Imperial College London

MSc EXAMINATION May 2019

This paper is also taken for the relevant Examination for the Associateship

THE STANDARD MODEL AND BEYOND

For Students in Quantum Fields and Fundamental Forces

Wednesday, May 29$^{\text{th}}$ 2019: 14:30 to 17:30

Answer TWO out of the following THREE questions.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

Complete the front cover of each of the TWO answer books provided.

If an electronic calculator is used, write its serial number at the top of the front cover of each answer book.

USE ONE ANSWER BOOK FOR EACH QUESTION.

Enter the number of each question attempted in the box on the front cover of its corresponding answer book.

Hand in TWO answer books even if they have not all been used.

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

Go to the next page for questions

1. (i) In the form developed by Feynman and Gell-Mann, the Fermi theory of the weak interactions had a basic 4-fermion interaction of the form $\frac{1}{\sqrt{2}}G_F J_\rho J^\rho$, where G_F is the Fermi constant and the leptonic part ℓ_ρ of the current J_ρ is of the form $\ell_\rho = \frac{1}{2}\bar{\psi}_{(e)}\gamma_\rho(1+\gamma^5)\psi_{\nu_{(e)}} + \frac{1}{2}\bar{\psi}_{(\mu)}\gamma_\rho(1+\gamma^5)\psi_{\nu_{(\mu)}}$, where $\psi_{(e)}$ and $\psi_{(\mu)}$ are the electron and muon spinor fields and $\psi_{\nu(e)}$ and $\psi_{\nu(\mu)}$ are the corresponding neutrino fields. (Note that (μ) and (ν) here denote particle types.)

(a) The Fermi theory of the weak interactions describes well such low-energy phenomena as neutron decay. Explain why it is nonetheless unacceptable as a theory at high energies. Derive the dimensions of the Fermi spinor fields ψ and of the Fermi constant G_F in units of mass.

(b) Explain how the electroweak Standard Model as developed by Salam and Weinberg resolves the problems of the Fermi theory. [10 marks]

(ii) Show that a general complex but nonsingular (*i.e.* having a nonvanishing determinant) matrix $\mathbf{M}$ can be made to be purely diagonal with positive real entries by use of a biunitary transformation of the form $\mathbf{M} \to \mathbf{V}^\dagger\mathbf{M}\mathbf{U}$ where $\mathbf{V}$ and $\mathbf{U}$ are unitary matrices. One approach is as follows.

(a) Explain why $\mathbf{M}^\dagger\mathbf{M}$ is straightforward to diagonalise by a unitary transformation $\mathbf{U}$. Show that the resulting diagonal entries are real and positive. Call this diagonal matrix $\mathbf{D}^2$, *i.e.* let $\mathbf{U}^\dagger\mathbf{M}^\dagger\mathbf{M}\mathbf{U} = \mathbf{D}^2$, where the entries of $\mathbf{D}$ are the positive square roots of the corresponding entries in $\mathbf{D}^2$.

(b) Let $\mathbf{H} = \mathbf{U}\mathbf{D}\mathbf{U}^\dagger$ and let $\tilde{\mathbf{U}} = \mathbf{M}\mathbf{H}^{-1}$. Show that $\mathbf{V} = \tilde{\mathbf{U}}\mathbf{U}$ is unitary and that $\mathbf{V}^\dagger\mathbf{M}\mathbf{U} = \mathbf{D}$. [20 marks]

(iii) The Yukawa couplings between the Higgs field and the Standard Model fermions are of the form

$$\mathcal{L}_{\text{Yukawa}} = -\mathrm{i}(f_{mn}\overline{L_m}e_{Rn}\phi + h_{mn}\overline{Q_{Lm}}d'_{Rn}\phi + k_{mn}\overline{Q_{Lm}}u'_{Rn}\tilde{\phi}) + \text{hermitian conj.},$$

where the vacuum values of ϕ and $\tilde{\phi}$ (where $\tilde{\phi}_a = \epsilon_{ab}(\phi^*)^b$) are $\phi^T_{\text{vac}} = (0, \frac{1}{\sqrt{2}}v)$ and $\tilde{\phi}^T_{\text{vac}} = \frac{1}{\sqrt{2}}(v, 0)$ (in which v is a constant and the T notation denotes a transpose), $L^T_m = (\nu_m, e_{Lm})$ are the left-handed lepton doublets and $Q^T_{Lm} = (u'_{Lm}, d'_{Lm})$ are the left-handed quark doublets in generations $m = 1, 2, 3$. The right-handed electrons e_{Rm} and the right-handed quarks u'_{Rm} and d'_{Rm} are SU(2) singlets. The coefficients f_{mn}, h_{mn} and k_{mn} describe the mixing between generations.

(a) Explain why bare mass terms are not allowed in the $\mathrm{SU}(3)\times\mathrm{SU_L}(2)\times\mathrm{U_Y}(1)$ gauge-invariant action (for a theory without neutrino masses). Explain how the Yukawa couplings nonetheless give rise to mass terms for the fermions after shifts in fields occasioned by symmetry breaking.

(b) Explain how biunitary transformations $U^{\text{field-type}}_{mn}$ such as those of part (ii) may be used to diagonalise the mass terms for the fermions in the m, n indices without disturbing the diagonal structure of the fermion kinetic terms (*i.e.* terms of the form $\overline{(\text{spinor})}\gamma^\mu\partial_\mu(\text{same spinor})$).

(c) Show why Lagrangian coupling terms involving the neutral A_μ and Z_μ vector fields remain invariant under such unitary transformations. Explain why Lagrangian terms coupling the *charged* vector fields $W^\pm_\mu$ to the Standard Model quarks become mixed between the $m = 1, 2, 3$ generations and find the form of the Cabbibo-Kobayashi-Maskawa (CKM) matrix V_{mn} as a result of the biunitary basis transformations of part (ii), since one cannot simultaneously diagonalise in the m indices both the fermion mass terms and the charged-vector-field couplings to the quarks. Why does one not need to introduce such a generation-mixing matrix for electrons and neutrinos couplings to $W^\pm_\mu$ in the original minimal Standard Model without right-handed neutrinos? [20 marks]

[Total 50 marks]

2. (i) In a seesaw mechanism model for two Majorana spinors, N and n, with Lagrangian mass terms

$$\mathcal{L}_{\text{mass}} = -i(M\bar{N}N + \frac{\mu}{2}(\bar{n}N + \bar{N}n)) ,$$

suppose that $M \gg \mu$ and show that one finds a state with a large mass eigenvalue $m_+ \simeq M$ and a state with a small mass eigenvalue $m_- \simeq -\frac{\mu^2}{4M}$ with magnitude $|m_-| \ll \mu$. Find the corresponding low-mass Majorana spinor field n_- as a combination of N and n to leading order in $\frac{\mu}{M}$. [9 marks]

(ii) Show that the negative sign in m_- is not physically meaningful by making a field redefinition of the corresponding spinor $n_- \to \tilde{n}_- = i\gamma_5 n_-$ which a) preserves the Majorana condition for $\tilde{n}_-$ and b) flips the sign of the $-im_-\bar{\tilde{n}}_-\tilde{n}_-$ mass term while leaving invariant the kinetic term $-i\bar{\tilde{n}}_-\gamma^\mu\partial_\mu\tilde{n}_-$. [9 marks]

(iii) In a system with N generations and PMNS unitary matrix U which diagonalises the neutrino mass matrix, the relation between flavour eigenstates $|\nu_\alpha\rangle$ and mass eigenstates $|i\rangle$ is $|\nu_\alpha\rangle = \sum_i U^*_{\alpha i}|i\rangle$. Consider ultrarelativistic neutrinos with momentum $p = |\vec{p}| >> m_i$ for any of the mass eigenvalues m_i, whose energies can accordingly be approximated by $E_i = \sqrt{p_i^2 + m_i^2} \simeq p_i + \frac{m_i^2}{2p_i} \approx E + \frac{m_i^2}{2E}$. For ultrarelativistic neutrinos and $c = 1$, one has a time of flight T to distance travelled L relation $T \approx L$.

(a) Show that the probability for a neutrino originally of flavour α to be later observed with flavour β is to leading order

$$P_{\alpha\to\beta} = |\langle\nu_\beta|\nu_\alpha(T)\rangle|^2 = |\sum_i U^*_{\alpha i}U_{\beta i}e^{-im_i^2L/2E}|^2 \quad .$$

[10 marks]

(b) Show that this probability may be rewritten as

$$P_{\alpha\to\beta} = \delta_{\alpha\beta} - 4\sum_{i>j}\text{Re}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin^2(\frac{\theta_{ij}}{2}) + 2\sum_{i>j}\text{Im}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin(\theta_{ij})$$

where $\Delta m^2_{ij} = m_i^2 - m_j^2$ and $\theta_{ij} = \left(\frac{\Delta m^2_{ij}L}{2E}\right)$. [10 marks]

(c) The CP asymmetry is $A^{\alpha\beta}_{\text{CP}} = P_{\alpha\to\beta} - P_{\bar{\alpha}\to\bar{\beta}} = 4\sum_{i>j}\text{Im}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin(\theta_{ij})$. In terms of the Jarlskog invariant J, determined in the $N = 3$ case by $\text{Im}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}) = -J\sum_{\gamma,k}\epsilon_{\alpha\beta\gamma}\epsilon_{ijk}$, use the identity $\sin(a+b)\sin(a+c)\sin(c+b) = \frac{1}{4}\left(-\sin(2a+2b+2c) + \sin(2a) + \sin(2b) + \sin(2c)\right)$ to show that the CP asymmetry is given by

$$A^{\alpha\beta}_{\text{CP}} = 16J\sum_\gamma \epsilon_{\alpha\beta\gamma}\sin(\frac{\theta_{31}}{2})\sin(\frac{\theta_{32}}{2})\sin(\frac{\theta_{21}}{2}) \ .$$

[12 marks]

Some relations: $\gamma_5^\dagger = \gamma_5$, $\Psi_C = C(\bar{\psi})^T$, $C^2 = \mathbb{1}$, $C\gamma_5^TC = \gamma_5$.

[Total 50 marks]

3. The standard Dynkin-index convention $C_F = \frac{1}{2}$ for the fundamental representation of a Lie group G corresponds to the fundamental-representation trace relation $\mathrm{tr}(T^I T^J) = \frac{1}{2}\delta^{IJ}$. For the group $SU(N)$, the corresponding symmetrized product relation for the fundamental-representation T^I generators is

$$\{T^I, T^J\} = \frac{1}{N}\delta^{IJ}\mathbb{1} + d^{IJK}T^K$$

which also defines the $SU(N)$ anomaly symbol d^{IJK}.

For a general representation R, one defines the anomaly coefficient $A(R)$ by the relation

$$\mathrm{tr}(T^I\{T^J, T^K\}) = \frac{1}{2}A(R)d^{IJK} \ .$$

The anomaly coefficient $A(R)$ is independent of the particular choice of generators T^I, T^J, T^k and it is normalised to one for the fundamental representation: $A(F) = 1$. One can therefore make a simple choice of generators or combination of generators in a given representation R in order to calculate the anomaly coefficient $A(R)$ for that representation and the result will be the same for any choice of three generators in that representation, when multiplied by d^{IJK}.

This problem considers the cancellation of gauge anomalies in the Georgi-Glashow grand-unified model based on the gauge group $SU(5)$. For simplicity, consider just the first generation.

(i) Show that d^{IJK} is totally symmetric in I, J, K. [6 marks]

(ii) Explain which Standard Model (SM) fermions fit into the fundamental **5** representation of $SU(5)$. Use the conventional Standard Model Y hypercharge eigenvalue assignments for the quarks and leptons to write the fundamental representation hypercharge generator Y as a diagonal 5×5 $SU(5)$ matrix. Find the necessary rescaling factor to rescale Y into a generator $\hat{Y}$ which obeys the standard $SU(5)$ normalisation. [10 marks]

(iii) Using the convention that the unbroken electromagnetic charge in the Standard Model is taken to be $Q = T^3 + Y$ where T^3 is the diagonal $i = 3$ generator of the $SU(2)$ weak interaction subgroup, show that in the standard $SU(5)$ normalisation, one obtains a rescaled electromagnetic charge generator $\hat{Q} = \sqrt{\frac{3}{8}}(T^3 + \sqrt{\frac{5}{3}}\hat{Y})$. [6 marks]

(iv) Treating all the left-handed and right-handed Standard Model fermions as left-handed after charge conjugation of the right-handed fields, show how the Standard Model fermions fit precisely into a $\bar{\mathbf{5}} + \mathbf{10}$ representation of $SU(5)$. Take into account the decomposition of the $SU(5)$ **5** into all the relevant Standard Model $SU(3) \times SU(2) \times U(1)$ representations. [10 marks]

(v) In order to calculate the anomaly coefficients for the left-handed $\bar{\mathbf{5}}$ and **10** representations of the $SU(5)$ grand-unified theory, write out $\hat{Q}$ acting on the independent components of each of these representations and then calculate $\mathrm{tr}\,\hat{Q}^3$ for each. Hence show that

$$A(\bar{\mathbf{5}}) + A(\mathbf{10}) = 0$$

so that the combination $\bar{\mathbf{5}}+\mathbf{10}$ of left-handed fermion representations in the $SU(5)$ grand-unified theory is free from gauge-symmetry anomalies. [12 marks]

(vi) Show that the gauge anomaly cancellation persists when one includes a right-handed neutrino in order to generate neutrino masses by a see-saw mechanism. [6 marks]

Note: the quark content of the proton is *uud*; that of the neutron is *udd*.

[Total 50 marks]

End of examination paper