8. The Higgs Effect

Now investigate what happens when one combines local gauge symmetry with a "matter" scalar sector displaying spontaneous symmetry breaking. We return once more to the simple Abelian $U(1)$ symmetric scalar model, but now coupled to a Maxwell vector field. This is, in fact, the illustrative model discussed by Peter Higgs in his first 1964 paper, following less specific work by Brout and Englert. Take the action

$$I = \int d^4x \left(-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - (\partial_\mu - ie A_\mu)\varphi^* (\partial^\mu + ie A^\mu)\varphi - V(\varphi^*,\varphi)\right)$$

where $V(\varphi^*,\varphi) = -\mu^2 \varphi^*\varphi + \lambda(\varphi^*\varphi)^2 \qquad \mu^2 \;\&\; \lambda > 0$

Comparing to the ungauged (σ,π) $U(1)$ symmetry-breaking model, we have $\varphi = \frac{1}{\sqrt{2}}(\sigma + i\pi)$.

The local $U(1)$ gauge symmetry is

$$\varphi(x) \longrightarrow \varphi'(x) = \exp(-i\theta(x))\varphi(x)$$
$$\varphi^*(x) \longrightarrow \varphi^{*\prime}(x) = \exp(i\theta(x))\varphi^*(x)$$
$$A_\mu(x) \longrightarrow A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\theta(x)$$

As in the ungauged case, the potential $V(\varphi^*,\varphi)$ gives rise to spontaneous symmetry breaking, with the scalar vacuum lying on the circle with $\varphi^*\varphi = \mu^2/2\lambda$.

$\bar{\varphi}$: VeV
φ^*: conjugate

Without loss of generality, we may take the vacuum to lie in the $\sigma \leftrightarrow \mathrm{Re}(\varphi)$ direction, with $\bar{\varphi} = \frac{v}{\sqrt{2}}$, $v = (\mu^2/\lambda)^{1/2}$.

Now we will shift φ to account for the non-zero VeV, but we shall choose to parametrize the shifted field somewhat differently from our ungauged $U(1)$ discussion. Parametrize $\varphi(x)$ exponentially: $\varphi(x) = \frac{1}{\sqrt{2}}\left(\exp(i\,\xi(x)/v)\right)(v + \eta(x))$, $\xi(x)$ and $\eta(x)$ real.
Expand this for small $\xi(x)$ and $\eta(x)$: $\varphi = \frac{1}{\sqrt{2}}(v + \eta + i\xi) + O(\eta^2, \xi^2, \eta\xi)$
So this exponential parametrization starts out with a shift of the real part of $\varphi(x)$ by $v/\sqrt{2}$, but differs from a simple shift in higher orders. The field $\xi(x)$, giving the $U(1)$ phase of $\varphi(x)$, is the analogue of $\pi(x)$ in our earlier discussion. In the absence of the $U(1)$ coupling to A_μ, $\xi(x)$ would be the massless Goldstone boson from the symmetry breaking.

Write out the leading terms of the coupled action in terms of $A_\mu(x)$, $\xi(x)$ and $\eta(x)$. Note that $\varphi^*\varphi = \frac{1}{2}(v+\eta)^2$.

$$I = \int d^4x \left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \frac{1}{2}\partial_\mu\xi\partial^\mu\xi - \frac{1}{2}e^2v^2A_\mu A^\mu - evA_\mu\partial^\mu\xi - \mu^2\eta^2 \right.$$
$$\left. + \text{ cubic \& higher order}\right)$$

$\eta(x)$: Higgs field

Now try to recognize what we have here, for weak field excitations. As before, there is a regular non-tachyonic mode $\eta(x)$, with $(\text{mass})^2$ equal to $2\mu^2$. The $A_\mu\partial^\mu\xi$ term is off-diagonal between the gauge field A_μ and ξ, however, so it's hard to interpret.

Recall at this point that we still have the local $U(1)$ gauge symmetry in the theory. Reparametrizing the complex scalar field $\varphi(x)$ in terms of $\xi(x)$ and $\eta(x)$ changes the expression of the symmetry in terms of the scalar fields, but it is still a valid symmetry of the action and field equations.

field-dependent gauge transformation

So make a gauge transformation with parameter $\theta(x) = \xi(x)/v$. Then since $\varphi(x) \to e^{-i\theta(x)}\varphi(x)$, we have $\varphi'(x) = \exp(-i\xi(x)/v)\varphi$. But for $\varphi(x) = \frac{1}{\sqrt{2}}\exp(i\xi(x)/v)(v+\eta(x))$, we obtain $\varphi'(x) = \frac{1}{\sqrt{2}}(v+\eta(x))$, with the apparent Goldstone field $\xi(x)$ gauged to zero.

At the same time, we have $A'_\mu(x) = A_\mu + \frac{1}{ev}\partial_\mu\xi$.

Rewrite I in terms of the transformed fields, remembering that $F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu$ is a $U(1)$ invariant:

$$I = \int d^4x \left(-\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} - \frac{1}{2}\left[(\partial_\mu - ieA'_\mu)(v+\eta)\right]\left[(\partial^\mu + ieA'^\mu)(v+\eta)\right] + \frac{\mu^2}{2}(v+\eta)^2 - \frac{\lambda}{4}(v+\eta)^4\right)$$

Now expand the transformed action and reorganize:

$$I = \int d^4x \left(-\frac{1}{4}F'_{\mu\nu}F'^{\mu\nu} - \frac{1}{2}e^2v^2A'_\mu A'^\mu - \frac{1}{2}\partial_\mu\eta\partial^\mu\eta - \mu^2\eta^2\right.$$
$$\left. - \frac{1}{2}e^2A'_\mu A'^\mu\eta(2v+\eta) - \lambda v^2\eta^3 - \frac{\lambda}{4}\eta^4 + \frac{\mu^2v^2}{4}\right)$$

The second line contains, aside from the constant $\frac{\mu^2v^2}{4}$, cubic and higher interactions and may be disregarded for weak fields.

Note what has happened in this "unitary" gauge: the Goldstone field $\xi(x)$ has disappeared. In particular, the confusing bilinear term $A_\mu\partial^\mu\xi$ is now absent, so the spectrum can simply be read off from the quadratic terms in the action.

The remaining fields in unitary gauge are the scalar Higgs field $\eta(x)$, with a normal non-tachyonic mass $\sqrt{2\mu^2}$, and a <u>massive</u> vector field $A'_\mu(x)$, with mass ev.

Where has the erstwhile Goldstone field $\xi(x)$ gone? It has been absorbed into the $U(1)$ gauge field $A_\mu(x)$, leading to the appearance of a mass term for this field. This is the <u>Higgs effect</u>: the disappearance of a Goldstone boson after gauge coupling, with the concomitant appearance of a mass for the corresponding local gauge field.

Count the numbers of on-shell degrees of freedom from two perspectives:

- massless vector plus $\eta(x)$ and $\xi(x)$ prior to symmetry breaking:

  $2+1+1=4$ degrees of freedom per spacetime point

- massive vector plus $\eta(x)$ after symmetry breaking:

  $3+1=4$ degrees of freedom per spacetime point

So no degree of freedom has actually been lost. What has happened is that the Goldstone field $\xi(x)$ has been exchanged for the longitudinal mode $A_i \sim k_i$ $i=1,2,3$ of the massive vector field. A traditional joke is that the gauge field eats the Goldstone field, and becomes heavy.

The unitary gauge is defined by the condition $\vec{\xi}(x)=0$, i.e. the vanishing of the Goldstone mode. After imposing this gauge condition, the remnants of the theory naturally do not display any more gauge invariance, and in this simple $U(1)$ model there is no residual unbroken gauge group. One can, of course, consider the model in other gauges and the physical content will be equivalent, as a consequence of the local gauge symmetry. But in other gauges the physical particle spectrum is less apparent.

Higgs Effect in a non-abelian model

Now consider a model with local $SO(3)$ symmetry. Take the scalar fields in this model to transform in the triplet $SO(3)$, i.e. adjoint, representation. For infinitesimal transformations with parameters $\delta\theta^i(x)$, the scalar fields transform as

$$\Delta\phi_j = i\,\delta\theta^i (T^i)^{jk}\phi_k = \delta\theta^i \epsilon_{ijk}\phi_k .$$

Let the scalar matter action be

$$I_{matter} = \int d^4x \left(-\frac{1}{2}\left(\partial_\mu\phi_i + g\epsilon_{ijk}A_\mu^j\phi_k\right)\left(\partial^\mu\phi_i + g\epsilon_{ilm}A^{\mu l}\phi_m\right) - V(\phi_k\phi_k)\right)$$

where $V(\phi)$ is an $SO(3)$ invariant polynomial, a function of the $SO(3)$ invariant $\phi_k\phi_k = \phi^2$. If $V(\phi^2)$ has its minimum at $\phi_i = 0$, then the $SO(3)$ symmetry is unbroken and we have an ordinary local $SO(3)$ gauge-coupled system with a triplet of massless vector fields $A_\mu{}^i$ and a matter sector with whatever masses are determined by $V(\phi^2)$.

Our interest here, however, is the spontaneous symmetry breaking case in which $V(\phi^2)$ has a minimum for non-zero $\bar{\Phi}_i$. As in the ungauged case, one can always make an $SO(3)$ rotation to put the nonzero VeV into the third component $\bar{\Phi}_3$: $\bar{\Phi} = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$. For this VeV, the T^3 generator of $SO(3)$ remains unbroken, but T^1 and T^2 are broken: $T^1\bar{\Phi} \neq 0$ and $T^2\bar{\Phi} \neq 0$ while $T^3\bar{\Phi} = 0$.

To handle this case, we extend our exponential parametrization of the $U(1)$ example to the $SO(3)$ case:

Broken generators only in exp

$$\phi = \exp\left(\frac{i}{v}\left(\xi_1 T^1 + \xi_2 T^2\right)\right)\begin{pmatrix} 0 \\ 0 \\ v+\eta \end{pmatrix}$$

Expanding, find $\phi(x) = \bar{\Phi} + \begin{pmatrix} -\xi_2(x) \\ \xi_1(x) \\ \eta(x) \end{pmatrix}$ + higher order

As in the Abelian $U(1)$ case, we can make a local $SO(3)$ gauge transformation to go into unitary gauge:

$$\phi' = \exp\left(\frac{-i}{v}\left(\xi_1 T^1 + \xi_2 T^2\right)\right)\phi = \begin{pmatrix} 0 \\ 0 \\ v+\eta(x) \end{pmatrix} .$$

We also need to transform the $SO(3)$ gauge field. Recall that $A'_\mu \cdot T = \mathcal{U}(A_\mu \cdot T - \frac{i}{g}\mathcal{U}^{-1}\partial_\mu \mathcal{U})\mathcal{U}^{-1}$ and here we have $\mathcal{U}(\xi) = \exp\left(\frac{-i}{v}(\xi_1 T^1 + \xi_2 T^2)\right)$, so

$$A'_\mu \cdot T = \exp\left(-\tfrac{i}{v}(\xi_1 T^1 + \xi_2 T^2)\right) A_\mu \cdot T \exp\left(\tfrac{i}{v}(\xi_1 T^1 + \xi_2 T^2)\right)$$
$$-\frac{i}{g}\partial_\mu\left(\exp\left(-\tfrac{i}{v}(\xi_1 T^1 + \xi_2 T^2)\right)\right)\exp\left(\tfrac{i}{v}(\xi_1 T^1 + \xi_2 T^2)\right)$$

Writing out then the action in terms of $A'^{\,i}_\mu$ and $\phi' = \begin{pmatrix} 0 \\ 0 \\ v+\eta \end{pmatrix}$, the Goldstone fields $\xi_1(x)$ and $\xi_2(x)$ completely disappear. What remains is a matter sector containing a massive scalar Higgs field $\eta(x)$ and a vector sector. The Higgs $(\text{mass})^2$ is obtained from $\left(\partial^2 V / \partial\phi_3 \partial\phi_3\right)_{|\phi = \bar\phi}$ in the matter action

$$I^{\text{unitary}}_{\text{matter}} = \int d^4x \left(-\tfrac{1}{2}\partial_\mu\eta\,\partial^\mu\eta - V((v+\eta)^2)\right) + \begin{array}{l}\text{vector-field} \\ \text{interactions with} \\ A'^{1}_\mu \text{ and } A'^{2}_\mu\end{array}$$

In the vector-field sector, a mass term appears out of the covariant derivatives acting on ϕ_i: $-\frac{g^2 v^2}{2}\epsilon^{ij3}\epsilon^{il3}A'^{j}_\mu A'^{\mu l}$, where the "3" indices on the ϵ structure constants/$SO(3)$ generators occur because in unitary gauge ϕ_i has only a $\phi_3 = v+\eta$ component. Combining this with the Yang-Mills kinetic term for the $SO(3)$ gauge fields, one obtains the vector-field action

$$I^{\text{unitary}}_{\text{vectors}} = \int d^4x \left(-\tfrac{1}{4}F'^{\,i}_{\mu\nu}F'^{\,i\mu\nu} - \tfrac{1}{2}M^2\left(A'^{2}_\mu A'^{\mu 2} + A'^{1}_\mu A'^{\mu 1}\right)\right)$$

where $M^2 = g^2 v^2$. So indeed, in this non-abelian example we also find mass terms for the vector fields corresponding to the <u>broken</u> symmetry generators. But we now also have an <u>unbroken</u> subgroup $H = SO(2)$ — here generated by T^3, which leaves the $\bar\phi = \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix}$ vacuum invariant. The vector field A'^{3}_μ correspondingly remains <u>massless</u>. Moreover, A'^{3}_μ still acts as a gauge field maintaining the $H = SO(2)$ unbroken symmetry as a <u>local</u> symmetry. The fields that are charged under this $SO(2)$ are A'^{1}_μ and A'^{2}_μ, but not the real $\eta(x)$.

Non-abelian Higgs effect due to Tom Kibble

Now consider the non-abelian Higgs effect in general. Consider an action that is invariant under local gauge transformations of some symmetry group G, so there are $\dim(G)$ Yang-Mills gauge fields A_μ^i $i=1,\dots,\dim(G)$. We also include scalar matter fields. As in our general Goldstone theorem discussion, we will consider the representation carried by the scalar fields in real form, so we take n real scalars ϕ_a, $a=1..n$. The action is then

$$I=\int d^4x\left(-\frac{1}{4}F^i_{\mu\nu}F^{i\mu\nu}-\frac{1}{2}\left[(\partial_\mu-ig_T A_\mu\cdot T)\phi\right]_a\left[(\partial^\mu-ig_T A^\mu\cdot T)\phi\right]_a-V(\phi)\right)$$

where the g_T coupling constants are the same for all T^i generators within any simple subgroup of G, but may differ between simple subgroups, e.g. between the factors of $SU(3)\times SU(2)\times U(1)$.

The potential $V(\phi)$ is assumed to be a polynomial with up to quartic terms in ϕ_a ("up to quartic" limitation comes from renormalizability considerations). Let the potential be minimized for ϕ_a at a vacuum value $\phi_a=\bar\phi_a$, constant in x^μ. Suppose $\bar\phi_a$ is left invariant by a subgroup $H\subset G$: this is the stability subgroup / little group of the system.

Divide the $\dim(G)$ generators T^i into $t^{\hat i}$, $\hat i=1,\dots,\dim(H)$ little group H generators annihilating the vacuum, $t^{\hat i}\bar\phi=0$, plus $(\dim(G)-\dim(H))$ broken generators $k^{\check i}$ that do not annihilate the vacuum, $k^{\check i}\bar\phi\neq 0$. In our discussion of Goldstone's theorem, we saw that the $k^{\check i}\bar\phi$ are independent and span a $(\dim(G)-\dim(H))$ dimensional subspace of the original n dimensional representation space of the ϕ_a fields.

In the absence of gauge field coupling, there would be $(\dim(G)-\dim(H))$ Goldstone bosons corresponding to the broken $k^{\check i}$ generators i.e. $(\dim(G)-\dim(H))$ massless scalars.

Now parametrize ϕ by $\phi=\exp\left(\frac{i}{v}\xi_{\check i}k^{\check i}\right)(\bar\phi+\eta)$, $v^2=\bar\phi_a\bar\phi_a$ where the Higgs fields $\eta_{\tilde a}$ belong to the $n-(\dim(G)-\dim(H))$ complement to the $k^{\check i}\bar\phi$ subspace of the original n-dimensional representation space.

In general, the $\eta_{\hat a}(x)$ Higgs fields will be massive (barring accidental vanishing), with a $(\text{mass})^2$ matrix $\left(\partial^2 V/\partial\phi_{\hat a}\partial\phi_{\hat b}\right)_{\phi=\bar\Phi}$.

To eliminate the Goldstone fields $\xi_{\check i}$, we go to unitary gauge: $\phi \to \phi' = \exp\left(-\frac{i}{v}\xi_{\check i}k^{\check i}\right)\phi = \mathcal{U}(\xi_{\check i})\phi$

$$A_\mu\cdot T \to A'_\mu\cdot T = \mathcal{U}(\xi_{\check i})\left(A_\mu\cdot T - \frac{i}{g}\mathcal{U}^{-1}(\xi_{\check i})\partial_\mu(\mathcal{U}(\xi_{\check i}))\right)\mathcal{U}^{-1}(\xi_{\check i})$$

In the unitary gauge, the action depends only on the $\eta_{\hat a}(x)$ Higgs fields and the vector fields $A'^{\,i}_\mu$, of which $(\dim(G)-\dim(H))$, corresponding to the $k^{\check i}$ broken generators, are now massive. The term in the Lagrangian responsible for the vector field masses is $\frac{1}{2}(g_T T^i\phi, g_T T^j\bar\phi)A'^{\,i}_\mu A'^{\mu j}$, where (v,w) is the inner product in the n-dimensional scalar representation space. Thus, the vector-field $(\text{mass})^2$ matrix is

$$(M^2)^{ij} = -g_T^2(T^i\bar\Phi, T^j\bar\Phi) = -g_T^2(\bar\Phi, T^iT^j\bar\Phi),$$

the second form following since the T^i are Hermitean. We remind again that the g_T must be the same for all T^i within any simple subgroup of the overall symmetry group G, but may differ between simple subgroups. As in the previous examples, the M^2 vector mass matrix structure is taken directly from the covariant derivatives of the scalar fields in the scalar kinetic term, with the scalars then projected to their VeVs.

Since we are considering the scalar matter field representation in real form (doubling the number of complex fields to obtain real fields as necessary), the T^i generators are purely imaginary and antisymmetric. Consequently, as in our Goldstone theorem discussion, $(M^2)^{ij}$ is <u>real and symmetric</u>, just like the matrix S^{ij} in the Goldstone theorem discussion. In fact, S^{ij} is just $(M^2)^{ij}$ without the $(-g_T^2)$ coupling constant factor.

minus sign in $(-g_T^2)$ compensates for imaginary T^i

Restricting $(M^2)^{ij}$ to the $(M^2)^{\check i\check j}$ submatrix corresponding to the broken $k^{\check i}\bar\Phi \neq 0$, one finds a positive definite $(\text{mass})^2$ matrix for $(\dim(G)-\dim(H))$ vector fields corresponding to the broken symmetries.

We see that in a gauge-coupled theory with spontaneous symmetry breaking, the $(\dim(G) - \dim(H))$ Goldstone scalars do not correspond to physical scalar particles, but instead appear as the longitudinal modes of $(\dim(G) - \dim(H))$ massive vector fields. This is seen explicitly from the leading term in the expansion of $A'^{\check{i}}_\mu$:

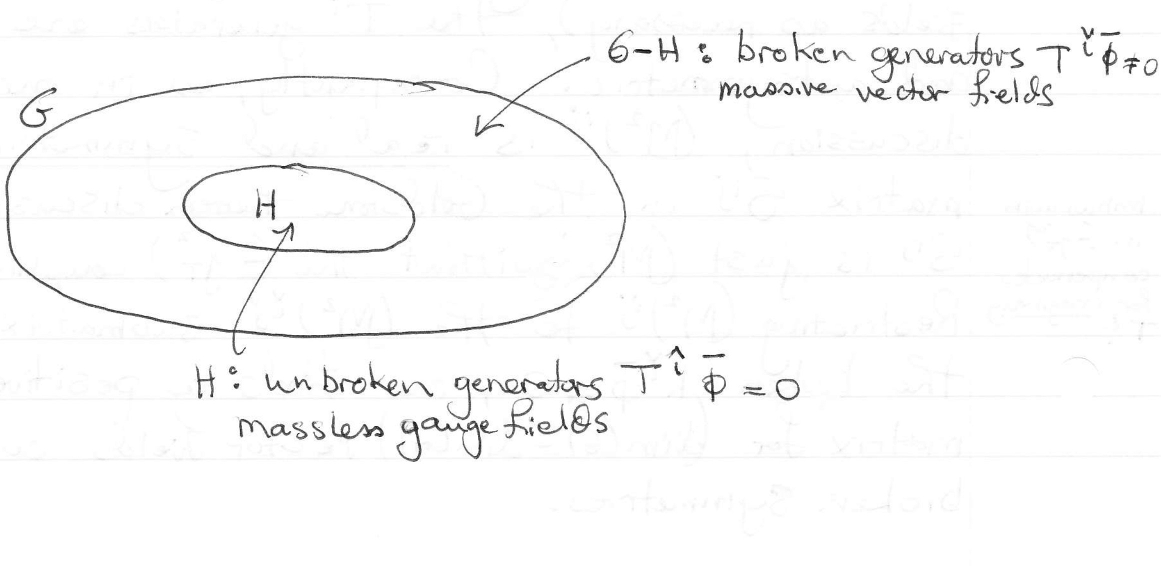
$$A'^{\check{i}}_\mu = A^{\check{i}}_\mu - \frac{1}{g_T v}\partial_\mu \xi_{\check{i}} + O(\xi^2).$$

The unitary gauge is characterized by the condition that the scalar field ϕ' has no components in the subspace spanned by the Goldstone modes, i.e. the subspace spanned by the $T^{\check{i}}\bar{\phi}$. Accordingly, a gauge-fixing condition that determines the unitary gauge is $(T^{\check{i}}\bar{\phi}, \phi'(x)) = 0$. In the quantum theory, one uses this gauge to display the physical spectrum and show that there are no bad negative norm/negative energy modes. It is easier to show renormalizability, however, in Lorenz gauge $\partial_\mu A^{\mu i} = 0$.

$A^{\hat{i}}_\mu$ belong to H

Gauge fields belonging to the unbroken subgroup $H \subset G$ remain <u>massless</u>. They continue to enter into covariant derivatives for the surviving Higgs $\eta_{\hat{a}}$ fields, and also couple to the now massive $A'^{\check{i}}_\mu$ for the broken generators.

We may sketch the general picture for the vector fields as follows:



9. Spinor Fields

P.A.M. Dirac 1928

The Dirac theory of relativistic spin ½ particles is based upon a 4-component complex spinor field $\Psi_{\underset{\sim}{\alpha}}(x) \leftrightarrow \begin{pmatrix} \Psi_1 \\ \Psi_2 \\ \Psi_3 \\ \Psi_4 \end{pmatrix}$, where $\underset{\sim}{\alpha}$ is a new kind of 4-component index; often these spinor indices are suppressed and one just has to remember that one is dealing with spinors $\Psi(x)$ and matrices multiplied into them. The Dirac equation for a massive spin ½ field is $(\gamma^\mu_{\underset{\sim}{\alpha}\underset{\sim}{\beta}}\partial_\mu + m\,\delta_{\underset{\sim}{\alpha}\underset{\sim}{\beta}})\Psi_{\underset{\sim}{\beta}}(x) = 0$ $\quad \underset{\sim}{\alpha} = 1,\dots,4$

or simply $(\not\partial + m)\Psi(x) = 0$, where $\not\partial = \gamma^\mu\partial_\mu$. The Dirac gamma matrices γ^μ (where the written index μ is a regular Lorentz 4-vector) satisfy the Dirac algebra

$\{A,B\} = AB+BA$

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,\mathbb{1}$$

where the γ^μ and $\mathbb{1}$ are 4x4 matrices

The spinor field $\Psi_{\underset{\sim}{\alpha}}(x)$ carries a representation of the group Spin(3,1), which bears a similar relation to the "bosonic" Lorentz group SO(3,1) as SU(2) bears to SO(3); it is referred to as the double cover of SO(3,1). Spin(3,1) has an SU(2) subgroup that is the double cover of the spatial rotation group SO(3); within this "spatial" SU(2), one needs to make a 4π rotation to return a spinor field to its starting value.

A very useful accidental isomorphism of Lie group theory is Spin(3,1) $\cong$ SL(2,$\mathbb{C}$), the group of unimodular 2x2 matrices with complex elements. The dimension of SL(2,$\mathbb{C}$) is $2^2-1=3$ complex dimensions $\leftrightarrow$ 6 real. This isomorphism gives rise to the Weyl representation for the Dirac γ-matrices:

Weyl rep.: often used in Supersymmetry

$$\gamma_\mu = \begin{pmatrix} 0 & i(\sigma_\mu)_{\alpha\dot\beta} \\ i(\tilde\sigma_\mu)^{\dot\gamma\delta} & 0 \end{pmatrix}$$

2x2 blocks of 2x2 submatrices
$\alpha,\delta = 1,2$ undotted
$\dot\beta,\dot\gamma = 1,2$ dotted

in which the σ_μ, $\tilde\sigma_\mu$ are the Van der Waerden matrices:

$(\sigma_\mu)_{\alpha\dot\beta} = (\underset{\mu=0}{\mathbb{1}}, \underset{\mu=i=1,2,3}{\sigma_i})$ where the σ_i are the Pauli matrices:

$(\tilde\sigma_\mu)^{\dot\alpha\beta} = (\mathbb{1}, -\sigma_i)$ $\quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Like the Pauli matrices which they generalize, σ_μ and $\tilde{\sigma}_\mu$ are Hermitean, $\sigma_\mu^{\dagger} = \sigma_\mu$; $\tilde{\sigma}_\mu^{\dagger} = \tilde{\sigma}_\mu$. The "$\mu$" index is a regular Lorentz 4-vector index and can be raised with $\eta^{\mu\nu}$: $\sigma^\mu = \eta^{\mu\nu}\sigma_\nu$.

The undotted α, β indices refer to the complex 2×2 fundamental $SL(2,\mathbb{C})$ representation, acting on 2-component spinors λ_α or 2-component spinor fields $\lambda_\alpha(x)$. The dotted $\dot\alpha, \dot\beta$ indices refer to the complex conjugate of the fundamental $SL(2,\mathbb{C})$ representation, or conjugate fundamental. For $SL(2,\mathbb{C})$, the notation does not involve raising/lowering of indices upon complex conjugation — one adds/removes dots instead. There is still the basic rule for complex representations, however, that index contractions are only made between raised and lowered indices. For $A_\alpha{}^\beta \in SL(2,\mathbb{C})$, a fundamental spinor λ_α and a conjugate fundamental spinor $\bar\lambda_{\dot\alpha}$

$A^\dagger = A^{*T}$ transform as $\lambda_\alpha \to A_\alpha{}^\beta \lambda_\beta$; $\bar\lambda_{\dot\alpha} \to \bar\lambda_{\dot\beta} A^{\dagger\dot\beta}{}_{\dot\alpha}$

There can also be spinor fields with raised indices $\psi^\alpha, \bar\psi^{\dot\alpha}$ transforming contragrediently to the fields with lowered indices,

$$\psi^\alpha \to \psi^\beta (A^{-1})_\beta{}^\alpha \quad ; \quad \bar\psi^{\dot\alpha} \to (A^{-1})^{\dagger\dot\alpha}{}_{\dot\beta}\,\bar\psi^{\dot\beta}.$$

Thus, contractions like $\psi^\alpha\lambda_\alpha$ and $\bar\lambda_{\dot\alpha}\bar\psi^{\dot\alpha}$ are invariant:

$\psi^\alpha\lambda_\alpha \to \psi^\beta (A^{-1})_\beta{}^\alpha A_\alpha{}^\gamma \lambda_\gamma = \psi^\beta\lambda_\beta$, and similarly for $\bar\lambda_{\dot\alpha}\bar\psi^{\dot\alpha}$.

homomorphism is a map that respects the group composition law

The precise relationship between $SL(2,\mathbb{C})$ and $SO(3,1)$ is that there exists a <u>homomorphism</u> $\mu: SL(2,\mathbb{C}) \to SO(3,1)$ given by $\mu: A_\alpha{}^\beta \to \Lambda^\mu{}_\nu$, where $\Lambda^\mu{}_\nu(A) = -\frac{1}{2}\mathrm{Tr}(\tilde\sigma^\mu A \sigma_\nu A^\dagger)$. Note that the index structures of A, $A^\dagger$, σ^μ and $\tilde\sigma^\mu$ are just as are needed to make this construction. This homomorphism map μ is clearly $2\to 1$ since both $A_\alpha{}^\beta$ and $-A_\alpha{}^\beta$, both with determinant 1, map to the same $\Lambda^\mu{}_\nu \in SO(3,1)$. This is the precise meaning of the statement that $SL(2,\mathbb{C})$ is the double cover of $SO(3,1)$.

Using this formalism, one may write vectors of $SO(3,1)$ as <u>bispinors</u>. Define $V_{\alpha\dot\beta} = V^\mu(\sigma_\mu)_{\alpha\dot\beta} = \begin{pmatrix} V^0+V^3 & V^1-iV^2 \\ V^1+iV^2 & V^0-V^3 \end{pmatrix}$.

One can invert this relation using $\tilde{\sigma}^\mu = \eta^{\mu\nu}\tilde{\sigma}_\nu$:

$$V^\mu = -\tfrac{1}{2}\mathrm{Tr}(\tilde{\sigma}^\mu V)$$

owing to the trace relation $\mathrm{Tr}(\tilde{\sigma}^\mu \sigma_\nu) = -2\delta^\mu{}_\nu$.
More generally, the σ_μ and $\tilde{\sigma}_\nu$ satisfy the relations

$$\sigma_\mu\tilde{\sigma}_\nu + \sigma_\nu\tilde{\sigma}_\mu = -2\eta_{\mu\nu}\mathbb{1} \quad (\text{i.e. } (\sigma_\mu)_{\alpha\dot{\beta}}(\tilde{\sigma}_\nu)^{\dot{\beta}\gamma} + (\sigma_\nu)_{\alpha\dot{\beta}}(\tilde{\sigma}_\mu)^{\dot{\beta}\gamma} = -2\eta_{\mu\nu}\delta_\alpha{}^\gamma)$$

$$\tilde{\sigma}_\mu\sigma_\nu + \tilde{\sigma}_\nu\sigma_\mu = -2\eta_{\mu\nu}\mathbb{1} \quad (\text{i.e. } (\tilde{\sigma}_\mu)^{\dot{\alpha}\beta}(\sigma_\nu)_{\beta\dot{\gamma}} + (\tilde{\sigma}_\nu)^{\dot{\alpha}\beta}(\sigma_\mu)_{\beta\dot{\gamma}} = -2\eta_{\mu\nu}\delta^{\dot{\alpha}}{}_{\dot{\gamma}})$$

Under an $SL(2,\mathbb{C})$ transformation, the bispinor V transforms as $V \to V' = AVA^\dagger$. Inverting to obtain $V^\mu = -\frac{1}{2}\mathrm{Tr}(\tilde{\sigma}^\mu V)$, one finds that this transforms correctly under $SO(3,1)$: $V'^\mu = -\frac{1}{2}\mathrm{Tr}(\tilde{\sigma}^\mu V') = -\frac{1}{2}\mathrm{Tr}(\tilde{\sigma}^\mu A\sigma_\nu V^\nu A^\dagger) = \Lambda^\mu{}_\nu V^\nu$.
Moreover, although $A_\alpha{}^\beta$ is complex, $\Lambda^\mu{}_\nu$ is real:

$$(\Lambda^\mu{}_\nu)^* = \left(-\tfrac{1}{2}\mathrm{Tr}(\tilde{\sigma}^\mu A\sigma_\nu A^\dagger)\right)^* = -\tfrac{1}{2}\mathrm{Tr}(A\sigma_\nu A^\dagger\tilde{\sigma}^\mu) = -\tfrac{1}{2}\mathrm{Tr}(\tilde{\sigma}^\mu A\sigma_\nu A^\dagger) = \Lambda^\mu{}_\nu$$

using the Hermiticity of σ_μ and $\tilde{\sigma}^\mu$ and the cyclic property of Tr.
The unimodular character of A is important to have the correct dimension 6 of the $\mathrm{Spin}(3,1)$ group just like $SO(3,1)$. It is also necessary to form a standard invariant, $\det(V)$:

$$\det(V') = \det(V)\,|\det(A)|^2 = \det(V) \qquad \text{since } \det(A) = 1.$$

In terms of V^μ, one recognizes this as $\det(V) = (V^0)^2 - (V^3)^2 - (V^1)^2 - (V^2)^2 = -V^\mu V_\mu$.
Two important numerically invariant $SL(2,\mathbb{C})$ bispinors are

$$\epsilon_{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \; ; \quad \epsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

N.B. books vary in these conventions

They are invariant because

$$\epsilon_{\alpha\beta} \to \epsilon'_{\alpha\beta} = A_\alpha{}^\gamma A_\beta{}^\delta \epsilon_{\gamma\delta} = \epsilon_{\alpha\beta}\det(A) = \epsilon_{\alpha\beta}, \text{ since } \det(A) = 1$$

$$\epsilon_{\dot{\alpha}\dot{\beta}} \to \epsilon'_{\dot{\alpha}\dot{\beta}} = \epsilon_{\dot{\gamma}\dot{\delta}} A^{\dagger\dot{\gamma}}{}_{\dot{\alpha}} A^{\dagger\dot{\delta}}{}_{\dot{\beta}} = \epsilon_{\dot{\alpha}\dot{\beta}}\det(A^\dagger) = \epsilon_{\dot{\alpha}\dot{\beta}} \text{ since } \det(A^\dagger) = 1.$$

We can also define the raised-index bispinors

$$\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \; ; \quad \epsilon^{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

which are similarly numerically invariant under $SL(2,\mathbb{C})$.
Note that with these conventions, one has

$$\epsilon^{\alpha\gamma}\epsilon_{\gamma\beta} = \delta^\alpha{}_\beta \; ; \quad \epsilon_{\dot{\alpha}\dot{\gamma}}\epsilon^{\dot{\gamma}\dot{\beta}} = \delta_{\dot{\alpha}}{}^{\dot{\beta}}.$$

Although the group $SL(2,\mathbb{C})$ does not give rise to a symmetric metric, one may instead raise and lower indices covariantly using the ϵ bispinors:

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \quad , \qquad \bar{\psi}^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar{\psi}_{\dot\beta} .$$

One can similarly lower indices:

$$\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \; ; \qquad \bar{\psi}_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar{\psi}^{\dot\beta} .$$

Another set of relations between the Van der Waerden matrices can now be written:

$$(\sigma^\mu)_{\alpha\dot\alpha}(\sigma^\nu)_{\beta\dot\beta}\,\eta_{\mu\nu} = -2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}$$
$$(\tilde\sigma^\mu)^{\dot\alpha\alpha}(\tilde\sigma^\nu)^{\dot\beta\beta}\,\eta_{\mu\nu} = -2\epsilon^{\alpha\beta}\epsilon^{\dot\alpha\dot\beta} .$$

Using these, one can verify that the Van der Waerden matrices themselves are numerical invariants, provided each of the three index types is transformed according to its appropriate transformation law: $\sigma^\mu \longrightarrow \Lambda^\mu{}_\nu(A)\,A\sigma^\nu A^\dagger = \sigma^\mu$, sim for $\tilde\sigma^\mu$.

Using the algebraic relations satisfied by σ_μ and $\tilde\sigma_\mu$, one verifies that the Weyl representation γ matrices satisfy the Dirac algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbb{1}$. The Weyl representation has the virtue of making it transparent that a general 4-component Dirac spinor is actually a <u>reducible</u> representation of Spin(3,1). One may write a general, complex, Dirac spinor in terms of two 2-component $SL(2,\mathbb{C})$ spinors as $\psi^{Dirac}_{\alpha} = \begin{pmatrix} \lambda_\alpha \\ \bar\chi^{\dot\beta} \end{pmatrix}$ $\quad \alpha = 1,2 \quad \dot\beta = 1,2$.

The $SL(2,\mathbb{C})$ spinors λ_α and $\bar\chi^{\dot\beta}$ in this decomposition are trully independent and one or the other may vanish. Spinors $\psi_L = \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix}$ are called left-handed chiral spinors, while spinors $\psi_R = \begin{pmatrix} 0 \\ \bar\chi^{\dot\beta} \end{pmatrix}$ are called right-handed chiral spinors. The chiral spinors may be projected out from a general Dirac spinor by applying $P_\pm = \frac{1}{2}(\mathbb{1} \pm \gamma_5)$ where $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$. In any representation for the γ matrices, one has $(\gamma_5)^2 = \mathbb{1}_{4\times4}$, so $(P_\pm)^2 = P_\pm$ are projectors. In the Weyl representation, $\gamma_5 = \begin{pmatrix} \delta_\alpha{}^\beta & 0 \\ 0 & -\delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix}$.

The projectors $P_\pm$ project into the chiral eigenspaces with γ_5 eigenvalues ± 1 : $P_+\Psi^{Dirac} = \Psi_L$, $\gamma_5 \Psi_L = \Psi_L$ while $P_-\Psi^{Dirac} = \Psi_R$, $\gamma_5\Psi_R = -\Psi_R$. Chiral spinor fields are key to the construction of the electroweak Standard Model.

Another way to exploit the reducibility of the general Dirac spinor representation is a Spin(3,1) covariant analogue of separation into real and imaginary parts. In general, λ_α and $\chi_\alpha = (\bar{\chi}_{\dot\alpha})^*$, where $\bar{\chi}_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar{\chi}^{\dot\beta}$, are fully independent. However, an $SL(2,\mathbb{C})$ covariant condition can be imposed setting them equal, $\lambda_\alpha = \chi_\alpha$. Then $\Psi = \begin{pmatrix}\lambda_\alpha \\ \bar{\lambda}^{\dot\beta}\end{pmatrix}$ is a <u>Majorana spinor</u>.

This is the Weyl-representation form of a Majorana spinor. Other Dirac algebra representations for the γ matrices exist (in which chiral spinors become less transparent). In the Majorana representation, the γ^μ matrices have only real matrix elements, and a Majorana spinor is really real.

In the litterature on spinor fields, one encounters many different representations of the Dirac γ matrices. Any set obtained from the Weyl representation by a similarity transformation $\gamma^\mu \to S\gamma^\mu S^{-1}$ is an equally good representation of the Dirac algebra. In a general spinor representation, one needs a few more constructs to build Spin(3,1) invariant theories. For a general Dirac spinor, one can form the <u>Dirac conjugate</u> $\bar{\Psi} = \Psi^\dagger\gamma^0$, which is a row object. The matrix γ^0 which stands here is numerically γ^0 in the given representation (in essentially all representations, following the analysis given in the appendix to Jauch and Rohrlich). It is not to be transformed under Spin(3,1) transformations.

Jauch + Rohrlich A matrix $= \gamma^0$

Using the Dirac conjugate, one may make Spin(3,1) scalar products such as $\bar{\Psi}\Upsilon$, where Ψ and Υ are Dirac spinors. In the Weyl representation, $\bar{\Psi} = -i(\chi^\alpha, \bar{\lambda}_{\dot\beta})$ for $\Psi = \begin{pmatrix}\lambda_\alpha \\ \bar{\chi}^{\dot\beta}\end{pmatrix}$.

So if $\Psi = \begin{pmatrix} \zeta_\alpha \\ \bar{\rho}^{\dot\beta} \end{pmatrix}$, one has $\bar{\Psi}\Psi = -i\left(\chi^\alpha \zeta_\alpha + \bar{\lambda}_{\dot\beta}\bar{\rho}^{\dot\beta}\right)$ in Weyl representation, which is manifestly $Spin(3,1) \cong SL(2,\mathbb{C})$ invariant.

From the Dirac conjugate together with another special matrix, this one much more representation-specific, called the <u>charge conjugation matrix</u> C, one can form the <u>charge conjugate</u>: $\Psi_c = C(\bar{\Psi})^T$. In the Weyl representation, $C = -\gamma^0\gamma^2$ numerically, so $C = \begin{pmatrix} i\epsilon_{\alpha\beta} & 0 \\ 0 & i\epsilon^{\dot\alpha\dot\beta} \end{pmatrix}$, and consequently $\Psi_c = \begin{pmatrix} \chi_\alpha \\ \bar{\lambda}^{\dot\beta} \end{pmatrix}$ for $\Psi = \begin{pmatrix} \lambda_\alpha \\ \bar{\chi}^{\dot\beta} \end{pmatrix}$.

A Majorana spinor is one that is equal to its charge conjugate, $\Psi = \Psi_c$. In the Weyl representation, this becomes $\lambda_\alpha = \chi_\alpha$ (or equivalently $\bar{\chi}^{\dot\beta} = \bar{\lambda}^{\dot\beta}$) as we have seen.

Thus we see that both Weyl, e.g. $\Psi_L = \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix}$ and Majorana $\Psi_{Maj} = \begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}^{\dot\alpha} \end{pmatrix}$ spinors are determined by a single 2-component $SL(2,\mathbb{C})$ spinor λ_α. When written as 4-component spinors $\Psi_{\tilde\alpha}$, they are differently organized.

The charge conjugate of a left-handed chiral spinor $\Psi_L = \begin{pmatrix} \lambda_\alpha \\ 0 \end{pmatrix}$ is the right-handed spinor $(\Psi_L)_c = \begin{pmatrix} 0 \\ \bar{\lambda}^{\dot\beta} \end{pmatrix}$, and vice versa. Hence, one can express a Majorana spinor as the sum of a chiral spinor and its charge conjugate:

$$\Psi_{Maj} = \Psi_L + (\Psi_L)_c \quad \text{or} \quad \Psi_R + (\Psi_R)_c,$$

since for a right-handed chiral spinor $\Psi_R = \begin{pmatrix} 0 \\ \bar{\chi}^{\dot\beta} \end{pmatrix}$ in Weyl representation one has $(\Psi_R)_c = \begin{pmatrix} \chi_\alpha \\ 0 \end{pmatrix}$. The chiral spinors $\Psi_{L,R}$ can be reobtained by projection: $P_+\Psi_{Maj} = \Psi_L$, $P_-\Psi_{Maj} = \Psi_R$ ($\overset{\text{also}}{=} (\Psi_L)_c$).

<u>The Dirac action</u>

Now consider how one might write an action for a spinor field $\Psi(x)$. An immediate difficulty arises, best illustrated by supposing that $\Psi(x)$ is Majorana. The obvious $Spin(3,1)$ candidate for a Lagrangian kinetic term is $\frac{1}{2}\bar{\Psi}_{Maj}\gamma^\mu\partial_\mu\Psi_{Maj}$. Write this out in two-component $SL(2,\mathbb{C})$ notation:

$$\frac{1}{2}\left(\lambda^\alpha \sigma^\mu_{\alpha\dot\beta}\partial_\mu\bar{\lambda}^{\dot\beta} + \bar{\lambda}_{\dot\alpha}\tilde{\sigma}^{\mu\,\dot\alpha\beta}\partial_\mu\lambda_\beta\right) \qquad \Psi_{Maj} = \begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}^{\dot\beta} \end{pmatrix}$$

In the second term, raise the indices on $\bar\lambda_{\dot\alpha}$ and λ_β using $\epsilon^{\dot\alpha\dot\beta}$ and $\epsilon^{\alpha\beta}$ and at the same time lower the 2-component indices on $\tilde\sigma^{\mu\dot\alpha\beta}$: $\frac{1}{2}\bar\lambda^{\dot\alpha}\tilde\sigma^{\mu}_{\dot\alpha\beta}\partial_\mu\lambda^\beta$. Now note that $\tilde\sigma^{\mu}_{\dot\alpha\beta} = (\sigma^{\mu}_{\beta\dot\alpha})^T$, so that <u>if</u> the components of λ^β and $\bar\lambda^{\dot\alpha}$ were ordinary complex-number-valued fields, then one could rewrite the second term as $\frac{1}{2}\partial_\mu\lambda^\beta\sigma^{\mu}_{\beta\dot\alpha}\bar\lambda^{\dot\alpha}$. Hence, the whole kinetic term for a Majorana spinor would be, changing summation indices,

$$\tfrac{1}{2}(\lambda^\alpha\sigma^{\mu}_{\alpha\dot\beta}\partial_\mu\bar\lambda^{\dot\beta} + \partial_\mu\lambda^\alpha\sigma^{\mu}_{\alpha\dot\beta}\bar\lambda^{\dot\beta}) = \tfrac{1}{2}\partial_\mu(\lambda^\alpha\sigma^{\mu}_{\alpha\dot\beta}\bar\lambda^{\dot\beta}),$$

so the action, being the integral of a total derivative, would produce no field equations.

Thus, $\frac{1}{2}\bar\Psi_{Maj}\gamma^\mu\partial_\mu\Psi_{Maj}$ is useless as a kinetic Lagrangian term for a Majorana spinor field if the components of $\Psi_{Maj}(x)$ are ordinary fields. At this point, one has to confront the fact that spinor fields are not ordinary classical objects. At the quantum level, they become field operators obeying equal-time anticommutation relations $\{\lambda_\alpha(t,\vec{x}), \lambda_\beta(t,\vec{y})\} = \hbar\,\delta_{\alpha\beta}\,\delta^3(x-y)$. At the "classical" level, one takes the limit $\hbar\to 0$, so the component fields $\lambda_\alpha(x)$ must become complex <u>anticommuting-number</u> valued functions of x^μ, requiring $\lambda_\alpha(x)\lambda_\beta(y) = -\lambda_\beta(y)\lambda_\alpha(x)$.

If one accepts the algebraic prescription that the λ_α and $\bar\lambda_{\dot\beta}$ are anticommuting, then $\frac{1}{2}\bar\lambda^{\dot\alpha}\tilde\sigma^{\mu}_{\dot\alpha\beta}\partial_\mu\lambda^\beta = -\frac{1}{2}\partial_\mu\lambda^\beta\sigma^{\mu}_{\beta\dot\alpha}\bar\lambda^{\dot\alpha}$ and the Majorana field's kinetic term becomes

$$\tfrac{1}{2}(\lambda^\alpha\sigma^{\mu}_{\alpha\dot\beta}\partial_\mu\bar\lambda^{\dot\beta} - \partial_\mu\lambda^\alpha\sigma^{\mu}_{\alpha\dot\beta}\bar\lambda^{\dot\beta})$$

which is no longer a total derivative.

<u>Aside: anticommuting "numbers"</u>

The anticommuting nature of spinor-field components λ_α is really a persistence at the "classical" $\hbar\to 0$ level of an underlying structure. Quantum mechanics also suggests a way to realize such algebraic behavior in terms of matrices.

Consider the group $Spin(2N)$, the double cover of $SO(2N)$. N here is to be thought of as a large integer.

The group Spin$(2\mathcal{N})$ has γ matrices satisfying an Euclidean-signature Dirac algebra, $\{\gamma_i, \gamma_j\} = 2\delta_{ij}\mathbb{1}$ $i,j=1,\cdots,2\mathcal{N}$ where the γ_i and $\mathbb{1}$ are (huge) $2^{\mathcal{N}} \times 2^{\mathcal{N}}$ matrices.

Next define $\Theta_n = \gamma_{2n} + i\gamma_{2n-1}$, $n=1,\cdots,\mathcal{N}$. These matrices anticommute, $\{\Theta_n, \Theta_m\} = 0$ $\forall n,m$, and so can form a basis for an algebra of anticommuting numbers and their products, known as a Grassmann algebra. As with the case of complex numbers z, which can be viewed as ordered pairs $(\mathrm{Re}\, z, \mathrm{Im}\, z)$, leading to the corresponding multiplication rule, the matrix realization of Grassmann numbers is not needed in practice. One just has to get used to the algebraic rule $\Theta_n\Theta_m = -\Theta_m\Theta_n$.

Components of anticommuting spinor fields $\lambda_\alpha(x)$ may be expanded in a Grassmann algebra basis with ordinary spacetime functions as expansion coefficients: $\lambda_\alpha(x) = \lambda_{\alpha n}^{(x)}\Theta^n$, where the $\lambda_{\alpha n}(x)$ are ordinary functions valued in the complex numbers.

Conjugation of spinor field products

Given the quasi-quantum behavior of spinor fields $\Psi(x)$, even in the "classical" limit $\hbar \to 0$, it is convenient to define conjugation of spinor products as Hermitian conjugation, i.e. complex conjugation together with transposition. Thus, if $(\Psi_\alpha)^* = \bar{\Psi}_{\dot\alpha}$ and $(\lambda_\alpha)^* = \bar\lambda_{\dot\alpha}$, then $(\Psi_\alpha\lambda_\beta)^* = \bar\lambda_{\dot\beta}\bar\Psi_{\dot\alpha} = -\bar\Psi_{\dot\alpha}\bar\lambda_{\dot\beta}$.

"$*$" $\Leftrightarrow$ $\dagger$

This convention at the "classical" level allows for actions and field equations at this level to be promoted to quantum field theory without major reorganization of the formalism.

With this definition of conjugation, and noting that the $\sigma^\mu_{\alpha\dot\beta}$ are Hermitian, one has

$(\sigma^\mu_{\alpha\dot\beta})^* = \sigma^\mu_{\beta\dot\alpha}$

$$
\begin{aligned}
\tfrac{1}{2}(\bar\Psi\gamma^\mu\partial_\mu\Psi)^* &= \tfrac{1}{2}\left(\lambda^\alpha\sigma^\mu_{\alpha\dot\beta}\partial_\mu\bar\lambda^{\dot\beta} - \partial_\mu\lambda^\alpha\sigma^\mu_{\alpha\dot\beta}\bar\lambda^{\dot\beta}\right)^* \\
&= \tfrac{1}{2}\left(\partial_\mu\lambda^\beta\sigma^\mu_{\beta\dot\alpha}\bar\lambda^{\dot\alpha} - \lambda^\beta\sigma^\mu_{\beta\dot\alpha}\partial_\mu\bar\lambda^{\dot\alpha}\right) \\
&= -\tfrac{1}{2}(\bar\Psi\gamma^\mu\partial_\mu\Psi)
\end{aligned}
$$

So, as written, the kinetic term for $\Psi(x)$ would be "imaginary".

Accordingly, in order to have a "real" $\leftrightarrow$ Hermitian kinetic term for a Majorana spinor field, one should include a factor of i:

$\overleftrightarrow{A\partial_\mu B} = A\partial_\mu B - \partial_\mu A B$

$$-\frac{i}{2}\bar{\Psi}\gamma^\mu\partial_\mu\Psi = -\frac{i}{2}\lambda^\alpha\sigma^\mu_{\alpha\dot\beta}\overleftrightarrow{\partial_\mu}\bar\lambda^{\dot\beta}, \text{ for } \Psi_{Maj} = \begin{pmatrix}\lambda_\alpha\\ \bar\lambda^{\dot\beta}\end{pmatrix}$$

Similarly, one writes a "real" mass term for a Majorana spinor field as $-\frac{i}{2}m\bar\Psi\Psi = -\frac{m}{2}(\lambda^\alpha\lambda_\alpha + \bar\lambda_{\dot\alpha}\bar\lambda^{\dot\alpha})$. Check "reality" by taking the conjugate:

$$(-\tfrac{i}{2}m\bar\Psi\Psi)^* = -\frac{m}{2}(\bar\lambda_{\dot\alpha}\bar\lambda^{\dot\alpha} + \lambda^\alpha\lambda_\alpha) = -\tfrac{i}{2}m\bar\Psi\Psi .$$

Note another problem that would arise if the spinor field Ψ were taken to be commuting instead of anticommuting: the mass term would vanish. This would happen because $\bar\lambda_{\dot\alpha}\bar\lambda^{\dot\alpha} = \bar\lambda_{\dot\alpha}\bar\lambda_{\dot\beta}\epsilon^{\dot\alpha\dot\beta}$ and $\lambda^\alpha\lambda_\alpha = \epsilon^{\alpha\beta}\lambda_\beta\lambda_\alpha$, both of which would vanish. So the existence of a mass term also requires the components of Ψ to anticommute.

Accordingly, we take the Lagrangian for a Majorana anticommuting spinor field with mass m to be

$$\mathcal{L}_{\Psi_{Maj}} = -\frac{i}{2}\bar\Psi\gamma^\mu\partial_\mu\Psi - \frac{i}{2}m\bar\Psi\Psi \qquad \Psi_{Maj} = \begin{pmatrix}\lambda_\alpha\\ \bar\lambda^{\dot\beta}\end{pmatrix}$$

Varying the action $I_{\Psi_{Maj}} = \int d^4x\, \mathcal{L}_{\Psi_{Maj}}$ with respect to Ψ gives the same result from $\delta\bar\Psi$ as from $\delta\Psi$ (after dropping surface terms), so the contributions just add and we find, dropping a factor of $-i$, $(\gamma^\mu\partial_\mu + m)\Psi(x) = 0$, the <u>Dirac equation</u> for a Majorana field. Write this out in 2-component notation:

$$\begin{pmatrix} 0 & i\sigma^\mu_{\alpha\dot\beta} \\ i\tilde\sigma^{\mu\dot\alpha\beta} & 0\end{pmatrix}\partial_\mu\begin{pmatrix}\lambda_\beta\\ \bar\lambda^{\dot\beta}\end{pmatrix} + m\begin{pmatrix}\lambda_\alpha\\ \bar\lambda^{\dot\alpha}\end{pmatrix} = 0$$

i.e.

$$i\sigma^\mu_{\alpha\dot\beta}\partial_\mu\bar\lambda^{\dot\beta} + m\lambda_\alpha = 0$$

$$i\tilde\sigma^{\mu\dot\alpha\beta}\partial_\mu\lambda_\beta + m\bar\lambda^{\dot\alpha} = 0$$

forming a conjugated pair of equations for the 2 complex spinor components λ_α. Conjugating the first and using $(\sigma^\mu_{\alpha\dot\beta})^* = \tilde\sigma^\mu_{\dot\alpha\beta}$, one has $(i\sigma^\mu_{\alpha\dot\beta}\partial_\mu\bar\lambda^{\dot\beta} + m\lambda_\alpha)^* = -i\tilde\sigma^\mu_{\dot\alpha\beta}\partial_\mu\lambda^\beta + m\bar\lambda_{\dot\alpha} = 0$, and upon raising the $\dot\alpha$ index with $\epsilon^{\dot\alpha\dot\beta}$ and noting that $\chi_\beta\lambda^\beta = \chi_\beta\lambda_\gamma\epsilon^{\beta\gamma} = -\chi^\beta\lambda_\beta$, one finds the 2nd equation

$$i\tilde\sigma^{\mu\dot\alpha\beta}\partial_\mu\lambda_\beta + m\bar\lambda^{\dot\alpha} = 0 .$$

/ to \ contraction costs a sign

Be careful with signs!

Now generalize the above discussion to the case of a Dirac (i.e. complex) spinor field $\Psi^{Dirac} = \begin{pmatrix} \lambda_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix}$, and at the same time minimally couple it to a Maxwell vector potential A_μ:

$$\mathcal{L}^{gauged}_{\Psi_{Dirac}} = -i\,\bar\Psi\,\gamma^\mu(\partial_\mu + ieA_\mu)\Psi - im\bar\Psi\Psi .$$

This is invariant under local $U(1)$ symmetry transformations of Ψ together with gauge transformations of A_μ:

$$\Psi \to \Psi' = e^{-i\theta(x)}\Psi \qquad \bar\Psi \to \bar\Psi' = e^{i\theta(x)}\bar\Psi$$

$$A_\mu \to A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta$$

So $D_\mu\Psi = (\partial_\mu + ieA_\mu)\Psi \to e^{-i\theta(x)}D_\mu\Psi$ is a good covariant derivative.

Varying the action $I^{gauged}_{Dirac} = \int d^4x\, \mathcal{L}^{gauged}_{\Psi_{Dirac}}$ with respect to $\Psi(x)$, which can now be treated as independent of the $\bar\Psi(x)$ variation since Ψ is a general Dirac spinor, one obtains the complex $U(1)$ gauge field coupled Dirac equation from $\delta I/\delta\bar\Psi$:

$$(\gamma^\mu D_\mu + m)\Psi(x) = 0 \qquad \Psi^{Dirac} = \begin{pmatrix} \lambda_\alpha \\ \bar\chi^{\dot\alpha} \end{pmatrix}.$$

Writing this out in 2-component notation now gives two independent equations

2x2=4 complex equations

$$i(\partial_{\alpha\dot\beta} + eA_{\alpha\dot\beta})\bar\chi^{\dot\beta} + m\lambda_\alpha = 0$$

$$i(\partial^{\dot\alpha\beta} + eA^{\dot\alpha\beta})\lambda_\beta + m\bar\chi^{\dot\alpha} = 0$$

where $\partial_{\alpha\dot\beta} = \sigma^\mu_{\alpha\dot\beta}\partial_\mu$, $\partial^{\dot\alpha\beta} = \tilde\sigma^{\mu\,\dot\alpha\beta}\partial_\mu$ and similarly for $A_{\alpha\dot\beta}$, $A^{\dot\alpha\beta}$. Varying the action with respect to Ψ yields the complex conjugates of these equations.

The action I^{gauged}_{Dirac} is of physical significance, as it describes, when taken together with the Maxwell action $-\frac{1}{4}\int d^4x\, F_{\mu\nu}F^{\mu\nu}$, the interaction of a charged spinor field such as the electron field with the electromagnetic field. This is the starting point for quantum electrodynamics (QED). Note that the complex Dirac spinor field is parametrized by the four complex anticommuting components $\lambda_\alpha(x)$, $\chi_\alpha(x)$ — twice as many as for a Majorana $\leftrightarrow$ "real" field. A Majorana field has no $U(1)$ symmetry, so no couplings to the gauge field A_μ.

Spinor fields carrying non-abelian representations

Now consider a spinor field carrying a non-abelian group G representation T^k_{ab}. For now, consider just Majorana spinors and let T^k_{ab} be a real representation. The generalization from the $U(1)$ case is straightforward for spinors Ψ_a (spinor indices suppressed):

$A_\mu \cdot T_{ab} = A^k_\mu T^k_{ab}$

$$\mathcal{L}_{\Psi_a} = -\frac{i}{2}\bar{\Psi}_a \gamma^\mu (D_\mu \Psi)_a \,, \quad (D_\mu\Psi)_a = \partial_\mu \Psi_a - ig A_\mu \cdot T_{ab}\Psi_b \,.$$

For real representations T, exactly the same construction works for Dirac (general complex 4-component) or Weyl spinors — real representations aren't picky about the spinor species.

If the group G possesses an invariant tensor δ_{ab}, e.g. for $SO(n)$, then one can also form a G-invariant mass term, e.g. for a Dirac spinor: $-im\bar{\Psi}_a\Psi_a = -m(\chi^\alpha_a \lambda_{\alpha a} + \bar{\lambda}_{\dot\beta a}\bar{\chi}^{\dot\beta}_a)$.

For complex representations $(T^k)_a{}^b$, however, things are more delicate. For one thing, one cannot endow a Majorana spinor with a complex G representation any more than one can for real scalars. In other words, $(\Psi_{Maj} + i\chi_{Maj})$ is not a Majorana, but a general complex Dirac spinor. On the other hand, Dirac spinors have no problems carrying complex representations, just like complex scalars.

The more subtle case is that of Weyl spinors with complex representations. An $SL(2,\mathbb{C})$ spinor can naturally carry a complex representation $(T^k)_a{}^b$, e.g. an $SU(n)$ fundamental $\lambda_{\alpha b}$; $\alpha = 1,2$ $b = 1,2,\ldots,n$. The conjugate spinor $(\lambda_{\alpha b})^* = \bar{\lambda}_{\dot\alpha}{}^b$ must transform according to conjugated representations of <u>both</u> groups: undotted $\leftrightarrow$ dotted, fundamental $\leftrightarrow$ conjugate fundamental. This can equally well express this in terms of left- and right-handed 4-component spinors: Ψ_{Lb} conjugates to $(\Psi_L)^b_C$, where C denotes charge conjugation. In consequence, note again ~~that~~ one cannot construct a Majorana spinor G-covariantly for a complex representation: $\Psi_{Lb} + (\Psi_L)^b_C$ is not G-covariant.

• For the same reason, <u>Weyl spinors with complex G-reps cannot have mass terms</u>: $\lambda^\alpha{}_a \lambda_{\alpha a}$ is not G-invariant.

One can summarize the situation for mass terms as follows:

Spinor type	Real G-rep	Complex G-rep
Majorana	✓	×
Weyl	$\otimes$	$\boxed{\times}$
Dirac	✓	✓

$\otimes$: essentially semantic

$\boxed{\times}$: Crucial for what follows

As we have seen, for Weyl spinors carrying complex G-representations, there is no way covariantly to build a mass term — $\lambda^\alpha{}_a \lambda_{\alpha a}$ is not G-invariant for complex reps, and $\bar\lambda_{\dot\alpha}{}^a \lambda_{\alpha a}$ is a Lorentz vector, not a scalar — there is no covariant way to contract the $\dot\alpha, \alpha$ indices.

For real G-reps, there is no problem for Majorana or Dirac spinors. In principle, for Weyl spinors $\lambda_{La} = \begin{pmatrix}\lambda_{\alpha a}\\ 0\end{pmatrix}$ one has $\lambda_{Ra} = (\lambda_{La})_C = \begin{pmatrix}0\\ \bar\lambda^{\dot\alpha}{}_a\end{pmatrix}$ and one could build $\lambda_{La} + \lambda_{Ra} = \lambda_{Maj\,a}$ and then form $\bar\lambda_{Maj\,a}\lambda_{Maj\,a}$, but this
$\otimes$ is simply a repackaging of $(\text{Weyl}) + (\text{Weyl})_C$ as a Majorana spinor in order to form the Majorana mass term $\bar\lambda_{Maj\,a}\lambda_{Maj\,a}$.

<u>Yukawa couplings and higher-order spinor couplings</u>

Given appropriate representation content, couplings of spinor fields to scalar fields and other spinor fields proceeds straightforwardly, provided one makes legal index contractions. For example, one could have a chiral spinor $\lambda_{\alpha a}$ in the $\square$ fundamental of SU(n) and a complex scalar $\varphi_{(ab)}$ in the $\square\!\square$ symmetric representation. Then one can legally form spinor-spinor-scalar <u>Yukawa interactions</u> like $(\lambda^\alpha{}_a \lambda_{\alpha b}\bar\varphi^{ab} + \bar\lambda_{\dot\alpha}{}^a \bar\lambda^{\dot\alpha b}\varphi_{ab})$. Note that φ_{ab} must be (ab) symmetric in order for this to work: $\bar\lambda_{\dot\alpha}{}^a \bar\lambda_{\dot\beta}{}^b$ must be overall <u>anti</u>symmetric under exchange of index pairs ${}_{\dot\alpha}{}^a \leftrightarrow {}_{\dot\beta}{}^b$. The $\epsilon^{\dot\alpha\dot\beta}$ used to make the $SL(2,\mathbb{C})$ contraction is antisymmetric, so on the ab SU(n) indices, one must have <u>symmetry</u>.

Similarly, one can make higher-order spinor couplings like $\lambda^\alpha{}_a \lambda_{\alpha b} \bar\chi_{\dot\beta}{}^a \bar\chi^{\dot\beta b}$. A particularly important example of this type consists in current-current interactions $J_\mu{}^k J^{\mu k}$ with
$J_\mu{}^k = \bar\lambda^b \tilde\sigma_\mu (T^k)_b{}^e \lambda_e$.