Unification - The Standard Model

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Outline:

- Local gauge symmetries & coupling to scalar matter
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1. Gauge Symmetry : Maxwell Theory & Lorentz invariance

The key example of unification of previously separate theories in modern physics is Maxwell theory. The basic idea of unification is symmetry, which relates the previously separate sectors of the theory.

Maxwell's equations in vacuum:

let $\mu = \epsilon = c = 1$ $\quad (\vec{D} = \epsilon\vec{E}\,,\ \vec{B} = \mu\vec{H})$

$$\boxed{\begin{array}{lll} \vec{\nabla}\cdot\vec{E} = 4\pi\rho & \vec{\nabla}\times\vec{B} - \dfrac{\partial\vec{E}}{\partial t} = 4\pi\vec{J} & \text{Sourced eqns} \\ \vec{\nabla}\cdot\vec{B} = 0 & \vec{\nabla}\times\vec{E} + \dfrac{\partial\vec{B}}{\partial t} = 0 & \text{Sourceless eqns} \end{array}}$$

Although not manifest in this presentation, Maxwell's equations possess Lorentz symmetry, extending the manifest invariance under spatial rotations (the group $SO(3)$, or $O(3)$ if one includes parity transformations).

To make the Lorentz covariance manifest, introduce 4-component notation: $\mu = 0, i\ ;\ i = 1,2,3$

Let $j^0 = 4\pi\rho$ $j^i = 4\pi J^i$ for the sources, making up a 4-vector j^μ.

The electric E^i and magnetic B^i fields combine to form a rank-2 antisymmetric tensor <u>field strength</u> $F^{\mu\nu} = -F^{\nu\mu}$, which therefore has $\frac{4.3}{2} = 6$ independent components. Specifically, let
$$F^{0i} = E^i \quad \text{and} \quad B^1 = F^{23},\ B^2 = F^{31},\ B^3 = F^{12}.$$

One may summarise the magnetic B^i field identification by $B^i = \frac{1}{2}\epsilon^{ijk}F^{jk}$, where repeated indices are summed over the corresponding range (Einstein summation). The factor of $1/2$ corrects for the jk, kj oversumming.

$\epsilon^{123}=1$ totally antisymmetric in 3 indices

With this notation, the <u>sourced</u> Maxwell equations become simply $\partial_\mu F^{\mu\nu} = -j^\nu$ $\quad (\partial_\mu = \frac{\partial}{\partial x^\mu})$.

The <u>sourceless</u> Maxwell equations can be written covariantly if we introduce the <u>dual</u> of the field strength
$$*F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$
where $\epsilon^{0123} = 1$ and $\epsilon^{\mu\nu\rho\sigma}$ is totally antisymmetric in 4 indices (the Levi-Civita tensor). Indices are raised and lowered covariantly using the <u>Minkowski metric</u> $\eta_{\mu\nu}$: $\eta_{00} = -1$, $\eta_{ij} = \delta_{ij}$, $\eta_{0i} = 0$ $\begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

$\eta_{\mu\nu} = \eta_{\nu\mu}$ symmetric

Thus, $F^\rho{}_\sigma = \eta_{\sigma\lambda}F^{\rho\lambda}$, etc. Note that one makes contractions <u>only</u> between "up" and "down" indices.

Armed with this notation, the sourceless Maxwell equations become simply $\partial_\mu {*F^{\mu\nu}} = 0$.

The set of Maxwell equations
$$\partial_\mu F^{\mu\nu} = -j^\nu, \quad \partial_\mu {*F^{\mu\nu}} = 0$$
is now manifestly covariant with respect to Lorentz transformations
$$x^{\mu'} = \Lambda^{\mu'}{}_\nu x^\nu, \quad \text{so} \quad \partial_{\mu'} = (\Lambda^{-1})^\nu{}_{\mu'}\partial_\nu$$

where $\Lambda^\mu{}_\nu$ is a constant 4x4 matrix that preserves, by definition, the Minkowski metric:
$\eta_{\rho\sigma}\Lambda^\rho{}_\mu\Lambda^\sigma{}_\nu = \eta_{\mu\nu}$. Defining $\eta^{\mu\nu}$ by $\eta^{\mu\sigma}\eta_{\sigma\nu} = \delta^\mu{}_\nu$, one has similarly $\eta^{\rho\sigma}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma = \eta^{\mu\nu}$.

These conditions on $\Lambda^\mu{}_\nu$ define the group $O(3,1)$, generalising $O(3)$ to 4-dimensional spacetime. Taking the determinant of the defining condition for $\Lambda^\mu{}_\nu$, and noting that $\det(\eta_{\mu\nu}) = -1$, one finds $(\det\Lambda)^2 = 1$, so $(\det\Lambda) = \pm 1$. If we restrict $\Lambda^\mu{}_\nu$ to the subgroup with $(\det\Lambda) = +1$, we have the group $SO(3,1)$ (proper Lorentz transformations).

Multi-index tensors (like $\eta_{\mu\nu}$) transform with a matrix Λ contracted with each index, according to the basic pattern of x^μ (contravariant indices) or ∂_μ (covariant indices). By definition & by construction, $\eta_{\mu\nu}$ and $\eta^{\mu\nu}$ are thus numerically invariant tensors. (Since we are talking about a flat-space metric, $\eta^{\mu\nu}$ is numerically the same as $\eta_{\mu\nu}$: $\eta^{00} = -1$, etc. This would not be the case for a curved-space metric.)

Under $SO(3,1)$, but not under $O(3,1)$, the Levi-Civita tensor $\epsilon^{\mu\nu\rho\sigma}$ is also numerically invariant. Under parity transformations with $(\det\Lambda) = -1$, $\epsilon^{\mu\nu\rho\sigma}$ flips sign.

Fields transform both on their indices and also in their arguments. Thus, a simple scalar field $\phi(x)$ transforms into itself, $\phi'(x') = \phi(x)$, while a contravariant vector field like $j^\mu(x)$ transforms into

$$j^{\mu'}(x') = \Lambda^{\mu'}{}_\nu\, j^\nu(x)$$

and a contravariant 2nd rank tensor field like $F^{\mu\nu}(x)$ transforms into

$$F^{\mu'\nu'}(x') = \Lambda^{\mu'}{}_\rho\,\Lambda^{\nu'}{}_\sigma\, F^{\rho\sigma}(x)$$

The Maxwell theory action

Symmetry properties of a theory typically are most manifest in the action, whose functional variation yields the field equations. Invariance of the action leads to covariant field equations (covariance: LHS and RHS are not necessarily invariant, but they transform the same way). Now, for Maxwell's theory a curious situation obtains: there is no local construction for an action in terms of the field strength $F^{\mu\nu}$ (or in terms of E^i and B^i). Writing an action requires solving the sourceless field equations $\partial_\mu {}^*F^{\mu\nu} = 0$ by writing

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

Since $[\partial_\mu, \partial_\nu] = \partial_\mu\partial_\nu - \partial_\nu\partial_\mu = 0$ when acting on suitably smooth functions, this construction clearly solves the sourceless Maxwell equations. Locally in a patch around some spacetime point, one can always find a vector potential $A_\mu(x)$ that yields a given $F_{\mu\nu}(x)$. Whether this can be done globally leads one into the subjects of magnetic monopoles and fibre bundles – but we will not delve into those issues here.

Although one may always find a vector potential $A_\mu(x)$ for a given $F_{\mu\nu}(x)$ locally, the assignment is not unique: there is redundancy in the vector potential formulation. One may change $A_\mu(x)$ by a local gauge transformation

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x)$$

with a spacetime-dependent gauge parameter Λ.

Once again, since $[\partial_\mu, \partial_\nu] = 0$ for continuously second-differentiable functions, it follows that $F_{\mu\nu}(x)$ is gauge invariant.

We are now in a position to write the Maxwell theory action:

$$I_{max} = -\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}$$

This is manifestly Lorentz invariant and gauge invariant. For Lorentz invariance, note that d^4x transforms by $\left|\frac{\partial x'}{\partial x}\right| = \det(\Lambda) = 1$ for $SO(3,1)$ transformations, but $\int d^4x$ is invariant even under parity transformations, so one has full $O(3,1)$ invariance. (Compare $\int_{-\infty}^{\infty} dx$ under $x \to -x$).

In varying I_{max} to look for the condition of stationary action, we consider $A_\mu(t,x^i)$ evolving from an initial configuration $A_\mu(t_0,x^i)$ to a final configuration $A_\mu(t_f,x^i)$. We look for the condition that I_{max} be stationary with respect to infinitesimal variations $\delta A_\mu(t,x^i)$ for $t_0<t<t_f$ but we hold the initial and final configurations fixed,

$$\delta A_\mu(t_0,x^i) = 0 = \delta A_\mu(t_f,x^i) .$$

Moreover, we consider only vector potentials $A_\mu(t,x^i)$ and variations $\delta A_\mu(t,x^i)$ that tend to zero as $r=\sqrt{x^ix^i}\to\infty$ at least as fast as $1/r$.

Now vary:

(1) varying each $F_{\mu\nu}$ gives same result

(2) antisymmetry of $F^{\mu\nu}$

$$\delta I_{max} = -\frac{1}{2}\int d^4x\, F^{\mu\nu}(\partial_\mu\delta A_\nu - \partial_\nu\delta A_\mu) \quad (1)$$

$$= -\int d^4x\, F^{\mu\nu}\partial_\mu\delta A_\nu \quad (2)$$

$$= \int d^4x\left(\partial_\mu F^{\mu\nu}\delta A_\nu - \partial_\mu(F^{\mu\nu}\delta A_\nu)\right)$$

$$= \int d^4x\, \partial_\mu F^{\mu\nu}\delta A_\nu - \underbrace{\int d^3x\, F^{0\nu}\delta A_\nu\Big|_{t_0}^{t_f}}_{\mu=0} - \underbrace{\int dt\int_{\partial(\infty)} d^2\Sigma_i\, F^{i\nu}\delta A_\nu}_{\mu=i}$$

vanishes since $\delta A_\nu(t_0,x^i) = \delta A_\nu(t_f,x^i) = 0$

Gauss's law surface integral vanishes by $r\to\infty$ asymptotic conds

Hence $\delta I_{max} = \int d^4x\, \partial_\mu F^{\mu\nu}\delta A_\nu$

Requiring δI_{max} to vanish for arbitrary $\delta A_\nu(t,x^i)$, $t_0<t<t_f$ then implies the remaining "sourced" Maxwell equations

$$\partial_\mu F^{\mu\nu} = 0$$

which, however, do not yet have a source on the RHS since we haven't built a source into the system yet.

<u>Degrees of freedom per spacetime point or momentum-space point</u>

Now we wish to establish just how many degrees of freedom there are in the Maxwell field. We have seen that the vector potential formulation in terms of $A_\mu(x)$ is redundant, so the ostensible 4 d.o.f. is an overcount. In terms of the vector potential, the free-space Maxwell equation is $\Box A_\mu - \partial_\mu \partial^\nu A_\nu = 0$; $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$.

Fourier transform now to momentum space:

$$A_\mu(x) = \int d^4k \, e^{ik\cdot x} a_\mu(k) \qquad k\cdot x = k_\mu x^\mu$$

Noting that $\partial_\nu A_\mu(x) = i\int d^4k \, k_\nu a_\mu(k) e^{ik\cdot x}$, one finds the momentum-space version of the field equation:

$$k^2 a_\mu(k) - k_\mu k^\nu a_\nu(k) = 0 \qquad k^2 = k_\nu k^\nu.$$

Now, transforming also the Maxwell gauge transformation parameter into momentum space, $\Lambda(x) = \int d^4k \, e^{ik\cdot x}\lambda(k)$, the vector-potential gauge transformation becomes

$$\delta a_\mu(k) = i\lambda(k) k_\mu .$$

This motivates decomposing a general momentum-space $a_\mu(k)$ into a "gauge" part parallel to k_μ and a remainder $\tilde{a}_\mu(k)$, about which all we require at this point is that it be <u>linearly</u> independent from k_μ in the sense of ordinary vector-space linear independence, at each point k_ν in momentum space.

So we write $a_\mu(k) = \tilde{\lambda}(k)\, k_\mu + \tilde{a}_\mu(k)$ where k_μ and $\tilde{a}_\mu(k)$ are linearly independent.

Substitute this decomposition into the momentum-space field equation: $k^2 \tilde{a}_\mu(k) - k_\mu k^\nu \tilde{a}_\nu(k) = 0$ — the gauge part "drops out". But now recall that $\tilde{a}_\mu(k)$ was required to be linearly independent from k_μ in the 4-dimensional vector space $\mu = 0,1,2,3$, and this at every point in momentum space for the field $\tilde{a}_\mu(k)$. Consequently, by linear independence, we learn two things:

$$1)\; k^2 \tilde{a}_\mu(k) = 0$$
$$2)\; k^\nu \tilde{a}_\nu(k) = 0$$

photons are massless

Condition 1) implies that the support of the vector field $\tilde{a}_\mu(k)$ (i.e. the domain where $\tilde{a}_\mu(k)$ is non-vanishing) is confined to the momentum-space light-cone subspace where $k^2 = k_\nu k^\nu = 0$.

Condition 2) then implies that "on shell", i.e. when one restricts to the $k^2=0$ subsurface of momentum space, the field $a_\mu(k)$ satisfies $k^\mu a_\mu(k)\big|_{k^2=0} = 0$. Note that this condition follows purely from the Maxwell field equation. We have not yet done any gauge fixing. One calls the condition $k^\mu a_\mu(k) = 0$ "4-transversality".

photons are 4-transverse

Off-shell, i.e. in regions of momentum space where $k^2 \neq 0$, one has $a_\mu(k) = \tilde{\lambda}(k)\, k_\mu$. One doesn't have to, but it is convenient now to <u>choose</u> to impose the Lorenz condition $k^\mu a_\mu(k) = 0 \Rightarrow \tilde{\lambda}(k)\big|_{k^2\neq 0} = 0$. Then the support of the full $a_\mu(k)$ is only on the $k^2=0$ shell, i.e. the light cone.

N.B. Lorenz ≠ Lorentz
Danish ≠ Dutch

So we choose to impose the Lorenz condition, but need to recognise that it is not a complete gauge-fixing condition, because we obtain the condition $k^\mu a_\mu(k) = 0$ on the light-cone "mass shell" automatically from Maxwell's equations. Hence, we still can do some residual gauge-fixing on $a_\mu(k)$ within the $k^2=0$ momentum-space subspace.

light-cone: "mass shell" for massless fields

Accordingly, we may impose a refinement of the Lorenz condition, which is known as radiation gauge. One may impose gauge conditions simply by picking values or conditions for the gauge part $\tilde{\lambda}(k)$ of $a_\mu(k)$.
So pick $\tilde{\lambda}(k)\big|_{k^2=0} = -\frac{\tilde{a}_0(k)}{k_0}$. Then with this refined gauge choice, one finds immediately $a_0(k) = 0$ everywhere in momentum space. Combining this with the 4-transversality condition $k^\mu a_\mu(k) = 0$, one then finds that $a_i(k)$, $i=1,2,3$ satisfies "3-transversality": $k^i a_i(k) = 0$ — the 3-longitudinal part of $a_i(k)$ vanishes.

photons are also 3-transverse

There is a motto that summarizes this situation: "The gauge condition shoots twice". Instead of the ostensible four degrees of freedom per spacetime point from the components of $A_\mu(x)$ or $a_\mu(k)$, there are in fact only two.

Putting this into a conventional framework, specialize to wave solutions propagating along the $x^3 = z$ axis so $\vec{k} = (0,0,k)$ and $k^\mu = (|k|, 0, 0, k)$. The surviving transverse components of $a_\mu(k)$ then involve only $a_1(k)$ and $a_2(k)$. The (a_1, a_2) basis is the linear polarization basis. One also can form the circular polarization basis (a_+, a_-):

$$a_+(k) = a_1(k) - i\,a_2(k)$$
$$a_-(k) = a_1(k) + i\,a_2(k)$$

The complex $a_\pm(k)$ are the <u>helicity</u> ± 1 components of the wave. What we have found here in classical Maxwell theory are the right and left circular polarizations corresponding to helicity ± 1 massless <u>photons</u> in the quantum theory.

<u>Proca Theory</u>

Now, for comparison, consider what happens when one adds a <u>mass term</u> for A_μ to the Maxwell action. This is known as <u>Proca theory</u>

$$I_{Proca} = \int d^4x \left(-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{m^2}{2} A_\mu A^\mu \right)$$

Once again, we have a Lorentz invariant action, but now <u>without gauge invariance</u>. We derive the Proca field equations as before, by varying A_μ. From the "kinetic" term $\int -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$, we get the same result as before, while the "mass" term gives $-m^2 \int d^4x\, A^\mu \delta A_\mu$, i.e.

$$\delta I_{Proca} = \int d^4x \left(\partial_\mu F^{\mu\nu} - m^2 A^\nu \right) \delta A_\nu$$

So the Proca field equation is

$$\partial_\mu F^{\mu\nu} - m^2 A^\nu = 0 .$$

Although we have lost the Maxwell gauge invariance by including the mass term, the number of physical degrees of freedom per spacetime point is still less than the ostensible four. To see this, take the divergence of the field equation, noting that $F^{\mu\nu}$ is antisymmetric, so $\partial_\nu \partial_\mu F^{\mu\nu} \equiv 0$. Thus, one finds $m^2 \partial_\nu A^\nu = 0$: Proca theory is also 4-transverse, even though there is no gauge symmetry. There are $4-1 = 3$ degrees of freedom per spacetime point, in contrast to Maxwell theory's 2 degrees of freedom. In addition to helicities ± 1, Proca theory has helicity 0: these are the 3 helicity states of massive spin-one theory.

Proca theory is also 4-transverse

2. Scalar matter fields and internal symmetry

Now we consider how charged sources may be provided for the Maxwell field. First consider the action for an uncharged scalar field: a single, real, scalar field $\phi(x)$ with a potential energy density $V(\phi)$

$$I_{\substack{\text{real}\\\text{scalar}}} = \int d^4x \left(-\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)\right)$$

This is once again a Lorentz invariant action, so the field equations will be Lorentz covariant. We derive them using the standard initial & final restrictions on $\delta\phi$, $\delta\phi(t_0, x^i) = 0 = \delta\phi(t_f, x^i)$, and the same asymptotic falloff conditions, $\phi(t, x^i)$ and $\delta\phi(t, x^i)$ being required to fall off at least as fast as $1/r$ when $r = \sqrt{x^i x^i} \to \infty$.

Then $\delta I_{\text{real sc.}} = \int d^4x \left(\Box\phi(x) - \frac{\partial V}{\partial\phi(x)}\right)\delta\phi(x)$

So requiring stationarity of $I_{\text{r.sc.}}$ with respect to arbitrary $\delta\phi(t, x^i)$, $t_0 < t < t_f$ yields the scalar field equation

$$\Box\phi(x) - \frac{\partial V}{\partial\phi(x)} = 0 .$$

If one chooses the potential to contain simply a mass term, $V = \frac{1}{2}m^2\phi^2$, then one obtains the Klein-Gordon equation:

first derived by Schrödinger - dissuaded from publication by Pauli

$$(\Box - m^2)\,\phi(x) = 0 .$$

Note that if one transforms to momentum space, $\phi(x) = \int d^4p\, e^{ip\cdot x}\phi(p)$, then one obtains the mass-shell condition $(p^2 + m^2)\phi(p) = 0$, so the support of $\phi(p)$ is only on the mass shell $p^2 = -m^2$ (N.B. $p^2 = -E^2 + \vec{p}\cdot\vec{p}$).

Now add a second scalar field, so one has $\phi_a(x)$, $a = 1, 2$. At this point, the index "a" is just a label to distinguish the two scalar fields.

The action for two scalar fields is

$\partial^\mu = \eta^{\mu\nu}\partial_\nu$

$$I_{\text{scalars}} = \int d^4x \left(-\tfrac{1}{2}\partial^\mu \Phi_a \partial_\mu \Phi_b \delta_{ab} - V(\Phi_1, \Phi_2)\right) \qquad a,b = 1,2$$

This yields the field equations

$$\Box\Phi_1(x) - \frac{\partial V}{\partial \Phi_1(x)} = 0 \qquad \Box\Phi_2(x) - \frac{\partial V}{\partial\Phi_2(x)} = 0 .$$

Now, another way to organise this system is to introduce the complex scalar field $\varphi = \frac{1}{\sqrt{2}}(\Phi_1 + i\Phi_2)$, for which one has the action

$$I_{\substack{\text{complex}\\ \text{scalar}}} = \int d^4x \left(-\partial^\mu\varphi^* \partial_\mu \varphi - \mathcal{U}(\varphi^*, \varphi)\right)$$

Note conventional normalization for complex fields

where the potential $\mathcal{U}(\varphi^*, \varphi)$ must be a real function of the complex field φ and its conjugate φ^*. Relating this to the previous formulation, one has $\mathcal{U}(\varphi^*, \varphi) = V(\Phi_1, \Phi_2)$.

In varying $I_{\text{complex sc.}}$ to obtain the field equations, the long-winded way to do it is to go back to the formulation in terms of Φ_a, $a = 1,2$, vary as above, and then recombine the equations for the real Φ_a fields into a complex equation for the complex field $\varphi(x)$. However, the result of this longer process is the same as one obtains by effectively treating $\delta\varphi(x)$ and $\delta\varphi^*(x)$ as independent fields:

$$\delta I_{c.sc.}/\delta\varphi \Rightarrow \Box\varphi^* - \frac{\partial\mathcal{U}}{\partial\varphi} = 0$$

$$\delta I_{c.sc.}/\delta\varphi^* \Rightarrow \Box\varphi - \frac{\partial\mathcal{U}}{\partial\varphi^*} = 0 .$$

Now let's consider a special kind of potential $\mathcal{U}(\varphi, \varphi^*)$. Suppose $\mathcal{U}(\varphi, \varphi^*) = \tilde{\mathcal{U}}(\varphi^*\varphi)$, i.e. $\mathcal{U}$ depends only on the $(\text{modulus})^2$ of the complex field $\varphi(x)$. In that case, we have for $I_{\text{complex sc.}}$ a Symmetry:

phase transformation

$$\varphi \to \varphi' = e^{-i\alpha}\varphi \qquad \varphi^* \to \varphi^{*\prime} = e^{i\alpha}\varphi^*$$

where the parameter α is a constant: "rigid symmetry".

The invariance of $I_{\text{complex sc.}}$ under this rigid phase transformation leads to covariance of the field equations

$\delta I_{c.sc.}/\delta\varphi \Rightarrow \quad \Box\varphi^* - \dfrac{\partial\tilde{\mathcal{U}}}{\partial(\varphi^*\varphi)}\varphi^* = 0$ transfs with $e^{i\alpha}$

$\delta I_{c.sc.}/\delta\varphi^* \Rightarrow \quad \Box\varphi - \dfrac{\partial\tilde{\mathcal{U}}}{\partial(\varphi^*\varphi)}\varphi = 0$ transfs with $e^{-i\alpha}$

Noether's theorem

Consider the implications of the above rigid symmetry more closely. The action is the spacetime integral of the Lagrangian density $\mathcal{L}(\partial_\mu\varphi, \partial_\nu\varphi^*, \varphi, \varphi^*)$. In terms of $\mathcal{L}$, the field equations (the Euler-Lagrange equations) are

N.B. $\dfrac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}$ is contravariant

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) - \frac{\partial\mathcal{L}}{\partial\varphi} = 0$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)}\right) - \frac{\partial\mathcal{L}}{\partial\varphi^*} = 0 .$$

The statement that $\mathcal{L}$ is invariant under the above rigid phase transformation, which we identify as a $U(1)$ rigid symmetry, may be written explicitly for an infinitesimal transformation $\Delta\varphi(x) = \varphi'(x) - \varphi(x)$ as

$$\Delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\varphi}\Delta\varphi + \frac{\partial\mathcal{L}}{\partial\varphi^*}\Delta\varphi^* + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Delta(\partial_\mu\varphi) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)}\Delta(\partial_\mu\varphi^*) = 0,$$

where for an infinitesimal transformation with phase $e^{-i\delta\alpha} = 1 - i\delta\alpha + \cdots$, one has $\Delta\varphi = \varphi' - \varphi = -i\delta\alpha\,\varphi$.

Now, since the $U(1)$ phase transformation is rigid, i.e. since the transformation parameter $\delta\alpha$ is constant in spacetime, one has $\Delta(\partial_\mu\varphi) = \partial_\mu\Delta\varphi$ and $\Delta(\partial_\mu\varphi^*) = \partial_\mu\Delta\varphi^*$.

One can use the field equations now to rewrite

$$\frac{\partial\mathcal{L}}{\partial\varphi} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right); \quad \frac{\partial\mathcal{L}}{\partial\varphi^*} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi^*)}\right).$$

Using these to rewrite the invariance statement for $\mathcal{L}$, one has

$$\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \Delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^*)} \Delta\varphi^* \right] = 0 ,$$

which prompts one to make this definition of a current:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \frac{\Delta\varphi}{\delta\alpha} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^*)} \frac{\Delta\varphi^*}{\delta\alpha} .$$

Noether's theorem is the statement that, associated to the rigid symmetry of $\mathcal{L}$, there exists a conserved current J^μ,

$$\partial_\mu J^\mu = 0 \qquad \text{conserved by virtue of the field equations}$$

The current J^μ is called the Noether current. For the $U(1)$ symmetry of our example, one has $\Delta\varphi = -i\delta\alpha\,\varphi$, $\Delta\varphi^* = i\delta\alpha\,\varphi^*$ so the $U(1)$ Noether current is

$$J^\mu_{U(1)} = -i \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \varphi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^*)} \varphi^* \right)$$

$$= i \left(\varphi \partial^\mu \varphi^* - \partial^\mu \varphi\, \varphi^* \right)$$

$A \overleftrightarrow{\partial_\mu} B = A\partial_\mu B - \partial_\mu A\, B$

$$= i \varphi \overleftrightarrow{\partial^\mu} \varphi^*$$

Check now directly that the field equations imply the conservation of $J^\mu_{U(1)}$:

$$\partial_\mu J^\mu = i \left(\cancel{\partial_\mu \varphi \partial^\mu \varphi^*} + \varphi \Box \varphi^* - \Box\varphi\, \varphi^* - \cancel{\partial^\mu \varphi \partial_\mu \varphi^*} \right)$$

then use the field equations,

$$\partial_\mu J^\mu = i \left(\varphi \frac{\partial U}{\partial \varphi} - \frac{\partial U}{\partial \varphi^*} \varphi^* \right) ; \text{ then}$$

recall $U(\varphi, \varphi^*) = \tilde{U}(\varphi^* \varphi)$, so one has the result

$$\partial_\mu J^\mu = i \left(\varphi \frac{\partial \tilde{U}}{\partial(\varphi^* \varphi)} \varphi^* - \frac{\partial \tilde{U}}{\partial(\varphi^* \varphi)} \varphi \varphi^* \right) = 0 .$$

Conservation (i.e. the vanishing divergence) of the Noether current leads to conservation in time of an integrated quantity, which is the <u>Noether charge</u>:

$$Q = \int d^3x\, J^0 .$$

Check conservation in time:

$$\frac{dQ}{dt} = \int d^3x \frac{\partial J^0}{\partial t}$$

now use conservation of the current: $\partial_0 J^0 = -\partial_i J^i$

so $$\frac{dQ}{dt} = -\int d^3x\, \partial_i J^i = -\int_{\partial(\infty)} d^2\Sigma_i\, J^i$$

and the $\int_{\partial(\infty)}$ surface integral vanishes provided the fields from which the current J^i is constructed fall off fast enough at infinity — the scalar field $\varphi(x)$ should fall off at least as fast as $1/r$ as $r = \sqrt{x^i x^i} \to \infty$.

So, subject to the field equations and the asymptotic fall-off conditions, one obtains conservation in time of the Noether charge, $\frac{dQ}{dt} = 0$.

The "space integral over a time component" structure of the Noether charge does not reveal manifestly the Lorentz covariance properties of the integrated charge Q. In fact, Q is a <u>Lorentz scalar</u>. To see this, write

$\Theta(s)$: Heaviside step function

$$Q = \int d^4x\, \partial_\nu \Theta(\eta_\rho x^\rho) J^\nu \quad \text{where } \Theta(s) = 1\ s \geq 0,\ \Theta(s) = 0\ s < 0$$
$$\text{so } \frac{d\Theta(s)}{ds} = \delta(s)$$

Then start from $x'^\mu = \Lambda^\mu{}_\nu x^\nu$, take $\det(\Lambda^\mu{}_\nu) = 1$, $\partial_{\nu'} = (\Lambda^\nu{}_\mu)^{-1}\partial_\nu = \Lambda_\mu{}^\nu \partial_\nu$

so $Q' = \int d^4x'\, \partial_{\nu'} \Theta(\eta_\rho x^{\rho'}) J^{\nu'}(x')$. Then change integration variables:

$$Q' = \int d^4x\, \Lambda_\nu{}^\beta \partial_\beta \Theta(\eta_\rho \Lambda^\rho{}_\alpha x^\alpha) \Lambda^\nu{}_\sigma J^\sigma(x) = \int d^4x\, \partial_\sigma \Theta(\eta'_\alpha x^\alpha) J^\sigma(x)\, ; \quad \eta'_\alpha = \eta_\rho \Lambda^\rho{}_\alpha .$$

Since J^σ is <u>conserved</u>, we may write $Q' = \int d^4x\, \partial_\sigma\left[\Theta(\eta'_\rho x^\rho) J^\sigma\right]$.

Then $$Q' = \underbrace{\int d^4x\, \partial_\sigma\left[\Theta(\eta_\rho x^\rho) J^\sigma\right]}_{Q} + \underbrace{\int d^4x\, \partial_\sigma\left[\left(\Theta(\eta'_\rho x^\rho) - \Theta(\eta_\rho x^\rho)\right) J^\sigma\right]}_{\text{remainder}}$$

$\Lambda^0{}_0 \geq 1$ orthochronous Lorentz transformation

The remainder surface term vanishes because $J^i \to 0$ as $|x| \to \infty$ at fixed t while $\Theta(\eta'_\rho x^\rho) - \Theta(\eta_\rho x^\rho) = 0$ for $|t| \to \infty$ at fixed x^i, so $Q' = Q$, a Lorentz scalar.

3. Matrix Symmetries

$SO(2) \cong U(1)$

Return to the initial formulation of the two-scalar model as written in terms of two scalar fields. Invariance of the potential means that it is a function of the invariant $\phi_a \phi_b \delta_{ab}$, so the action is

$$I_{\text{scalars}} = \int d^4x \left(-\tfrac{1}{2}\partial^\mu \phi_a \, \partial_\mu \phi_b \, \delta_{ab} - \mathcal{U}(\phi_c \phi_d \delta_{cd})\right).$$

The $U(1)$ transformations that we have been considering now take the $SO(2)$ form

$$\phi_a \to \phi_a' = O_{ab}\,\phi_b$$

where $O_{ab} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ is an orthogonal 2×2 matrix $OO^T = O^T O = \mathbb{1}$

θ is a real, continuous parameter $0 \le \theta < 2\pi$.

Now restrict attention to infinitesimal transformations $\theta \to \delta\theta \ll 1$, for which

factor of i conventional to make T Hermitian

$$O_{ab} \simeq \begin{pmatrix} 1 & \delta\theta \\ -\delta\theta & 1 \end{pmatrix} + \text{terms of order } (\delta\theta)^2$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\,\delta\theta \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \mathbb{1} + i\,\delta\theta\, T$$

$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ where $T = \sigma_2$ is a Hermitian <u>generator</u>. Noting that $T^2 = \mathbb{1}$, one recognizes that finite $SO(2)$ transformation matrices can be written

$$O = e^{i\theta T}$$

O real & orthogonal $\Leftrightarrow$ T Hermitian & antisymmetric

The group $SO(2) \cong U(1)$ is <u>Abelian</u>, i.e. all the group transformation matrices commute:

$$OO' = e^{i\theta T} e^{i\theta' T} = e^{i(\theta+\theta')T} = e^{i\theta' T} e^{i\theta T} = O'O$$

Return now to Noether's theorem:

$$J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \frac{\Delta\phi_a}{\delta\theta} \quad \text{and} \quad \frac{\Delta\phi_a}{\delta\theta} = i\,T_{ab}\,\phi_b$$

$$\text{so } J^\mu = i\,\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} T_{ab}\,\phi_b = \phi_1 \partial^\mu \phi_2 - \partial^\mu \phi_1\, \phi_2 = \phi_1 \overleftrightarrow{\partial^\mu} \phi_2$$

Check once more conservation using the f. eq. $\Box\phi_a - 2\frac{\partial U}{\partial(\phi^2)}\phi_a = 0$

$$\partial_\mu J^\mu = \phi_1 \Box \phi_2 - \Box\phi_1 \phi_2 = 2\frac{\partial U}{\partial(\phi^2)}(\phi_1\phi_2 - \phi_1\phi_2) = 0$$

where $\phi^2 = \phi_a \phi_b \delta_{ab}$.

General real symmetry groups

Now suppose we have a set of d real scalar fields which transform in a d-dimensional real representation of a symmetry group G:

$$\phi_a(x) \to \phi_a'(x) = O_{ab}\phi_b(x) = \left(e^{i\theta^k T^k}\right)_{ab}\phi_b(x)$$

where $(T^k)_{ab}$ are Hermitian generators of the Lie algebra for the group G. $k = 1, 2, \ldots, \dim G$ while $a = 1, 2, \ldots, d$. The parameters θ^k are a set of $\dim G$ constants, continuously chosen from some appropriate ranges of real numbers (e.g. $[0, 2\pi)$).

For infinitesimal transformations, $\theta^k \to \delta\theta^k \ll 1$ and $\left(e^{i\delta\theta^k T^k}\right)_{ab} \simeq \delta_{ab} + i\,\delta\theta^k (T^k)_{ab} + \mathcal{O}((\delta\theta^k)^2)$

So $\Delta\phi_a(x) = \phi_a'(x) - \phi_a(x) = i\,\delta\theta^k (T^k)_{ab}\phi_b(x)$.

Consequently, for a scalar field theory symmetric under the group G, one will have not just one, but $\dim G$ components of the Noether current:

$$\mathcal{J}^{k\mu} = i\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi_a)}(T^k)_{ab}\phi_b, \quad k = 1, \ldots, \dim G$$

Specialize now to the group $SO(d)$, whose defining, or fundamental, representation consists of $d\times d$ real & orthogonal matrices with unit determinant. Consider the orthogonality conditions $OO^T = O^TO = \mathbb{1}$ in the case of infinitesimal transformations $O_{ab} = \delta_{ab} + i\delta\theta^k (T^k)_{ab}$:

$$O_{ab}O^T_{bc} = O_{ab}O_{cb} = (\delta_{ab} + i\delta\theta^k(T^k)_{ab})(\delta_{cb} + i\delta\theta^\ell (T^\ell)_{cb})$$
$$= \delta_{ac} + i\delta\theta^k (T^k)_{ac} + i\,\delta\theta^k (T^k)_{ca} + \mathcal{O}(\delta\theta^2)$$

requiring this to equal δ_{ac}, we find, since the $\delta\theta^k$ are arbitrary,

$(T^k)_{ac} = -(T^k)_{ca}$: the $SO(d)$ generators are antisymmetric.

Correspondingly, the dimension of the group SO(d) (i.e. the number of independent continuously choosable parameters Θ^k) is $\frac{d(d-1)}{2}$: the number of independent components of a real antisymmetric $d\times d$ matrix.

• A basic example is SO(3), for which the group dimension $\frac{3\cdot 2}{2}=3$ accidentally happens to equal the dimension of the defining fundamental ("vector") representation. One can work out explicitly the three Hermitian generators by considering infinitesimal rotations about the x, y and z axes in ordinary 3 dimensional space:

$$T^1=\begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}\quad T^2=\begin{pmatrix}0&0&i\\0&0&0\\-i&0&0\end{pmatrix}\quad T^3=\begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix}$$

<u>The Lie algebra</u>

The structure of a Lie group G is determined, in the vicinity of the identity element (which for a matrix group is just the corresponding unit matrix $\mathbb{1}$), by the <u>commutation relations</u> of the generators T^k:

$$[T^i,T^j]=if^{ijk}T^k$$

$[A,B]=AB-BA$

where the array of real numbers f^{ijk} $i,j,k=1,\cdots,\dim G$ gives the structure constants of the Lie algebra $\underline{g}$ of the group G. For the semisimple groups with which we shall principally be dealing, the f^{ijk} are totally $[ijk]$ antisymmetric.

The simplest nonabelian example is SO(3), for which we compute the Lie algebra commutation relations directly from the fundamental representation above:

$$[T^i,T^j]=i\epsilon^{ijk}T^k\qquad i,j,k=1,2,3$$

So for SO(3), $f^{ijk}=\epsilon^{ijk}$.

Representations

Any set of matrices T^i, $i=1,\cdots,\dim G$ that satisfy a Lie algebra's commutation relation $[T^i,T^j]=if^{ijk}T^k$ is called a representation of the algebra. Lie algebras and the corresponding Lie groups have infinite numbers of different representations. For a group G, the notion of a representation is a map from group elements $g\in G$ to matrices $M(g)$ that forms a homomorphism. A homomorphism map must preserve the group composition law: if $g_1g_2=g_3$, then one must have $M(g_1)M(g_2)=M(g_3)$, where the composition of matrices is just ordinary matrix multiplication.

There is no requirement that the homomorphism map be 1 to 1. So in particular, every group has a trivial representation where all group elements g are represented by the number 1. For a matrix group like $SO(d)$, the defining, i.e. fundamental, representation is, by definition, 1 to 1, or faithful.

The Lie algebra describes the structure of the group near the identity. Representations of the Lie algebra must respect the commutation relation — this the Lie algebra requirement corresponding to preservation of the composition law in the full group G. Corresponding to the trivial identity representation of the group, one has the trivial Lie algebra representation by a zero matrix. But then there is always an infinite number of non-trivial representations as well.

The adjoint representation

The structure constants f^{ijk} themselves provide a representation of the Lie algebra which is in general different from the defining fundamental representation (but for $SO(3)$, they turn out accidentally to be the same).

• The <u>Jacobi identity</u>: matrix multiplication is associative, $(AB)C = A(BC)$, so one always has the identity

$$[[T^i, T^j], T^k] + [[T^j, T^k], T^i] + [[T^k, T^i], T^j] \equiv 0.$$

Proof: just write out all 12 terms and watch them cancel.

Now insert the basic commutation relation, finding

$$(f^{ijl} f^{lkm} + f^{jkl} f^{lim} + f^{kil} f^{ljm}) T^m = 0.$$

Now, the T^m matrices $m = 1 \cdots, \dim G$ are by construction <u>linearly independent</u>. So the coefficients of the T^m in this equation must separately vanish:

$$f^{ijl} f^{lkm} + f^{jkl} f^{lim} + f^{kil} f^{ljm} = 0.$$

Rewrite this using the antisymmetry of f^{ijk} as

$$f^{ijl} f^{kem} - f^{kjl} f^{iem} + f^{ikl} f^{ljm} = 0.$$

So one can define a particular set of matrices $(T^i)_{ab} = -i f^{iab}$ with, in this special case, i and also $a, b = 1, \cdots, \dim G$. The above relation for the structure constants becomes

$$[T^i, T^k] = i f^{ikl} T^l$$

which is precisely the Lie algebra commutation relation. Hence the matrices $(T^i)_{ab} = -i f^{iab}$ form a representation of the Lie algebra which is of dimension $\dim G$: this is the <u>adjoint representation</u>.

<u>Exponentiating the Lie algebra: connected groups</u>

A <u>connected group</u> is one where every element of the group can be joined to the identity element by a continuous path within the group. For example, $SO(d)$ is connected, but $O(d)$ is not: one cannot reach a group element p with $\det p = -1$ by a continuous path of matrices starting from the identity matrix $\mathbb{1}$.

For connected groups, the full group structure can be obtained by exponentiating the Lie algebra, representing general group elements in the form $e^{i\theta^k T^k}$ $k=1,\dots,\dim G$.
For a connected group, the group multiplication laws can be obtained perturbatively from the Lie algebra:

$$\exp(i\alpha^k T^k)\exp(i\beta^l T^l) = \exp(i\gamma^r T^r)$$

where by expanding and using the Lie algebra commutation relations, one finds

$$\gamma^r = \alpha^r + \beta^r - \frac{1}{2} f^{ijr}\alpha^i\beta^j + \dots$$

Non-connected groups always have a connected subgroup (like the $SO(d)$ subgroup of $O(d)$). The whole group can be reached by multiplying selected disconnected elements together with general connected elements. So, for example, an arbitrary element of $O(3)$ may be obtained by combining $SO(3)$ (connected to the identity $\mathbb{1}$) with elements obtained from a chosen $\det P = -1$ matrix like $P = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ multiplied together with general $SO(3)$ elements.

4. Complex symmetry groups

The important symmetry groups in particle physics are often inherently complex. Particularly important is the $\mathcal{U}(n)$ series, for which the defining representation is formed from unitary $n\times n$ matrices. Let $\mathcal{U}$ be such a matrix, i.e. $\mathcal{U}\mathcal{U}^+ = \mathcal{U}^+\mathcal{U} = \mathbb{1}$, where $\mathcal{U}^+ = (\mathcal{U}^*)^T$.
Taking the determinant of this and using $\det(M) = \det(M^T)$, one finds $|\det(\mathcal{U})|^2 = 1$. Thus, $\det(\mathcal{U}) = e^{i\theta}$. There is a corresponding Abelian subgroup of matrices of the form $e^{i\varphi}\mathbb{1}_{n\times n}$. The corresponding generator for this $\mathcal{U}(1)$ subgroup clearly commutes with all the other $\mathcal{U}(n)$ generators, since diagonal matrices commute with all matrices.

an Abelian ideal

This $U(1)$ subalgebra is thus an invariant Abelian subalgebra. This means that $U(n)$ is not semisimple, since a semisimple algebra is one that has no invariant Abelian subalgebras. However, if one restricts attention to the subgroup $SU(n)$ of $U(n)$ matrices with $\det(U) = 1$, then the $SU(n)$ Lie algebra is semisimple.

Consider the Lie algebra of $U(n)$, letting $U = \mathbb{1} + iq$ where q is an infinitesimal matrix; we will ignore $\mathcal{O}(q^2)$ terms. Then $UU^\dagger = \mathbb{1} + iq - iq^\dagger + \mathcal{O}(q^2, q^{\dagger 2}, qq^\dagger)$; requiring $UU^\dagger = \mathbb{1}$, one learns that $q - q^\dagger = 0$, so $q = q^\dagger$ is <u>Hermitian</u>. Unlike for the orthogonal groups, however, the $U(n)$ generators are more general than the antisymmetric & imaginary $O(d)$ generators: one can now also have real & symmetric $U(n)$ generators. The antisymmetric & imaginary generators belong to the $O(n)$ subgroup of $U(n)$.

Count independent generator matrices:

(imaginary & antisymmetric): $\frac{n(n-1)}{2}$

(real & symmetric): $\frac{n(n+1)}{2}$

Adding these, we find that the dimension of $U(n)$ is n^2. Of these, the generator proportional to the $\mathbb{1}_{n\times n}$ unit matrix, one of the real & symmetric subset, corresponds to the $U(1)$ invariant subalgebra. The remaining <u>traceless</u> generators (all $\frac{n(n-1)}{2}$ imaginary & antisymmetric plus the $\frac{(n+2)(n-1)}{2}$ traceless real & symmetric, so (n^2-1) independent traceless generators in total) correspond to <u>unimodular</u> group elements. To see this, note that $e^{i\,\mathrm{tr}(T)} = \det(e^{iT})$ so $\mathrm{tr}(T) = 0 \Rightarrow \det(e^{iT}) = 1$. Thus, the generators of $SU(n)$ are <u>traceless Hermitian</u> $n\times n$ matrices, (n^2-1) independent.

Raised & lowered indices

For $\mathcal{U}(n)$, a convenient notational convention distinguishes between upper and lower indices, but this distinction is different from that obtaining for Lorentz indices. For $\mathcal{U}(n)$, one wants to distinguish between representations and their complex conjugates.

• The fundamental representation of $\mathcal{U}(n)$ acts on n-dimensional columns $\begin{pmatrix} \psi_1 \\ \psi_2 \\ \vdots \\ \psi_n \end{pmatrix}$. We denote the complex components ψ_a, $a=1\cdots n$.

The Lie algebra generators are written $(T^i)_a{}^b$, where $i = 1, 2, \ldots, \dim(G) = n^2$ and $a, b = 1, 2, \ldots, n$.

Thus, for ψ_a transforming in the fundamental representation of $\mathcal{U}(n)$, one has $\psi_a \to \psi_{a'} = \mathcal{U}_a{}^b \psi_b = \left[\exp(i\theta^k T^k)\right]_a{}^b \psi_b$.

• The conjugate fundamental representation is formed from the complex conjugates of the ψ_a. One may write the set of conjugated components as a row $((\psi_1{}^*), \ldots, (\psi_n{}^*))$ and transform by multiplying into T^i on the left. However, in order to make the notation fit with that of $(T^i)_a{}^b$ and preserve the rule of making contractions only between raised and lowered indices, one should <u>raise</u> the indices of the conjugated components: $(\psi_a)^* = \psi^{*a}$. Then we can write $\psi^{*a}(T^i)_a{}^b$.

— This notation works equally well for the $\mathcal{U}(n)$ group elements themselves: $(\mathcal{U}_a{}^b)^* = \mathcal{U}^{*a}{}_b$, so $[(\mathcal{U}_a{}^b)^*]^T = \mathcal{U}^{\dagger}{}_b{}^a$.

The conjugate fundamental representation then transforms according to
$$\psi^{*a} \to \psi^{*a'} = (\mathcal{U}_a{}^b \psi_b)^* = \mathcal{U}^{*a}{}_b \psi^{*b} = \psi^{*b}(\mathcal{U}^*)^T{}_b{}^a = \psi^{*b}\mathcal{U}^{\dagger}{}_b{}^a,$$
i.e. the conjugate fundamental representation matrices are the $\mathcal{U}^\dagger$, taking the transformation pattern from the ψ^{*a}:
$$\chi^a \to \chi^{a'} = \chi^b \mathcal{U}^{\dagger}{}_b{}^a.$$
Correspondingly, upper – lower contractions yield <u>invariants</u>:
$$\chi^a\psi_a \to \chi^{a'}\psi_{a'} = \chi^a \mathcal{U}^{\dagger}{}_a{}^e \mathcal{U}_e{}^b \psi_b = \chi^a\psi_a \quad \text{since } \mathcal{U}\mathcal{U}^\dagger = \mathbb{1}.$$

Invariant tensors of $\mathcal{U}(n)$ and $SU(n)$

The invariance of upper–lower contractions under $\mathcal{U}(n)$ can also be expressed in terms of invariance of Kronecker deltas with raised & lowered indices:

$$\delta_a{}^b \to \delta_{a'}{}^{b'} = \mathcal{U}_a{}^d\, \delta_d{}^e\, \mathcal{U}^\dagger{}_e{}^b = (\mathcal{U}\mathcal{U}^\dagger)_a{}^b = \delta_a{}^b$$

$$\delta^a{}_b \to \delta^{a'}{}_{b'} = \mathcal{U}_b{}^e\, \delta^d{}_e\, \mathcal{U}^\dagger_d{}^a = (\mathcal{U}\mathcal{U}^\dagger)_b{}^a = \delta_b{}^a \quad (= \delta^a{}_b).$$

For $\mathcal{U}(n)$, these are the only invariant tensors. Note that they do not allow one to raise or lower indices, since

$$\delta_a{}^b \psi_b = \psi_a .$$

- Under $SU(n)$, however, there is another kind of invariant tensor: $\epsilon_{a_1 a_2 \cdots a_n}$ and $\epsilon^{b_1 b_2 \cdots b_n}$, totally antisymmetric in n lowered or n raised indices.

To see why these are $SU(n)$ invariant, just transform each index:

$$\epsilon_{a_1 \cdots a_n} \to \epsilon_{a_1' \cdots a_n'} = \mathcal{U}_{a_1}{}^{b_1} \mathcal{U}_{a_2}{}^{b_2} \cdots \mathcal{U}_{a_n}{}^{b_n} \epsilon_{b_1 b_2 \cdots b_n} = A_{a_1 \cdots a_n}.$$

What can $A_{a_1 \cdots a_n}$ be? The antisymmetry of $\epsilon_{b_1 \cdots b_n}$ implies that $A_{a_1 \cdots a_n}$ is also totally antisymmetric in n indices. These indices take values ranging from 1 to n. The only possibility is for $A_{a_1 \cdots a_n}$ to be proportional to $\epsilon_{a_1 \cdots a_n}$:

$A_{a_1 a_2 \cdots a_n} = D\, \epsilon_{a_1 a_2 \cdots a_n}$. To find out what D is, let $a_1 a_2 \cdots a_n = 1\,2 \cdots n$. Then $D = \mathcal{U}_1{}^{b_1} \mathcal{U}_2{}^{b_2} \cdots \mathcal{U}_n{}^{b_n} \epsilon_{b_1 b_2 \cdots b_n}$, i.e. $D = \det(\mathcal{U})$. So for $SU(n)$, $D = 1$ and one has $\epsilon_{a_1 a_2 \cdots a_n}$ as an invariant tensor.

- Similarly, $\epsilon^{b_1 \cdots b_n}$ is invariant, since $\det(\mathcal{U}) = 1 \Rightarrow \det(\mathcal{U}^\dagger) = 1$.
- Under $\mathcal{U}(n)$, $\epsilon_{a_1 \cdots a_n}$ and $\epsilon^{b_1 \cdots b_n}$ are not invariant, since the $\mathcal{U}(1)$ invariant subgroup transforms these multi-index tensors by $e^{in\varphi}$ or $e^{-in\varphi}$, where $e^{i\varphi}$ is the transformation phase for a single-index V_a.

• For the adjoint representation indices $i = 1, 2, \cdots, \dim(G)$, we shall not need to be careful about raised / lowered index positions — at least for the internal symmetry groups we shall be concerned with. Strictly speaking, one should perhaps write the Lie algebra $[T^i, T^j] = i f^{ij}{}_k T^k$. But then one can construct the Cartan-Killing metric, which when normalized for $SU(n)$ is $g^{ij} = -\frac{1}{n} f^{ik}{}_\ell f^{j\ell}{}_k$. It then turns out that for compact semi simple groups, $g^{ij} = \delta^{ij}$, so raising with g^{ij} has no effect on a tensor's value.

$SU(2)$

Now let's consider in detail the simplest nonabelian complex group, $SU(2)$. The defining / fundamental representation acts on 2-component columns $\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ or doublets ψ_a, $a = 1, 2$. The 2×2 matrices for this representation can be written

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \quad \text{with} \quad \det(U) = 1 \Rightarrow |a|^2 + |b|^2 = 1 .$$

One can check directly that these are unitary. The dimension of the $SU(2)$ group is $2^2 - 1 = 3$, corresponding to the 3 independent real numbers determining a and b subject to the determinental condition $|a|^2 + |b|^2 = 1$. Considering $a = x + iy$ and $b = w + iz$ and embedding (x, y, w, z) in Euclidean $\mathbb{R}^4$, we see what the $SU(2)$ group manifold is: the $|a|^2 + |b|^2 = 1$ condition defines a 3-sphere S^3: $x^2 + y^2 + w^2 + z^2 = 1$.

Another way to describe S^3 is in terms of an S^1 bundle over S^2. Concretely, this can be realized by a parametization (φ, n^1, n^2, n^3) where $\vec{n} \in \mathbb{R}^3$ subject to $(n^1)^2 + (n^2)^2 + (n^3)^2 = 1$, so that (n^1, n^2, n^3) determine a point on S^2.

We let $a = \cos\varphi + i n^3 \sin\varphi$, $b = (n^2 + i n^1)\sin\varphi$ and one can check directly that $|a|^2 + |b|^2 = 1$.

Now consider the SU(2) Lie algebra. Matrices near the identity $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, i.e. $a=1, b=0 \leftrightarrow \varphi = 0$, will be those with $|\varphi| \ll 1$. For such SU(2) elements, one has

$$\mathcal{U} \simeq \mathbb{1} + i\varphi\left(n^1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + n^2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + n^3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right) + \dots$$

Thus, we identify a basis for the SU(2) Lie algebra in terms of the <u>Pauli matrices</u>

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The traditional notation for the SU(2) generators is $T^i = J^i = \frac{1}{2}\sigma^i$, $i = 1, 2, 3$ and the conventional normalization is chosen so that $\mathrm{Tr}(J^i J^k) = \frac{1}{4}\mathrm{Tr}(\sigma^i \sigma^k) = \frac{1}{2}\delta^{ik}$.

Taking commutators, we find the SU(2) Lie algebra:

$$[J^i, J^j] = i\epsilon^{ijk} J^k,$$

and we recognize that this is <u>precisely</u> the same Lie algebra as for SO(3).

— We have found an intimate relationship between SU(2) and SO(3). They share the same Lie algebra, but SU(2) is a "<u>double covering</u>" of SO(3). One can see this in terms of the corresponding group manifolds: SU(2) is a full S^3, while SO(3) is an S^3 with antipodal points identified. (So one could represent SO(3) as an S^3 hemisphere.) For infinitesimal transformations near the group identity, they have the same commutation relations and hence the same Lie algebra.

• Despite having the same Lie algebra, the difference in global structure has an impact on the representations: every SO(3) representation is an SU(2) representation, but the converse is not true.

The 2-component fundamental representation of $SU(2)$ is not an $SO(3)$ representation — the smallest nontrivial representation (i.e. leaving aside the trivial rep. by the unit matrix) of $SO(3)$ is the $SO(3)$ fundamental triplet. Now, as we have seen, $SU(2)$ is a dimension-three group and so its adjoint representation is 3-dimensional. This turns out to be equivalent to the fundamental triplet of $SO(3)$:

$$\text{adjoint of } SU(2) \Longleftrightarrow \text{fundamental of } SO(3)$$

(and, as we have seen, this rep. is also equivalent to the adjoint of $SO(3)$, which is also 3-dimensional).

The richer set of $SU(2)$ representations is sometimes referred to as $SO(3)$ representations "up to a sign". — When applied to rotations in the 3 spatial directions, the factor of $1/2$ in the normalization of the $SU(2)$ generators causes the doublet rep. to return to its original value half as quickly as for the $SO(3)$ triplet. So one requires a 4π rotation instead of a 2π rotation to return a spinor (i.e. a doublet) to its original value, while a 3-vector of $SO(3)$ returns to its original value after just a 2π rotation.

<u>$SU(3)$</u>

$SU(3)$ has $3^2-1=8$ independent Hermitian generators $T^i = \frac{1}{2}\lambda^i$ where

$$\lambda^1 = \begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&0 \end{pmatrix} \quad \lambda^2 = \begin{pmatrix} 0&-i&0\\ i&0&0\\ 0&0&0 \end{pmatrix} \quad \lambda^3 = \begin{pmatrix} 1&0&0\\ 0&-1&0\\ 0&0&0 \end{pmatrix} \quad \lambda^4 = \begin{pmatrix} 0&0&1\\ 0&0&0\\ 1&0&0 \end{pmatrix}$$

$$\lambda^5 = \begin{pmatrix} 0&0&-i\\ 0&0&0\\ i&0&0 \end{pmatrix} \quad \lambda^6 = \begin{pmatrix} 0&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix} \quad \lambda^7 = \begin{pmatrix} 0&0&0\\ 0&0&-i\\ 0&i&0 \end{pmatrix} \quad \lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1&0&0\\ 0&1&0\\ 0&0&-2 \end{pmatrix}$$

are the <u>Gell-Mann matrices</u>, traditionally normalized so that $\underset{i,j=1,\dots,8}{\mathrm{Tr}(\lambda^i\lambda^j) = 2\delta^{ij}}$, similarly to $\underset{i,j=1,2,3}{\mathrm{Tr}(\sigma^i\sigma^j) = 2\delta^{ij}}$ for $SU(2)$.

The structure constants can then be calculated for the Lie algebra in this basis. The nonvanishing structure constants are determined by the total $[ijk]$ $(i=1..8)$ antisymmetry of f^{ijk} and the values

$$f^{123}=1 \quad f^{147}=1/2 \quad f^{156}=-1/2 \quad f^{246}=1/2 \quad f^{257}=1/2$$
$$f^{345}=1/2 \quad f^{367}=-1/2 \quad f^{458}=\sqrt{3}/2 \quad f^{678}=\sqrt{3}/2 .$$

From these, and their variously permuted index values, one can verify that the Cartan-Killing metric is $g^{ij}=-\frac{1}{3}f^{ikl}f^{jlk}=\delta^{ij}$.

Noether's Theorem for Lie group symmetries

Now consider some set of (in general) complex scalar fields Φ_A carrying a generalized index A corresponding to some possibly reducible representation of a Lie group G. This generalized index could correspond, for example, to a multi-index representation like $\varphi_{(ab)}=\frac{1}{2}(\varphi_{ab}+\varphi_{ba})$ (denoted by the Young tableau ☐☐). Or it could correspond to a reducible direct sum of irreducible representations like $(\chi_a, \varphi_{(bc)})$ (in Young tableaux: ☐ $\oplus$ ☐☐). Repeated A indices will denote Einstein summation with the correct combinatoric factors to account for oversummation, e.g.

$\varphi_{(ab)}$ in $SU(3)$: $\frac{3\cdot 4}{2}=6$ dim

$$\Phi_A\Phi^{*A} = \chi_a\chi^{*a}+\frac{1}{2}\varphi_{bc}\varphi^{*bc}, \text{ etc.}$$

Raised indices as on Φ^{*A} denote transformation in the representation conjugate to a lowered index representation like Φ_A.

Now assume ~~that~~ we have an action invariant under G,

$$I=\int d^4x\, \mathcal{L}(\partial_\mu\Phi_A, \partial_\nu\Phi^{*B}, \Phi_C, \Phi^{*D}).$$

For example, this could be

note combinatoric factors

$$I=\int d^4x\left(-\frac{1}{2}\partial^\mu\varphi^{*ab}\partial_\mu\varphi_{ab}-\partial^\mu\chi^{*c}\partial_\mu\chi_c-V(\varphi_{ab},\varphi^{*cd},\chi_e,\chi^{*f})\right)$$

where, e.g. $V=\frac{m}{4}\varphi^{*cd}\varphi_{cd}+\frac{m'}{2}\chi^{*c}\chi_c+\frac{g}{2}(\varphi^{*ab}\chi_a\chi_b+\varphi_{ab}\chi^{*a}\chi^{*b})$

To ensure G invariance, one simply has to follow the rules of the notation and contract only between raised and lowered indices, forming a Lagrangian density $\mathcal{L}$ that is a scalar invariant – both under the Lorentz group and also G.

G transformations for the fields ϕ_A are effected using $\mathcal{U}_A{}^B = [\exp(i\Theta^k T^k)]_A{}^B$ where the generators $(T^i)_A{}^B$, $i=1\dots\dim G$ are the correct ones for the (possibly reducible) representation ϕ_A.

Now consider an infinitesimal transformation $\Theta^k \to \delta\Theta^k$, so

$$\mathcal{U}_A{}^B \simeq \delta_A{}^B + i\,\delta\Theta^k (T^k)_A{}^B + \mathcal{O}(\delta\Theta^2)$$

where $\delta_A{}^B$ is the appropriate identity matrix, e.g.

$$\begin{bmatrix} \delta_{(ab)}{}^{(cd)} & 0 \\ 0 & \delta_e{}^f \end{bmatrix} \quad \text{with} \quad \delta_{(ab)}{}^{(cd)} = \tfrac{1}{2}\left(\delta_a{}^c \delta_b{}^d + \delta_a{}^d \delta_b{}^c\right).$$

The Noether current for this system is

$$J^{i\mu} = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_A)} \frac{\Delta\phi_A}{\delta\Theta^i} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu \phi^{*A})} \frac{\Delta\phi^{*A}}{\delta\Theta^i}$$

where $\Delta\phi_A = i\,\delta\Theta^i (T^i)_A{}^B \phi_B$, $\Delta\phi^{*A} = -i\,\delta\Theta^i \phi^{*B}(T^i)_B{}^A$

so one has

$$J^{i\mu} = i\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)}(T^i)_A{}^B\phi_B - \phi^{*B}(T^i)_B{}^A \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^{*A})}\right)$$

i.e. $$J^i_\mu = i\left(\phi^{*A}(T^i)_A{}^B \partial_\mu\phi_B - \partial_\mu\phi^{*A}(T^i)_A{}^B\phi_B\right)$$
$$= i\,\phi^{*A}(T^i)_A{}^B \overleftrightarrow{\partial_\mu}\phi_B$$

Noether current conservation, $\partial_\mu J^{i\mu}=0$, follows from $\mathcal{L}$ invariance,

$$\Delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_A}\Delta\phi_A + \frac{\partial\mathcal{L}}{\partial\phi^{*A}}\Delta\phi^{*A} + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)}\partial_\mu(\Delta\phi_A) + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^{*A})}\partial_\mu\Delta\phi^{*A} = 0$$

together with the Euler-Lagrange field equations

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_A)}\right) = \frac{\partial\mathcal{L}}{\partial\phi_A}, \quad \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^{*A})}\right) = \frac{\partial\mathcal{L}}{\partial\phi^{*A}}.$$