Imperial College London

MSc EXAMINATION May 2018

This paper is also taken for the relevant Examination for the Associateship

THE STANDARD MODEL AND BEYOND

For Students in Quantum Fields and Fundamental Forces

Friday, May 25$^{\text{th}}$ 2018: 14:00 to 17:00

Answer TWO out of the following THREE questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the TWO answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in TWO answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

1. The Standard Model of weak and electromagnetic interactions is based upon the gauge group $\mathrm{SU}(3) \times \mathrm{SU_L}(2) \times \mathrm{U_Y}(1)$ with asymmetrical representation assignments to left- and right-handed fermionic fields such as electrons and neutrinos. In the Standard Model, the unbroken $U(1)_{em}$ electromagnetic generator is conventionally taken to be $Q_{\rm em} = T^3_L + Y$, where $T^3_L = \frac{1}{2}\sigma^3$ and Y is weak hypercharge, *i.e.* it is the generator of the $U(1)_Y$ factor in the $SU(2)_L \times U(1)_Y$ gauge group. In the minimal Standard Model, neutrinos are not given masses.

(i) State the $\mathrm{SU}(3) \times \mathrm{SU_L}(2) \times \mathrm{U_Y}(1)$ gauge-group representations in which the left- and right-handed components of the electron or its muon or tau generation copies and all other fields required in the minimal Standard Model transform. Explain why a bare mass term for the electron is not allowed in the $\mathrm{SU}(3) \times \mathrm{SU_L}(2) \times \mathrm{U_Y}(1)$ gauge-invariant action. Explain how the incorporation of Yukawa couplings nonetheless allows mass terms to arise for the electron after shifts in fields occasioned by symmetry breaking. What is the physical reason for constructing the Standard Model in such a way that bare mass terms are ruled out in the minimal Standard Model? [10 marks]

(ii) How does one ensure that neutrinos can have no masses in the minimal Standard Model? How is the representation content of the model changed, and which types of additional terms are included when one includes neutrino masses via the see-saw mechanism? Explain how small masses for the known neutrino species can be obtained. [6 marks]

(iii) The left-handed spinorial (u_L, d_L) quarks transform in the $SU(2)_L$ doublet representation. Given that the left-handed component of the proton $(uud)_L$ has electromagnetic charge $+1$ in units of electric charge, while the left-handed component of the neutron $(udd)_L$ has electromagnetic charge zero, work out the electromagnetic charge and weak hypercharge assignments for the u_L and d_L quarks. Given that the right-handed spinorial versions of these quarks are $SU(2)_L$ singlets, work out the weak hypercharge assignments for the right-handed u_R and d_R quarks. [12 marks]

(iv) The d quark and the "strange" s quark have the same $SU(2)_L$ and weak hypercharge quantum numbers. Two hadronic resonance particles that played important rôles in the development of the quark model are the positively charged Δ and the negatively charged Ω particles. The Δ is formed purely out of three u quarks in the $m = 1$ first generation, while the Ω is formed purely out of three s quarks in the $m = 2$ second generation. What consequence for the electromagnetic charges of the theory follows from the fact that the overall symmetry group is a direct product of the Poincaré (Lorentz plus translations) group and the internal symmetry group? Find the electromagnetic charges of the Δ and Ω particles. [10 marks]

(v) Consider the purely left-handed, extremal helicity, component of the Δ and Ω particles (whose 2-component field representation has only undotted indices). Show that if these quarks did not carry any other symmetry indices *i.e.* if they were singlets with respect to other possible symmetries, then they could not exist. Show that these particles can exist, however, provided they carry fundamental representation triplet indices with respect to an additional $SU(3)$ colour symmetry. What symmetry property must such states have in the three colour indices in order for the particles to be overall $SU(3)$ invariant? What symmetry property must they have in the Lorentz spinor $\mathrm{SL}(2, \mathbb{C})$ indices? Find the maximal helicity eigenvalues for the Δ and Ω particles in units of $\hbar$. [12 marks]

[Total 50 marks]

2. (i) Show that solutions to the momentum-space Dirac equation for a massless spinor field satisfy

$$p^\mu p_\mu \psi(p) = 0 \quad \text{and} \quad \psi(p) = -\zeta \frac{p_i}{|\mathbf{p}|}\gamma^0\gamma^i\psi(p)$$

with $\zeta = 1$ if $p^0 > 0$ and $\zeta = -1$ if $p^0 < 0$, where the p^i are components of three-momentum, $i = 1, 2, 3$, and $|\mathbf{p}| = \sqrt{p^i p^i}$. In the Fourier expansion of $\psi(x)$, both $p^0 \gtrless 0$ parts of a wavefunction are present – the $p^0 > 0$ terms are particle solutions while the $p^0 < 0$ terms are antiparticle solutions. [5 marks]

(ii) The helicity Λ of a particle is the projection of its angular momentum 3-vector in the direction of the spatial momentum. Given that the Lorentz generators for spinors are $M_{\mu\nu} = -\frac{\mathrm{i}}{2}\gamma_{\mu\nu} = -\frac{\mathrm{i}}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$ and the spatial angular-momentum 3-vector operator is $S_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$, show that the helicity operator Λ can be written as $\Lambda = -\frac{1}{2}\gamma^5\gamma^0\gamma^i\frac{p_i}{|p|}$ where $\gamma^5 = \mathrm{i}\gamma^0\gamma^1\gamma^2\gamma^3$. Hence, show that massless chiral fermions are eigenstates of helicity, and specifically that

$$\left.\begin{aligned} \Lambda\psi_L(p) &= \frac{\zeta}{2}\psi_L(p) \\ \Lambda\psi_R(p) &= -\frac{\zeta}{2}\psi_R(p) \end{aligned}\right\} \quad \zeta = \pm 1 \text{ for } p^0 \gtrless 0 \quad \text{where} \quad \gamma^5\psi_{\substack{L\\R}} = \pm\psi_{\substack{L\\R}}\,.$$

[10 marks]

The ζ factors should be interpreted this way: for elementary particles, which have $p^0 > 0$, helicity is, up to a factor of $\frac{1}{2}$, the same as chirality, while for antiparticles, which have $p^0 < 0$, helicity is, up to a factor of $\frac{1}{2}$, opposite to chirality.

(iii) Now consider a massive spinor field satisfying $(\gamma_\mu\partial^\mu + m)\psi = 0$ and repeat the above analysis to show

$$(p^\mu p_\mu + m^2)\psi(p) = 0 \quad \text{and} \quad \psi(p) = \zeta\left(\frac{-p_i\gamma^0\gamma^i + im\gamma^0}{\sqrt{\mathbf{p}^2 + m^2}}\right)\psi(p)$$

[5 marks]

(iv) Hence, show that the helicity operator Λ acting on the left- or right-chirality projections of a massive spinor particle satisfies

$$\left.\begin{aligned} \Lambda\psi_L(p) &= \left(\frac{\zeta\sqrt{\mathbf{p}^2+m^2}\psi_L(p) + im\gamma^0\psi_R(p)}{2|\mathbf{p}|}\right) \\ \Lambda\psi_R(p) &= -\left(\frac{\zeta\sqrt{\mathbf{p}^2+m^2}\psi_R(p) + im\gamma^0\psi_L(p)}{2|\mathbf{p}|}\right) \end{aligned}\right\} \quad \zeta = \pm 1 \text{ for } p^0 \gtrless 0 \quad \text{where} \quad \gamma^5\psi_{\substack{L\\R}} = \pm\psi_{\substack{L\\R}}\,.$$

Hence, show that in the ultrarelativistic limit $|p| \to \infty$, one obtains the results of part (ii). [10 marks]

(v) Now consider positively charged pion decay into a positively charged lepton in the m^{th} generation and its corresponding neutrino, $\pi^+ \to e^+_m + \nu_m$, in the original Standard Model, where neutrinos are massless.

(a) Give the form of the interaction between leptons in the m^{th} generation and the $\mathrm{SU(2)_L}$ gauge fields of the Standard Model and use this to show that if the lepton were massless, the helicities of the lepton and the neutrino would have to be opposite. [10 marks]

(b) For a π^+ initially at rest, show from kinematics and conservation laws that the helicities of the lepton and the neutrino would have to be the same. [10 marks]

Consequently conclude that for massless leptons, such decay of a charged pion would be ruled out. This phenomenon is called helicity suppression. It has the effect that the charged pion decay into lighter generation leptons is disfavoured as compared to decay into heavier leptons. Consequently, π^+ decay into μ^+ dominates over π^+ decay into e^+.

[Total 50 marks]

3. A Lie group symmetry G of a quantum field theory present at the classical level may be violated at the quantum level by anomalies. All the fermion fields ψ^m of a given theory may be catalogued as left-handed spinors – either by taking them as left-handed spinors in their original form or by cataloguing their left-handed charge conjugates $(\psi^m)_C$ if they are originally right-handed. Then one may write infinitesimal G transformations as $\delta\psi^m = \mathrm{i}\epsilon^a (T_a)_m{}^n \psi_n$ where the $(T_a)_m{}^n$ are hermitian generators of G ($a = 1, \ldots, \dim(\mathrm{G})$). The theory will then be anomaly-free if the anomaly coefficients $A_{abc} = \mathrm{tr}(T_a\{T_b, T_c\})$ vanish for all a, b, c, where $\{A, B\} = AB + BA$ and the trace is taken over all m, *i.e.* over the full spectrum of left-handed spinor fields in the theory's spinor catalogue.

(i) Explain why the mixed axial anomaly $A_{T_{3A}\,EM\,EM}$ between the $T_{3A} = \tau_3\gamma_5$ axial symmetry generator in the pion effective action and two T_{EM} electromagnetic generators is not threatening to the integrity of the pion effective theory (and in fact gives important physical information about π^0 decay), while anomalies in the Standard Model's $\mathrm{SU}(3) \times \mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)$ symmetry would be disastrous. [15 marks]

(ii) A pseudoreal representation of G is one for which the generators T_a satisfy $(T_a)^* = -S T_a S^{-1}$ for some invertible matrix S. The Lie algebra generators T_a of a compact G are taken to be hermitian.

(a) Show that the anomaly coefficients A_{abc} are totally symmetric in general and that they all vanish for a pseudoreal representation. [10 marks]

(b) Show that the anomaly coefficients A_{abc} vanish for a theory where left-handed and right-handed fermions transform in the same unitary representation with hermitian generators t_a (for this purpose, consider the right-handed fermions in their original right-handed forms, prior to charge conjugation). As a consequence, such a left-right symmetric theory is anomaly-free. [5 marks]

(iii) Show that the $\mathrm{SU}(3) \times \mathrm{SU}_\mathrm{L}(2) \times \mathrm{U}_\mathrm{Y}(1)$ anomaly coefficients all vanish in the Standard Model, so the theory is anomaly-free.

(a) First show the vanishing of all the $A(3,3,2)$ and $A(3,X,Y)$ anomaly coefficients, where $X, Y = 2$ or 1, in which $A(3,2,1)$ stands for the set of anomaly coefficients between an $\mathrm{SU}(3)$, an $\mathrm{SU}(2)_\mathrm{L}$ and the $\mathrm{U}(1)_\mathrm{Y}$ generator, *etc.*. [5 marks]

(b) Then show that the $A(3,3,3)$, $A(3,3,1)$, $A(2,2,2)$, $A(2,2,1)$ and $A(1,1,1)$ anomaly coefficients all vanish, where $A(3,3,3)$ stands for the set of anomaly coefficients $A_{\vartheta\varphi\varkappa}$ between three $\mathrm{SU}(3)$ generators, *etc.* [15 marks]

It may be helpful to note that the fundamental representation $\mathrm{SU}(3)$ generators are $t_\vartheta = \frac{1}{2}\lambda_\vartheta$ where the λ_ϑ, $\vartheta = 1, \ldots, 8$ are the Gell-Mann matrices, which satisfy $\{\lambda_\vartheta, \lambda_\varphi\} = \frac{4}{3}\delta_{\vartheta\varphi} + 2d_{\vartheta\varphi\varkappa}\lambda_\varkappa$. The $\mathrm{U}(1)_\mathrm{Y}$ hypercharges for the left-handed spinor fields are $-\frac{1}{2}$ for the leptons and $\frac{1}{6}$ for the quarks, while for the right-handed spinor fields one has -1 for the leptons, $\frac{2}{3}$ for the up quarks and $-\frac{1}{3}$ for the down quarks.

Note also the following for representations of non-simple groups. When considering representations of a product group such as the six-dimensional $(\mathbf{3}, \mathbf{2})$ representation of $\mathrm{SU}(3) \times \mathrm{SU}(2)$, generators of simple subgroup factors such as $T_\vartheta^{(3)}$ for $\mathrm{SU}(3)$ will have a product structure $T_\vartheta^{(3)} = t_\vartheta \otimes \mathbb{1}_2$, where the t_ϑ are the ordinary 3×3 matrix fundamental representation generators of $\mathrm{SU}(3)$ as above and $\mathbb{1}_2$ is a 2×2 unit matrix acting on the $\mathrm{SU}(2)$ indices; similarly the $T_i^{(2)}$ generators of $\mathrm{SU}(2)$ will have a product structure $T_i^{(2)} = \mathbb{1}_3 \otimes t_i$, where the t_i are the ordinary 2×2 matrix fundamental representation generators of $\mathrm{SU}(2)$ and $\mathbb{1}_3$ is a 3×3 unit matrix acting on the $\mathrm{SU}(3)$ indices

[Total 50 marks]

End of examination paper