Imperial College London

MSc EXAMINATION May 2020

This paper is also taken for the relevant Examination for the Associateship

THE STANDARD MODEL AND BEYOND

For Students in Quantum Fields and Fundamental Forces

Friday, May 22$^{\text{nd}}$ 2020: 10:00 to 14:00

Answer TWO out of the following THREE questions.
Given that this exam is being carried out open-book, full credit will only be given for properly derived answers, not for simple quotation of known results.
Marks shown on this paper are indicative of those the Examiners anticipate assigning.

General Instructions

QFFF examinations in the Spring 2020 examination session may be taken open-book.

At the top of each page of your answers, write your CID number, module code, question number and page number. Scan and upload your answers to the Turnitin dropboxes as described in the guidance documents in the Blackboard module for this exam. Upload each answer to the dropbox provided for that specific question.

Your uploaded file name should be of the form CID_ModuleCode_QuestionNumber(s).pdf

For each answer you should prepare a coversheet which should be the first page of your scanned answer. The coversheet should contain the following:

- your CID
- module name and code
- the question number
- the number of pages in your answer

You are reminded that Examiners attach great importance to legibility, accuracy and clarity of expression.

1. (i) Derive the electromagnetic charge and weak hypercharge assignments of the u_L, u_R, d_L and d_R quarks. In doing this, you may use the following information. The unbroken electromagnetic $U(1)_{em}$ generator in the Standard Model is conventionally taken to be $Q_{\text{em}} = T_L^3 + Y$, where $T_L^3 = \frac{1}{2}\sigma^3$ and Y is weak hypercharge, *i.e.* the generator of the $U(1)_Y$ factor in the $SU(2)_L \times U(1)_Y$ gauge group. The left-handed spinorial (u_L, d_L) quarks transform as a doublet under $SU(2)_L$, while the right-handed spinorial versions of these quarks are $SU(2)_L$ singlets. As for electromagnetic charges, note that the left-handed component of the proton $(uud)_L$ has electromagnetic charge $+1$ while the left-handed component of the neutron $(udd)_L$ has electromagnetic charge zero. [15 marks]

(ii) The Standard Model gauge group $SU(3) \times SU(2)_L \times U(1)_Y$ fits nicely as a subgroup into the Georgi-Glashow candidate Grand Unification gauge group $SU(5)$.

(a) How does the fundamental **5** representation ψ_A $(A = 1, 2, 3, 4, 5)$ of $SU(5)$ decompose into (R_3, R_2) representations of $SU(3) \times SU(2)$ where R_3 and R_2 are representations of the respective simple groups?

(b) Write the $\mathbf{5}^T$ representation of $SU(5)$ as a row $(f_1, f_2, f_3, h_1, h_2)$. Given that the Standard Model gauge group is the direct product $SU(3) \times SU(2)_L \times U(1)_Y$, explain why the $f_{\tilde{a}}$ $(\tilde{a} = 1, 2, 3)$ must correspond to the same $U(1)_Y$ hypercharge y_f and the h_b $(b = 1, 2)$ correspond to the same hypercharge y_h. From the requirement that the hypercharge Y generator must be chosen from among the $SU(5)$ generators, derive a linear relation involving y_f and y_h.

(c) Given the value for the hypercharge assignment for the d_R quarks found in part (i) above, find the corresponding values of y_f and y_h for the $SU(5)$ representation that contains the d_R quarks.

(d) Show that the h_b part of the **5** representation containing the d_R quarks can only be $(\tilde{L}_C)_b = \epsilon_{bd}(L_C)^d$, where $(L_C)^b$ is the charge conjugate of the Standard Model $L_b = (\nu_L, e_L)$ doublet (suppressing Lorentz spinor indices) containing the left-handed neutrino and electron components.

(e) The conventional normalisation of the non-abelian $SU(5)$ generators T^i is such that $\text{Tr}(T^i T^j) = \frac{1}{2}\delta^{ij}$. Find the rescaled $\tilde{Y}$ that satisfies this condition. [15 marks]

(iii) Show how the charge-conjugated right-handed electron singlet $(e_R)_C$, the charge-conjugated $SU(3)$ triplet of right-handed $(u_R)_C$ quarks and the ($SU(3)$ triplet, $SU(2)$ doublet) of left-handed $(Q_L)_{\tilde{a}b}$ quarks (again suppressing Lorentz spinor indices) fit into a two-index $SU(5)$ tensor representation χ_{AB}. Find the symmetry of χ_{AB} needed to agree with the Standard Model content. Show that the Y hypercharge values for the components of this two-index representation agree with those required in the Standard Model. [10 marks]

(iv) Explain how a Higgs mechanism involving scalar fields carrying two different representations of $SU(5)$ can account for spontaneous symmetry breaking of the $SU(5)$ gauge symmetry: first down to $SU(3) \times SU(2)_L \times U(1)_Y$ and then down to $SU(3) \times U(1)_{EM}$. [5 marks]

(v) Explain why, despite the attractive representation composition of the $SU(5)$ Grand Unified model, this theory is not acceptable as a physical theory. [5 marks]

[Total 50 marks]

2. The standard Dynkin-index convention $C_F = \frac{1}{2}$ for the fundamental representation of a Lie group G corresponds to the fundamental-representation trace relation $\mathrm{tr}(T^I T^J) = \frac{1}{2}\delta^{IJ}$. For the group $SU(N)$, the corresponding symmetrised product relation for the fundamental-representation T^I generators is

$$\{T^I, T^J\} = \frac{1}{N}\delta^{IJ}\mathbb{1} + d^{IJK}T^K$$

which also defines the $SU(N)$ anomaly symbol d^{IJK}. For a given representation R, one defines the anomaly coefficient $A(R)$ by the relation

$$\mathrm{tr}(T^I\{T^J, T^K\}) = \frac{1}{2}A(R)d^{IJK} \ ,$$

where the trace is over all fermion fields carrying the representation R. The anomaly coefficient is normalised to unity for the fundamental representation.

In computing anomalies in a given theory, chiral fields are all taken to be left-handed. Accordingly, fields which are intrinsically right-handed are included in the above trace through their left-handed charge conjugates, with appropriately conjugated group representations R^*.

(i) Explain why anomalies can give important physical information about the rigid symmetries of effective theories developed after integrating out higher energy/mass scale physics, but that they are inadmissible for gauge symmetries in a fundamental theory. [6 marks]

(ii) Show that d^{IJK} is totally symmetric in I, J, K. [3 marks]

(iii) Define a *pseudoreal* representation (not necessarily irreducible) of a compact Lie group and show that the anomaly coefficient for such a representation must vanish. Hence show that the anomaly coefficient vanishes for a symmetry group representation carried by both a left-handed and a right-handed fermion field, *i.e.* for a left-right symmetric representation. [6 marks]

(iv) Show that the anomaly coefficients $A(3,3,3)$ (denoting all anomaly coefficients between three $SU(3)$ generators, etc), $A(3,3,Q_{\mathcal{EM}})$, $A(3,Q_{\mathcal{EM}},Q_{\mathcal{EM}})$ and $A(Q_{\mathcal{EM}},Q_{\mathcal{EM}},Q_{\mathcal{EM}})$ involving the unbroken $SU(3)$ and $U(1)_{\mathcal{EM}}$ *gauge* symmetry generators all cancel in the Standard Model. [10 marks]

(v) The effective-theory chiral symmetry model of pions is based on an $SU(2)$ representation $U = \exp\left(\frac{i}{F_\pi}\begin{pmatrix}\pi^0 & \sqrt{2}\pi^+ \\ \sqrt{2}\pi^- & -\pi^0\end{pmatrix}\right)$, where $\pi^- = (\pi^+)^*$, transforming under *rigid* $SU(2)_L \times SU(2)_R$ as $U \to g_L U g_R^\dagger$ where the nonlinearly-realised axial symmetries have $g_R = g_L^\dagger$. F_π is the pion decay constant. This pion effective-theory symmetry is partially broken by electromagnetic coupling in the chiral Lagrangian $\mathcal{L}_{\mathrm{chiral}} = -\frac{F_\pi^2}{4}\,\mathrm{tr}\left[(D^\mu U)(D_\mu U)^\dagger\right]$ where $D_\mu = \partial_\mu - iqA_\mu$ is the usual electromagnetic covariant derivative for a field of electromagnetic charge q.

(a) For $g_L = \exp(\frac{i}{2}\omega^k\sigma^k) \in SU(2)_L$, expand the axial $g_R = g_L^\dagger$ symmetry transformation of (π^+, π^-, π^0) to lowest order in the transformation parameters ω^k and show that the axial symmetry transformations of π^+ and π^- are broken by the electromagnetic coupling but the ω^3 transformation of π^0 remains unbroken and is given by $\pi^0 \to \pi^0 + \omega^3 F_\pi$. [9 marks]

(b) At the quark level, for vanishing quark masses and Yukawa interactions and considering just the first generation, show that the axial generator of the rigid $SU(2)_L \times SU(2)_R$ symmetry that is compatible with the electromagnetic coupling is $T_{3A} = \frac{1}{2}\sigma^3\gamma_5$. [8 marks]

(vi) Experimentally, the decay of a π^0 neutral pion into photons is well described by an interaction term

$$\mathcal{L}_{\pi^0 AA} = \frac{e^2}{32\pi^2 F_\pi}\pi^0\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}(A)F^{\rho\sigma}(A) \ .$$

Show that the electromagnetic-coupling compatible symmetry found in part v(a) is violated in this interaction by the same amount as that arising from the anomaly coefficient $A(3A, Q_{\mathcal{EM}}, Q_{\mathcal{EM}})$ for precisely $N_C = 3$ colours, given that the anomalous axial symmetry violation is given by $\delta\mathcal{L}_{SM} = \omega\partial_\mu J^\mu_{3A} = \frac{\omega e^2}{64\pi^2}A(3A, Q_{\mathcal{EM}}, Q_{\mathcal{EM}})\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}(A)F^{\rho\sigma}(A)$. [8 marks]

[Total 50 marks]

3. (i) In a seesaw mechanism model for two Majorana spinors, N and n, with Lagrangian mass terms

$$\mathcal{L}_{\text{mass}} = -i(M\bar{N}N + \frac{\mu}{2}(\bar{n}N + \bar{N}n)) ,$$

suppose that $M \gg \mu$ and show that one finds a state with a large mass eigenvalue $m_+ \simeq M$ and a state with a small mass eigenvalue $m_- \simeq -\frac{\mu^2}{4M}$ with magnitude $|m_-| \ll \mu$. Find the corresponding low-mass Majorana spinor field n_- as a combination of N and n, to leading order in $\frac{\mu}{M}$. [8 marks]

(ii) Show that the negative sign in m_- is not physically meaningful by making a field redefinition of the corresponding low-mass spinor n_- involving γ_5 that preserves the Majorana condition for n_- but flips the sign of the $-im_-\bar{n}_-n_-$ mass term while leaving invariant the kinetic term $-i\bar{n}_-\gamma^\mu\partial_\mu n_-$. [7 marks]

(iii) In a system with N_g generations and PMNS unitary matrix U diagonalising the neutrino mass matrix, the relation between flavour eigenstates $|\nu_\alpha\rangle$ and mass eigenstates $|i\rangle$ is $|\nu_\alpha\rangle = \sum_i U^*_{\alpha i}|i\rangle$. Consider ultrarelativistic neutrinos with momentum $p = |\vec{p}| >> m_i$ for any of the mass eigenvalues m_i, whose energies can accordingly be approximated by $E_i = \sqrt{p_i^2 + m_i^2} \simeq p_i + \frac{m_i^2}{2p_i} \approx E + \frac{m_i^2}{2E}$. For ultrarelativistic neutrinos and $c = 1$, one has a time of flight T to distance travelled L relation $T \approx L$.

(a) Show that the probability for a neutrino originally of flavour α to be later observed with flavour β is to leading order

$$P_{\alpha\to\beta} = |\langle\nu_\beta|\nu_\alpha(T)\rangle|^2 = |\sum_i U^*_{\alpha i}U_{\beta i}e^{-im_i^2L/2E}|^2 \quad .$$ [8 marks]

(b) Show that this probability may be rewritten as

$$P_{\alpha\to\beta} = \delta_{\alpha\beta} - 4\sum_{i>j}\text{Re}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin^2(\frac{\omega_{ij}}{2}) + 2\sum_{i>j}\text{Im}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin(\omega_{ij})$$

where $\Delta m^2_{ij} = m_i^2 - m_j^2$ and $\omega_{ij} = \left(\frac{\Delta m^2_{ij}L}{2E}\right)$. [9 marks]

(c) The CP asymmetry is $A^{\alpha\beta}_{\text{CP}} = P_{\alpha\to\beta} - P_{\bar{\alpha}\to\bar{\beta}} = 4\sum_{i>j}\text{Im}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin(\omega_{ij})$. In terms of the Jarlskog invariant J, determined in the $N = 3$ case by $\text{Im}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}) = -J\sum_{\gamma,k}\epsilon_{\alpha\beta\gamma}\epsilon_{ijk}$, use the identity $\sin(a+b)\sin(a+c)\sin(c+b) = \frac{1}{4}\left(-\sin(2a+2b+2c) + \sin(2a) + \sin(2b) + \sin(2c)\right)$ to show that the CP asymmetry is given by

$$A^{\alpha\beta}_{\text{CP}} = 16J\sum_\gamma \epsilon_{\alpha\beta\gamma}\sin(\frac{\omega_{31}}{2})\sin(\frac{\omega_{32}}{2})\sin(\frac{\omega_{21}}{2}) \ .$$ [9 marks]

(iv) The disappearance of ν_e neutrinos coming from the Sun gives a measurement of $\Delta m^2_{12} = \Delta m^2_\odot \sim 7.5\times10^{-5}\text{eV}^2$ between two neutrino mass eigenstates labelled ν_1 and ν_2. Either ν_1 or ν_2 must have a significant overlap with the ν_e flavour basis neutrino; this is conventionally taken to be ν_1. The third mass eigenstate is ν_3; information about it is obtained from the disappearance of ν_μ and $\bar{\nu}_\mu$ neutrinos from upper atmospheric cosmic-ray showers, giving $\Delta m^2_{23} = \Delta m^2_{\text{atm}} \sim 2.4\times10^{-3}\text{eV}^2$.

Explain the difference between the "normal" and the "inverted" neutrino mass hierarchies given this data pattern. [9 marks]

Some relations: $\gamma_5^\dagger = \gamma_5$, $\Psi_C = C(\bar{\psi})^T$, $C^2 = \mathbb{1}$, $C\gamma_5^TC = \gamma_5$.

[Total 50 marks]