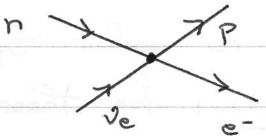
10. Towards the Standard Model: Weak Interactions

The best-known weak-interaction process is neutron decay $n \longrightarrow p + e^- + \bar{\nu}_e$, where $\bar{\nu}_e$ is the electron antineutrino. Related processes are $n + \nu_e \rightleftarrows p + e^-$ and $n + e^+ \rightleftarrows p + \bar{\nu}_e$. These can all be described in an <u>effective</u> theory by Feynman diagrams like



Cross sections for such processes involve $|M|^2$, the $(\text{modulus})^2$ of an amplitude M, e.g. $|M|^2 \sim G_F^2 (k_n \cdot k_{\nu_e})(k_p \cdot k_e)$, where $G_F = 1.17 \times 10^{-5} (\text{GeV})^{-2}$ is the <u>Fermi coupling constant</u> and $k_i \cdot k_j$ are scalar products of the 4-momenta of the various fields entering the 4-point vertex.

also, $G_F \simeq 1.01 \times 10^{-5} (m_p)^{-2}$

From this, one obtains a neutron half-life $\tau_n \simeq 886$ seconds, very long by nuclear physics standards. So the relevant process is <u>weak</u>.

In addition to such lepton-hadron interactions, there are also purely leptonic weak-interaction processes such as $\bar{\nu}_e + e^- \longrightarrow \mu^- + \bar{\nu}_\mu$, which is related to muon decay, $\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu$. At low energies, all known weak interaction processes are described to acceptable accuracy by local 4-point interactions of the relevant spinorial fermionic fields. This is <u>Fermi theory.</u>

Following the discovery by Lee and Yang that weak decays violate parity conservation, Feynman and Gell-Mann proposed a more specific form for the 4-fermion interactions: $\frac{G_F}{\sqrt{2}} J_\rho(x) J^\rho(x)$, where $J_\rho(x) = l_\rho(x) + h_\rho(x)$ is a current constructed from fermionic fields; $l_\rho(x)$ is the lepton part and $h_\rho(x)$ is the hadron part. The key feature of the Feynman & Gell-Mann proposal is the <u>chiral</u> nature of these currents. In particular, the lepton part of the current is purely left-handed:

leptons: e^-, ν_e, μ^-, ν_μ, etc

hadrons: p, n, etc

$$l_\rho(x) = \tfrac{1}{2} \bar{\Psi}_e(x) \gamma_\rho (1+\gamma_5) \Psi_{\nu_e} + \tfrac{1}{2} \bar{\Psi}_\mu \gamma_\rho (1+\gamma_5) \Psi_{\nu_\mu}$$

(where all indices on the spinor fields are suppressed; Ψ_e and Ψ_μ are electron and muon spin ½ fields while Ψ_{ν_e} and Ψ_{ν_μ} are their associated neutrino spinor fields). The explicit form of the hadronic current $h_\rho(x)$ is complicated by features of the strong interactions, but one can say much about the theory from the structure of the leptonic part $l_\rho(x)$ alone.

Chiral currents like $l_\rho(x)$ are not parity invariant, and would not have been considered prior to the discovery that parity is not conserved by the weak interactions. Under spatial reflections $x^i \to -x^i$, $i=1,2,3$, one finds that $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is no longer invariant as it is under Spin (3,1) transformations, but instead $\gamma_5 \to -\gamma_5$. Aside: for theories involving spinors that are invariant under parity, there exists an extension of Spin (3,1) called Pin (3,1), whose purely bosonic analogue is O(3,1). So the precise statement is that weak-interaction theory is not Pin(3,1) invariant.

The Fermi four-spinor interaction theory, and in particular its refinement by Feynman and Gell-Mann, works well at low energies. Nowadays, it is what we call an effective theory, good for a certain limited energy range. However, the Fermi theory runs into serious problems when considered as a fundamental theory. An important indication of trouble lies in the high-energy behavior of reactions such as $\bar{\nu}_e + e^- \to \mu^- + \bar{\nu}_\mu$. The cross section σ for such a process is $\sigma \sim |M|^2/E^2$, where M is the 4-particle amplitude and E is the energy in the center of mass frame. Consequently, $\sigma \sim G_F^2 E^2$ grows quadratically with energy in the Fermi theory

The unbounded growth of cross sections is contradicted, however, by an important physical principle: unitarity of the S matrix, $S^\dagger S = \mathbb{1}$. This does not allow for the unbounded growth of probability for any particular physical process; the sum or integral of probabilities for anything at all to happen must equal unity. The coefficient G_F^2 of the quadratic growth in E also gives a hint of the energy range where the Fermi theory must start to break down. Above energies of order $(G_F)^{-1/2} \approx 300$ GeV, something else must happen.

A problem closely related to that of unitarity is that of renormalizability, which also concerns high-energy behavior. Relativistic quantum field theories are plagued by the problem of ultraviolet divergences in Feynman diagrams containing loops of particle lines. Consider a rough estimate of the behavior of a 4-fermion amplitude like 

in a momentum-space calculation. Since the Dirac equation $(\gamma^\rho\partial_\rho+m)\psi$ is of first order in derivatives, fermionic propagators in momentum space are $\sim (-i\gamma^\rho k_\rho+m)/(k^2+m^2)$, i.e. at high momenta k_ρ they go like $(\text{momentum})^{-1}$. Consider the behavior of a loop amplitude where the loop-momentum integral $\int^{\Lambda} d^4k$ is cut off at some scale Λ (e.g. after Wick rotating and limiting the Euclideanized integral to momenta inside a 4-ball of radius Λ). Consequently, for a loop amplitude with two fermionic propagators, the overall

$\underset{\int}{\Lambda^4}\ \underset{\text{ferm}}{\Lambda^{-1}}\ \underset{\text{ferm}}{\Lambda^{-1}}=\Lambda^2$ high-momentum behavior will be like $\Lambda^{4-2}=\Lambda^2$, quadratically divergent. The correction to the "tree-level" amplitude 

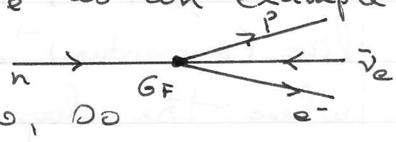
then goes like $G_F(G_F\Lambda^2)$. This is dimensionally consistent since $G_F\Lambda^2$ is dimensionless.

The problems with ultraviolet divergences get worse and worse as one goes up in loop order. For example, at the two-loop order, the correction 

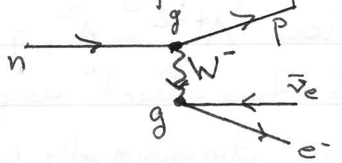
behaves at large Λ like $G_F(G_F\Lambda^2)^2$, and so on. Detailed study of the quantum corrections also reveals divergences at the one-loop level of structure $G_F(\ln\Lambda)\,p^i_\tau p^{\tau j}$ where the p^i_τ are the momenta on the external lines. At higher loops, one then gets also behaviors like $G_F(\ln\Lambda)(G_F\Lambda^2)\,p^i_\tau p^{\tau j}$ or $G_F^2(\ln\Lambda)\,p^i_\tau p^{\tau j} p^k_\sigma p^{\sigma l}$ with various contractions on the i,j indices labelling external lines. In sum, the ultraviolet behavior of the Fermi theory, when considered as a fundamental theory, is a mess. The proliferation of ultraviolet divergences is clearly linked to the $(\text{mass})^{-2}$ dimensionality of the Fermi constant G_F.

Theories with only a limited number of ultraviolet divergence structures that can be removed by a process of a finite number of subtractions of infinite terms are called renormalizable. Fermi theory is decidedly non-renormalizable.

A suggestion for how to attack the renormalizability problems was to replace four-fermion interactions with a split process involving only three-point vertices without coupling constants carrying inverse-mass dimensions, but joined by an intermediate massive vector field to reproduce the $J_\rho{}^* J^{\rho *}$ form of the interaction. Consider neutron decay $n \to p + e^- + \bar{\nu}_e$ as an example. Instead of the 4-point process



(where arrows indicate natural time flow, so the inflowing arrow on the neutrino line is interpreted as an outgoing antineutrino), one can have a sequence of two three-point processes:



a charged intermediate vector field W_ρ^-, not seen in the initial or final states, splits the 4-point interaction into two 3-point interactions with dimensionless coupling g

The coupling terms in the Lagrangian describing such 3-point interactions are natural current-vector couplings $\frac{g}{\sqrt{2}}\left(J_\rho^\dagger W^\rho + J_\rho W^{\rho\dagger}\right)$ where the coupling constant g is <u>dimensionless</u> and $J_\rho = l_\rho + h_\rho$ is the fermionic current introduced by Feynman and Gell-Mann. Now, in our earlier discussion of the massive vector Proca theory, we saw that the field equations are $\partial_\rho F^{\rho\sigma} - M^2 W^\sigma = J^\sigma$ (providing now a source J^σ on the RHS). In momentum space these become $\left(k_\sigma k_\rho - (k^2+M^2)\eta_{\sigma\rho}\right)W^\rho(k) = J_\sigma(k)$. Solve this for $W^\rho(k)$ to obtain the Feynman propagator:

$$W^\rho = -\left(J^\rho + \frac{k^\rho k^\sigma}{M^2} J_\sigma\right)\Big/(k^2+M^2) = -\left(\eta^{\rho\sigma} + \frac{k^\rho k^\sigma}{M^2}\right)J^\sigma \Big/ (k^2+M^2) \,.$$

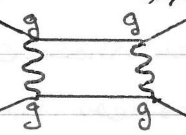
Accordingly, at lowest order the neutron decay amplitude is

$$\mathcal{M} = -\frac{1}{4} g^2 \bar{\psi}_e \gamma_\rho (1+\gamma_5)\psi_{\nu_e}\left[-\left(\eta^{\rho\sigma} + \frac{k^\rho k^\sigma}{M^2}\right)\Big/(k^2+M^2)\right]\langle p_{out} | h_\sigma | n_{in}\rangle$$

where evaluation of the hadronic current matrix element $\langle p_{out} | h_\sigma | n_{in}\rangle$ involves strong-interaction physics.

When one considers the neutron decay amplitude at low 4-momenta $|k^\rho| << M$, one has $\frac{1}{k^2+M^2} \approx \frac{1}{M^2}$ and $\eta^{\rho\sigma} + \frac{k^\rho k^\sigma}{M^2} \approx \eta^{\rho\sigma}$, so the propagator [ ] becomes just $-\eta^{\rho\sigma}/M^2$. Accordingly, if one makes the identification $G_F = 2^{-5/2} g^2/M^2$, the Fermi-theory neutron decay emerges in the low-momentum limit of the W_ρ^- exchange amplitude. (The $2^{-5/2}$ factor has a historical origin — G_F was defined before the recognition of parity violation in the weak interactions.)

Introducing the intermediate vector boson field W_ρ^- removes the most obvious source of nonrenormalizability, the dimensional character of G_F. However, UV problems remain. At large momenta $\overset{|k|\to\infty}{,}$ the massive vector propagator $\left[-\left(\eta^{\rho\sigma} + k^\rho k^\sigma/M^2\right)/(k^2+m^2)\right]$ tends to $-k^\rho k^\sigma/M^2 k^2$. This couples to the longitudinal part $J_\sigma^{long.}$ of the fermionic current and remains of order $1/M^2$ for large 4-momenta. This asymptotic behavior is worse than that of scalar propagators $\frac{1}{k^2+M^2}$ or spinor propagators $(\gamma^\sigma k_\sigma + im)/(k^2+m^2)$. Consider what happens in diagrams like



—— solid lines: fermions
〰〰 wavy lines: vector bosons

The worst behavior as the cutoff Λ is taken to infinity is $\underset{\int}{\Lambda^4}\,\underset{ferm}{\Lambda^{-1}}\,\underset{ferm}{\Lambda^{-1}}\,\underset{vec}{\Lambda^0}\,\underset{vec}{\Lambda^0} = \Lambda^2$, i.e. similar to the $G_F \!>\!\!<\! G_F$ one-loop Fermi-theory contribution.

A <u>hint</u> of how the renormalizability problem might be cured comes from the observation that the worst UV behavior above involves $k_\sigma J^\sigma(k)$ couplings to the fermionic currents; in position space, this becomes $\partial_\sigma J^\sigma(x)$. So the suggestion arises that perhaps one might cure the bad UV behavior by ensuring that $\partial_\sigma J^\sigma = 0$, i.e that the fermionic current is <u>conserved</u>. Indeed, Abdus Salam showed that this works for neutral intermediate vector boson exchange.

Showing renormalizability in models with massive vectors carrying non-abelian indices was much harder. The ultimate resolution of the UV problem requires masses obtained via the Higgs effect and spontaneous symmetry breaking. The proof of Yang-Mills renormalizability was due to G.'t Hooft and M.Veltman (1971; Nobel Prize 1999).

Electroweak Theory: $SU(2) \times U(1)$ symmetry

One part of the solution to the renormalizability problem is to limit the terms in the Lagrangian to terms of mass dimension no more than 4, so that there are no coupling constants with negative mass dimensions. Thus, one allows $m^2\phi^2$, $\lambda\phi^4$ and $f\phi\bar{\psi}\chi$ (where m is of dimension 1 and λ & f dimensionless), but terms like ϕ^6 or $\bar{\psi}\psi\bar{\chi}\chi$ are excluded.

ϕ scalar
ψ, χ spinors

Conservation of the fermion currents requires local gauge symmetries that are broken only spontaneously, with vector field masses arising from the Higgs effect. Putting this all together and showing that one can preserve renormalizability in the presence of spontaneous symmetry breaking required the contributions of several people. After 't Hooft and Veltman, the proof was completed in the work of S. Weinberg and B. Lee.

After the discovery by T.D. Lee and C.N. Yang that parity is not conserved in the weak interactions, and in view of the original impression that neutrinos could be massless, they can be separated in the massless limit into chiral spinors. Initially, all neutrinos are left-handed, i.e. they all have +1 eigenvalues of γ_5. Later on, we will discuss neutrino masses, which may be handled by introducing right-handed neutrino states. In any case, the experimentally confirmed theory is a chiral theory involving Weyl spinors.

Initially, neutrinos are all left-handed
antineutrinos are all right-handed

In weak interaction processes like muon decay, $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$, charged leptons of a given generation (here, the muon generation) can convert into their corresponding electrically neutral neutrinos (here ν_μ). Consequently, the intermediate vector boson W_ρ^- must carry electric charge, and its antiparticle W_ρ^+ must carry the opposite electric charge. Thus, at least two gauge bosons are needed. Moreover, since $W_\rho^\pm$ are charged, the corresponding group generators cannot commute with the electromagnetic $U(1)$ generator, i.e. the kinetic terms for $W_\rho^\pm$ must involve the electromagnetic field A_μ in the overall Yang-Mills field strength. Thus, the gauge group cannot be Abelian.

The simplest gauge group that can accommodate $W_p^{\pm}$ is $SU(2)$. Indeed, it plays a central role in the weak interactions; it is called <u>weak isospin</u>. Now, $SU(2)$ has three generators, and the third generator must correspond to an electrically neutral vector field. Could this be the photon? No — all three $SU(2)$ generators act on neutrinos, so the gauge field corresponding to the third $SU(2)$ generator must couple to neutrinos, which the photon doesn't do. But we have just seen that the weak isospin group doesn't commute with the electromagnetic $U(1)$, since $W_p^{\pm}$ are charged. So the photon's generator must somehow be incorporated into the overall gauge group. The next simplest group is $SU(2) \times U(1)$, which turns out to be the correct choice.

One is led to consider the following structure. The left-handed electron e_L^- can be converted into the left-handed electron neutrino ν_e. These two spinor fields form an $SU(2)$ doublet, transforming as $\Psi_L^e = \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \to U \Psi_L^e \qquad U \in SU(2)$.
As far as the Lorentz structure is concerned, one may represent these left-handed chiral spinor fields either as Weyl 4-component or 2-component spinors, according to taste. In most of the litterature, one sees them considered as 4-component Weyl spinors, so we shall do the same in writing Lagrangians here.

Although the neutrino ν_L is initially being treated as a massless field, the electron e^- definitely has a mass. Accordingly, unlike for a massless neutrino, there is always a Lorentz boost that will turn the left-handed electron e_L^- into a right-handed electron e_R^-. Only massless fields can be purely of just one chirality. However, with the neutrino ν_e massless, e_R^- has no $SU(2)$ partner. Thus, e_R^- must be an $SU(2)$ <u>singlet</u>. It's not a beautiful picture, but it works.

11. Salam-Weinberg Theory: The Standard Model

The $U(1)$ factor in the Standard Model electroweak symmetry group $SU(2)\times U(1)$ is known as <u>weak hypercharge</u>. We have seen in the discussion of the Higgs effect that vector fields will remain massless for all generators of the unbroken stability subgroup H. Vector fields corresponding to the generators of the full symmetry $G = SU(2)\times U(1)$ that lie outside H will acquire masses. In electroweak theory, we want to retain only one massless vector field, for the photon. So $SU(2)\times U(1)$ must break down to $H = U(1)_{em}$. This unbroken $U(1)$ group is generated by a <u>mixture</u> of the hypercharge generator, denoted Y, and some $SU(2)$ generator, which one may take to be $T_L^3 = \frac{1}{2}\sigma^3$ for the left-handed spinors.

Since $U(1)_Y$ commutes with all the $SU(2)$ generators in the direct product $SU(2)\times U(1)_Y$, the $U(1)_Y$ charges of all members of an $SU(2)$ multiplet must be the same. Since e_L^- and ν_e belong to the same $SU(2)$ doublet, it is clear that $U(1)_Y$ <u>cannot</u> be $U(1)_{em}$. Let the surviving $U(1)_{em}$ generator Q_{em} be $Q_{em} = T_L^3 + Y$. (Warning: conventions vary — older books often took $Q_{em} = T_L^3 + \frac{1}{2}Y_{old}$.) Thus, for the $\psi_L^e = \begin{pmatrix}\nu_e\\ e_L^-\end{pmatrix}$ left-handed doublet, one can use the known electromagnetic charges of ν_e

$T^3_{doublet} = \frac{1}{2}\sigma^3$ and e_L^- (viz. 0 and -1 in units of e) to read off the Y eigenvalues:

$\frac{1}{2}-\frac{1}{2}=0$
$-\frac{1}{2}-\frac{1}{2}=-1$

$Y\begin{pmatrix}\nu\\ e_L^-\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}\nu_e\\ e_L^-\end{pmatrix}$. Likewise, for the e_R^- right-handed singlet electron, for which $T_R^3 = 0$, one must have $Y e_R^- = (-1)e_R^-$, since $y = q_{em}$ for e_R^-.

Similarly, for the quarks, one notes that the proton (uud quarks) has $q_{em} = +1$ in units of e, while the neutron (udd quarks) has $q_{em} = 0$, so the u quark must have $q_{em} = +2/3$ and the d quark must have $q_{em} = -1/3$. Accordingly, the left-handed quark doublet $\begin{pmatrix}u_L\\ d_L\end{pmatrix}$

$\frac{1}{2}+\frac{1}{6}=\frac{2}{3}$
$-\frac{1}{2}+\frac{1}{6}=-\frac{1}{3}$

must have $y = +1/6$. The right-handed quarks are singlets just like e_R^-, so using $T_R^3 = 0$ one finds that u_R has $y = q_{em} = +2/3$ while d_R has $y = q_{em} = -1/3$.

Although the strong interactions, with gauge group $SU(3)$, are not our main focus at this point, we note the $SU(3)$ representation assignments for the leptons and quarks: the leptons $(\nu_e, e^-_L ; e^-_R)$ are $SU(3)$ singlets, while the quarks $(u_L, d_L ; u_R ; d_R)$ are $SU(3)$ fundamental $\underline{3}$ reps, i.e. triplets (Young tableau $\square$).

An incompletely understood feature of elementary particle physics is the fact that all the above structure, called the first generation, gets repeated twice over with leptons and quarks at progressively higher mass levels, making three generations in all (as far as we know). Moreover, there are small amounts of mixing between the three generations, so the "physical" quarks of the various generations are slightly misaligned with respect to the generational assignments of the leptons. Thus, we need another label for the generation, $m=1,2,3$, and also need to be alert to the effects of quark mixing between generations. So, in all, we have an (initial, before introducing neutrino masses) Standard Model spectrum consisting of left-handed leptons $L_m = \begin{pmatrix} \nu_m \\ e_{Lm} \end{pmatrix}$, right-handed leptons e_{Rm}, left-handed quarks $Q_{LmA} = \begin{pmatrix} u'_{LmA} \\ d'_{LmA} \end{pmatrix}$ and right-handed quarks u'_{RmA} and d'_{RmA}. The primes on the quarks anticipate eventual small realignments to account for the generational mixing.

$m=1$: electron gen.
$m=2$: muon gen.
$m=3$: tau gen.

capitals: $SU(2)$ doublets
lower-case: $SU(2)$ singlets

Summarize the $SU(3) \times SU(2) \times U(1)_Y$ representation assignments:

Field	$SU(3)$	$SU(2)$	$U(1)_Y$ (y eigenvalue)		
$L_m = \begin{pmatrix} \nu_m \\ e_{Lm} \end{pmatrix}$	$\underline{1}$	$\underline{2}$	$-\frac{1}{2}$	leptons	$m=1,2,3$ generation
e_{Rm}	$\underline{1}$	$\underline{1}$	-1		
$Q_{LmA} = \begin{pmatrix} u'_{LmA} \\ d'_{LmA} \end{pmatrix}$	$\underline{3}$	$\underline{2}$	$+\frac{1}{6}$	quarks	$A=1,2,3$ colour $SU(3)$ index
u'_{RmA}	$\underline{3}$	$\underline{1}$	$+\frac{2}{3}$		
d'_{RmA}	$\underline{3}$	$\underline{1}$	$-\frac{1}{3}$		

drop the "$-$" on e^-_{Lm} and e^-_{Rm} now

Dirac conjugates of all these spinor fields are necessary to write the Standard Model Lagrangian. The Dirac conjugates of these chiral fields have opposite chirality (opposite γ_5 eigenvalues) and transform in conjugated $SU(3)\times SU(2)\times U(1)_Y$ representations. Thus, for the left-handed lepton doublet, one has $\overline{\Psi_{Lb}} = (\overline{\Psi})_R{}^b$, $b=1,2$ and the $U(1)_Y$ eigenvalue is flipped $y\to -y$, giving $+\frac{1}{2}$ for the conjugated field. The left-handed conjugate term is an $SU(2)$ (as well as $SU(3)$) singlet, but its $U(1)_Y$ eigenvalue flips to $+1$. For the quarks, one has to take care also with the strong-interaction $SU(3)$ representations as well. Thus, $\overline{Q_{LmA}} = ((\overline{u'_m})_R{}^A, (\overline{d'_m})_R{}^A)$ (i.e. $\overline{Q_{LmbA}} = (\overline{Q}_m)^{bA}$) transforms in the

(SU(3), SU(2)) reps $U(1)_Y$ charge

$(\underline{3}^*, \underline{2}^*, -\frac{1}{6})$ representation of $SU(3)\times SU(2)\times U(1)_Y$, while $\overline{u'_{RmA}} = (\overline{u'_m})_L{}^A$ and $\overline{d'_{RmA}} = (\overline{d'_m})_L{}^A$ transform in the $(\underline{3}^*, \underline{1})_{-2/3}$ and $(\underline{3}^*, \underline{1})_{+1/3}$ representations respectively.

The detailed transformations of these spinor fields follow the general pattern that we have described earlier. Thus, under $SU(3)$, $Q_A \to U^{(3)}{}_A{}^B Q_B$ with $U^{(3)}{}_A{}^B = \exp(\frac{i}{2}\theta\cdot\lambda)_A{}^B$; under $SU(2)$, $L_b \to U^{(2)}{}_b{}^e L_e$ with $U^{(2)}{}_b{}^e = \exp(\frac{i}{2}\zeta\cdot\sigma)_b{}^e \Psi_e$ and under $U(1)_Y$, $\Psi \to \exp(iy\rho)\Psi$, where y is the $U(1)_Y$ eigenvalue of the field Ψ.

The Standard-Model gauge group $SU(3)\times SU(2)\times U(1)_Y$ is non-simple, so in constructing covariant derivatives one can have different coupling constants g_3, g_2, g_1 for the three simple factor groups. To construct covariant derivatives, one needs gauge fields for $SU(3)$, $SU(2)$ and $U(1)_Y$. Denoting these by the corresponding adjoint representations (but recalling that they really need to be gauge fields, not just adjoint tensors), one has $G_\mu{}^i$ for $SU(3) \sim (\underline{8},\underline{1})_0$

$i=1,\dots,8$

$W_\mu{}^{i'}$ for $SU(2) \sim (\underline{1},\underline{3})_0$ and B_μ for $U(1)_Y \sim (\underline{1},\underline{1})_0$

$i'=1,2,3$

Then one can construct covariant derivatives for all the spinor fields, noting that $(T^{(3)i})_A{}^B = \frac{1}{2}(\lambda^i)_A{}^B$, $i=1,\dots,8$ and $(T^{(2)i'})_a{}^b = \frac{1}{2}(\sigma^{i'})_a{}^b$, $i'=1,2,3$:

$m = 1,2,3$ generation, not a symmetry index (so far)

$$D_\mu L_m = \partial_\mu L_m + \left(-\tfrac{i}{2} g_2 W_\mu \cdot \sigma + \tfrac{i}{2} g_1 B_\mu\right) L_m$$
$$D_\mu e_{Rm} = \partial_\mu e_{Rm} + i g_1 B_\mu e_{Rm}$$
$$D_\mu Q_{Lm} = \partial_\mu Q_{Lm} + \left(-\tfrac{i}{2} g_3 G_\mu \cdot \lambda - \tfrac{i}{2} g_2 W_\mu \cdot \sigma - \tfrac{i}{6} g_1 B_\mu\right) Q_{Lm}$$
$$D_\mu u'_{Rm} = \partial_\mu u'_{Rm} + \left(-\tfrac{i}{2} g_3 G_\mu \cdot \lambda - \tfrac{2i}{3} g_1 B_\mu\right) u'_{Rm}$$
$$D_\mu d'_{Rm} = \partial_\mu d'_{Rm} + \left(-\tfrac{i}{2} g_3 G_\mu \cdot \lambda + \tfrac{i}{3} g_1 B_\mu\right) d'_{Rm}$$

The kinetic terms for the gauge fields are built using standard Yang-Mills field strengths for each gauge factor:

$$G_{\mu\nu}^i = \partial_\mu G_\nu^i - \partial_\nu G_\mu^i + g_3 f^{ijk} G_\mu^j G_\nu^k \qquad i,j = 1,\cdots,8 \quad SU(3)$$
$$W_{\mu\nu}^{i'} = \partial_\mu W_\nu^{i'} - \partial_\nu W_\mu^{i'} + g_2 \epsilon^{i'j'k'} W_\mu^{j'} W_\nu^{k'} \qquad i',j' = 1,2,3 \quad SU(2)$$
$$B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

The gauge-field and spinor-field kinetic terms are then standard:

$$\mathcal{L}^{\text{gauge/spinor}} = -\tfrac{1}{4} G_{\mu\nu}^i G^{i\mu\nu} - \tfrac{1}{4} W_{\mu\nu}^{i'} W^{i'\mu\nu} - \tfrac{1}{4} B_{\mu\nu} B^{\mu\nu}$$

$\not{D} = \gamma^\mu D_\mu$

$$- \tfrac{i}{2} \overline{L_m} \not{D} L_m - \tfrac{i}{2} \overline{e_{Rm}} \not{D} e_{Rm} - \tfrac{i}{2} \overline{Q_{Lm}} \not{D} Q_{Lm} - \tfrac{i}{2} \overline{u'_{Rm}} \not{D} u'_{Rm} - \tfrac{i}{2} \overline{d'_{Rm}} \not{D} d'_{Rm}$$

So far, all we have done is to construct a gauge-coupled system for the non-simple gauge group $SU(3) \times SU(2) \times U(1)_Y$ with a particular set of massless fermions. For the real Standard Model, the e_m, u'_m and d'_m will need to be massive. However, with the choices of group representations that have been made, it is <u>impossible</u> to construct $SU(3) \times SU(2) \times U(1)_Y$ invariant mass terms for these chiral fields. Every attempt to do so involves something like $\overline{\chi_R} \psi_L$ for some spinors χ_R and ψ_L and the gauge-group index structures are never such as to permit a legal construction of a gauge invariant expression. For example, $\overline{e_{Rm}} L_m$ is mismatched in $SU(2)$ and $U(1)_Y$ transformation types, and similarly for all other candidate mass terms.

The Higgs sector and Yukawa terms

To generate masses for the Standard Model fermions and the intermediate vector bosons, we employ the Higgs effect to break the electroweak sector $SU(2)\times U(1)_Y$ down to $U(1)_{em}$. The simplest system that can do this involves a complex doublet of scalar fields $\phi_a = \begin{pmatrix}\varphi^+ \\ \varphi^0\end{pmatrix}$ transforming as $(\underline{1},\underline{2})_{1/2}$ under $SU(3)\times SU(2)\times U(1)_Y$.

$a=1,2$

The complex conjugate doublet $\phi^{*a} = (\varphi^{+*}, \varphi^{0*})$ transforms as $(\underline{1},\underline{2}^*)_{-1/2}$ but in $SU(2)$, ~~we~~ can lower the "a" index: $\tilde{\phi}_a = \epsilon_{ab}\phi^{*b} = \begin{pmatrix}\varphi^{0*} \\ -\varphi^{+*}\end{pmatrix}$ which transforms as $(\underline{1},\underline{2})_{-1/2}$. More complicated systems involving multiple Higgs fields are also possible. So far, only one Higgs state has been seen — presumably a residue of symmetry breaking.

use "Higgs" to refer to whole ϕ_a, not just the residue

The Lagrangian for the Higgs doublet is standard:

$$\mathcal{J}_{Higgs} = -D_\mu\phi^\dagger D^\mu\phi - V(\phi^\dagger\phi) \qquad \phi^\dagger = (\varphi^{+*}, \varphi^{0*})$$

$$V(\phi^\dagger\phi) = \lambda\left(\phi^\dagger\phi - \frac{\mu^2}{2\lambda}\right)^2$$

$$= \lambda(\phi^\dagger\phi)^2 - \mu^2(\phi^\dagger\phi) + \frac{\mu^4}{4\lambda}$$

where, for the ϕ_a gauge transformations

$$\phi_a \to \exp\left(\tfrac{i}{2}\xi\cdot\sigma\right)_a{}^b\phi_b \qquad SU(2)$$

$$\phi_a \to \exp\left(\tfrac{i}{2}\rho\right)\phi_a \qquad U(1)_Y$$

the corresponding covariant derivative is

$$(D_\mu\phi)_a = \partial_\mu\phi_a - \tfrac{i}{2}g_2 W_\mu\cdot\sigma_a{}^b\phi_b - \tfrac{i}{2}g_1 B_\mu\phi_a$$

In order to generate masses for the fermions, we introduce $SU(3)\times SU(2)\times U(1)_Y$ gauge invariant Yukawa interaction terms. With the Higgs doublet field, we can construct three gauge-invariant types of terms: $\overline{L_m} e_{Rn}\phi = \overline{L_m}^a e_{Rn}\phi_a$ (hypercharge $\frac{1}{2}-1+\frac{1}{2}=0$),

$A=1,2,3$ $SU(3)$

$\overline{Q_{Lm}} d'_{Rn}\phi = \overline{Q_{Lm}}^{Aa} d'_{RnA}\phi_a$ (hypercharge $-\frac{1}{6}-\frac{1}{3}+\frac{1}{2}=0$)

$a=1,2$ $SU(2)$

and $\overline{Q_{Lm}} u'_{Rn}\tilde{\phi} = \overline{Q_{Lm}}^{Ab} u'_{RnA}\tilde{\phi}_b$ (hypercharge $-\frac{1}{6}+\frac{2}{3}-\frac{1}{2}=0$). Note that we have left the m,n generation labels open. All one can say a priori is that there should be some matrices of coefficients f_{mn}, h_{mn} and k_{mn} for the Yukawa interactions:

$$\mathcal{J}_{Yukawa} = -i\left(f_{mn}\overline{L_m}e_{Rn}\phi + h_{mn}\overline{Q_{Lm}}d'_{Rn}\phi + k_{mn}\overline{Q_{Lm}}u'_{Rn}\tilde{\phi}\right) + \text{h.c.}$$

The full Standard Model Lagrangian is then

$$\mathcal{L}_{SM} = \mathcal{L}^{gauge/spinor} + \mathcal{L}_{Higgs} + \mathcal{L}_{Yukawa} .$$

Symmetry breaking

For the chosen potential $V = \lambda\left(\phi^\dagger\phi - \frac{\mu^2}{2\lambda}\right)^2$, the vacuum expectation value ϕ_0 of the scalar doublet field may, without loss of generality, be taken to be

$$\phi_0 = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} v \end{pmatrix} \quad \text{with } v^2 = \mu^2/\lambda, \quad v \text{ real}.$$

Now find the stability subgroup H of unbroken gauge symmetry by making infinitesimal $SU(2) \times U(1)_Y$ transformations of the vacuum ϕ_0:

$$\delta\phi_0 = \frac{i}{2}(\zeta\cdot\sigma + \rho)\,\phi_0 = \frac{i}{2}\begin{pmatrix} \zeta_3 + \rho & \zeta_1 - i\zeta_2 \\ \zeta_1 + i\zeta_2 & -\zeta_3 + \rho \end{pmatrix}\begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}.$$

Requiring $\delta_H\phi_0 = 0$, one finds $\zeta_1 = \zeta_2 = 0$ and $\zeta_3 = \rho$. Accordingly, the stability subgroup $H = U(1)_{em}$ is generated by $T^3 + Y$, as anticipated.

Now expand ϕ about its ϕ_0 vacuum value using the exponential parametrization used earlier:

$$\phi = \exp\left(\frac{i}{\sqrt{2}v}\,\xi_i(x)\,k^i\right)\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{pmatrix}$$

where the k^i are the broken generator combinations $\frac{1}{2}\sigma^1$, $\frac{1}{2}\sigma^2$ and $\frac{1}{2}\sigma^3 - Y$ and the $\xi_i(x)$ are the would-be Goldstone bosons. The $\xi_i(x)$ are eliminated by going into unitary gauge where $\xi_i(x) = 0$ and

$$\phi \underset{\text{unitary}}{=} \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + H(x)) \end{pmatrix}$$

To work out the boson field masses, we need the covariant derivative of ϕ in detail in the unitary gauge:

$$D_\mu\phi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ \partial_\mu H \end{pmatrix} - \frac{i}{2\sqrt{2}}\begin{pmatrix} g_2 W_\mu^3 + g_1 B_\mu & g_2 W_\mu^1 - i g_2 W_\mu^2 \\ g_2 W_\mu^1 + i g_2 W_\mu^2 & -g_2 W_\mu^3 + g_1 B_\mu \end{pmatrix}\begin{pmatrix} 0 \\ v + H \end{pmatrix}$$

Expanding the kinetic term for ϕ, one has

$$-D_\mu\phi^\dagger D^\mu\phi =$$
$$-\frac{1}{2}\partial_\mu H\partial^\mu H - \frac{1}{8}(v+H)^2 g_2^2 (W_\mu^1 - iW_\mu^2)(W^{1\mu}+iW^{2\mu})$$
$$-\frac{1}{8}(v+H)^2(-g_2 W^{3\mu}+g_1 B^\mu)(-g_2 W_\mu^3 + g_1 B_\mu)$$

while the potential term becomes

$$-V = -\frac{\lambda}{4}\left[(v+H)^2 - \mu^2/\lambda\right]^2 = -\frac{\lambda}{4}(2vH+H^2)^2$$
$$= -\left(\lambda v^2 H^2 + \lambda v H^3 + \frac{\lambda}{4}H^4\right)$$

From these results, we read off the masses of the bosonic fields. For the Higgs field $H(x)$, compare the expansion of $-V$ out to second order with a standard mass term $-\frac{1}{2}m_H^2 H^2$ giving $m_H^2 = 2\lambda v^2 = 2\mu^2$.

We now know $M_H \sim 125\,\text{GeV}$

For the spin-one particles, the relevant terms in $\mathcal{L}_{Higgs}$ are

$$-\frac{1}{8}g_2^2 v^2(W_\mu^1 - iW_\mu^2)(W^{1\mu}+iW^{2\mu}) - \frac{1}{8}v^2(-g_2 W_\mu^3 + g_1 B_\mu)(-g_2 W^{3\mu} + g_1 B^\mu).$$

The fields W_μ^1 and W_μ^2 appear nicely here in the diagonal combination $-\frac{1}{8}g_2^2 v^2(W_{1\mu}W^{1\mu} + W_{2\mu}W^{2\mu})$. Comparing to the standard form $-\frac{1}{2}M_1^2 W_\mu^1 W^{1\mu} - \frac{1}{2}M_2^2 W_\mu^2 W^{2\mu}$, we identify $M_1^2 = M_2^2 = g_2^2 v^2/4$.

It is no accident that W_μ^1 and W_μ^2 have the same mass: they transform into each other under the unbroken stability subgroup $H = U(1)_{em}$ generated by $T^3 + Y$: set $\xi_1 = \xi_2 = 0$ and $\xi_3 = \rho$ to find $\delta\begin{pmatrix} W_\mu^1 \\ W_\mu^2 \end{pmatrix} = \delta\rho\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix} W_\mu^1 \\ W_\mu^2\end{pmatrix}$.

Accordingly, $W_\mu^{1,2}$ are charged under $U(1)_{em}$. Combine them into a conjugated pair of complex vector fields $W_\mu^\pm = \frac{1}{\sqrt{2}}(W_\mu^1 \mp iW_\mu^2)$. These transform as $W_\mu^\pm \to e^{\pm i\rho}W_\mu^\pm$ for finite $U(1)_{em}$ transformations. Compare the mass term to the standard $-M_W^2 W_\mu^\pm W^{\mp\mu}$ to obtain again $M_W = M_1 = M_2 = g_2 v/2$.

The remaining vector fields W_μ^3 and B_μ appear in the combination $-g_1 B_\mu + g_2 W_\mu^3$ in the mass term. So define a normalized combination

$$Z_\mu = \frac{-g_1 B_\mu + g_2 W_\mu^3}{\sqrt{g_1^2 + g_2^2}} .$$

$\sin^2\theta_W = .23120 \pm .00015$ @ M_Z momentum transfer

One can also write $Z_\mu = W_\mu^3 \cos\theta_W - B_\mu \sin\theta_W$ where $\cos\theta_W = \frac{g_2}{\sqrt{g_1^2+g_2^2}}$ $\sin\theta_W = \frac{g_1}{\sqrt{g_1^2+g_2^2}}$.

θ_W is the <u>weak mixing angle</u>, or Weinberg angle. In terms of Z_μ, the kinetic term has the standard form

$$-\tfrac{1}{4}(\partial_\mu Z_\nu - \partial_\nu Z_\mu)(\partial^\mu Z^\nu - \partial^\nu Z^\mu)$$

while the mass term $-\frac{1}{2}M_Z^2 Z_\mu Z^\mu$ is $-\frac{1}{8}v^2(g_1^2+g_2^2)Z_\mu Z^\mu$,

$M_Z = 91.2$ GeV/c^2 so we identify $M_Z = \frac{1}{2}v\sqrt{g_1^2+g_2^2}$.

lifetime 3×10^{-25} sec.

The remaining vector field combination, orthogonal to Z_μ, is

$$A_\mu = W_\mu^3 \sin\theta_W + B_\mu\cos\theta_W = \frac{g_1 W_\mu^3 + g_2 B_\mu}{\sqrt{g_1^2+g_2^2}} .$$

$A_\mu(x)$ <u>has no mass term</u>. This vector field combination is the gauge field corresponding to the unbroken $H = U(1)_{em}$ gauge symmetry. So we identify A_μ as the <u>photon</u> field

<u>Fermion masses and vector-field couplings</u>

In addition to the masses of $W_\mu^\pm$ and Z_μ, spontaneous symmetry breaking also generates masses for the fermions, via the Yukawa interactions. Inserting the unitary-gauge values $\Phi = \begin{pmatrix} 0 \\ \frac{1}{\sqrt2}(v+H)\end{pmatrix}$ and $\tilde\Phi = \begin{pmatrix}\frac{1}{\sqrt2}(v+H) \\ 0\end{pmatrix}$ into the Yukawa terms in the SM Lagrangian, one obtains

$$\mathcal{L}^{\text{unitary}}_{\text{Yukawa}} = -\frac{1}{\sqrt2}(v+H)\left(f_{mn}\bar e_{Lm} e_{Rn} + h_{mn}\bar d'_{Lm} d'_{Rn} + k_{mn}\bar u'_{Lm} u'_{Rn}\right) + \text{h.c.}$$

The fermion masses arise from the $v/\sqrt2$ terms, while the other terms are trilinear Higgs-(lepton)^2 and Higgs-(quark)^2 couplings.

The mass terms $-\frac{iv}{\sqrt{2}}\left(f_{mn}\overline{e_{Lm}}\,e_{Rn} + k_{mn}\overline{u'_{Lm}}\,u'_{Rn} + h_{mn}\overline{d'_{Lm}}\,d'_{Rn}\right)$ involve general mixtures of the three $m=1,2,3$ generations for general f_{mn}, k_{mn} and h_{mn}. However, we can simplify the structure of the mass terms by making non-symmetry $U(3)$ unitary transformations independently for all the lepton and quark species: $e_{Lm} \to U^{eL}_{mn} e_{Ln}$, $e_{Rm} \to U^{eR}_{mn} e_{Rn}$, $u'_{Lm} \to u_{Lm} = U^{uL}_{mn} u'_{Ln}$, $u'_{Rm} \to u_{Rm} = U^{uR}_{mn} u'_{Rn}$, $d'_{Lm} \to d_{Lm} = U^{dL}_{mn} d'_{Ln}$, $d'_{Rm} \to d_{Rm} = U^{dR}_{mn} d'_{Rn}$. Note that we are now defining the unprimed quarks. The Dirac conjugates of each of these transform contragrediently: $\overline{e_{Lm}} \to \overline{e_{Ln}}\,U^{+}_{nm}$, etc. The effects of these field-redefinition transformations on the Yukawa coupling matrices are biunitary transformations $f_{mn} \to U^{eL+}_{ms} f_{st} U^{eR}_{tn}$, and similarly for k_{mn} and h_{mn}. Since the unitary matrices acting on the first and second indices are independent, one can use these transformations to make f_{mn}, k_{mn} and h_{mn} diagonal and with positive real values: $f_{mn} = \mathrm{diag}(f_1, f_2, f_3)$, $k_{mn} = \mathrm{diag}(k_1, k_2, k_3)$, $h_{mn} = \mathrm{diag}(h_1, h_2, h_3)$. Then the mass terms become simply

$$-\frac{iv}{\sqrt{2}}\left(f_{\hat{m}}\overline{e_{Lm}}\,e_{Rm} + k_{\hat{m}}\overline{u_{Lm}}\,u_{Rm} + h_{\hat{m}}\overline{d_{Lm}}\,d_{Rm}\right)$$

$\hat{m}$ indices correlated with repeated m indices causing summation

So the lepton and quark masses are $m^{(e)}_n = \frac{1}{\sqrt{2}} f_n v$, $m^{(u)}_n = \frac{1}{\sqrt{2}} k_n v$, $m^{(d)}_n = \frac{1}{\sqrt{2}} h_n v$. Note that the neutrinos have not so far developed masses because we have not yet introduced right-handed species for them.

The six non-symmetry $U(3)$ transformations used to diagonalize the mass terms cannot be made scott-free in the rest of the Lagrangian, however – we need to take account of their effects on the kinetic and gauge-coupling terms. The pure derivative kinetic terms like $\frac{-i}{2}\overline{e_{Lm}}\,\partial\!\!\!/\, e_{Lm}$ or $\frac{-i}{2}\overline{u'_{Lm}}\,\partial\!\!\!/\, u_{Lm}$ are actually invariant under such $U(3)$ transformations. Moreover, although no neutrino masses have yet been generated, there is no reason not to transform ν_{Lm} in exactly the same way as e_{Lm}. Then, for the left-handed leptons one has a uniform transformation $L_m \to U^{eL}_{mn} L_n$, and for such a rigid $U(3)$ transformation, $D_\rho L_m \to U^{eL}_{mn} D_\rho L_n$.

Consequently the full left-handed lepton kinetic term $-\frac{i}{2}\bar{L}_m \not{D} L_m$, including the gauge-field interactions, is invariant under $\mathcal{U}^{eL}_{mn}$. This is the reason why we did not bother putting primes on e_{Lm}. Similarly, for the right-handed leptons one finds that $-\frac{i}{2}\bar{e}_{Rm}\not{D} e_{Rm}$ is invariant under $\mathcal{U}^{eR}_{mn}$.

For the quarks, one needs to be more careful. To diagonalize the mass matrix, one needs to make <u>independent</u> $\mathcal{U}(3)$ transformations $\mathcal{U}^{uL}_{mn}$ and $\mathcal{U}^{dL}_{mn}$, so for the left-handed quarks, there is no <u>uniform</u> transformation of Q_{Lm}. The pure derivative kinetic terms $-\frac{i}{2}\bar{u}'_{Lm}\not{\partial} u'_{Lm}$ and $-\frac{i}{2}\bar{d}'_{Lm}\not{\partial} d'_{Lm}$, being naturally diagonal in the u'_{Lm} and d'_{Lm} quarks are themselves invariant. So are the other naturally u'_{Lm} and d'_{Lm} diagonal terms coupling the quarks to W_ρ^3 and B_ρ (i.e. to Z_ρ and A_ρ). But the gauge-coupling terms involving $W^\pm_\rho$ are not all invariant. These charged-current interactions are

$$\mathcal{L}_{cc} = \frac{ig_2}{\sqrt{2}}\left[W_\rho^+\left(\bar{\nu}_{Lm}\gamma^\rho e_{mL} + \bar{u}'_{Lm}\gamma^\rho d'_{Lm}\right) + W_\rho^-\left(\bar{e}_{Lm}\gamma^\rho \nu_{Lm} + \bar{d}'_{Lm}\gamma^\rho u'_{Lm}\right)\right].$$

As we have seen, transforming ν_{Lm} and e_{Lm} by the same $\mathcal{U}^{eL}_{mn}$ leaves the lepton terms invariant. For the quark terms, define $V_{mn} = \left((\mathcal{U}^{uL})^\dagger\,\mathcal{U}^{dL}\right)_{mn}$. $V^\dagger V = \mathbb{1}$, so V_{mn} is also unitary. Then, after diagonalizing the fermion mass matrices, the charged-current interactions become

$$\mathcal{L}_{cc} = \frac{ig_2}{\sqrt{2}}\left[W_\rho^+\left(\bar{\nu}_{Lm}\gamma^\rho e_{Lm} + V_{mn}\bar{u}_{Lm}\gamma^\rho d_{Ln}\right) + W_\rho^-\left(\bar{e}_{Lm}\gamma^\rho \nu_{Lm} + V^\dagger_{mn}\bar{d}_{Lm}\gamma^\rho u_{Ln}\right)\right].$$

The matrix V_{mn} is the Cabbibo-Kobayashi-Maskawa (CKM) matrix. A priori, such a 3×3 unitary matrix would depend on 9 real parameters. However there is still some simplification one can make with non-symmetry field redefinitions. One can make pure phase $e^{i\alpha}$ transformations of the $3\times 2 = 6$ quark species, leaving the pure derivative $\not{\partial}$ kinetic terms and the mass terms invariant (since for these, e.g., e_R and e_L transform the same way, so $\bar{e}_{Lm}e_{Rm}$ is invariant). These can be used to remove some of the parameters determining V_{mn}. But only the non-symmetry phase transformations can be used this way, and there exists one phase symmetry of the whole Lagrangian: an overall transformation of all six quarks by the same phase. So only $6-1=5$ parameters determining V_{mn} can be removed. Left over, we have 4 physically important parameters. For unknown reasons, they are small: V_{mn} is near δ_{mn}.

Had V_{mn} turned out to be a real, orthogonal 3×3 matrix, it would have depended on $3\cdot 2/2 = 3$ real parameters. Since V_{mn} actually depends on 4 physically significant parameters, there is just one that reveals the complex nature of V_{mn} — a phase. The non-zero value of this phase is related to the breakdown of CP symmetry, and was a key element in the 2008 Nobel Prize for Nambu, Kobayashi and Maskawa.

Count the physically important parameters in the original (ie, prior to neutrino masses) Standard Model. One has 10 masses (3 leptons, 6 quarks and 1 Higgs); 4 CKM matrix parameters; the Higgs self-coupling λ and 3 coupling constants g_3, g_2 & g_1. That makes 18 visible at the classical level. There is one more that arises at the quantum level (the "vacuum angle" θ_3 in the term $-\frac{g_3^2}{32\pi^2}\theta_3 G^{*A}_{\mu\nu} G^{\mu\nu A}$, $G^{*A}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}G^{\rho\sigma A}$). So there are 19 parameters in total. That may seem a lot, but consider the precision of the resulting theory. Taking electroweak SM effects into account, the anomalous gyromagnetic ratio $\frac{g-2}{2}$ for the electron is calculated to be 0.0011596521594 (230). This should be compared with the experimental value 0.0011596521884 (43), a fantastic agreement between theory and experiment.

Neutrino masses

In 1967, when the Standard Model was formulated, there was no evidence for neutrino masses, so they were treated as massless. Subsequent experiments by Ray Davis (Homestake Gold Mine, Lead SD, 10^5 gallons of dry cleaning fluid) and Masatoshi Koshiba (Super Kamiokande) led to the discovery of oscillations between

Davis + Koshiba: Nobel Prize 2002

$\nu_e \to \nu_\mu$ & ν_τ — the three generations of neutrinos. This explained why the Davis experiment saw only about 1/3 of the expected flux of solar neutrinos, because it was sensitive only to ν_e. In 1968, Bruno Pontecorvo and V.N. Gribov had proposed that neutrino masses would lead to such oscillations. Since 2001, results from the Sudbury Neutrino Observatory show that only about 35% of solar neutrinos are ν_e when they reach the earth.

Pontecorvo: defected to Russia 1950

These observations give evidence for neutrino masses, in particular for differences in the ν_e, ν_μ and ν_τ masses, but don't give any single value. Nonetheless, we have from them evidence that at least one mass is at least .04 eV : miniscule with respect to the other masses in electroweak physics.

One popular way to incorporate neutrino masses into the Standard Model is to add right-handed spinor fields ν_{Rm} that are singlets with respect to all S.M. factor groups: $(\underset{\sim}{1}, \underset{\sim}{1}, 0)$. This allows for two new types of term in the Lagrangian. The first is a new type of Yukawa coupling: $-i\, p_{mn} \bar{L}_m \nu_{Rn} \tilde{\Phi}$ + h.c. (hyperchange: $+\frac{1}{2} - \frac{1}{2} = 0$). Taken alone, this is not very good physically: if p_{mn} are of the same order as the other Yukawa coefficients f_{mn}, k_{mn} & h_{mn}, then neutrino masses come out far too large.

The second new type of term in the Lagrangian has a paradoxical effect. Since the ν_{Rm} are $SU(3) \times SU(2) \times U(1)_Y$ singlets, we can, for them and only for them, construct a normal bilinear mass term:

$(\nu_R)_c = C(\bar{\nu}_R)^T$ Charge conjugate

$$-M_{\hat{n}} \left(\nu_{Rn}^{\alpha} \nu_{Rn\alpha} + \bar{\nu}_{Rn\dot{\alpha}} \bar{\nu}_{Rn}^{\dot{\alpha}} \right) = -i M_{\hat{n}} \left(\bar{\nu}_{Rn} (\nu_{Rn})_c + \overline{(\nu_{Rn})_c}\, \nu_{Rn} \right).$$

Another way to write this mass term is to repackage the ν_{Rn} as Majorana spinors $\nu_{Maj\,n} = \begin{pmatrix} \nu_{Rn\alpha} \\ \bar{\nu}_{Rn}^{\dot{\alpha}} \end{pmatrix}$ (a covariant step for ν_R since it is a gauge group singlet) and write $-i M_{\hat{n}} \bar{\nu}_{maj\,n} \nu_{maj\,n}$. This "Majorana mass" has no reason to be small — it could for example be at the TeV scale.

Now consider what happens after symmetry breaking and choice of the unitary gauge. The mass term arising from $-i\, p_{mn} \bar{L}_m \nu_{Rn} \tilde{\Phi}$ is $-i\, p_{mn} (v/\sqrt{2})\, \bar{\nu}_{Lm} \nu_{Rn} = -p_{mn}(v/\sqrt{2})\, \bar{\nu}_{Lm\dot{\alpha}} \bar{\nu}_{Rn}^{\dot{\alpha}}$. Adding to this its conjugate, one has $-\frac{v}{\sqrt{2}} \left(p_{mn} \bar{\nu}_{Lm\dot{\alpha}} \bar{\nu}_{Rn}^{\dot{\alpha}} + p_{mn}^{*} \nu_{Lm}^{\alpha} \nu_{Rn\alpha} \right)$

Thus, the Yukawa interaction produces <u>off-diagonal</u> mass terms between left and right neutrino species after electroweak symmetry breaking.

To understand better what one will see physically with such a system, consider just a single generation, e.g. $m=1$. Phase rotation of ν_{L1} can be used to make $p_{11}=p$ real and positive. For simplicity of presentation, repackage both ν_{L1} and ν_{R1} spinors into Majorana spinors: $n_1 = \begin{pmatrix} \nu_{L1\alpha} \\ \bar{\nu}_{L1}^{\dot{\alpha}} \end{pmatrix}$, $n_2 = \begin{pmatrix} \nu_{R1\alpha} \\ \bar{\nu}_{R1}^{\dot{\alpha}} \end{pmatrix}$. Then combine the two types of mass term:

$$-i\left(M\bar{n}_2 n_2 + \frac{vp}{2\sqrt{2}}\left(\bar{n}_1 n_2 + \bar{n}_2 n_1\right)\right), \text{ giving a mass matrix}$$

$$\begin{array}{cc} & \begin{matrix} \quad\quad & \quad\quad \end{matrix} \\ \begin{matrix} 1 \\ \\ 2 \end{matrix} & \begin{pmatrix} 0 & \frac{vp}{2\sqrt{2}} \\ & \\ \frac{vp}{2\sqrt{2}} & M \end{pmatrix} \\ & \begin{matrix} 1 & \quad\quad 2 \end{matrix} \end{array}$$

The vanishing element in the 11 position reflects the impossibility of having a mass for ν_L spinors in the original Standard Model. The true eigenvalues λ of such a mass matrix are given by solving $-\lambda(M-\lambda) - \frac{v^2p^2}{8} = 0$, giving $\lambda = \frac{M}{2}\left(1 \pm \sqrt{1 - \frac{p^2v^2}{2M^2}}\right)$.

Thus, if M is very <u>large</u> compared to $\frac{pv}{\sqrt{2}}$, i.e. for $\frac{p^2v^2}{2M^2} \ll 1$, we have one <u>large</u> eigenvalue $\lambda_+ \simeq M$, corresponding to a very heavy neutrino species (so far undiscovered) and also a <u>small</u> eigenvalue $\lambda_- \simeq \frac{M}{2}\left(1 - \left(1 - \frac{p^2v^2}{4M^2}\right)\right) = \frac{p^2v^2}{8M}$. So if $\frac{pv}{4} \ll M$, one has $\lambda_- \simeq \frac{p^2v^2}{8M} \ll \frac{pv}{2}$, i.e. this mechanism can give a mass that is nonzero but much <u>smaller</u> than the other lepton masses. This mechanism is called the <u>Seesaw mechanism</u>. It remains speculative — much remains to be understood about neutrino masses and the mixing of neutrino generations.

Some resources on neutrinos:

John N. Bahcall, "Solving the Mystery of the Missing Neutrinos"
www.nobelprize.org/nobel_prizes/physics/articles/bahcall

Hitoshi Murayama, "The origin of neutrino mass"
hitoshi.berkeley.edu/neutrino/PhysicsWorld.pdf

12. Grand Unification

The Standard Model successfully accounts for known electroweak phenomena, but together with the strong interactions the $SU(3) \times SU(2)_L \times U(1)_Y$ structure is only marginally a "unified" theory. A much more appealing unification would be obtained if the gauge group were a <u>simple</u> group. Then there would be only a single gauge coupling and one could hope to obtain the Standard Model's g_3, g_2 and g_1 couplings from some dynamical mechanism after a high energy symmetry breaking down to $SU(3) \times SU(2)_L \times U(1)_Y$.

The simplest group into which $SU(3) \times SU(2)_L \times U(1)_Y$ can be embedded is $SU(5)$. This is the basis for the <u>Georgi - Glashow model</u> (1974). Consider the $SU(5)$ fundamental (defining) representation; schematically the hermitean generators are:

$$(T^I)_A{}^B = \begin{pmatrix} a_1 & a_4 & a_5 & b_1 & b_2 \\ a_4^* & a_2 & a_6 & b_3 & b_4 \\ a_5^* & a_6^* & a_3 & b_5 & b_6 \\ b_1^* & b_3^* & b_5^* & c_1 & c_3 \\ b_2^* & b_4^* & b_6^* & c_3^* & c_2 \end{pmatrix} \qquad \begin{array}{l} I = 1, \ldots, 24 \\ A, B = 1, \ldots, 5 \end{array}$$

Identify the upper-left "a" block as the $SU(3)$ subgroup generators, the lower-right "c" block as the $SU(2)_L$ generators. The $SU(5)$ generators must be traceless; there is then a diagonal traceless generator of $SU(5)$ that commutes with both the $SU(3)$ and $SU(2)$ subgroups: $Y = \mathrm{diag}\left(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2}\right)$. (The normalization of Y is chosen to give the correct hypercharge assignments for quarks and leptons.)

It is striking that the known leptons and quarks can be assembled into representations of $SU(5)$. With the above identification of the $SU(3) \times SU(2)_L \times U(1)_Y$ subgroup, the fundamental $SU(5)$ 5 representation decomposes as

$5 \rightarrow (3,1) \oplus (1,2)$ under $SU(3) \times SU(2)_L$. Label the 5 rep. components as $(f_1, f_2, f_3, h_1, h_2)^T$ (written as a row^T). The tracelessness of the $SU(5)$ generators requires that the diagonal $U(1)_Y$ generator in this representation satisfy $3y_f + 2y_h = 0$. Thus, choosing $y_f = -\frac{1}{3}$ so as to agree with the hypercharge assignment for known SM fermions transforming under $SU(3)\times SU(2)_L$ as $(3,1)$, i.e. for the d'_{Rm} quarks, one finds $y_h = +\frac{1}{2}$ as given above. Which particles can these be? The whole internal symmetry structure needs to be in a direct product with the $SL(2,\mathbb{C})$ Lorentz symmetry, so the (h_1, h_2) doublet must also correspond to right-handed spinors. Looking at the available SM species, we recall that the L_m leptons transform as $(1,2)_{-\frac{1}{2}}$, so their charge conjugates $(L_m)_c$ are right-handed $(1,2^*)_{+\frac{1}{2}}$. However, in $SU(2)$ one can lower an index with ϵ_{ab}: $(\tilde{L}_{mc})_a = \epsilon_{ab}(L_{mc})^b$. So the fundamental $SU(5)$ representation is $(d'_{Rm}, (\tilde{L}_m)_c)$.

What about the right-handed electron singlet and the u'_{Rm} and Q_{Lm} quarks? Consider Q_{Lm}, transforming in the SM representation $(3,2)_{1/6}$. Since this has both $SU(3)$ and $SU(2)_L$ indices, it cannot come from a single-index $SU(5)$ fundamental rep. Try instead a $SU(5)$ tensor-product representation $\chi_{[AB]}$. This is a $5\cdot 4/2 = 10$ dimensional complex representation of $SU(5)$, and it does the trick: $A \to (\tilde{a}, a)$; $\tilde{a} = 1,2,3$ $a = 1,2$

$$\chi_{ABm} = \begin{pmatrix} \epsilon_{\tilde{a}\tilde{b}\tilde{d}}(u'_{Rm})_c^{\tilde{d}} & -Q_{Lm\tilde{a}b} \\ Q_{Lm\tilde{a}b} & \epsilon_{ab}(e_{Rm})_c \end{pmatrix}$$

$m = 1,2,3$ generation label

where every component is $SL(2,\mathbb{C})$ left-handed

Let's see how the hypercharge assignments work out for this representation. For $M_A{}^B \in SU(5)$, χ_{AB} transforms as $\chi_{AB} \to M_A{}^E M_B{}^F \chi_{EF}$, so for a transformation $M_A{}^B = (e^{i\rho Y})_A{}^B$ with $Y = \mathrm{diag}(-\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{2}, \frac{1}{2})$, the hypercharges corresponding to the A and B indices on χ_{AB} simply add, so for the components of χ_{AB} we get hypercharge assignments

$$\begin{pmatrix} -\frac{1}{3}-\frac{1}{3} = -\frac{2}{3} & -\frac{1}{3}+\frac{1}{2} = \frac{1}{6} \\ -\frac{1}{3}+\frac{1}{2} = \frac{1}{6} & \frac{1}{2}+\frac{1}{2} = 1 \end{pmatrix}$$

which are correct for the above SM fermions, remembering that $(u_{Rm})_c$ has $y = -\frac{2}{3}$ and $(e_{Rm})_c$ has $y = +1$.

Thus, taken together with the fundamental rep $\psi_{Am} = \left(d'_{Rm\tilde{a}},\ \epsilon_{ab}(L_{mc})^b\right)^T$, which when charge conjugated gives a $\bar{5}$ representation $(\psi_{mc})^A = \left((d'_{Rmc})^{\tilde{a}},\ L_m^a\right)^T$ which is composed of left-handed $SL(2,\mathbb{C})$ spinors, the ordinary Standard-Model fermions fit precisely into $SU(5)$ representations $\bar{5} \oplus 10$ for each $m = 1,2,3$ generation.

If one wants to generate neutrino masses by the see-saw mechanism, one then needs to include an additional $SU(5)$ singlet for the right-handed neutrinos ν_{Rm}. Then the $SU(5)$ fermion representations are $\bar{5} \oplus 10 \oplus 1$. Can one make things even more unified by picking a larger grand unification gauge group? Well, yes. Before the Georgi-Glashow $SU(5)$ model, Howard Georgi realized that the group $Spin(10)$ (double covering of $SO(10)$) has a spinor representation 16, which decomposes under $SU(5) \times U(1)$ as $16 \to 10_1 \oplus \bar{5}_{-3} \oplus 1_5$, giving the right $SU(5)$ representations, including all the left-handed SM fermions of a given generation ($m = 1, 2$ or 3) into a single irreducible representation of $Spin(10)$. The $Spin(10)$ adjoint $\leftrightarrow$ adjoint of $SO(10)$ decomposes as $45 \to 24_0 \oplus 10_{-4} \oplus \overline{10}_4 \oplus 1_0$

Before one gets too excited about all of this, one must recognise the potential downside to all this unification dreaming. Unification is not cost-free: in positing the existence of more and more gauge fields for these larger groups, one must take care that they don't produce unwanted physical effects.

Consider the gauge fields of the $SU(5)$ theory. The $SU(5)$ adjoint decomposes under $SU(3) \times SU(2)_L \times U(1)_Y$ as

$$24 \to \underbrace{(8,1)_0}_{SU(3)} \oplus \underbrace{(1,3)_0}_{SU(2)_L} \oplus \underbrace{(1,1)_0}_{U(1)_Y} \oplus \underbrace{(3,2)_{-5/6} \oplus (\bar{3},2)_{5/6}}_{\text{new } X_{\mu \tilde{a} b} \text{ vector fields}}$$

The art of making unified models is to arrange for further symmetry breaking to make the extra non-SM fields massive.

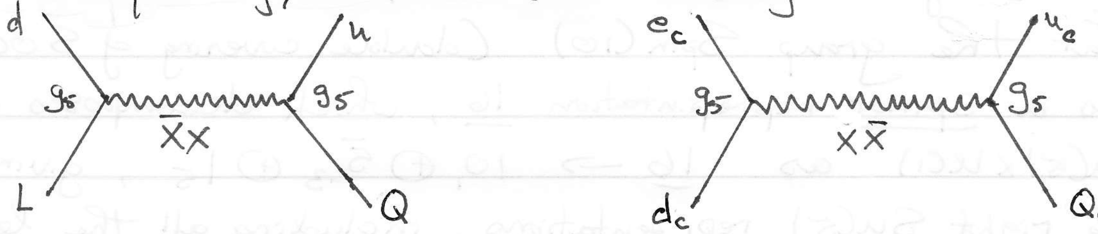
In order to break the $SU(5)$ gauge symmetry down to $SU(3) \times SU(2)_L \times U(1)_Y$ in the first instance (with $SU(2)_L \times U(1)_Y$ subsequently breaking to $U_{em}(1)$), one needs to incorporate further Higgs-type fields. As in the Standard Model itself, the resulting massive X vector fields will mediate dimension 6 processes, with an effective coupling $\frac{g_5^2}{M^2}$ where M is the mass of the X vector field.

Now comes the dangerous part of this game: the $X_{\mu \tilde{a} b}$ vector bosons carry both $SU(3)$ colour and $SU(2)_L$ indices. So there can be interactions like

$X_\mu \overline{u_{Rc}} \gamma^\mu Q_L \epsilon$ $\quad X_{\mu \tilde{a} a}\, u_{R \dot{\alpha} \tilde{b}}\, \tilde{\sigma}^{\mu \dot{\alpha} \beta}\, Q_{L \beta \tilde{d} b}\, \epsilon^{\tilde{a} \tilde{b} \tilde{d}}\, \epsilon^{ab}$ $\left(Y: -\frac{5}{6} + \frac{2}{3} + \frac{1}{6} = 0\right)$ and

$\bar{X}_\mu \overline{d_{Rc}} \gamma^\mu L$ $\quad \bar{X}_\mu{}^{\tilde{a} a}\, d_{R \dot{\alpha} \tilde{a}}\, \tilde{\sigma}^{\mu \dot{\alpha} \beta}\, L_{\beta a}$ $\quad \left(Y: 5/6 - \frac{1}{3} - \frac{1}{2} = 0\right)$.

Consequently, one can have diagrams like



which give rise to processes like $uu \to d_c e^+$ $\left(\mathcal{E}m: \frac{2}{3} + \frac{2}{3} \to \frac{1}{3} + 1\right)$ which can give rise to proton decay: $p \leftrightarrow uud \to d d_c e^+ \leftrightarrow \pi^0 e^+$, with the π^0 pion then decaying electromagnetically to gamma rays, $\pi^0 \to \gamma\gamma$.

The problem with the Georgi-Glashow model was thus proton decay. Detailed analysis shows that the proton lifetime in the $SU(5)$ model can't be more than 10^{31} years. However, proton-decay observation experiments at the Super

1996-2008 Kamiokande lab in Japan (172.8 kilotons of pure water) shows the proton lifetime to have a lower limit of 1.01×10^{34} years. So the $SU(5)$ Georgi-Glashow model disappointingly fails because it predicts too short a proton lifetime.

Another way to view the chances for $SU(5)$ grand unification is from the bottom up. In order to try to push the proton lifetime as high as possible,

10^{15} GeV: grand unification scale

keeping the g_5 coupling constant in a reasonable range for perturbation theory, $g_5 \sim 0.1$, the mass of the X vector boson must be very large, $M_X \gtrsim 10^{15}$ GeV.

Instead of assuming a unified model based on some gauge group corresponding to the 10^{15} GeV scale unification, consider just the Standard Model but evolved in energy scale from the 100 GeV electroweak scale up to 10^{15} GeV. As a consequence of renormalization, gauge coupling constants effectively change with energy scale μ:

$$\frac{1}{\alpha(\mu)} = \frac{1}{\alpha(\mu_0)} - b \ln\left(\frac{\mu^2}{\mu_0^2}\right) \qquad \alpha = \frac{g^2}{4\pi}$$

This is in fact the main reason why one can even attempt to make a grand unified model. At energy scales larger than the grand unification scale $\sim 10^{15}$ GeV, the grand unification gauge group would be effectively unbroken and there would be a single overall coupling constant, eg. g_5. At energy scales below the grand unification scale, this gauge group would be broken to the Standard Model gauge group $SU(3) \times SU(2)_L \times U(1)_Y$. With the broken symmetry, the three gauge coupling constants evolve differently:

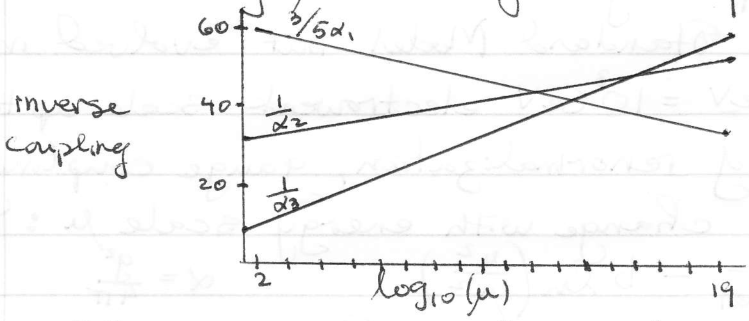
$$\frac{1}{\alpha_1(\mu)} = \frac{1}{\alpha_1(\mu_0)} - b_1 \ln\left(\frac{\mu^2}{\mu_0^2}\right)$$

$$\frac{1}{\alpha_2(\mu)} = \frac{1}{\alpha_2(\mu_0)} - b_2 \ln\left(\frac{\mu^2}{\mu_0^2}\right)$$

$$\frac{1}{\alpha_3(\mu)} = \frac{1}{\alpha_3(\mu_0)} - b_3 \ln\left(\frac{\mu^2}{\mu_0^2}\right)$$

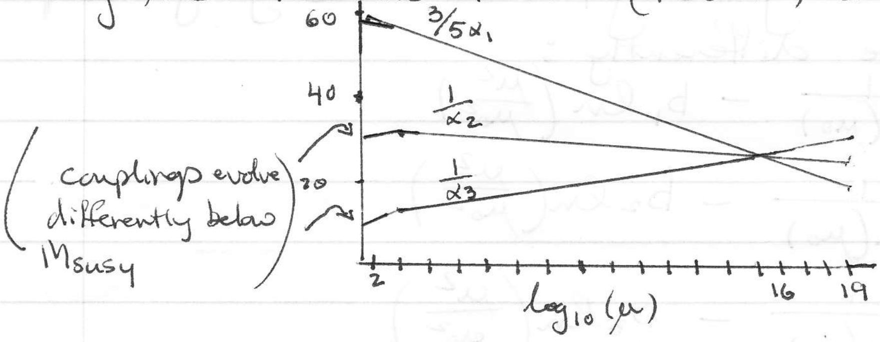
Because M_X is so large, g_1, g_2 and g_3 can evolve to very different values at the electroweak scale. Conversely, in a "bottom-up" approach, one can take the known experimental values for α_1, α_2 and α_3 at the electroweak scale (say $\mu = M_Z = 91.2$ GeV) and see what values they run to at 10^{15} GeV. If there is a unified theory based upon a simple gauge group, there should be some scale μ where α_1, α_2 and α_3 <u>cross at a point</u>.

In making this exercize, it is important to get the normalization right for the $U(1)_Y$ coupling as compared to g_2 and g_3. The upshot is that one should compare $5\alpha_1/3$ to α_2 and α_3. Plotting the inverses of these versus $\log(\mu^2)$ one gets this picture:



Standard Model couplings <u>fail</u> to cross at a point

Is this the end for grand unified models? No: <u>Supersymmetric</u> gauge theories introduce a "superpartner" for every species present in the Standard Model (selectrons, sneutrinos, gauginos, higgsinos, ...). These change the renormalization coefficients b_1, b_2, b_3 in a useful way providing the scale of <u>supersymmetry breaking</u> is not too much larger than the electroweak scale (current hope: $M_{susy} \sim 1$ TeV). Here is the picture in the Minimal Supersymmetric Extension of the Standard Model (MSSM) with $M_{susy} = 1$ TeV:

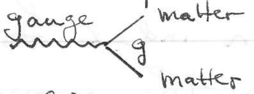


MSSM couplings <u>do</u> cross at a point at about 10^{16} GeV

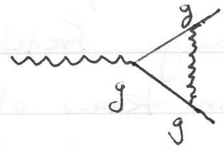
Thus, the best hopes for a truly unified theory of particle physics currently lie in the supersymmetric models. Supersymmetric $SU(5)$ and $Spin(10)$ models manage to have sufficiently long proton lifetimes to be consistent with current data.

Of course, that scenario would require eventually discovering superpartners in the TeV range: exactly the energies now accessible in the LHC.

The Running of Coupling Constants

At tree level in quantum field theory, one has vertices like 

with coupling constant g. At one loop, there are diagrams contributing to the same process (ie with the same external lines) like



These generally are divergent and require renormalization of g.

In the procedure of renormalization, one introduces a regulator, e.g. a maximum scale Λ for $\int d^4\hat{k}$ momentum integrals, and expresses the renormalized coupling in terms of an unrenormalized coupling g_0 and Λ, referred to a subtraction scale μ. For example, in electrodynamics one has $e_R^2(\mu) = e_0^2 - \frac{e_0^4}{12\pi^2}\ln\left(\frac{\Lambda^2}{\mu^2}\right)$. When amplitudes are rewritten in terms of e_R (and other renormalized quantities like the electron mass), the Λ dependence of the divergent diagrams cancels against the Λ dependence introduced by e_R. Consequently, one may take the limit $\Lambda \to \infty$ and obtain non-divergent results. However, this procedure introduces dependence on the subtraction scale μ.

Once again considering the example of electrodynamics, if one compares the values of $e_R^2(\mu)$ and $e_R^2(\mu_0)$, one finds (to one-loop order, further corrected at higher orders)

$$\frac{1}{e_R^2(\mu)} = \frac{1}{e_R^2(\mu_0)} + \frac{1}{12\pi^2}\ln\left(\frac{\mu_0^2}{\mu^2}\right),$$

or, defining $\alpha = e^2/4\pi$, $\quad \frac{1}{\alpha(\mu)} = \frac{1}{\alpha(\mu_0)} + b\ln\left(\frac{\mu_0^2}{\mu^2}\right)$

where b is the renormalization coefficient. Its value depends on the field species participating in the dynamics.

Returning to the running of the Standard Model couplings, one finds

$$b = \frac{1}{12\pi}\left[T(R_0) + 2T(R_{1/2}) - 11\,T(A)\right]$$

where $T(R)$ is the Dynkin index for a representation R and R_0, $R_{1/2}$ and A refer to the representations carried by spin-zero fields, spin-half fields, and the adjoint (for the contributions to b from the gauge fields themselves). The Dynkin index is defined in terms of the trace of group R representation generators T^i by $\mathrm{tr}(T^i T^j) = T(R)\,\delta^{ij}$. In these conventions, $T(F) = 1/2$ for the fundamental representation of $SU(N)$ and $T(A) = N$ for the $SU(N)$ adjoint.

When it comes to $U(1)_Y$ hypercharge, care needs to be taken to normalize the generator in calculating b_1 consistently with the conventions for the nonabelian factor groups $SU(2)_L$ and $SU(3)$ and eventually $SU(5)$. For them, in the fundamental representation one has $\mathrm{tr}(T^i T^j) = \frac{1}{2}\delta^{ij}$. So to make the Y normalization consistent with this, one should define $\tilde{Y} = \sqrt{3/5}\,Y$

$\mathrm{tr}\,YY = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$ (for $Y = \mathrm{diag}\left(-\frac{1}{3}\,-\frac{1}{3}\,-\frac{1}{3}\,\frac{1}{2}\,\frac{1}{2}\right)$), for which $\mathrm{tr}(\tilde{Y}\tilde{Y}) = \frac{3}{5}\cdot\frac{5}{6} = \frac{1}{2}$.

Consequently, the three couplings that should evolve to come together at some scale μ are g_3, g_2 and $\tilde{g}_1 = \sqrt{\frac{5}{3}}\,g_1$ (so that $g_1 Y = \tilde{g}_1 \tilde{Y}$). Or, in terms of the α's, one looks for a convergence at the same value for α_3, α_2 and $\frac{5\alpha_1}{3}$.