined by this relation, one has equivalent left-side and right-side versions of a general $X = \mathrm{diag}(l\mathbb{1}, r\mathbb{1}) \in G = \mathrm{SU}(N) \times \mathrm{SU}(N)$ transformation:

$$u' = Xuh^\dagger \qquad \leftrightarrow \qquad u' = huX_P^\dagger \tag{3.3a}$$

$$u^{\dagger\prime} = hu^\dagger X^\dagger \qquad \leftrightarrow \qquad u^{\dagger\prime} = X_P u^\dagger h^\dagger \tag{3.3b}$$

[5 marks]

(vi) Show that for constant $g_0 = h_0 = e^{i\theta\cdot V} = e^{i\theta\cdot T}\mathbb{1}_{4\times4} \in H$ transformations, the transformations (3.3) are linear and x^μ independent with $h = h_0$. [5 marks]

The standard left-sided transformation of the form (3.1) for a nonlinear realisation of G (or the equivalent right-sided transformation from (3.3a)) requires finding $h(g_0, u(x))$ from the implicit relation given in part (v). General G invariants are then built by requiring local h invariance for "matter" fields such as the $\tilde{q}$ quarks. Such constructions make use of the standard Maurer-Cartan form $udu^{-1} = udu^\dagger = Z\cdot A + M\cdot V$, where the G/H coset projection $Z = Z_\mu dx^\mu$ transforms according to $Z \to hZh^\dagger$, while the $M = M_\mu dx^\mu$ stability subgroup H projection transforms like a gauge field for H, *i.e.* $M \to hdh^\dagger + hMh^\dagger$.

In order to construct an invariant Lagrangian for the Goldstone fields $u = e^{i\xi\cdot A}$ themselves, however, a more convenient form of the realisation is obtained by defining $\mathcal{U} = u^2$.

(vii) Combine a left-side transformation from (3.3) for one u factor with an equivalent right-side transformation from (3.3) for the second u factor to obtain the transformation rule for $\mathcal{U}' = u'^2$ in terms of X. For $X = e^{i\xi_0\cdot A} \in G/H$, show that this transformation agrees with the result found in part (i) for a general symmetric-space G/H nonlinear realisation. [5 marks]

(viii) In order to construct a normalised Goldstone-field kinetic term, write now $\xi^k(x) = F_\pi \pi^k(x)$ so $\mathcal{U}(\pi(x)) = e^{\frac{2i\pi^k(x)}{F_\pi}\gamma_5 T_k}$ where $T_k = \frac{1}{2}\lambda_k$ and for $N = 2$ the λ_k are the SU(2) Pauli matrices σ_k while for $N = 3$ the λ_k are the SU(3) Gell-Mann matrices. Show that the standard nonlinear-realisation kinetic term $\mathcal{L}_{\mathrm{kin}} = \frac{F_\pi^2}{16}\,\mathrm{tr}(Z_\mu Z^\mu)$ can be written in the form

$$\mathcal{L}_{\mathrm{kin}} = \frac{F_\pi^2}{4}\,\mathrm{tr}(Z_\mu Z^\mu) = -\frac{F_\pi^2}{4}\,\mathrm{tr}(\partial_\mu \mathcal{U} \partial^\mu \mathcal{U}^\dagger)$$

where the trace is over both γ_5 and adjoint SU(N) indices.

[10 marks]

[Total 50 marks]

 End of examination paper