Supersymmetry MSc Course

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Books:

J. Bagger + J. Wess, "Supersymmetry and Supergravity" (Princeton)

P.C. West, "Introduction to Supersymmetry and Supergravity" (World Scientific)

P.G.O. Freund, "Introduction to Supersymmetry" (C.U.P.)

S. Weinberg, "The Quantum Theory of Fields III - Supersymmetry" (C.U.P.)

I.L. Buchbinder + S.M. Kuzenko, "Ideas and Methods of Supersymmetry and Supergravity" (IoP)

Unification problems arising from the Coleman-Mandula theorem:

Ordinary QFT symmetries relate particles of the same spin, e.g. the spin-0 pions

$$\underbrace{\pi^+ \; \pi^0 \; \pi^-}$$

form an $SU(2)$-isospin triplet

or spin $\frac{1}{2}$ leptons forming $SU(2)$ doublets:

$e_L \; \nu_e$	$\mu_L \; \nu_\mu$	$\tau_L \; \nu_\tau$
electron doublet	muon doublet	Tau doublet

Spin-statistics theorem: particles are either bosons or fermions, according to whether their spin is integral or integral + $\frac{1}{2}$

$s = 0, 1, 2, 3, \ldots$ bosons

$s = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$ fermions

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Coleman-Mandula theorem: example of a "no-go" theorem. These sometimes have the effect of focussing attention on the way one can evade their restrictions, rather than simply spelling out an impossibility.

Rough content: for a flat-space relativistic interacting field theory with a finite number of fields, the allowed algebras of conserved charges take the direct-product form

$$\mathcal{P} \times \mathcal{T} \times \mathcal{Z}$$

$\mathcal{P}$: Poincaré algebra (space-time symmetry)

$\mathcal{T}$: Semi-simple Lie algebra } "internal" symmetries

$\mathcal{Z}$: Abelian Lie algebra

In certain special $D=4$ cases, the Poincaré algebra may be extended to the conformal algebra $SO(4,2)$.

Since the theorem is based upon S-matrix considerations, it is not directly applicable to $D=2$ theories. So $D=2$ was seen as a special case.

Conserved charges ruled out by Coleman + Mandula: Q_{mn} with space-time indices other than those found among Poincaré generators, e.g. $Q_{(mn)}$ symmetric or $Q_{mn}{}^{ij}$ mixed space-time + internal indices.

$m,n = 0,1,2,3$ bosonic indices

Give a flavor of what goes wrong with $Q_{(mn)}$

Consider 2-particle elastic scattering + suppose that in addition to conserved momentum P_m and angular momentum $M_{[mn]}$ (antisymmetric) there were a conserved traceless symmetric charge $Q_{(mn)}$.

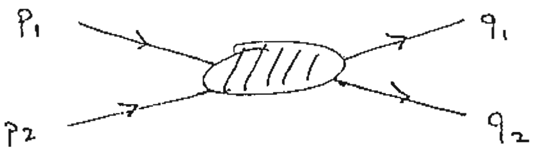
$\eta_{mn} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

Consider matrix elements of Q_{mn} between on-shell scalar one-particle states:

$$\langle p | Q_{mn} | p \rangle = \left(p_m p_n - \tfrac{1}{4} \eta_{mn} m^2 \right) f(m^2)$$

form dictated by Lorentz covariance $\qquad m^2 = -p^\sigma p_\sigma$

Now scatter particles 1 and 2:



From conservation of P_m and M_{mn}, know that only the scattering angle is undetermined.

Now claim conservation of Q_{mn} as well:
let $f(1) = f(m_1^2)$ $\quad f(2) = f(m_2^2)$

$$p_{1m} p_{1n} f(1) + p_{2m} p_{2n} f(2) - \tfrac{1}{4}\left(m_1^2 f(1) + m_2^2 f(2)\right)\eta_{mn}$$

$$= q_{1m} q_{1n} f(1) + q_{2m} q_{2n} f(2) - \tfrac{1}{4}\left(m_1^2 f(1) + m_2^2 f(2)\right)\eta_{mn}$$

$$\Rightarrow p_{1m} p_{1n} f(1) + p_{2m} p_{2n} f(2) = q_{1m} q_{1n} f(1) + q_{2m} q_{2n} f(2)$$

Go now to center of mass frame: $p_{2i} = -p_{1i}$, $q_{2i} = -q_{1i}$ $\quad i = 1,2,3$

$$\Rightarrow p_{1i} p_{1j} \left(f(1) + f(2) \right) = q_{1i} q_{1j} \left(f(1) + f(2) \right)$$

$$\Rightarrow q_{1i} = \pm p_{1i} \qquad \text{Scattering angle} = 0 \text{ or } \pi \text{ only}$$

In other words, non-trivial interactions would be ruled out. A basic aspect of QFT is that scattering amplitudes in interacting theories should be analytic functions of the scattering angle.

<u>Supersymmetry</u> evades the Coleman-Mandula Theorem by enlarging the notion of a symmetry algebra to a <u>Graded Lie Algebra</u> containing both <u>commutators</u> and <u>anticommutators</u>.

Ordinary Lie algebra:

$$[Q_a, Q_b] = i f_{ab}{}^c Q_c$$

commutator $\qquad$ $f_{ab}{}^c$ ← structure constants, can be zero

e.g. the Poincaré algebra:

$$i[M_{mn}, M_{pq}] = \eta_{np} M_{mq} - \eta_{mp} M_{nq} - \eta_{nq} M_{mp} + \eta_{mq} M_{np}$$

$$i[M_{mn}, P_p] = \eta_{np} P_m - \eta_{mp} P_n$$

$$i[P_m, P_n] = 0$$

index policy: use m, n for bosonic indices $m = 0,1,2,3$

This can be extended to the Super-Poincaré algebra by introducing a <u>spinorial</u> charge Q_α

$$\{Q_\alpha, Q_\beta\} = 2(\gamma^m C)_{\alpha\beta} P_m$$

index policy: α: spinorial index

C: charge conjugation matrix $= \gamma^0$ in Majorana rep.

anticommutator: $Q_\alpha Q_\beta + Q_\beta Q_\alpha$

γ_m: Dirac matrices, $\{\gamma_m, \gamma_n\} = 2\eta_{mn}$

$$i[M_{mn}, Q_\alpha] = -\tfrac{1}{2}(\gamma_{mn} Q)_\alpha \qquad \gamma_{mn} = \tfrac{1}{2}(\gamma_m\gamma_n - \gamma_n\gamma_m)$$

$\Rightarrow Q_\alpha$ is <u>spinorial</u> $\qquad i[P_m, Q_\alpha] = 0$

Although this Super-Poincaré, or "N=1 Supersymmetry" algebra succeeds in extending the notion of a symmetry algebra, the Coleman-Mandula theorem is still restrictive on the ordinary Lie subalgebra, which must be of the form $\mathcal{P} \times \mathcal{O}_J \times Z$.

The graded character of the supersymmetry algebra means that generators can be divided into even and odd types. This is correlated with the types of commutator/anticommutation relations:

$[\text{even}, \text{even}] = \text{even}$, $[\text{even}, \text{odd}] = \text{odd}$, $\{\text{odd}, \text{odd}\} = \text{even}$

or, letting A,B stand for adjoint generator labels, with $[A] = \begin{cases} 0 & \text{even gen.} \\ 1 & \text{odd gen.} \end{cases}$

the algebra can be written $M_A M_B - (-1)^{[A][B]} M_B M_A = f_{AB}^{\ \ C} M_C$, where $f_{AB}^{\ \ C}$ are numerical structure constants.

Even elements like $\{Q_\alpha, Q_\beta\}$ are still restricted by the Coleman-Mandula theorem. For the N=1 supersymmetry algebra, the only generator that can be Lorentz covariantly related to this is the momentum P_m.

So $\{Q_\alpha, Q_\beta\} = 2(\gamma^m C)_{\alpha\beta} P_m$ determined by Coleman-Mandula + Lorentz covariance

or $\{Q_\alpha, \bar{Q}_\beta\} = -2\gamma^m_{\alpha\beta} P_m$ $\qquad Q = C(\bar{Q})^T$

$= Q^\dagger \gamma^0$

Majorana spinor for N=1, D=4 supersymmetry

Representations: particle supermultiplets

Since Q_α carries a spinorial index, supersymmetry relates a boson of spin $S \Longleftrightarrow$ fermions of spin $S+\frac{1}{2}$ or $S-\frac{1}{2}$

e.g.

observed particles		anticipated superpartners
photon spin 1	$\Longleftrightarrow$	photino spin 1/2
quark spin 1/2	$\longleftrightarrow$	squark spin 0

The Hierarchy Problem

Despite initial hopes that supersymmetry might pair up known fermions and bosons, it turns out that this just doesn't work phenomenologically. No superpartner of a known particle has yet been found. So, what's the point?

In ordinary unified models like the Standard Model, spontaneous symmetry breaking $SU(2) \times U_Y(1) \longrightarrow U_{em}(1)$ occurs at a scale set by the Higgs mass ~ 100 GeV (expected).

Any attempt to introduce gravity brings in the Planck scale $M_{Pl} \sim 10^{19}$ GeV.

Owing to quadratic divergences in the mass renormalization procedure, the renormalized $(\text{mass})^2$ is subject to radiative corrections $\Delta m_H^2 \sim M_{Pl}^2 \Longrightarrow$ "natural" scale for the Higgs mass becomes the Planck scale, way far above 100 GeV.

This can be patched up by fine tuning the renormalization procedure, but such fine tuning is unnatural. Moreover, once the Higgs is discovered and information becomes available about the evolution of its dynamics with energy scale, the quadratic evolution of $(m_H)^2$ with Λ^2_{scale} could in principle be directly ruled out.

Supersymmetry brings about cancellation of quadratic divergences between bosonic and fermionic contributions: $\Delta m_H^2 = 0$.

Although supersymmetry must itself be broken, say at a scale $M_S \sim 100$ GeV, this brings about only mild corrections $\Delta m_H^2 \sim M_S^2 \ln\left(\frac{M_{Pl}}{M_S}\right) \sim 20 M_S^2$

So the natural scale for M_H is M_S, not M_{Pl}, in supersymmetric models.

Conserved charges, symmetries and Noether's theorem

Consider an action $I[\phi]$ for some set of fields $\phi^i(x)$ where "i" is any index.

Now vary: let $\phi^i \to \phi^i + \delta\phi^i(x)$

$$\delta I = \int d^4x\, \delta\phi^i \frac{\delta I}{\delta \phi^i} \qquad \frac{\delta I}{\delta\phi^i} \text{: variational derivative}$$

The classical field equations $\frac{\delta I}{\delta \phi^i} = 0$ are the conditions for $\delta I = 0$ regardless of what the $\delta\phi^i$ are.

Symmetries, by contrast, are special choices of $\delta\phi^i$ which one sometimes can find such that $\delta I = 0$ or $\delta I = \int d^4x\, \partial^m S_m$ without imposing the classical field equations.

So suppose $\delta_\alpha \phi^i = \alpha f^i(\phi)$ is a symmetry for some $f^i(\phi)$ with a constant infinitesimal parameter α. We will ignore terms of order α^2.

Now allow α to become spacetime dependent: $\alpha \to \alpha(x)$. In general, we now have $\delta I \neq 0$. But since $\delta_\alpha \phi^i$ becomes a symmetry again if we change our minds and set $\alpha = $ const. again, we must be able to write, up to a total derivative which we ignore,

$$\delta_\alpha I = \int d^4x\, J^m(\phi, \partial\phi)\, \partial_m \alpha$$

for some 4-vector J^m. J^m is the Noether current associated to the symmetry.

Noether's theorem states that J^m is conserved as a consequence of the classical field equations:

$$\frac{\delta I}{\delta\phi^i} = 0 \implies \partial_m J^m = 0$$

For a proof, ignoring boundary terms, integrate $\delta_\alpha I$ by parts:

$$\delta_\alpha I = -\int d^4x\, (\partial_m J^m)\, \alpha(x).$$

When $\frac{\delta I}{\delta\phi^i} = 0$, the L.H.S. vanishes (up to a boundary term) for any $\delta\phi^i$, including in particular $\delta_{\alpha(x)}\phi^i$. Since $\alpha(x)$ is arbitrary, this can only happen if $\partial_m J^m = 0$.

Moreover, a conserved current yields a conserved charge: define $Q = \int_V d^3x\, J^0$ $\quad V$ = volume of space.

Check conservation: $\frac{dQ}{dt} = \int_V d^3x\, \dot{J}^0 = -\int_V d^3x\, \partial_i J^i$

(∂V = boundary of V, generally at infinity) $\quad = -\int_{\partial V} d^2\Sigma^i J^i = 0$ if no current escapes through ∂V

Hamiltonian formalism

Poisson's theorem gives us another way to view Noether's theorem:

H: Hamiltonian
P.B: Poisson bracket

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + [H, Q]_{P.B.}$$

for any function $Q(q_i, p_j)$ on phase space.

Now, if $\frac{\partial Q}{\partial t} = 0$ (no explicit t dependence), then the statement that Q is conserved $\iff [H,Q]_{P.B.} = 0$.

Since Poisson brackets generate infinitesimal canonical transformations, $[H,Q]_{P.B.} = 0$ can be read as saying that Q is invariant under the infinitesimal canonical transformations generated by the Hamiltonian H, i.e. under infinitesimal time translations.

However, the equation $[H, Q]_{P.B.} = 0$ can also be read as saying that H is invariant under the infinitesimal canonical transformations generated by Q:

take $Q = \int d^3x\, J^0(\phi, \partial_k\phi, \dot\phi)$ and rewrite it in terms of ϕ^i and $\pi^i = \dfrac{\delta I}{\delta \dot\phi^i}$ (momentum conjugate to ϕ^i)

and generate transformations using Q,

$$\delta_\alpha \phi^i = -[\alpha Q, \phi^i]_{P.B.}$$

So we have two equivalent ways of seeing the relation between a conserved Q and H:

1) Q is time independent.
2) H, the dynamical generator, is invariant under the transformations generated by Q.

We have thus a triangular relationship in Noether's theorem.

$$\begin{tikzcd}
 & \delta\phi \arrow[dr] & \\
Q \arrow[ur] & & J^m \arrow[ll]
\end{tikzcd}$$

Finding symmetries is equivalent to finding conserved quantities.

In quantum mechanics, the Poisson brackets get replaced by commutators:

$$\delta_\alpha \phi^i = -\frac{i}{\hbar}[\alpha Q, \phi^i]$$

We will find it useful to use a quantum idiom in deriving supersymmetry transformations in the following.

Can there be spinorial conserved charges?

Take a simple model with a complex scalar field $\phi(x)$ and a Majorana spinor $\Psi = C\bar\Psi^T$ $\quad (\bar\Psi = \Psi^\dagger\gamma^0)$

$$I = \int d^4x\left[-\partial_m\phi^*\partial^m\phi - i\bar\Psi\not\partial\Psi\right] \qquad \not\partial = \gamma^m\partial_m$$

For this discussion, it helps to choose a spinor basis such that the γ^m are real: $\gamma^i = \gamma^{iT}$ hermitean, $\gamma^0 = -\gamma^{0T}$ anti-hermitean

In this (Majorana) basis, the charge conjugation matrix is simply $C = \gamma^0$, so the Majorana condition on Ψ becomes $\Psi = \gamma^0\gamma^{0T}\Psi^*$, i.e. $\Psi = \Psi^*$, a real spinor.

The momenta conjugate to ϕ and Ψ are:

$$\pi_\phi = \dot\phi^* \qquad \pi_\Psi = -i\bar\Psi\gamma^0 = i\Psi^T, \text{ i.e. } \pi_{\Psi_\alpha} = i\Psi_\alpha$$
$$\pi_{\phi^*} = \dot\phi$$

So quantization is effected by the equal-time commutators/anticommutator (setting $\hbar = 1$)

$$[\dot\phi^*(x), \phi(y)] = -i\delta^3(x-y) \qquad [\dot\phi(x), \phi(y)] = 0$$
$$[\dot\phi(x), \phi^*(y)] = -i\delta^3(x-y)$$

$$\{\Psi_\alpha(x), \Psi_\beta(y)\} = \delta_{\alpha\beta}\delta^3(x-y) \quad \text{or} \quad \{\bar\Psi_\alpha(x), \Psi_\beta(y)\} = \gamma^0_{\alpha\beta}\delta^3(x-y)$$

Anticommutation relations for the components of Ψ are required by the spin-statistics theorem, implementing the Pauli exclusion principle.

We will eventually want to return to the classical level, so the components of Ψ will become anticommuting:

$$\{\Psi_\alpha, \Psi_\beta\} = 0 \quad \text{classically}$$

Note that this is precisely what is needed to prevent the spinor part of the Lagrangian from becoming a total derivative : $\gamma^0\gamma^n$ is symmetric, so if ψ were a commuting field at the classical level, one would have, for Majorana ψ,

$$\bar{\psi}\gamma^n\partial_n\psi = \psi^T(\gamma^0\gamma^n)\partial_n\psi \Rightarrow \tfrac{1}{2}\partial_n\left[\psi^T\gamma^0\gamma^n\psi\right].$$

This is prevented by having ψ anticommuting.

Now consider the spinor current, projected into a right-handed Weyl spinor by applying $\frac{1+i\gamma_5}{2}$ $\quad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$

$$J^m_{(R)} = \left((\partial\!\!\!/\phi)\gamma^m\psi\right)_{(R)}$$

Check for conservation, using the classical field equations
$\Box\phi = 0 = \Box\phi^*$, $\partial\!\!\!/\psi = 0$:

$$\partial_m J^m = \partial_m\left(\partial_n\phi\,\gamma^n\gamma^m\psi\right) \qquad \text{assume (R) proj. here}$$

$\{\gamma^n,\gamma^m\} = 2\eta^{mn}$

$$= \tfrac{1}{2}\partial_m\partial_n\phi\left(\gamma^n\gamma^m + \gamma^m\gamma^n\right)\psi + \partial\!\!\!/\phi\,\partial\!\!\!/\psi$$

$$= \Box\phi\,\psi + \partial\!\!\!/\phi\,\partial\!\!\!/\psi = 0 \quad \text{by field equations}$$

So we have a conserved spinor current!

Construct the associated charge :

$$Q_{(R)} = \int d^3x\, J^0_{(R)} = \int d^3x\left[(\gamma^n\gamma^0)\partial_n\phi\,\psi\right]_{(R)}$$

$$= \int d^3x\left[-\dot\phi\psi + \partial_i\phi\,(\gamma^i\gamma^0)\psi\right]_{(R)}$$

One can reconstruct the Majorana spinor charge
$Q = Q_{(R)} + Q_{(L)}$ taking $Q_{(L)} = (Q_{(R)})^*$

$Q_{(L)}$ is projected with $\frac{1-i\gamma_5}{2}$. Taking $(Q_{(R)})^*$ exchanges $\phi \to \phi^*$ and changes the sign of $i\gamma_5$, so $(Q_{(R)})^* = Q_{(L)}$ is another characterisation of a Majorana spinor.

Supersymmetry transformations

Now try to use Q to try to generate a symmetry: supply an infinitesimal constant Majorana spinor parameter ϵ_α

$$\delta\phi(y) = \left[\bar\epsilon Q, \phi(y)\right] = \left[\bar\epsilon_L Q_L + \bar\epsilon_R Q_R, \phi(y)\right]$$

contains $\dot\phi^*$, contributes — contains ϕ, gives zero

$$= -\bar\epsilon_L\int d^3x\left[\dot\phi^*(x), \phi(y)\right]\psi_L(x)$$

$$= i\,\bar\epsilon_L\psi_L(y)$$

Similarly, $\delta\phi^*(y) = i\,\bar\epsilon_R\psi_R(y)$

Now try to generate a transformation of ψ:

$$\delta\psi(y) = \left[\bar\epsilon Q, \psi(y)\right] = \left[\bar\epsilon_L Q_L + \bar\epsilon_R Q_R, \psi(y)\right]$$

i.e. $\delta\psi_\beta(y) = \int d^3x\,(\gamma^n\gamma^0)_{\lambda\kappa}\left[\left(\partial_n\phi_{\alpha}\bar\epsilon_{(R)\lambda} + \partial_n\phi^*_{\alpha}\bar\epsilon_{(L)\lambda}\right)\psi_\alpha(x), \psi_\beta(y)\right]$

Looks like we are stuck: have a commutator instead of an anticommutator.

At this point, we realize that the parameters ϵ_α cannot be a set of ordinary numbers.

If we make ϵ anticommuting, then $\delta\psi(y)$ can be rewritten in terms of anticommutators, and we can use the quantum rules for ψ.

Take $\{\epsilon_\alpha, \epsilon_\beta\} = 0$ and $\{\epsilon_\alpha, \psi_\beta\} = 0$

then $[\bar{\epsilon}_\lambda \psi_\alpha(x), \psi_\beta(y)] = \bar{\epsilon}_\lambda \psi_\alpha(x)\psi_\beta(y) - \psi_\beta(y)\bar{\epsilon}_\lambda\psi_\alpha(x)$

$$= \bar{\epsilon}_\lambda \{\psi_\alpha(x), \psi_\beta(y)\}$$

$$= \bar{\epsilon}_\lambda \delta_{\alpha\beta}\delta^3(x-y)$$

So, noting that γ^0 is antisymmetric and $(\bar{\epsilon}_R \gamma^n \gamma^0)^T = \gamma^0\gamma^{nT}\gamma^0\epsilon_R = \gamma^n \epsilon_R$, $(\gamma^{nT}\gamma^0 = \gamma^{0T}\gamma^n)$
we have

$$\delta\psi_L = \partial_n \phi \gamma^n \epsilon_R$$

and similarly,

$$\delta\psi_R = \partial_n \phi^* \gamma^n \epsilon_L$$

i.e. $\delta\psi = \frac{1}{2}\left(\not\partial(\phi + \phi^*) + i\not\partial(\phi - \phi^*)\gamma_5\right)\epsilon$

The meaning of anticommuting numbers (Grassmann numbers)

Here is a construction to realise anticommuting numbers in terms of ordinary matrices (albeit very big ones, and only for a finite, or countable, basis of anticommuting numbers):

Take the Clifford algebra (algebra of Dirac γ-matrices) for the group $SO(2N)$, where N is to be thought of as a large integer.

The γ-matrices satisfy $\{\gamma_i, \gamma_j\} = 2\delta_{ij}\mathbb{1}$ $\quad i = 1 \dots 2N$
where the γ_i and $\mathbb{1}$ are $2^N \times 2^N$ matrices.

Then define
$$\Theta_n = \frac{1}{\sqrt{2}}(\gamma_{2n} + i\gamma_{2n-1}) \qquad n = 1 \dots N$$
$$d\Theta_n = \frac{1}{\sqrt{2}}(\gamma_{2n} - i\gamma_{2n-1})$$

The Θ_n will be the basis for our discrete Grassmann algebra.

The Θ_n anticommute, $\{\Theta_n, \Theta_m\} = 0$
(so also $\Theta_n^2 = 0$);

while the $d\Theta_n$ also anticommute amongst themselves,
$$\{d\Theta_m, d\Theta_n\} = 0,$$

but between the Θ_n and the $d\Theta_n$ we have the non-trivial anticommutator
$$\{d\Theta_m, \Theta_n\} = 2\delta_{mn}.$$

Use this fact to make a somewhat surprizing definition of Berezin integration using the trace over the (unwritten) γ-matrix indices:

$$\int d\Theta_m \Theta_n = 2^{-N}\,\mathrm{tr}\,(d\Theta_m \Theta_n) = \delta_{mn}$$

Expanding more general Grassmann numbers in this basis, $\Theta = a_n \Theta_n$ $\qquad a_n \in \mathbb{C}$
and the $d\Theta$ in a conjugate way,
$$d\Theta = \frac{a_n d\Theta_n}{a_m a_m}$$

One has the rules for Berezin integration
$$\int d\Theta = 0$$
$$\int d\Theta\, \Theta = 1$$
One may mnemonically summarise these by "integration & differentiation" for anticommuting variables.

And if one integrates some function $f(\Theta) = c_0 + c_1\Theta$, the integration picks out the linear term in Θ:
$$\int d\Theta\, f(\Theta) = c_1$$

Extended Supersymmetry

A further enlargement of the symmetry algebra is possible if one considers multiple spinorial generators: Q_α^i, $i = 1 \cdots N$. The index "i" may (depending on the field-theory model in question) be interpreted as belonging to a standard Lie algebra such as SO(N) or SU(N). In that case, one sees how supersymmetry truly evades the Coleman-Mandula theorem: the Q_α^i carry both "space-time" (the spinorial α) and "internal" symmetry indices (i), forming a "bridge" between these two types of symmetry.

Still, the Coleman-Mandula theorem is valid in the Lie subalgebra. The implications of this were studied by Haag, Łopuszanski and Sohnius. Their analysis made heavy use of the Super-Jacobi identities to restrict the coupling constants.

Writing the superalgebra's basic relation as

$:=$ definition

$$[M_A, M_B\} := M_A M_B - (-1)^{[A][B]} M_B M_A = f_{AB}{}^C M_C$$

where $[A] = \begin{cases} 0 & \text{even generator (bosonic)} \\ 1 & \text{odd generator (fermionic)} \end{cases}$

$\equiv$ identity

the Super-Jacobi identity, expressing associativity of the superalgebra, may be written

$$[M_A, [M_B, M_C\}\} + (-1)^{[A]([B]+[C])}[M_B, [M_C, M_A\}\} + (-1)^{[C]([A]+[B])}[M_C, [M_A, M_B\}\} \equiv 0$$

Thus, for example, for A, B and C all fermionic, one has

$$\left[Q_\alpha^i, \{Q_\beta^j, Q_\gamma^k\}\right] + \left[Q_\beta^j, \{Q_\gamma^k, Q_\alpha^i\}\right] + \left[Q_\gamma^k, \{Q_\alpha^i, Q_\beta^j\}\right] \equiv 0$$

The HLS analysis showed that the most general {odd, odd} anticommutator consistent with the Coleman-Mandula theorem in flat spacetime has the form

$$\{Q_\alpha^i, Q_\beta^j\} = 2\delta^{ij}(\gamma^m C)_{\alpha\beta} P_m + Z^{ij} C_{\alpha\beta} + \tilde{Z}^{ij}(\gamma_5 C)_{\alpha\beta}$$

where Z^{ij} and $\tilde{Z}^{ij}$ are central charges in the sense that they commute with the Q_α^i and the Poincaré generators.

In the following, we will generally consider situations without central charges. However, the importance of these for supersymmetric theories in diverse dimensions has now been recognised: supersymmetric branes (generalizations of membranes) are the basic carriers of central charges.

The HLS analysis shows the extent to which one can evade the Coleman + Mandula restrictions, permitting fermionic "bridges" Q_α^i between space-time and certain internal symmetries that can operate on the "i" index. One can, of course, also have further internal symmetries that commute with all of the N-extended super Poincaré generators; direct products of symmetry groups were always allowed by C+M.

Representations of supersymmetry on particle states

We will work directly from the supersymmetry algebra without central charges. The exercise can be repeated with the latter turned on, but the results are then more complicated.

In quantum language, each field can be Fourier decomposed into creation and annihilation operators. The creation operators create particle states from a vacuum state.

Begin by considering the true vacuum $|vac\rangle$, the state with no particles present. If supersymmetry is unbroken, one has $Q_\alpha^i|vac\rangle = 0$. Using the susy algebra, this implies $P^m|vac\rangle = 0$, i.e. the translations are unbroken $\Leftrightarrow$ the true vacuum is translation invariant.

Now consider particle states of mass M. Let $|M\rangle$ be a state representing a mass M particle at rest. For $P^m = (H, \vec{P})$, we have $H|M\rangle = M|M\rangle$, $\vec{P}|M\rangle = 0$.

Then, since $[P^m, Q_\alpha^i] = 0$, the states $Q_\alpha^i|M\rangle$ have the same 4-momentum as $|M\rangle$: they all represent mass M particles at rest. The full set of independent states of this sort spans a <u>representation</u> of supersymmetry.

Focus attention just on the subspace of the state vector space corresponding to this SUSY representation.

Within this subspace, replace P^m by their eigenvalues:

$$\{Q_\alpha^i, Q_\beta^j\} = 2M\delta^{ij}\delta_{\alpha\beta}$$

in Majorana representation since $\gamma^0\gamma_0 = \mathbb{1}_4$

We want to find the <u>irreducible</u> representations of this rest-frame superalgebra. Note that if one rescales the SUSY generators $Q_\alpha^i = \sqrt{M}\,\tilde{Q}_\alpha^i$, then the $\tilde{Q}_\alpha^i$ satisfy a <u>Clifford algebra</u> $\{\tilde{Q}_\alpha^i, \tilde{Q}_\beta^j\} = 2\delta^{ij}\delta_{\alpha\beta}$.

The fundamental representation of this algebra is well-known: we have $4N$ $\tilde{Q}_\alpha^i$ ($\alpha = 1\ldots4$, $i = 1\ldots N$) that satisfy the same algebra as the γ_A matrices $A = 1\ldots4N$ for the group $SO(4N)$. The fundamental representation for these Dirac matrices consists of $2^{2N}\times 2^{2N}$ matrices. So the space of states acted upon by these matrices has dimension 2^{2N}.

These states must be equally divided between bosons and fermions:

Every theory containing fermions has a conserved quantity corresponding to <u>fermion number modulo two</u> — fermions can only be created or annihilated in pairs. The corresponding symmetry is $\psi \to -\psi$: flip the sign of all spinors — the equations of motion must be covariant under this symmetry.

The corresponding quantum number (conserved quantity) is $(-1)^F$, F = fermion number

$(-1)^F = +1$ for bosons

$(-1)^F = -1$ for fermions

Now, Q_α^i are spinorial charges, so they exchange bosons and fermions. Thus, one must have

$$\{(-1)^F, Q_\alpha^i\} = 0.$$

For an operator $\mathcal{O}$ acting on the subspace of one-particle states of mass M within a given SUSY irrep, define the <u>trace</u> $\operatorname{tr}\mathcal{O} = \sum_k {}_k\langle|\mathcal{O}|\rangle_k$ where k runs over the basis.

From the above discussion of the $\tilde{Q}_\alpha^i$ Clifford algebra, we know that

$$\mathrm{tr}\,\mathbb{1} = 2^{2N} = \text{total dimension of the multiplet.}$$

However, we must have a zero trace for $(-1)^F$:

$$\{\tilde{Q}_\alpha^i, \tilde{Q}_\beta^j\} = 2\delta^{ij}\delta_{\alpha\beta} \quad \text{so } (\tilde{Q}_1^1)^2 = \mathbb{1} \quad (\text{unit op.})$$

thus
$$\begin{aligned} \mathrm{tr}(-1)^F &= \mathrm{tr}\left((-1)^F(\tilde{Q}_1^1)^2\right) \\ &= \mathrm{tr}\left(\tilde{Q}_1^1(-1)^F\tilde{Q}_1^1\right) \quad \text{cyclic property of trace} \\ &= -\mathrm{tr}\left((-1)^F\tilde{Q}_1^1\tilde{Q}_1^1\right) \quad \{(-1)^F, \tilde{Q}_\alpha^i\} = 0 \\ &= -\mathrm{tr}(-1)^F \end{aligned}$$

So we learn that $\mathrm{tr}(-1)^F = 0$, i.e. that for any multiplet of massive one-particle states

$$\boxed{\#\text{ of bosons} = \#\text{ of fermions}}$$

(True in general for all states of non-zero energy)

Thus, the total irrep. of 2^{2N} states must consist of 2^{2N-1} bosons + 2^{2N-1} fermions

Weyl representation and 2-component spinor notation.

Now build the fundamental mass M irreps in detail. Instead of the Majorana representation for the D=4 Dirac γ matrices, we now switch to the Weyl representation. Recall that when discussing spinors, the Lorentz group in D=4 is really the double cover of SO(3,1), the group

$$\mathrm{Spin}(3,1) \cong Sl(2,\mathbb{C}).$$

$Sl(2,\mathbb{C})$ is the complexification of SU(2), which is the double cover of SO(3). So the above isomorphism is the Minkowskian version for spinors of the Euclidean isomorphism $SO(4) \cong SO(3)\times SO(3)$ for integral spin reps.

The Weyl rep. accordingly uses a lot of SU(2) machinery.

In the Weyl representation, the γ matrices are

$$\gamma_m = \begin{pmatrix} 0 & i\sigma_{m\alpha\dot\beta} \\ i\bar\sigma_m^{\dot\alpha\beta} & 0 \end{pmatrix} \qquad \gamma_5 = \underbrace{\gamma_0\gamma_1\gamma_2\gamma_3}_{\text{diagonal}} = \begin{pmatrix} i\mathbb{1}_2 & 0 \\ 0 & -i\mathbb{1}_2 \end{pmatrix}$$

$\alpha, \dot\beta = 1,2$

where the Van der Waerden matrices $\sigma_{m\alpha\dot\beta}$ are 2×2 complex matrices related to the SU(2) Pauli matrices:

$$\sigma_{m\alpha\dot\beta} = (\mathbb{1}_2, \vec\sigma) \qquad (\text{pay attention to index positions!})$$
$$\bar\sigma_m^{\dot\alpha\beta} = (\mathbb{1}_2, -\vec\sigma)$$

The $\vec\sigma$ are the usual Pauli matrices $\begin{pmatrix}0&1\\1&0\end{pmatrix}$ $\begin{pmatrix}0&-i\\i&0\end{pmatrix}$ $\begin{pmatrix}1&0\\0&-1\end{pmatrix}$. Since these are Hermitean, we have

$$(\sigma^m_{\alpha\dot\beta})^* = \bar\sigma^m_{\dot\alpha\beta} \quad \text{where } \bar\sigma^{m\dot\alpha\beta} = \epsilon^{\dot\alpha\dot\gamma}\epsilon^{\beta\delta}\bar\sigma^m_{\dot\gamma\delta}$$

$$(\sigma^m_{\alpha\dot\beta})^T = \bar\sigma^m_{\dot\alpha\beta} \qquad \text{and } \epsilon^{\alpha\beta} = \begin{pmatrix}0&1\\-1&0\end{pmatrix} = \epsilon^{\dot\alpha\dot\beta}$$

In Weyl rep., the Charge Conjugation matrix is

$$C = -\gamma^0\gamma_2 = \begin{pmatrix} i\epsilon_{\alpha\beta} & 0 \\ 0 & i\epsilon^{\dot\alpha\dot\beta} \end{pmatrix}$$

where $\epsilon_{\alpha\beta}$ with lowered indices satisfies

$$\epsilon^{\alpha\gamma}\epsilon_{\gamma\beta} = \delta^\alpha{}_\beta \qquad \text{so } \epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta}$$

and similarly $\epsilon_{\dot\alpha\dot\beta} = -\epsilon^{\dot\alpha\dot\beta}$, $\epsilon_{\dot\alpha\dot\gamma}\epsilon^{\dot\gamma\dot\beta} = \delta_{\dot\alpha}{}^{\dot\beta}$ for the dotted indices.

The dotted and undotted indices correspond to Weyl spinor types. In Weyl rep., a general Dirac spinor is written $\Psi_{\text{Dirac}} = \begin{pmatrix}\psi_\alpha \\ \bar\chi^{\dot\alpha}\end{pmatrix}$ where ψ_α and $\bar\chi^{\dot\alpha}$ are independent 2-component spinors.

A right-handed spinor, projected with $\frac{1+i\gamma_5}{2}$, is represented by $\begin{pmatrix} 0 \\ \bar\chi^{\dot\alpha} \end{pmatrix} = \Psi_{(R)}$; $\left(\frac{1+i\gamma_5}{2}\right)\psi_{(R)} = \psi_{(R)}$ $(0, 1/2)$ Lorentz rep.

while a left-handed spinor projected with $\frac{1-i\gamma_5}{2}$ is represented by $\begin{pmatrix} \psi_\alpha \\ 0 \end{pmatrix} = \Psi_{(L)}$; $\left(\frac{1-i\gamma_5}{2}\right)\psi_{(L)} = \psi_{(L)}$ $(1/2, 0)$ Lorentz rep.

A Majorana spinor has conjugated left and right projections, so

$$\Psi_{\text{Majorana}} = \begin{pmatrix} \psi_\alpha \\ \bar\psi^{\dot\alpha} \end{pmatrix} \qquad \alpha, \dot\alpha = 1, 2$$

with $(\psi_\alpha)^* = \bar\psi_{\dot\alpha}$

where again raising and lowering of the indices is done with ϵ:

$$\psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \qquad \bar\psi^{\dot\alpha} = \epsilon^{\dot\alpha\dot\beta}\bar\psi_{\dot\beta}$$

$$\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \qquad \bar\psi_{\dot\alpha} = \epsilon_{\dot\alpha\dot\beta}\bar\psi^{\dot\beta}$$

So, instead of the 4-component Dirac notation, one can use 2-component ψ_α, $\bar\chi_{\dot\alpha}$ Van der Waerden notation for $D=4$ spinors. Moreover, one can write vectors in terms of $(\alpha,\dot\alpha)$ bi-spinors:

$$V_{\alpha\dot\alpha} = \sigma^m_{\alpha\dot\alpha} V_m \qquad \text{transforms in } (1/2, 1/2) \text{ Lorentz rep.}$$
$$= \begin{pmatrix} -V_0 + V_3 & V_1 - iV_2 \\ V_1 + iV_2 & -V_0 - V_3 \end{pmatrix}.$$

Using the relations

$$(\sigma^m\bar\sigma^n + \sigma^n\bar\sigma^m)_\alpha{}^\beta = -2\eta^{mn}\delta_\alpha^\beta$$
$$(\bar\sigma^m\sigma^n + \bar\sigma^n\sigma^m)^{\dot\alpha}{}_{\dot\beta} = -2\eta^{mn}\delta^{\dot\alpha}_{\dot\beta}$$

yielding the completeness relations

$$\operatorname{Tr}\sigma^m\bar\sigma^n = -2\eta^{mn} \qquad \sigma^m_{\alpha\dot\alpha}\,\bar\sigma_m^{\dot\beta\beta} = -2\,\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta}$$

one can go back to 4-component vectors:

$$V^m = -\tfrac{1}{2}\bar\sigma^{m\,\dot\alpha\alpha} V_{\alpha\dot\alpha}.$$

While we are on the subject of notation, let us also give the Lorentz generators in Weyl representation:

$$M^{nm}{}_\alpha{}^\beta = \Sigma^{nm}{}_\alpha{}^\beta = \tfrac{i}{4}\left(\sigma^n_{\alpha\dot\alpha}\bar\sigma^{m\,\dot\alpha\beta} - \sigma^m_{\alpha\dot\alpha}\bar\sigma^{n\,\dot\alpha\beta}\right)$$
$$M^{nm\,\dot\alpha}{}_{\dot\beta} = \Sigma^{nm\,\dot\alpha}{}_{\dot\beta} = \tfrac{i}{4}\left(\bar\sigma^{n\,\dot\alpha\alpha}\sigma^m_{\alpha\dot\beta} - \bar\sigma^{m\,\dot\alpha\alpha}\sigma^n_{\alpha\dot\beta}\right)$$

If one uses these to transform the undotted and dotted indices on $\sigma^m_{\alpha\dot\beta}$, and the usual vector rep. generator to transform the "m" index, one finds that $\sigma^m_{\alpha\dot\beta}$ is a Lorentz invariant.

Now return to the supersymmetry algebra and write it using the Van der Waerden notation:

$$\{Q_\alpha{}^i, Q_\beta{}^j\} = 0 = \{\bar Q_{\dot\alpha i}, \bar Q_{\dot\beta j}\}$$
$$\{Q_\alpha{}^i, \bar Q_{\dot\beta j}\} = 2\,\delta^i_j\,P_{\alpha\dot\beta}$$

Note here a convention relating to the i,j indices: we take $(Q_\alpha{}^i)^* = \bar Q_{\dot\alpha i}$ with lowered index. This is done to reveal an outer automorphism of the SUSY algebra that appears when using 2-component notation: the algebra is covariant if we transform the lowered "i" indices in the fundamental representation of $U(N)$ and the raised "i" indices in the conjugate fundamental rep. The Kronecker tensor $\delta^i{}_j$ is then a $U(N)$ invariant.

The other outer automorphism of the SUSY algebra is of course the Lorentz group.

Neither $U(N)$ nor Lorentz is required by supersymmetry invariance, because these are <u>outer</u> automorphisms (the $U(N)$ and Lorentz generators do not themselves appear in the Q, P algebra).

While there is little interest in non-Lorentz-invariant supersymmetric models, the choice whether to incorporate $U(N)$ invariance or one of its subgroups is very model dependent.

Going into the subspace of mass M particles at rest, the SUSY algebra becomes

$$\{Q_\alpha{}^i, Q_\beta{}^j\} = 0 = \{\bar{Q}_{\dot\alpha i}, \bar{Q}_{\dot\beta j}\}.$$

$$\{Q_\alpha{}^i, \bar{Q}_{\dot\beta j}\} = 2M\,\delta_{\alpha\dot\beta}\,\delta^i{}_j$$

Now rescale the Q, $\bar{Q}$ a bit differently from before:
let $\xi_\alpha{}^i = \frac{1}{\sqrt{2M}} Q_\alpha{}^i$. $\xi^+_{\alpha i} = \frac{1}{\sqrt{2M}} \bar{Q}_{\alpha i}$.

Note that since we are restricting here to the <u>rest</u> frame of the mass M particles, it makes no sense to keep track of covariance on the spinor indices beyond the Lorentz little group $SU(2)$ (double cover of $SO(3)$). This subgroup transforms dotted and undotted spinor indices equivalently, so in the rest frame there is no point in distinguishing them.

Thus, we have an algebra of $2N$ fermionic creation operators $\xi^+_{\alpha i}$ and $2N$ fermionic annihilation operators $\xi_\alpha{}^i$:

$$\{\xi_\alpha{}^i, \xi_\beta{}^j\} = 0 = \{\xi^+_{\alpha i}, \xi^+_{\beta j}\}$$

$$\{\xi_\alpha{}^i, \xi^+_{\beta j}\} = \delta_{\alpha\beta}\,\delta^i{}_j$$

<u>Massive particle supermultiplets in detail</u>

We start building the multiplet from a reference state or set of states that are annihilated by the annihilation operators. By analogy with the familiar bosonic oscillator construction, this state or set of states is called a "Clifford vacuum;" but it should be remembered that these are mass M single particle states, not the true vacuum.

Let $|\Omega\rangle$ be a Clifford vacuum; take it to be a single bosonic state to begin. Thus $\xi_\alpha{}^i|\Omega\rangle = 0$ for all α, i.

Create the rest of the multiplet by applying the $\xi^+_{\alpha i}$ creation operators as many times as possible. This process terminates with the application of all $2N$ $\xi^+_{\alpha i}$ together, since a $(2N+1)$st creation operator would necessarily be a repetition of one already applied, and $(\xi^+_{\alpha i})^2 = 0$ would then kill the result.

Accordingly, one has the sequence of states

	$\lvert\Omega\rangle$,	$\xi^+_{\alpha i}\lvert\Omega\rangle$,	$\xi^+_{\alpha i}\xi^+_{\beta j}\lvert\Omega\rangle$,	$\ldots$,	$\xi^+_{\alpha_1 i_1}\xi^+_{\alpha_2 i_2}\cdots\xi^+_{\alpha_{2N} i_{2N}}\lvert\Omega\rangle$
multiplicity:	1	$2N$	$\frac{2N(2N-1)}{2}$ $(\alpha,i)\neq(\beta,j)$		pairs (α_k, i_k) all different } 1 state only

These multiplicities sum up to 2^{2N} as expected for a SUSY irrep:

$$1 + 2N + \frac{2N(2N-1)}{2} + \cdots + 2N + 1 = (1+1)^{2N} = 2^{2N}$$

As a matter of notation, instead of writing down just the independent creation operator products in this construction, one tends to let the (α_k, i_k) index pairs run over all possible values, but then recalls that the resulting tensors in these "double indices" must be totally antisymmetric since $\{\xi^+_{\alpha i}, \xi^+_{\beta j}\} = 0$.

Illustrate the construction by the $N=1$ case: simple supersymmetry, generated by a single Majorana spinor charge Q, acting on a singlet bosonic Clifford vacuum $|\Omega\rangle$

	$\|\Omega\rangle$	$\xi^+_\alpha\|\Omega\rangle$	$\xi^+_\alpha \xi^+_\beta\|\Omega\rangle$	(no further possibilities since $\xi^+_\alpha\xi^+_\beta\xi^+_\gamma = 0$ for $N=1$)
mult.	1	2	$\frac{2\cdot 1}{2!} = 1$	totally antisymm, 3 indices in 2 values only → vanishes
type	bose	fermi	bose	

The final tensorial combination is antisymmetric in α, β. One of the great virtues of 2-component notation is that antisymmetric 2-tensors must be proportional to $\epsilon_{\alpha\beta}$. So write $\xi^+_\alpha \xi^+_\beta |\Omega\rangle = \epsilon_{\alpha\beta}|X\rangle$. To find $|X\rangle$, just take the "trace" with $\epsilon^{\beta\alpha}$: $2|X\rangle = \xi^{+\beta}\xi^+_\beta|\Omega\rangle$

'$\xi^+_\alpha = \xi^{+\beta}$

${}_{\alpha\beta} = 2$

So $\xi^+_\alpha \xi^+_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\xi^{+\gamma}\xi^+_\gamma|\Omega\rangle$

involves just one independent state.

Overall, we have $4 = 2^{2\cdot 1}$ states: 2 bose + 2 fermi; i.e. two scalar states and one Weyl LH (undotted) spinor. Writing $\psi_\alpha = \xi^+_\alpha|\Omega\rangle$ and then taking $\begin{pmatrix}\psi_\alpha \\ \bar\varphi^{\dot\alpha}\end{pmatrix}$, the 2-cpt Weyl spinor can be traded in for a 4-component Majorana spinor.

In the general case with N-extended supersymmetry $(Q_\alpha^i, \bar Q_{\dot\alpha i})$ starting from a singlet bosonic Clifford vacuum, application of even numbers of the $\xi^+_{\alpha i}$ will give bosonic states while application of odd numbers of the ξ^+_α will give fermionic states, with multiplicities.

$$1 + \frac{2N(2N-1)}{2} + \dots + 1 = 2^{2N-1} \quad \text{bosons} \quad \text{even terms}$$

$$2N + \frac{2N(2N-1)(2N-2)}{3!} + \dots + 2N = 2^{2N-1} \quad \text{fermions} \quad \text{odd terms}$$

To see how one separates such an extended multiplet into standard (Lorentz, internal) irreps, consider the $N=2$ multiplet, starting again from a singlet bosonic Clifford vacuum state $|\Omega\rangle$: we have 2×2 $\xi^+_{\alpha i}$, so can have up to $\xi^+\xi^+\xi^+\xi^+$ operators:

$$|\Omega\rangle,\ \xi^+_{\alpha i}|\Omega\rangle,\ \xi^+_{\alpha i}\xi^+_{\beta j}|\Omega\rangle,\ \xi^+_{\alpha i}\xi^+_{\beta j}\xi^+_{\gamma k}|\Omega\rangle,\ \xi^+_{\alpha i}\xi^+_{\beta j}\xi^+_{\gamma k}\xi^+_{\delta l}|\Omega\rangle$$

Now start pulling these apart into (Lorentz, $SU(2)$) irreps. It is easier to consider the $SU(2)$ automorphism group rather than $U(2)$ because ϵ^{ij} is then a numerical invariant and we can use it to pull things apart.

$|\Omega\rangle$: Lorentz scalar, $SU(2)$ singlet — 1 state — bose

$\xi^+_{\alpha i}|\Omega\rangle$: LH Weyl spinor, $SU(2)$ doublet — $2\times 2 = 4$ states — fermi

$\xi^+_{\alpha i}\xi^+_{\beta j}|\Omega\rangle$: $\frac{4\times 3}{2} = 6$ bose states. Now separate into irreps.

The double-index structure $(\alpha i, \beta j)$ must be overall antisymmetric since $\{\xi^+_{\alpha i}, \xi^+_{\beta j}\} = 0$.

Focus on just the α, β indices: these can be symmetric $(\alpha\beta)$ or antisymmetric $[\alpha\beta]$. Similarly, the i, j indices can be symmetric (ij) or antisymmetric $[ij]$. To have overall antisymmetry in double indices $[\alpha i\ \beta j]$ the only possibilities are $(\alpha\beta)[ij]$ and $[\alpha\beta](ij)$. Since the antisymmetric combinations are proportional to $\epsilon_{\alpha\beta}$ or ϵ_{ij}, we can extract the independent components by tracing with these:

$$\xi^+_{\alpha i}\xi^+_{\beta j}|\Omega\rangle \longrightarrow \underbrace{\xi^+_{(\alpha i}\xi^+_{\beta) j}\epsilon^{ij}|\Omega\rangle}_{(\alpha\beta):\ \text{spin } 1,\ SU(2)\ \text{singlet}} \oplus \underbrace{\epsilon^{\alpha\beta}\xi^+_{\alpha(i}\xi^+_{\beta j)}|\Omega\rangle}_{\text{spin } 0,\ SU(2)\ \text{triplet } (ij)}$$

$(\alpha\beta)$ Lorentz rep: $(1, 0)$ i.e. spin 1

2×2 symm. tensor: 3 indep. components

$3 + 3 = 6$, as required

If one insists on writing the decomposition of these states in detail, it is:

$$\xi^+_{\alpha i}\xi^+_{\beta j}|\Omega\rangle = -\tfrac{1}{2}\epsilon_{ij}\,\xi^+_{(\alpha k}\xi^+_{\beta)\ell}\,\epsilon^{k\ell}|\Omega\rangle + \tfrac{1}{2}\epsilon_{\alpha\beta}\,\xi^{+\gamma}{}_{(i}\,\xi^+_{\gamma j)}|\Omega\rangle$$

where ϵ_{ij} is defined similarly to $\epsilon_{\alpha\beta}$, i.e. $\epsilon^{ij}\epsilon_{jk}=\delta^i{}_k$

Now continue on to the cubic possibility, allowed for $N=2$ since there are a total of $2\times 2=4$ different $\xi^+_{\alpha i}$ now.

$\xi^+_{\alpha i}\xi^+_{\beta j}\xi^+_{\gamma k}|\Omega\rangle$ cannot be totally antisymmetric in $\alpha\beta\gamma$ nor in ijk, because each of these index types runs over just 2 values. So we have no spin 3/2 nor
III·4 SU(2) $\underline{4}$ reps. here. Instead, we must have mixed symmetry in both types of indices, i.e. terms proportional to $\epsilon^{\beta\gamma}\epsilon^{ij}\xi^+_{\alpha i}\xi^+_{\beta j}\xi^+_{\gamma k}|\Omega\rangle := |\lambda_{\alpha k}\rangle$.

Note that $\epsilon^{\alpha\beta}\epsilon^{ij}\xi^+_{\alpha i}\xi^+_{\beta j} = 0$ (symm. in $\xi^+_{\alpha i}, \xi^+_{\beta j}$) So the antisymmetrization in $|\lambda_{\alpha k}\rangle$ must be as shown.

Thus, the cubic $\xi^+\xi^+\xi^+|\Omega\rangle$ states form a spin 1/2, SU(2) doublet representation only.

If one insists on writing this out explicitly, one has

$$\xi^+_{\alpha i}\xi^+_{\beta j}\xi^+_{\gamma k}|\Omega\rangle = \frac{1}{8}\left[\begin{array}{l}\epsilon_{\beta\gamma}\epsilon_{ij}|\lambda_{\alpha k}\rangle + \epsilon_{\gamma\alpha}\epsilon_{jk}|\lambda_{\beta i}\rangle + \epsilon_{\alpha\beta}\epsilon_{ki}|\lambda_{\gamma j}\rangle \\ -\epsilon_{\gamma\beta}\epsilon_{ik}|\lambda_{\alpha j}\rangle - \epsilon_{\alpha\gamma}\epsilon_{ji}|\lambda_{\beta k}\rangle - \epsilon_{\beta\alpha}\epsilon_{kj}|\lambda_{\gamma i}\rangle\end{array}\right]$$

Proceeding on to the quartic level in ξ^+ operators, there is only one independent possibility because each of the four $\xi^+_{\alpha i}$ operators must appear once only. Consequently, $\xi^+_{\alpha i}\xi^+_{\beta j}\xi^+_{\gamma k}\xi^+_{\delta \ell}|\Omega\rangle$ must equal a numerically invariant tensor times

$\epsilon^{\alpha\beta}\epsilon^{\gamma\delta}\epsilon^{ik}\epsilon^{j\ell}\xi^+_{\alpha i}\xi^+_{\beta j}\xi^+_{\gamma k}\xi^+_{\delta\ell}|\Omega\rangle$, which is a spin 0, SU(2) singlet. Note again that the antisymmetrizations must be "staggered" as shown because $\epsilon^{\alpha\beta}\epsilon^{ij}\xi^+_{\alpha i}\xi^+_{\beta j} = 0$.

Writing out the full tensorial equation for the quartic level in ξ^+ oscillators is left as an exercise.

Summing up, we have the following content in the basic massive $N=2$ supermultiplet without central charges:

5 states	5 spin 0 in SU(2) reps. $\underline{1}+\underline{1}+\underline{3}$
$4\times 2 = 8$ states	4 spin 1/2, split into 2 spin doublets carrying SU(2) $\underline{2}$
3 states	1 spin 1, SU(2) singlet

Overall, we thus have $5+3=8$ bosonic and $4\times 2 = 8$ fermionic states; # bose = # fermi as expected, and with a total multiplicity for the entire susy irrep of $16 = 2^{2\cdot 2}$ as expected from our initial discussion.

So far, we have discussed only the fundamental supermultiplets in which the Clifford vacuum state $|\Omega\rangle$ is a bosonic singlet. One can generalize this to cases where $|\Omega\rangle$ carries <u>spin</u> or <u>internal symmetry</u> representations. The resulting supermultiplet is obtained by taking the <u>tensor product</u> of the rep. carried by $|\Omega\rangle$ with the reps. carried by the ξ^+ operators.

For example, if $|\Omega_j\rangle$ carries a spin j representation, then it really is a set of $2j+1$ states transforming under the same $SU(2)_{spin}$ that we used to classify the ξ^+ products.

Go back to $N=1$ and consider the $\xi^+_\alpha |\Omega_j\rangle$ states. Since we have the Clebsch-Gordon decomposition

$$j \otimes \tfrac{1}{2} = (j+\tfrac{1}{2}) \oplus (j-\tfrac{1}{2}) \qquad \text{for } j \neq 0$$

we find that our $N=1$ massive multiplet becomes

	$\lvert\Omega_j\rangle$	$\xi^+_\alpha \lvert\Omega_j\rangle$	$\epsilon^{\alpha\beta}\xi^+_\alpha \xi^+_\beta \lvert\Omega_j\rangle$
spin	j	$(j+\frac{1}{2}) \oplus (j-\frac{1}{2})$	j
mult.	$2j+1$	$2j+2 \;+\; 2j$	$2j+1$

i.e. a multiplet containing $4j+2$ bosons and $4j+2$ fermions organised into spins $(j-\frac{1}{2}, j, j, j+\frac{1}{2})$ for $j \neq 0$

Note that the Clifford vacuum states sit in the middle of the multiplet. The second spin j multiplet in the middle is obtained by applying the maximal product $\epsilon^{\alpha\beta}\xi^+_\alpha \xi^+_\beta$, and is hence annihilated by application of another ξ^+_γ operator. In this sense, it can also be considered to be a set of Clifford vacuum states, except with the ξ^+_α taken to be the annihilation operators. Starting from this "conjugate" Clifford vacuum set, one can reobtain the whole multiplet by applying products of the ξ_α which are thus seen as conjugate creation operators.

Things work similarly if $|\Omega\rangle$ carries an internal symmetry representation.

Overall, the total dimension of a multiplet is

$$D = 2^{2N} \times d_\Omega \qquad \text{where } d_\Omega = \text{dim. of Clifford vacuum}$$

Examples of massive Supersymmetry representations (without central charges)

Spin of the Clifford vacuum is j_Ω ("superspin")

$N \backslash j$	2	3/2	1	1/2	0	j_Ω
1				1	2	0
			1	2	1	1/2
		1	2	1		1
	1	2	1			3/2
2			1	4	5	0
		1	4	5+1	4	1/2
	1	4	5+1	4	1	1
3		1	6	14	14	0
	1	6	14+1	14+6	14	1/2
4	1	8	27	48	42	0

In general, spins range from 0 to $N/2$ for the fundamental $j_\Omega = 0$ irrep. For reps. starting from Clifford vacua with spin j_Ω, the spin range is $\max(0, j_\Omega - \frac{N}{2})$ to $j_\Omega + \frac{N}{2}$.

Massless particle supermultiplets

If $M=0$, there is no rest frame, so the above analysis needs to be changed. We still follow Wigner and choose a convenient standard momentum by making appropriate boosts and rotations:

$$P^{ref.}_m = \frac{1}{4}(-1,0,0,1) \qquad P_m P^m = 0 \text{ , light-like}$$

Then $\sigma^m P_m = \frac{1}{4}(\mathbb{1}+\sigma_3) = \begin{pmatrix} 1/2 & 0 \\ 0 & 0 \end{pmatrix}$, so the two-component form of the supersymmetry algebra, restricted to states with the above P_m, is

$$\{Q_1{}^i, \bar{Q}_{\dot{1}j}\} = \delta^i_j \qquad \{Q_2{}^i, \bar{Q}_{\dot{2}j}\} = 0$$

$$\{Q_1{}^i, \bar{Q}_{\dot{2}j}\} = \{\bar{Q}_{\dot{1}i}, Q_2{}^j\} = 0$$

algebra of just N fermionic creation & annihilation ops.

$$\{Q_\alpha{}^i, Q_\beta{}^j\} = \{\bar{Q}_{\dot{\alpha}i}, \bar{Q}_{\dot{\beta}j}\} = 0$$

The smallest irrep. of this algebra will have $Q_2{}^i$ and $\bar{Q}_{\dot{2}j}$ realized trivially, i.e. both of these $\alpha,\dot{\alpha}=2$ operator types will annihilate the states. This is now possible because of the zero eigenvalue of $\sigma^m P_m$.

The $Q_1{}^i$ will be taken to be annihilation operators on the reference Clifford vacuum state. The $\bar{Q}_{\dot{1}}{}^i$ will then give the other states upon repeated application.

To organise the states of the supermultiplet, we will need to keep track of the little group representations. An obvious part of the little group of the standard momentum $P^{ref.}_m = \frac{1}{4}(-1,0,0,1)$ consists of the internal symmetry acting on the i,j indices (which we are taking to be $SU(N)$ for simplicity). The space-time part is less obvious: the little group of $P^{ref.}_m$ is actually E_2, the Euclidean group of the plane, whose generators are composed of combinations of rotations and boosts that leave $P^{ref.}_m$ invariant. E_2 contains two "translational" generators and a $U(1) \cong SO(2)$ "rotational" compact subgroup. Only the latter provides discrete eigenvalues to classify representations by.

The normalized $U(1)$ little group generator is Helicity

$$\underline{h} = \frac{\epsilon^{ijk} p^i M^{jk}}{(p\cdot p)^{1/2}} \qquad i,j,k = 1,2,3$$

For the dotted spinor representation useful for classifying different states in the massless supermultiplet, and for P^{ref}_m as given, one has the helicity operator

$$\underline{h} = \frac{1}{2}\sigma_3 \quad \text{in the spin 1/2 representation,}$$

where $M^{nm\,\dot{\alpha}}{}_{\dot{\beta}} = \frac{1}{4}(\bar{\sigma}^n\sigma^m - \bar{\sigma}^m\sigma^n)^{\dot{\alpha}}{}_{\dot{\beta}}$

The creation operators $\bar{Q}_i = \begin{pmatrix} \bar{Q}_{\dot{1}i} \\ 0 \end{pmatrix}$ thus have a commutation relation

$$[\underline{h}, \bar{Q}_i] = \frac{1}{2}\bar{Q}_i$$

so application of a $\bar{Q}_{\dot{1}i}$ creation operator raises the helicity of a massless state by 1/2.

Massless particles of "spin j" in fact have only two degrees of freedom, corresponding to helicities $\pm j$.

Applying the creation operators repeatedly as many times as possible, we can thus organise the supermultiplet into various spin states.

Start building the massless supermultiplet thus from a Clifford vacuum state $|\Omega_h\rangle$ of helicity h:
$\underline{h}\,|\Omega_h\rangle = h\,|\Omega_h\rangle$; $Q_1{}^i\,|\Omega_h\rangle = 0$

Since $\bar{Q}_{1i}$ raise helicity, the Clifford vacuum state $|\Omega_h\rangle$ will be the state of lowest helicity in the supermultiplet. Obtain the rest of the supermultiplet by repeated application of the $\bar{Q}_{1i}$:

	$\vert\Omega_h\rangle$	$\bar{Q}_{1i}\vert\Omega_h\rangle$	$\bar{Q}_{1i}\bar{Q}_{1j}\vert\Omega_h\rangle$	...	$\bar{Q}_{1i_1}\bar{Q}_{1i_2}\cdots\bar{Q}_{1i_N}\vert\Omega_h\rangle$	
helicity	h	$h+1/2$	$h+1$		$h+\frac{N}{2}$	
multiplicity	1	N	$\frac{N(N-1)}{2}$	...	$\frac{N!}{N!}=1$	totally antisym in N indices $i_1\cdots i_N$

In total, we have here $(1+1)^N = 2^N$ states, half fermionic and half bosonic. We can prove the equality of boson and fermion numbers using
$1 = \frac{1}{2}\left(Q_1{}^1\bar{Q}_{11} + \bar{Q}_{11}Q_1{}^1\right)$
inserted into $\mathrm{tr}\,(-1)^F$ and repeating the argument made for the massive supermultiplets.

The simplest example of such a massless multiplet is for $N=1$, where the above basic supermultiplet consists just of two adjacent helicities: h, $h+1/2$.

Is there a corresponding massless field theory? No.

Reason: in field theory, one always gets balanced positive and negative helicities, because field theories always have PCT invariance and helicities are flipped by the PCT operator.

What one may have is a massless field theory describing a combination of two basic supermultiplets:
$(h, h+1/2) \oplus (-h-1/2, -h)$

Accordingly, for $N=1$ starting from $|\Omega_h\rangle$ with $h=1/2$ we obtain the <u>$N=1$ Super Maxwell multiplet</u>:

$$\underbrace{(-1,-\tfrac12)\oplus(\tfrac12,1)}$$

photino

photon

The Lagrangian formulation of this theory contains two fields: a Majorana spinor field $\psi_\alpha(x)$ (or equivalently a 2-cpt. Weyl spinor) and a real vector field $A_m(x)$. Since $A_m(x)$ contains more components than we want as physical degrees of freedom, the action must be invariant under Maxwell <u>gauge invariance</u> $\delta A_m(x) = \partial_m\Lambda(x)$. We shall return to the detailed Lagrangian formulation of this theory shortly.

The need to take a formally reducible combination of supermultiplets to construct massless field theories invariant under PCT is a general feature, but there are some exceptions.

Consider the $N=4$ massless multiplet starting from $h=-1$:

$\vert h\rangle$:	$\vert -1\rangle$	$\vert -1/2\rangle$	$\vert 0\rangle$	$\vert 1/2\rangle$	$\vert 1\rangle$
SU(4) rep.	·	□	$\boxminus$	$\bar{\square}$	▯
mult	1	4	6	$\bar{4}$	1

"$N=4$ Super Maxwell theory"

This massless multiplet is naturally PCT <u>self-conjugate</u>: it naturally balances positive and negative helicities without the need to add a conjugate multiplet.

Clearly, self-conjugate multiplets can only occur for $h_{max} = \frac{N}{4}$, so that helicities (and SU(N) reps.) are balanced in the N steps of $\Delta h = 1/2$ between $h_{min} = -\frac{N}{4}$ and $h_{max} = \frac{N}{4}$.

(Note that the middle 6 rep. of $SU(4)$ in the N=4 S. Maxwell mult. is pseudo-real, i.e. $(\phi_{ij})^* := \bar{\phi}^{ij} = \frac{1}{2}\epsilon^{ijkl}\phi_{kl}$. This is correlated with overall self-conjugacy of the N=4 Super Maxwell multiplet.)

Here is another famous self-conjugate theory, for N=8 starting from $h=-2$:

$\|h\rangle$	$\|-2\rangle$	$\|-\frac{3}{2}\rangle$	$\|-1\rangle$	$\|-\frac{1}{2}\rangle$	$\|0\rangle$	$\|\frac{1}{2}\rangle$	$\|1\rangle$	$\|\frac{3}{2}\rangle$	$\|2\rangle$
SU(8) rep.	·	□	⊟	⊟	⊟	⊟	⊟	⊟	⊟
mult.	1	8	28	56	70	$\overline{56}$	$\overline{28}$	$\overline{8}$	1

"N=8 Supergravity"

{middle 70 is also pseudo-real here: $(\phi_{ijkl})^* := \bar{\phi}^{ijkl} = \frac{1}{4!}\epsilon^{ijklmnpq}\phi_{mnpq}$

The two states with helicities ± 2 can be identified with the two polarization states of the graviton.
The 2×eight helicity $\pm 3/2$ states are called the gravitini.

Let's make a table of examples of massless representations. Multiplicities for $h \neq 0$ are to be understood as $\pm h$ PCT pairs, which one often roughly refers to as "spin h".

N \ $\|h\|$		2	3/2	1	1/2	0	
massless WZ mult.	1				1	1+1	
N=1 super Maxwell	1			1	1		
	1		1	1			
N=1 supergravity	1	1	1		.		
hypermultiplet	2				2	2+2	
N=2 S. Maxwell	2			1	2	1+1	
	2		1	2	1		
N=2 SG	2	1	2	1			
N=4 S. Maxwell	4			1	4	6	self-conjugate
N=4 SG	4	1	4	6	4	1+1	
N=8 SG	8	1	8	28	56	70	self-conjugate

Why isn't the N=2 hypermultiplet indicated to be self-conjugate? The corresponding multiplet would have helicities $\pm\frac{1}{2}$, 0, 0 ; however, there is no known N=2 supersymmetric theory with this content. A theory does exist for the doubled multiplet (hypermultiplet)

Similarly, one might have expected a self-conjugate N=6 theory starting from $h=-3/2$, but this runs into similar problems with a Lagrangian field theory realization in undoubled form.

So we see that it is not always enough to have an available formal supermultiplet in order to find a corresponding supersymmetric field theory. This is particularly true when we extend the search to interacting field theories.

In flat space, we have interacting supersymmetric theories for $N \leq 4$ extended supersymmetry. The self-conjugate N=4 Super Maxwell multiplet is the largest containing helicities $|h| \leq 1$. The interacting version of this is N=4 Super Yang-Mills, which we will discuss later.

For helicities $|h| \geq 1$, we have interacting supergravity theories. The maximal supergravity is based on the self-conjugate N=8 supermultiplet.

For helicities $|h| \geq 2$, we do not know any interacting massless field theories with finite numbers of fields. To see why, we are again in the domain of Coleman and Mandula:
a) massless fields of spin $j \geq 1$ must be described by gauge fields to exclude the unwanted middle helicities.
b) in an interacting theory, gauge invariance requires that gauge fields couple to conserved currents. For example, a spin-two field h_{mn} interacts via a conserved stress tensor

$\kappa h_{mn} T^{mn}$, $\kappa^2 = 8\pi G$ first interaction, perturbative expansion

For consistency with the $\delta h_{mn} = \partial_{(m}\xi_{n)}$ initial gauge symmetry, the stress tensor must be conserved, $\partial_m T^{mn} = 0$, as a consequence of the field equations. Continuing on perturbatively building a gauge-invariant theory for h_{mn}, one eventually obtains a generally covariant theory (the hard way!)

Similarly, interacting spin 1 massless theories require a conserved vector current, $\partial_m J^m$, as a consequence of the field equations. If J^m is built out of the set of vector gauge fields themselves, one can thus perturbatively obtain Yang-Mills theory.

c) For massless fields of spins $j \le 2$, interacting theories can be found, consistently with the strictures of Coleman and Mandula. Thus, for spin 1 couplings one has conserved charges $\Leftrightarrow$ symmetry generators $Q = \int d^3x\, J^0$ and for spin 2 couplings one has $P^m = \int d^3x\, T^{m0}$, conserved 4-momentum. This is allowed by Coleman and Mandula because it is a Poincaré group generator, but it is also thereby essentially <u>unique</u>: the interacting spin-two theory must be <u>General Relativity</u>.

d) What would happen for spins $j > 2$? Take $j = 3$ as an example. One would have a gauge field $h_{mnp}(x) = h^{(s)}_{(mnp)}$ with a gauge symmetry $\delta h_{mnp} = \partial_{(m}\xi_{np)}$ to remove the unwanted helicities $\pm 2, \pm 1$ and 0. An interacting theory would require couplings $h_{mnp} T^{mnp}$ to a conserved current, $\partial_m T^{mnp} = 0$, via the equations of motion. But if we had such a current T^{mnp}, then $Q^{mn} = \int d^3x\, T^{mn0}$ would be a conserved charge not corresponding to a Poincaré group generator. By Coleman and Mandula, this would not be compatible with an interacting field-theory.

e) Extending the framework to supersymmetric theories does not appear to help with the $j \le 2$ limitation. For example, attempts to build a "hypergravity" including spin 5/2 fail for reasons similar to the difficulties with spin 3.

Consequently, $N=8$ is the maximal extension of supersymmetry for an interacting $D=4$ field theory.

<u>N=1 Super-Maxwell Field Theory</u>

As we have seen, the fields needed to describe the massless $N=1$ supermultiplet containing spins $(1, 1/2)$ are

A_m – gauge vector $\qquad \psi$ – Majorana spinor

The free-field action is

$$I = \int d^4x \left(-\tfrac{1}{4} F_{mn} F^{mn} - \tfrac{i}{2} \bar{\psi} \not{\partial} \psi \right)$$

where $F_{mn} = \partial_m A_n - \partial_n A_m \qquad \not{\partial}\psi = \gamma^m \partial_m \psi$

Why the factor of "i" even though ψ is Majorana? Taking the cue from our treatment of anticommuting numbers as $SO(2N)$ gamma matrices, complex conjugation of products of anticommuting numbers is most naturally defined as hermitean conjugation of the underlying gamma matrices, so $(\Theta_1\Theta_2)^* = \Theta_2^*\Theta_1^* = -\Theta_1^*\Theta_2^*$, so even though $\Theta_1^* = \Theta_1$, $\Theta_2^* = \Theta_2$, one has $(\Theta_1\Theta_2)^* = -\Theta_1\Theta_2$.

Thus, $(\bar{\psi}\not{\partial}\psi)^* = \partial_m \psi^T (\gamma^0\gamma^m)^T \psi = -\psi^T\gamma^0\gamma^m\partial_m\psi$

in Majorana rep (also true in general) $\qquad = -\bar{\psi}\not{\partial}\psi$

So a factor of "i" is needed for reality of the action.

Now proceed to find the supersymmetry transformations.

<u>Dimensional analysis</u>:

Since $\{Q, Q\} \sim P$, Q has dimension $1/2$ in mass units. Correspondingly, the spinor supersymmetry parameter ϵ must have $\dim \epsilon = -1/2$ to make $\bar{\epsilon}Q$ dimensionless.

(with $\hbar = 1$)

In mass unit dimensions, we also have

$$\left.\begin{array}{l}\dim I = 0\\ \dim d^4x = -4\\ \dim \partial_m = 1\end{array}\right\} \quad \text{so} \quad \begin{array}{l}\dim A_m = 1\\ \dim \Psi = 3/2\end{array}$$

Gauge invariance : in a $U(1)$ gauge theory, the Majorana field Ψ must be gauge invariant.
Hence $\delta\Psi$ can only involve F_{mn}, and not "naked" A_m.

Only possibility : $\delta_\epsilon \Psi = c\, \gamma^{mn} F_{mn}\, \epsilon$ — c some real coefficient

dims: $3/2$ — $h\,1\ -1/2$

$\gamma^{mn} = \frac{1}{2}[\gamma^m, \gamma^n]$, antisymmetrized "strength one"
so if $A_{mn} = -A_{nm}$ then $A_{mn}\gamma^m\gamma^n = A_{mn}\gamma^{mn}$

As noted before, ϵ must be anticommuting, so the above form for $\delta\Psi$ preserves all essential properties of Ψ.

For δA_m, the only possibility consistent with dimensionality, reality, Lorentz invariance and Grassman parity (comm., anticomm.) is $\delta A_m = i\,\bar\epsilon\gamma_m\Psi$

dims: 1 — $-1/2$ — $+3/2$

The factor of "i" is necessary for reality, as above:
$(\bar\epsilon\gamma_m\Psi)^{*T} = -\bar\epsilon\gamma_m\Psi$ for ϵ, Ψ Majorana

The coefficient in δA_m has been set to 1 by convention (can absorb it into the parameter ϵ)

Thus, after considering the various covariances and properties to be preserved, only the coefficient c is undetermined.

Calculate :

both Ψ's contribute same expression after integration by parts

$$\delta_\epsilon I = \int d^4x \left[\partial_m F^{mn}\delta_\epsilon A_n - i\bar\Psi \partial\!\!\!/ \delta_\epsilon\Psi + \text{total derivative}\right]$$

discard the total derivative :

$$= \int d^4x \left[i\,\partial_m F^{mn}\bar\epsilon\gamma_n\Psi - ic\bar\Psi\gamma^p\gamma^{mn}\epsilon\,\partial_p F_{mn} - ic\bar\Psi\gamma^p\gamma^{mn}F_{mn}\partial_p\epsilon \right]$$

The last term is of course zero for $\partial_p\epsilon = 0$, but it is useful to do the calculation for non-constant ϵ as well.

γ-matrix lore : $\gamma^p\gamma^{mn} = \gamma^{pmn} + \eta^{pm}\gamma^n - \eta^{pn}\gamma^m$
where $\gamma^{pmn} = \frac{1}{6}[\gamma^p\gamma^m\gamma^n \pm 5 \text{ terms}]$ antisymmetrized strength one

Thus, $\gamma^p\gamma^{mn}\epsilon\,\partial_p F_{mn} = \gamma^{pmn}\epsilon\,\partial_{[p}F_{mn]} + 2\gamma^n\epsilon\,\partial^m F_{mn}$

however $\partial_{[p}F_{mn]} \equiv 0$ is the "Bianchi identity" for $F_{mn} = \partial_m A_n - \partial_n A_m$

So for $\partial\epsilon = 0$, one has

$$\delta_\epsilon I = \int d^4x \left[i\,\partial^m F_{mn}\,\bar\epsilon\gamma^n\Psi - 2ic\bar\Psi\gamma^n\epsilon\,\partial^m F_{mn}\right]$$

Moreover, $\bar\Psi\gamma^n\epsilon = \Psi^T\gamma^0\gamma^n\epsilon = -\epsilon^T(\gamma^0\gamma^n)^T\Psi = -\bar\epsilon\gamma^n\Psi$
since $\gamma^0\gamma^n$ is symmetric

so 2nd term can be written $+2ic\,\bar\epsilon\gamma^n\Psi\,\partial^m F_{mn}$
and we have cancellation for $c = -\frac{1}{2}$

Thus our transformation rules are

$$\delta_\epsilon A_m = i\,\bar\epsilon\gamma_m\Psi$$
$$\delta_\epsilon \Psi = -\tfrac{1}{2}\gamma^{mn}F_{mn}\,\epsilon$$

Supersymmetry Noether Current

Now recall the term in $\delta_\epsilon I$ that remains for $\partial_p \epsilon \neq 0$:

$$\delta_{\epsilon(x)} I = \int d^4x \left(\frac{i}{2} \bar{\Psi} \gamma^p \gamma^{mn} F_{mn} \partial_p \epsilon \right) = \frac{i}{2} \int d^4x\, \partial_p \bar{\epsilon}\, \gamma^{mn} F_{mn} \gamma^p \psi$$

using the rules for flipping the order of Majorana spinor products (Each γ^m contributes a (-1), but then one has to re-order $\gamma^{nm} = -\gamma^{mn}$, so net $(+1)$ factor here. Basic rule is $\bar{\Psi}\gamma^m \lambda = -\bar{\lambda}\gamma^m \psi$.)

Thus, the Noether current for supersymmetry is

$$J^m_\alpha = \frac{1}{2} \left(\gamma^{pq} F_{pq} \gamma^m \Psi \right)_\alpha$$

(suppress spinor α index from now, but remember this is a spinor-vector)

Let's check conservation using the Maxwell and Dirac equations:

$$\partial_m J^m = \frac{1}{2} \left(\gamma^{pq} \partial_m F_{pq} \gamma^m \psi \right) + \frac{1}{2} \left(\gamma^{pq} F_{pq} \underbrace{\partial\!\!\!/ \psi}_{\text{Dirac eqn.}} \right)$$

as before, use $\gamma^{pq}\gamma^m = \gamma^{pqm} + \eta^{qm}\gamma^p - \eta^{pm}\gamma^q$
and $\gamma^{pqm}\partial_m F_{pq} \equiv 0$

thus, $\partial_m J^m = -\partial^m F_{mq} \gamma^q \psi + \frac{1}{2}\left(\gamma^{pq} F_{pq} \partial\!\!\!/ \psi \right) = 0$
using the Maxwell and Dirac field equations.

Note the algebraic steps to verify conservation of the supersymmetry current are essentially identical to those used in verifying invariance of the action. This is a reflection of Noether's theorem.

The Supersymmetry Algebra

Now try to verify whether the symmetry we have found really satisfies the supersymmetry algebra.

For $\delta A_m = i\bar{\epsilon}\partial_m \Psi$, $\delta\Psi = -\frac{1}{2}\gamma^{mn} F_{mn} \epsilon$
consider transformations with parameters ϵ_1 and ϵ_2
and calculate the commutator $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]$ on A^m and Ψ.

Why the commutator? Because the parameters are anticommuting, so $\bar{\epsilon}Q$ is Grassmann even.

What do we expect?

$$[\bar{\epsilon}_1 Q, \bar{\epsilon}_2 Q] = -\bar{\epsilon}_{1\alpha}\bar{\epsilon}_{2\beta}\{Q_\alpha, Q_\beta\} \qquad \text{since } \{\bar{\epsilon}, Q\} = 0$$

$$= -2\bar{\epsilon}_{1\alpha}\bar{\epsilon}_{2\beta}(\gamma^q C)_{\alpha\beta} P_q$$

$$\{C_{\alpha\beta}\bar{\epsilon}_{2\beta} = \epsilon_{2\alpha}\} \qquad = -2\bar{\epsilon}_1\gamma^q\epsilon_2 P_q$$

So we expect to see a translation (generator $P_q = i\partial_q$) with parameter $-2\bar{\epsilon}_1\gamma^q\epsilon_2$.

Lets see what we actually get when acting on A^m:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] A^m = \delta_{\epsilon_1}\left(i\bar{\epsilon}_2 \gamma^m \Psi \right) - (1 \leftrightarrow 2)$$
$$= -\frac{i}{2}\bar{\epsilon}_2 \gamma^m \gamma^{pq} F_{pq} \epsilon_1 - (1 \leftrightarrow 2)$$

as before, use the identity $\gamma^m\gamma^{pq} = \eta^{mp}\gamma^q - \eta^{mq}\gamma^p + \gamma^{mpq}$

so $[\delta_{\epsilon_1}, \delta_{\epsilon_2}] A^m = -\frac{i}{2}\bar{\epsilon}_2\gamma^{mpq}\epsilon_1 F_{pq} - i\bar{\epsilon}_2\gamma^q\epsilon_1 F^m{}_q - (1 \leftrightarrow 2)$

$(C\gamma^{mpq})^T = C\gamma^{qpm}$; $\{\epsilon_1, \epsilon_2\} = 0$

Now, note that for Majorana spinors one has
$\bar{\epsilon}_2 \gamma^{mpq}\epsilon_1 = (-1)^3 \bar{\epsilon}_1 \gamma^{qpm}\epsilon_2 = (-1)^4 \bar{\epsilon}_1 \gamma^{mpq}\epsilon_2 = \bar{\epsilon}_1\gamma^{mpq}\epsilon_2$
so the first term is symmetric in labels 1,2, so cancels when one subtracts the $1 \leftrightarrow 2$ flip.

The other combination, $\bar{\epsilon}_2 \gamma^m \epsilon_1$, is naturally $1 \leftrightarrow 2$ antisymmetric, so it doubles.

Breaking up $F_{pq} = \partial_p A_q - \partial_q A_p$, one then has

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] A^m = 2i\, \bar{\epsilon}_2 \gamma^q \epsilon_1 \partial_q A^m - 2i \partial^m (\bar{\epsilon}_2 \gamma^q \epsilon_1 A_q) \qquad \epsilon\text{'s constant}$$

Flipping $\bar{\epsilon}_2 \gamma^q \epsilon_1 = -\bar{\epsilon}_1 \gamma^q \epsilon_2$, we see that the first term is exactly the translation we expected.

The second term is a field-dependent gauge transformation.

Why did this show up? $\delta\Psi$ must be a gauge singlet, as we have seen, so $[\delta_{\epsilon_1}, \delta_{\epsilon_2}] A^m$ can only depend on A^m through its field strength F_{mn}. So $-2i \bar{\epsilon}_1 \gamma^q \epsilon_2 \partial_q A^m$ alone could not have been expected.

The problem is that one is not yet treating the gauge symmetry supersymmetrically.

Something more unsettling happens when we check the $[\delta_{\epsilon_1}, \delta_{\epsilon_2}]$ commutator on Ψ:

$$[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \Psi = -\frac{i}{2} (\bar{\epsilon}_1 \gamma_n \partial_m \Psi - \bar{\epsilon}_1 \gamma_m \partial_n \Psi) \gamma^{mn} \epsilon_2 \quad - (1 \leftrightarrow 2)$$

Now we face the difficulty that the ϵ_1, ϵ_2 are not hooked together right. To proceed, we must make a typical maneuvre: a Fierz transformation.

The Fierz identity is $\frac{1}{4} (C\gamma_A)_{\alpha\beta} (C\gamma^A)_{\gamma\delta} = C_{\alpha\delta} C_{\gamma\beta}$ (C = charge conjugation matrix) or $\frac{1}{4} (\gamma_A)_{\alpha\beta} (\gamma^A)_{\gamma\delta} = \delta_{\alpha\delta} \delta_{\gamma\beta}$ where "A" is a generalized index standing for the set

$$\gamma_A = \mathbb{1}, \gamma_5, \gamma_m, \gamma_{5m} := \gamma_5 \gamma_m, \tfrac{1}{\sqrt{2}} \gamma_{mn}$$

and $\gamma^A = \mathbb{1}, -\gamma_5, \gamma^m, \gamma_5{}^m = \gamma_5 \gamma^m, -\frac{1}{\sqrt{2}} \gamma^{mn}$

Accordingly, if one has anticommuting spinors λ, φ, ψ (which can carry also additional indices unrelated to the Fierz, e.g. φ could stand for $\gamma^{\mu\nu} \partial_\rho \psi$), the Fierz identity yields

$$\lambda \bar{\varphi} \psi = -\frac{1}{4} (\bar{\varphi} \gamma_A \lambda) \gamma^A \psi$$

The extra factor of (-1) comes from anticommuting λ and $\bar{\varphi}$.

Fierzing involves some judgement on what to take for λ, φ, ψ in this model transformation. Put dividers $|$ to indicate choices:

In our case, we have $\underset{``\bar{\varphi}"}{\bar{\epsilon}_1} \Big| \underset{``\psi"}{\gamma_n \partial_m \Psi} \Big| \underset{\text{leave in front}}{\gamma^{mn}} \Big| \underset{``\lambda"}{\epsilon_2}$

thus, $[\delta_{\epsilon_1}, \delta_{\epsilon_2}] \Psi = \frac{i}{2} \bar{\epsilon}_{[1} \gamma_A \epsilon_{2]} \gamma^{mn} \gamma^A \gamma_n \partial_m \Psi$

$\gamma^m \gamma^n = \eta^{mn} + \gamma^{mn}$

$$= \frac{i}{2} \bar{\epsilon}_{[1} \gamma_A \epsilon_{2]} \gamma^m \gamma^n \gamma^A \gamma_n \partial_m \Psi - \frac{i}{2} \bar{\epsilon}_{[1} \gamma_A \epsilon_{2]} \gamma^A \not{\partial} \Psi$$

At this point, notice that a number of terms are killed off either by

1) having antisymmetrized $\epsilon_{1,2}$ surrounding a γ_A that makes for a symmetric combination
2) "pinching" $\gamma_p \gamma_{mn} \gamma^p = 0 \qquad \gamma_p \gamma_m \gamma^p = -2\gamma_m$

For the symmetry properties, note that for Majorana spinors,

$$\bar{\Psi}\lambda = \bar{\lambda}\Psi$$
$$\bar{\Psi}\gamma_5\lambda = \bar{\lambda}\gamma_5\Psi$$
$$\bar{\Psi}\gamma_m\lambda = -\bar{\lambda}\gamma_m\Psi$$
$$\bar{\Psi}\gamma_{5m}\lambda = \bar{\lambda}\gamma_{5m}\Psi$$
$$\bar{\Psi}\gamma_{mn}\lambda = -\bar{\lambda}\gamma_{mn}\Psi$$

So, out of the 5 Fierz combinations, only γ_m and γ_{mn} survive. Moreover, γ_{mn} is killed in the first term above by pinching.

We are left with:

$$[\delta_{\epsilon_1},\delta_{\epsilon_2}]\psi = -i\bar\epsilon_1\gamma_a\epsilon_2\gamma^m\gamma^a\partial_m\psi - \frac{i}{2}\bar\epsilon_1\gamma_a\epsilon_2\gamma^a\partial\!\!\!/\psi + \frac{i}{4}\bar\epsilon_1\gamma_{ab}\epsilon_2\gamma^{ab}\partial\!\!\!/\psi$$

$$= \underbrace{-2i\bar\epsilon_1\gamma_a\epsilon_2\partial^a\psi}_{\text{this is the desired translation term}} + \frac{i}{2}\bar\epsilon_1\gamma_a\epsilon_2\gamma^a\partial\!\!\!/\psi + \frac{i}{4}\bar\epsilon_1\gamma_{ab}\epsilon_2\gamma^{ab}\partial\!\!\!/\psi$$

On-shell versus off-shell mode counts

As we have seen, supersymmetry maps boson states to fermion states, requiring the same numbers of degrees of freedom of each type (at least for non-zero energy).

Thus, in the super Maxwell multiplet, we have as physical states helicity $\pm 1 \oplus$ helicity $\pm 1/2$.

However, in the off-shell counting of fields, there is an imbalance:

A_m: count 4 components minus one corresponding to the gauge degree of freedom, eliminated by gauge symmetry of the action: 3 fields $\sim A_m^{\text{transverse}}$

ψ: 4 components

So: effectively 3 bosonic vs 4 fermionic fields, so far.

Auxiliary fields

The cure to the field imbalance is to add a bosonic "auxiliary" field that does not contribute to the on-shell count of states.

Such a field can occur in the action in the form $\int d^4x\, D^2$, with an algebraic field equation $D=0$.

Why play this seemingly silly game? Consider how the dim 2 field D can enter the SUSY transformations. On dimensional grounds, we have $\underset{\dim\ 3/2}{\delta\psi} \sim \underset{-1/2\ 1\ 1}{\epsilon\partial A}$; we could include a term ϵD here.

Since D is not supposed to have any dynamical content of its own, it had better transform into a field equation. It also transpires that D should be a pseudoscalar.

Here are the transformations that work:

$$\delta A_m = i\bar\epsilon\gamma_m\psi$$
$$\delta\psi = -\tfrac{1}{2}F_{mn}\gamma^{mn}\epsilon + D\gamma_5\epsilon$$
$$\delta D = i\bar\epsilon\gamma_5\partial\!\!\!/\psi$$

First check invariance of the action including the new terms:

$$I = \int d^4x\left(-\tfrac{1}{4}F_{mn}F^{mn} - \tfrac{i}{2}\bar\psi\partial\!\!\!/\psi + \tfrac{1}{2}D^2\right)$$

Let $\delta' I$ denote the new variation terms:

$$\delta' I = -i\int\bar\epsilon\gamma_5\partial\!\!\!/\psi D + i\int D\bar\epsilon\gamma_5\partial\!\!\!/\psi = 0$$

using $\delta\bar\psi = \bar\epsilon\gamma_5 D$ and doubling, after integration by parts

Next, check the new realization of the SUSY algebra on A_m:

$$[\delta_{\epsilon_1},\delta_{\epsilon_2}]A_m = i\bar\epsilon_2\gamma_m\left(-\tfrac{1}{2}F_{pq}\gamma^{pq}\epsilon_1 + D\gamma_5\epsilon_1\right) - (1\leftrightarrow 2)$$
$$= (\text{previous terms}) - i\bar\epsilon_2\gamma_5\cancel{\gamma_m}\epsilon_1 D + i\bar\epsilon_1\gamma_5\cancel{\gamma_m}\epsilon_2 D$$
$$= \text{previous terms unchanged}$$

Now check the new realization of the algebra on Ψ:

$$[\delta\epsilon_1,\delta\epsilon_2]\Psi = (\text{previous terms}) + i\underbrace{\bar\epsilon_1}_{\text{"}\bar\varphi\text{"}}\Big|\gamma_5\underbrace{\partial\!\!\!/\Psi}_{\text{"}\psi\text{"}}\Big|\underbrace{\gamma_5\,\epsilon_2}_{\text{"}\lambda\text{"}} - (1\leftrightarrow 2)$$

Fierz: $= (\text{previous terms}) \frac{-i}{4}\bar\epsilon_1\gamma_A\epsilon_2\,\gamma_5\gamma^A\gamma_5\partial\!\!\!/\Psi \;-(1\leftrightarrow 2)$

$\gamma_5^2=-1$ $\quad = (\text{previous terms}) - \frac{i}{2}\bar\epsilon_1\gamma_a\epsilon_2\gamma^a\partial\!\!\!/\Psi - \frac{i}{4}\bar\epsilon_1\gamma_{ab}\epsilon_2\gamma^{ab}\partial\!\!\!/\Psi$

The new equation-of-motion terms exactly <u>cancel</u> the unwanted previous terms, so now one has the clean algebra realization

$$[\delta\epsilon_1,\delta\epsilon_2]\Psi = -2i\bar\epsilon_1\gamma_a\epsilon_2\partial^a\Psi .$$

Finally, one should check closure also on the field D now; this is left as an exercize.

The only unresolved issue with the super Maxwell multiplet A_m, λ, D is the occurrence of the field-dependent gauge transformations in $[\delta\epsilon_1,\delta\epsilon_2]A_m$. This will be cleared up when we reformulate everything in superspace.

The off-shell Wess–Zumino multiplet

Return now to our original example of a supermultiplet, contain propagating states with helicity $\pm\frac12$ $\Psi(x)$ and two scalars. $\phi(x)=\frac{1}{\sqrt2}(A(x)+iB(x))$

Counting field components, we again see an imbalance off-shell:

$A(x), B(x)$: 2 bose fields ; $\Psi(x)$: 4 fermi fields

dim 1 1 3/2

Computing the closure relations in the SUSY algebra, we again get $\partial\!\!\!/\Psi$ terms unless we include two auxiliary fields: $F(x), G(x)$

dim 2 2

The Wess-Zumino (or "chiral") multiplet transformations are:

$$\delta A(x) = i\bar\epsilon\Psi(x) \qquad \delta B(x) = i\bar\epsilon\gamma_5\Psi(x) \quad \text{pseudoscalar}$$
$$\delta\Psi(x) = \partial_m(A(x)-\gamma_5B(x))\gamma^m\epsilon + (F(x)+\gamma_5G(x))\epsilon$$
$$\delta F(x) = i\bar\epsilon\partial\!\!\!/\Psi(x) \qquad \delta G(x) = i\bar\epsilon\gamma_5\partial\!\!\!/\Psi(x)$$

The $\{Q,Q\}$ algebra realized as $[\delta\epsilon_1,\delta\epsilon_2]$ closes correctly on all fields of this multiplet. The check is left as an exercize.

Lagrangian invariant densities for WZ model

Since the $\{Q,Q\}$ anticommutator gives momenta $2\gamma^mCP_m$, one cannot hope for strictly invariant Lagrangians — as for the translations, the most one can have is a Lagrangian that transforms by a total derivative under the SUSY transformations: invariant densities.

With our clean linear realization of supersymmetry in the WZ multiplet, we now can form a variety of invariant densities, <u>including interactions:</u>

$$\mathcal{L}_{\text{kinetic}} = -\tfrac12\partial_mA\partial^mA - \tfrac12\partial_mB\partial^mB - \tfrac{i}{2}\bar\Psi\partial\!\!\!/\Psi + \tfrac12F^2 + \tfrac12G^2$$

$$\mathcal{L}_{\text{mass}} = m\left(FA + GB - \tfrac{i}{2}\bar\Psi\Psi\right)$$

$$\mathcal{L}_{\phi^3} = g\left(FA^2 - FB^2 + 2GAB - i\bar\Psi(A-\gamma_5B)\Psi\right)$$

We leave as exercizes the checks that the $\int d^4x$ integrals of these terms are separately invariant under the WZ supersymmetry transformations.

Elimination of auxiliary fields

From the combined action

$$I_{WZ} = \int d^4x \left(\mathcal{L}_{kinetic} + \mathcal{L}_{mass} + \mathcal{L}_{\phi^3} \right)$$

derive the field equations for the auxiliary fields:

$$\left.\begin{aligned} F + mA + g(A^2 - B^2) &= 0 \\ G + mB + 2gAB &= 0 \end{aligned}\right\} \begin{array}{l} \underline{\text{algebraic}}: \\ \text{no indep. degrees} \\ \text{of freedom} \end{array}$$

Strictly speaking, one should analyze the dynamics of the full system by combining these with the field equations for A, B and Ψ. Since they are algebraic, it turns out that one does no harm, however, by simply substituting them back into the action. (Note: this procedure would give wrong results for differential equations.)

Thus one obtains an action for the propagating fields only:

$$\begin{aligned} I_{WZ}^{prop} = \int d^4x \Big(& -\tfrac{1}{2}\partial_m A \partial^m A - \tfrac{1}{2}\partial_m B \partial^m B - \tfrac{i}{2}\bar{\Psi}\not{\partial}\Psi \\ & - \tfrac{1}{2}m^2A^2 - \tfrac{1}{2}m^2B^2 - \tfrac{i}{2}m\bar{\Psi}\Psi \\ & - gmA(A^2+B^2) \\ & - \tfrac{1}{2}g^2(A^2+B^2)^2 - ig\bar{\Psi}(A - \gamma_5 B)\Psi \Big) \end{aligned}$$

This action is still invariant under a form of supersymmetry, even though we have eliminated some component fields of the multiplet. Substituting the auxiliary field equations into the linear supersymmetry WZ transformations, one obtains the nonlinear transformations

$$\delta^{NL}A = i\bar{\epsilon}\Psi \qquad \delta^{NL}B = i\bar{\epsilon}\gamma_5\Psi$$
$$\delta^{NL}\Psi = \partial_m(A - \gamma_5 B)\gamma^m\epsilon - m(A + \gamma_5 B)\epsilon - g\left[(A^2-B^2) + 2gAB\gamma_5\right]\epsilon$$

Although it involves some work, one can verify that I_{WZ}^{prop} is invariant under these infinitesimal transformations.

If one checks the closure of $[\delta^{NL}_{\epsilon_1}, \delta^{NL}_{\epsilon_2}]$, one finds expected translation terms plus disagreeable terms involving the now-nonlinear fermionic equation of motion $\not{\partial}\Psi + m\Psi + 2g(A - \gamma_5 B)\Psi$. Thus, as in the Super-Maxwell case prior to the introduction of the auxiliary field D, one obtains a realization of supersymmetry only "on-shell", i.e. modulo equations of motion.

Aside from the problem of closing only on-shell, the nonlinear δ^{NL} transformations are specific to the action I_{WZ}^{prop}. If one changes parameters m and g, then the transformations change. Moreover, the $\mathcal{L}_{\phi^3}$ interaction is not the most general one can have – there exist supersymmetric densities of all orders in fields. For each choice of mass and interaction terms, there is a specific δ^{NL} transformation, good for the chosen action only.

By contrast, the full set of Wess-Zumino transformations including auxiliary fields is linear in the component fields and universal: this one set works for all invariant densities built from the full multiplet.

Thus, although they do not have any independent dynamics of their own, the auxiliary fields play an essential rôle in giving a linear realization of supersymmetry.

Given a linear realization, one can formulate general rules for the construction of invariant Lagrangian densities. This tensor calculus for supersymmetry is most naturally given a geometrical interpretation by extending Minkowski spacetime to superspace.

Superspace

Let us first review the construction of Minkowski space as a coset space: Poincaré/Lorentz $= G/H$

Consider a Poincaré group element $k = e^{-ix^m P_m}$ where x^m are a set of four parameters. Let this group element be the representative of the coset $e^{-ix^m P_m}H$, where H is the Lorentz group.

The coset parameters x^m transform linearly under H: for $h \in H$ corresponding to the Lorentz transformation $\Lambda^m{}_n$ (specifically, $h = e^{\frac{i}{2}\omega^{mn}M_{mn}}$ for M_{mn} in the vector rep.), one has $hk = he^{-ix^m P_m} = (he^{-ix^m P_m}h^{-1})h$ and since $hH = H$, the coset representative is transformed to $hkh^{-1} = he^{-ix^m P_m}h^{-1} = e^{-ix'^m P_m}$ with $x'^m = \Lambda^m{}_n x^n$. Verifying this for $h = e^{\frac{i}{2}\omega^{mn}M_{mn}}$ is an instructive exercize.

Under group elements outside H, the coset parameters transform nonlinearly. The general pattern is obtained by noting that any group element can be factorized into (coset) $\times$ (subgroup H) parts: $g = kh$ with k a coset representative, $h \in H$.

Then for a coset representative k and transforming group element g_0 one has $g_0 k = \tilde{k} h_1$ where $\tilde{k}$ is some other coset representative and $h_1 \in H$. In the case where $g_0 = h \in H$ above, one has $\tilde{k} = e^{-ix'^m P_m}$ and $h_1 = h$. For elements g_0 not in H, the transformation of the coset parameters will be nonlinear.

The easiest example of a nonlinear (meaning here just "not linear") transformation of the x^m is given by $g_0 = e^{-ic^m P_m}$, c^m = constants. Then one has $g_0 k = \tilde{k} e_H$ where $\tilde{k} = e^{-i(x^m + c^m)P_m}$, e_H = identity in $H \subset G$

In this way, one gives the Minkowski space Poincaré transformations

$x^m \to x^m + c^m$ translations

$x^m \to \Lambda^m{}_n x^n$ Lorentz transf.

a group-theoretical interpretation.

Now return to supersymmetry. Consider the coset space Super Poincaré/Lorentz and take as coset representatives the group elements

$$G(x^m, \theta^\alpha, \bar\theta_{\dot\alpha}) = e^{i[-x^m P_m + \theta^\alpha Q_\alpha + \bar\theta_{\dot\alpha}\bar Q^{\dot\alpha}]}$$

(using 2-component notation, related to 4-component notation with Majorana spinors by $-i(\theta^\alpha Q_\alpha + \bar\theta_{\dot\alpha}\bar Q^{\dot\alpha}) = \bar\Theta Q$. Note that this product is purely imaginary for Majorana spinors.)

One gets a set of linear transformations of the coset parameters $(x, \theta, \bar\theta)$ for $h \in H$; x^m transforms in the vector rep. $(\frac12,\frac12)$ and $\theta, \bar\theta$ in the corresponding chiral spinor reps. $(\frac12, 0), (0, \frac12)$.

For transformations g_0 by members of cosets outside H, the coset parameters transform nonlinearly. The easiest case to work out is when g_0 is itself a coset representative. For $g_0 = e^{-ic^m P_m} = G(c^m, 0, 0)$ the transformation is just $x^m \to x^m + c^m$; $\theta^\alpha \to \theta^\alpha$, $\bar\theta_{\dot\alpha} \to \bar\theta_{\dot\alpha}$, a translation of x^m alone.

For transformations $G(0, \xi^\alpha, \bar\xi_{\dot\alpha})$, one can work out the action on the coset parameters using the Baker-Campbell-Hausdorff theorem:

$$e^A e^B = e^{A+B + \frac12[A,B] + \frac{1}{12}[A-B,[A,B]] + \cdots}$$

For $G(0, \xi^\alpha, \bar\xi_{\dot\alpha})$ acting on $G(x^m, \theta^\alpha, \bar\theta_{\dot\alpha})$, the double commutators and higher vanish: $[\xi_1 Q, [\xi_2 Q, \xi_3 Q]] = 0$, etc.

With this simplification, one finds directly

$$G(0, \xi^\alpha, \bar{\xi}_{\dot\alpha})\, G(x^m, \theta^\alpha, \bar{\theta}_{\dot\alpha}) = G(x^m + i\theta\sigma^m\bar{\xi} - i\xi\sigma^m\bar{\theta},\ \theta^\alpha + \xi^\alpha,\ \bar{\theta}_{\dot\alpha} + \bar{\xi}_{\dot\alpha})$$

$(\theta\sigma^m\bar{\xi} = \theta^\alpha\sigma^m_{\alpha\dot\beta}\bar{\xi}^{\dot\beta}$, etc.)

Thus, group multiplication on the left by $G(0, \xi^\alpha, \bar{\xi}_{\dot\alpha})$ induces the superspace transformation

$(x^m, \theta^\alpha, \bar{\theta}_{\dot\alpha})$: superspace coordinates

$$(x^m, \theta^\alpha, \bar{\theta}_{\dot\alpha}) \longrightarrow (x^m + i\theta\sigma^m\bar{\xi} - i\xi\sigma^m\bar{\theta},\ \theta^\alpha + \xi^\alpha,\ \bar{\theta}_{\dot\alpha} + \bar{\xi}_{\dot\alpha})$$

Note that these transformations are not in fact restricted to the infinitesimal case. This is a construction in the full group coset space, and is valid for arbitrary transformation parameters c^m, ξ^α, $\bar{\xi}_{\dot\alpha}$.

In practise, however, we are mostly concerned with infinitesimal transformations $\xi^\alpha = \epsilon^\alpha$, $\bar{\xi}_{\dot\alpha} = \bar{\epsilon}_{\dot\alpha}$. Just as infinitesimal translations $e^{-i\delta x^m P_m} = (\mathbb{1} - i\delta x^m P_m)$ give rise to shifts $x^m \to x^m + \delta x^m$ for which the Minkowski space generators are realized as differential operators $P_m \leftrightarrow i\partial_m$, so we now have differential operator realizations of $Q^\alpha, \bar{Q}_{\dot\alpha}$:

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\sigma^m_{\alpha\dot\alpha}\bar{\theta}^{\dot\alpha}\partial_m$$

$$\bar{Q}^{\dot\alpha} = \frac{\partial}{\partial\bar{\theta}_{\dot\alpha}} + i\theta^\alpha\sigma^m_{\alpha\dot\beta}\epsilon^{\dot\alpha\dot\beta}\partial_m$$

We may lower the $\dot\alpha$ index on $\bar{Q}^{\dot\alpha}$, but care is needed, since this involves raising the index on $\partial\bar{\theta}_{\dot\alpha}$, and $\epsilon_{\dot\alpha\dot\beta} = -\epsilon^{\dot\alpha\dot\beta}$. We obtain

$$\bar{Q}_{\dot\alpha} = -\frac{\partial}{\partial\bar{\theta}^{\dot\alpha}} + i\theta^\alpha\sigma^m_{\alpha\dot\alpha}\partial_m$$

$\theta_\alpha \theta^\alpha \theta_\alpha$

$\bar\theta = \theta_{\dot\beta}\bar\theta^{\dot\beta}$

These transformation generators satisfy the required supersymmetry algebra:

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2i\sigma^m_{\alpha\dot\alpha}\partial_m = 2\sigma^m_{\alpha\dot\alpha}P_m$$

$$\{Q_\alpha, Q_\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = [Q_\alpha, P_m] = [\bar{Q}_{\dot\alpha}, P_m] = [P_m, P_n] = 0$$

Superfields

Following the pattern of Minkowski space field theory, we now define superfield representations of the super Poincaré group: $F(x, \theta, \bar{\theta})$.

One may expand this into a set of ordinary component fields by making a Taylor expansion in $\theta, \bar{\theta}$. Since these anticommute, this expansion truncates at a finite order:

$$\begin{aligned} F(x,\theta,\bar\theta) = {} & f(x) + \theta^\alpha\chi_\alpha(x) + \bar\theta_{\dot\alpha}\bar\varphi^{\dot\alpha}(x) \\ & + \theta\theta\, m(x) + \bar\theta\bar\theta\, n(x) + \theta\sigma^m\bar\theta\, V_m(x) \\ & + \theta\theta\bar\theta_{\dot\alpha}\bar\lambda^{\dot\alpha}(x) + \bar\theta\bar\theta\theta^\alpha\psi_\alpha(x) + \theta\theta\bar\theta\bar\theta\, d(x) \end{aligned}$$

This most general scalar superfield (i.e. without Lorentz rep. overall on F) is complex, and is reducible in several different senses.

Clearly, one thing one could do would be to impose a reality condition $F \overset{!}{=} F^*$, with corresponding constraints on the component fields.

More interesting constraints on a general scalar superfield are possible, for which one should leave F complex. ...

Covariant derivatives

The partial derivatives $\frac{\partial}{\partial\theta}$, $\frac{\partial}{\partial\bar\theta}$ used in making the above Taylor expansion of $F(x,\theta,\bar\theta)$ are not the most convenient operators to use in superspace because they have non-trivial anticommutation properties with the $Q,\bar{Q}$ generators, e.g.

$$\left\{\frac{\partial}{\partial\theta^\alpha}, \bar{Q}_{\dot\beta}\right\} = i\sigma^m_{\alpha\dot\beta}\partial_m$$

This is awkward because $\frac{\partial}{\partial\theta^\alpha}F$ does not transform under SUSY by the same simple rule as F, i.e. just by operation with $(\epsilon Q + \bar\epsilon\bar{Q})$.

Much better to use are the fermionic covariant derivatives

$$D_\alpha := \frac{\partial}{\partial\theta^\alpha} + i\sigma^m_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_m$$

$$\bar{D}_{\dot\alpha} := -\frac{\partial}{\partial\bar\theta^{\dot\alpha}} - i\theta^\alpha\sigma^m_{\alpha\dot\alpha}\partial_m$$

Note that these differ from the $Q,\bar{Q}$ in the signs of the second terms.

The most important property of $D,\bar{D}$ is that they anticommute with the $Q,\bar{Q}$:

$$\{D_\alpha, Q_\beta\} = \{D_\alpha, \bar{Q}_{\dot\beta}\} = \{\bar{D}_{\dot\alpha}, Q_\beta\} = \{\bar{D}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0.$$

Note: $[\partial_m, Q_\alpha] = [\partial_m, \bar{Q}_{\dot\alpha}] = 0$, so ∂_m is also covariant in this sense.

Among themselves, they satisfy the algebra

$$\{D_\alpha, \bar{D}_{\dot\alpha}\} = -2i\sigma^m_{\alpha\dot\alpha}\partial_m$$

This indicates that superspace has torsion, but not curvature

$$\{D_\alpha, D_\beta\} = \{\bar{D}_{\dot\alpha}, \bar{D}_{\dot\beta}\} = 0$$

$$[D_\alpha, \partial_m] = [\bar{D}_{\dot\alpha}, \partial_m] = 0$$

[Aside: the similarity of the algebra of the $D,\bar{D}$ to the supersymmetry algebra realized by the $Q,\bar{Q}$ is a consequence of the flatness of superspace. One may actually view the $D,\bar{D}$ operators as generating multiplication on the right of $G(x,\theta,\bar\theta)$.]

Using the superspace covariant derivatives, one now reobtains superfields after differentiation, in the sense of quantities transforming by application of $\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar{Q}^{\dot\alpha}$:

if $\delta F = (\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar{Q}^{\dot\alpha})F$ then

$$\delta(D_\beta F) = D_\beta\big((\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar{Q}^{\dot\alpha})F\big) = (\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar{Q}^{\dot\alpha})D_\beta F$$

and similarly

$$\delta(\bar{D}_{\dot\beta}F) = (\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar{Q}^{\dot\alpha})\bar{D}_{\dot\beta}F$$

Accordingly, differential constraints made using $D,\bar{D}$ that have non-trivial solutions allow one to reduce the general scalar superfield representation.

Chiral superfields

Consider a complex superfield $\phi(x,\theta,\bar\theta)$ which is subject to the differential constraint

$$\bar{D}_{\dot\alpha}\phi = 0 \qquad \text{chiral superfield constraint}$$

As one can see by considering the general scalar superfield expansion, this restricts the component fields but it does not make all of them trivial.

An antichiral superfield $\lambda(x,\theta,\bar\theta)$ is one satisfying the conjugate constraint

$$D_\alpha\lambda = 0.$$

The $D,\bar{D}$ style of component field extraction makes it very easy to work out supersymmetry transformations, since every component field is defined as the lowest component of some superfield.

Thus, for the low-level component fields in $F(x,\theta,\bar\theta)$ that we have defined, we can directly read off their susy transformations from the superfield transformat

$$\delta F = (\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar{Q}^{\dot\alpha})F$$

Evaluate this at $\theta=\bar\theta=0$:

$$\delta f(x) = \left((\epsilon^\alpha Q_\alpha + \bar\epsilon_{\dot\alpha}\bar{Q}^{\dot\alpha})F(x,\theta,\bar\theta)\right)_{|\theta=\bar\theta=0}$$

Recall that the $Q_\alpha, \bar{Q}_{\dot\alpha}$ differ from the $D_\alpha, \bar{D}_{\dot\alpha}$ only in the signs of the second terms $i\partial_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}$, $-i\theta^\beta\partial_{\beta\dot\alpha}$. Since we are evaluating $\delta f(x)$ at $\theta=\bar\theta=0$, the second terms are irrelevant, so we can also write

$$\delta f(x) = \left((\epsilon^\alpha D_\alpha + \bar\epsilon_{\dot\alpha}\bar{D}^{\dot\alpha})F(x,\theta,\bar\theta)\right)_{|\theta=\bar\theta=0} = \epsilon^\alpha\chi_\alpha(x) + \bar\epsilon_{\dot\alpha}\bar\varphi^{\dot\alpha}(x) .$$

Obviously, this way of rewriting $\delta f(x)$ is a special thing we can do only because $f(x)$ is the lowest component of a superfield.

However, the point of the $D,\bar{D}$ expansion is that <u>every</u> component field is defined as the lowest component of some superfield. So writing susy transformations using $D,\bar{D}$ can in fact be done generally

For example, $\delta\chi_\alpha = \left((\epsilon^\beta D_\beta + \bar\epsilon_{\dot\beta}\bar{D}^{\dot\beta})D_\alpha F\right)_{|\theta=\bar\theta=0}$

in which one substitutes the component fields defined at the $D^2, D\bar{D}$ levels and lower (the lower components appear differentiated with ∂_m).

Trying to make a superfield chiral and antichiral at the same time makes it a trivial constant, since $\{D_\alpha,\bar{D}_{\dot\alpha}\} = -2i\sigma^m_{\alpha\dot\alpha}\partial_m$ so $D_\alpha\phi = \bar{D}_{\dot\alpha}\phi = 0 \Rightarrow \partial_m\phi = 0$.

For this reason, a chiral superfield cannot be real without being restricted to a trivial constant: for $\bar{D}_{\dot\alpha}\phi=0$, one has $D_\alpha\bar\phi = 0$ $(\bar\phi = \phi^*)$, so $\phi=\bar\phi \Rightarrow \partial_m\phi = 0$.

<u>Component field expansions</u>

The straightforward Taylor expansion is not the most convenient way to expand a superfield into component fields. It is easier to use the $D,\bar{D}$ covariant derivatives for this. The procedure is similar to a Taylor expansion, though: one evaluates successive differentiations of $F(x,\theta,\bar\theta)$ at $\theta=\bar\theta=0$.

Thus, for a general scalar superfield $F(x,\theta,\bar\theta)$, one can define $f(x) := F_{|\theta=\bar\theta=0}$ to start,

and then

$$\chi_\alpha(x) := (D_\alpha F)_{|\theta=\bar\theta=0}$$
$$\bar\varphi^{\dot\alpha}(x) := (\bar{D}^{\dot\alpha}F)_{|\theta=\bar\theta=0} \quad \text{etc.}$$

Although the $D,\bar{D}$ expansion and the ordinary $\theta,\bar\theta$ Taylor expansion agree in the definitions of the component fields at these low orders, the two styles of expansion are different at higher orders.

$\partial_{\alpha\dot\alpha} = \sigma^m_{\alpha\dot\alpha}\partial_m$

Note that since $\{D_\alpha,\bar{D}_{\dot\alpha}\} = -2i\partial_{\alpha\dot\alpha}$, the expansion in $D,\bar{D}$ derivatives never strictly terminates (i.e. at every order, there are still some non-vanishing derivatives that can be taken). However, what happens eventually is that the components extracted by differentiation and evaluation at $\theta=\bar\theta=0$ all become ∂_m derivatives of components already defined at lower orders, so one stops.

Component field expansion of a chiral superfield

Let's apply the $D, \bar{D}$ style of component field expansion to $\phi(x,\theta,\bar{\theta})$, constrained by $\bar{D}_{\dot{\alpha}}\phi = 0$.

Since ϕ is complex, at lowest order define

$$a(x) = \frac{A(x) + iB(x)}{\sqrt{2}} := \left(\phi(x,\theta,\bar{\theta})\right)_{|\theta=\bar{\theta}=0}$$

then $\chi_\alpha(x) = \sqrt{2}\,\psi_\alpha(x) := \left(D_\alpha \phi(x,\theta,\bar{\theta})\right)_{|\theta=\bar{\theta}=0}$

and finally

$$h(x) = 2\sqrt{2}\left(F(x) - iG(x)\right) := \left(D^2\phi(x,\theta,\bar{\theta})\right)_{|\theta=\bar{\theta}=0}$$

where $D^2 = D^\alpha D_\alpha$. The redefinitions in terms of A, B, ψ, F, G involve conventional rescalings that will give canonical normalizations in the action. [One of the awkward things about superspace is that it doesn't naturally give conventional normalizations, so one must rescale in this way.]

The above definitions of component fields are all that have to be made. Other differentiations are possible, but they give ∂_m derivatives of the components defined above. For example, $\left(\bar{D}_{\dot{\beta}} D_\alpha \phi\right)_{|\theta=\bar{\theta}=0}$ does not vanish, but since $\bar{D}_{\dot{\beta}}\phi = 0$, one can rewrite

$$\left(\bar{D}_{\dot{\beta}} D_\alpha \phi\right)_| = -\left(D_\alpha \bar{D}_{\dot{\beta}}\phi\right)_| + \left(\{\bar{D}_{\dot{\beta}}, D_\alpha\}\phi\right)_|$$

$$= 0 - 2i\,\partial_{\alpha\dot{\beta}}(\phi)_|$$

$$= -2i\,\partial_{\alpha\dot{\beta}}\, a$$

{letting $(\ )_|$ denote evaluation at $\theta=\bar{\theta}=0$}

From the above definitions and use of the $D, \bar{D}$ algebra to extract ∂_m derivatives, one can now work out the component-field SUSY transformations.

Thus,

$$\delta a = \left((\epsilon^\alpha D_\alpha + \cancel{\bar{\epsilon}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}})\phi\right)_| = \epsilon^\alpha \chi_\alpha$$

and then

$$\delta\chi_\alpha = \left((\epsilon^\beta D_\beta + \bar{\epsilon}_{\dot{\beta}}\bar{D}^{\dot{\beta}})D_\alpha\phi\right)_| = \left(-\tfrac{1}{2}\epsilon_\alpha D^2\phi - 2i\,\bar{\epsilon}_{\dot{\beta}}\,\partial_\alpha{}^{\dot{\beta}}\phi\right)_|$$

using $D_\beta D_\alpha = \frac{1}{2}\epsilon_{\beta\alpha}D^\gamma D_\gamma = -\frac{1}{2}\epsilon_{\alpha\beta}D^2$ $\qquad \epsilon^{\gamma\beta}\epsilon_{\beta\alpha} = \delta^\gamma{}_\alpha$

and $\bar{D}^{\dot{\beta}}D_\alpha\phi = -\cancel{D_\alpha\bar{D}^{\dot{\beta}}\phi} + \{\bar{D}^{\dot{\beta}}, D_\alpha\}\phi = -2i\,\partial_\alpha{}^{\dot{\beta}}\phi$ as above

(but with $\partial_\alpha{}^{\dot{\beta}} = \epsilon^{\dot{\beta}\dot{\gamma}}\partial_{\alpha\dot{\gamma}} = \epsilon^{\dot{\beta}\dot{\gamma}}\sigma^m_{\alpha\dot{\gamma}}\partial_m$)

Change of slant: for 2-component contractions like $\bar{\psi}_{\dot{\beta}}\bar{\lambda}^{\dot{\beta}}$, changing the slanting direction of the contraction costs a minus sign. Thus,

$$\bar{\psi}_{\dot{\beta}}\bar{\lambda}^{\dot{\beta}} = \epsilon_{\dot{\beta}\dot{\gamma}}\bar{\psi}^{\dot{\gamma}}\epsilon^{\dot{\beta}\dot{\delta}}\bar{\lambda}_{\dot{\delta}} = -\epsilon_{\dot{\gamma}\dot{\beta}}\epsilon^{\dot{\beta}\dot{\delta}}\bar{\psi}^{\dot{\gamma}}\bar{\lambda}_{\dot{\delta}} = -\bar{\psi}^{\dot{\gamma}}\bar{\lambda}_{\dot{\gamma}}$$

So one can rewrite $\delta\chi_\alpha$ as $\delta\chi_\alpha = \left(-\frac{1}{2}\epsilon_\alpha D^2\phi + 2i\,\partial_{\alpha\dot{\beta}}\phi\,\bar{\epsilon}^{\dot{\beta}}\right)_|$

so $\quad \delta\chi_\alpha = -\frac{1}{2}h\epsilon_\alpha + 2i\,\partial_{\alpha\dot{\beta}}\,a\,\bar{\epsilon}^{\dot{\beta}}$.

Finally, we have

$$\delta h = \left((\epsilon^\beta D_\beta + \bar{\epsilon}_{\dot{\beta}}\bar{D}^{\dot{\beta}})D^2\phi\right)_| = \bar{\epsilon}_{\dot{\beta}}\left([\bar{D}^{\dot{\beta}}, D^2]\phi\right)_|$$

$D_\alpha D_\beta D_\gamma \equiv 0$

$\bar{D}_{\dot{\beta}}\phi = 0$

Now,

$$[\bar{D}^{\dot{\beta}}, D^2] = \bar{D}^{\dot{\beta}}D^\alpha D_\alpha - D^\alpha D_\alpha \bar{D}^{\dot{\beta}}$$
$$= -D^\alpha\cancel{\bar{D}^{\dot{\beta}}D_\alpha} - 2i\,\partial^{\alpha\dot{\beta}}D_\alpha + D^\alpha\cancel{\bar{D}^{\dot{\beta}}}D_\alpha + 2i\,D^\alpha\partial_\alpha{}^{\dot{\beta}}$$
$$= -4i\,\partial^{\alpha\dot{\beta}}D_\alpha$$

$D^\alpha\partial_\alpha{}^{\dot{\beta}} = -D_\alpha\partial^{\alpha\dot{\beta}}$

So $\quad \delta h = -4i\,\bar{\epsilon}_{\dot{\beta}}\left(\partial^{\alpha\dot{\beta}}D_\alpha\phi\right)_|$

i.e. $\quad \delta h = -4i\,\bar{\epsilon}_{\dot{\beta}}\,\partial^{\alpha\dot{\beta}}\chi_\alpha$,

completing the basic supersymmetry transformation rules for the chiral multiplet.

Chiral basis

Sometimes it is convenient to write a chiral superfield as an unconstrained superfield on a subspace of the full $x, \theta, \bar{\theta}$ superspace.

Observe that $\bar{D}_{\dot{\alpha}} \theta_\beta = 0$ $\qquad \bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}} - i\theta^\alpha \partial_{\alpha\dot{\alpha}}$

and $\bar{D}_{\dot{\alpha}} (x^m + i\theta\sigma^m\bar{\theta})$

$$= -i\theta^\alpha \sigma^m_{\alpha\dot{\alpha}} + i\theta^\beta \sigma^m_{\beta\dot{\alpha}} = 0$$

So the subspace (y^m, θ_α) with $y^m = x^m + i\theta\sigma^m\bar{\theta}$ is the one on which the $\bar{D}_{\dot{\alpha}}$ derivative naturally vanishes. Accordingly, one can write

$$\bar{D}_{\dot{\alpha}} \phi = 0 \implies \phi = \phi(y^m, \theta_\alpha).$$

One can also rewrite $D, \bar{D}$ in chiral basis as

$$D_\alpha = \frac{\partial}{\partial \theta^\alpha} + 2i\sigma^m_{\alpha\dot{\alpha}} \bar{\theta}^{\dot{\alpha}} \frac{\partial}{\partial y^m}$$

$$\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial \bar{\theta}^{\dot{\alpha}}}$$

using $D'_N = \frac{\partial z^M}{\partial z'^N} D_M$ with $z^M = (x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$, $z'^M = (y^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$

Component fields are also written as functions of y^m in this basis.

Superspace Actions

The Berezin integration rules, motivated by our earlier discussion of the $SO(2N)$ γ-matrices, are

$$\int d\theta \,(\text{constant}) = 0$$

$$\int d\theta\, \theta = 1 .$$

From these rules, one has it that integration and differentiation are equivalent for fermionic variables:

$$\int d\theta = \frac{\partial}{\partial\theta}$$

This simple rule plays a major rôle in the evaluation of path integrals over fermionic fields.

The simplest supersymmetric invariant is the integral of a general scalar superfield over the full superspace:

$$I_{full} = \int d^4x\, d^2\theta\, d^2\bar{\theta}\, F(x, \theta, \bar{\theta}) .$$

Since we are integrating over $\int d^4x$ and ignoring surface terms, we may add irrelevant total derivatives in $\frac{\partial}{\partial x^m}$ to the integrand, rewriting the above integral as

$$I_{full} = \int d^4x\, D^2 \bar{D}^2 F$$

This may seem to contain θ and $\bar{\theta}$ dependence, but the highest order in $\theta, \bar{\theta}$ at which one finds an independent component is the $\theta^2\bar{\theta}^2$ order. Accordingly, all components of $D^2\bar{D}^2F$ above the lowest $\theta^0, \bar{\theta}^0$ order are total ∂_m derivatives of other components of F. For example, one has

$$\bar{D}^{\dot{\alpha}} D^2 \bar{D}^2 F = \cancel{D^2 \bar{D}^{\dot{\alpha}} \bar{D}^2 F} + [\bar{D}^{\dot{\alpha}}, D^2] \bar{D}^2 F$$
$$= -4i\partial_m (\bar{\sigma}^m)^{\dot{\alpha}\beta} D_\beta \bar{D}^2 F .$$

Thus, the effect of integrating over $\int d^4x$ and ignoring surface terms is equivalent to evaluating the $D^2\bar{D}^2F$ integrand at $\theta = \bar{\theta} = 0$:

$$I_{full} = \int d^4x \left(D^2\bar{D}^2F\right)\Big|_{\theta=\bar{\theta}=0} . \qquad \text{"D term invariant"}$$

In other words, the full superspace integral of F gives the $\int d^4x$ integral of the highest independent component of the superfield integrand F (hence "D term").

The check of supersymmetry invariance for I_{full} makes direct use of the fact that all components of $D^2\bar{D}^2F$ above the lowest are total ∂_m derivatives:

$$\delta \left(D^2\bar{D}^2F\right)\Big|_{\theta=\bar{\theta}=0} = \left((\cancel{\epsilon^\alpha D_\alpha} + \bar{\epsilon}_{\dot{\alpha}} \bar{D}^{\dot{\alpha}}) D^2 \bar{D}^2 F \right)\Big|_{\theta=\bar{\theta}=0} \qquad D_\alpha D_\beta D_\gamma = 0$$

$$= -4i\, \bar{\epsilon}_{\dot{\alpha}} \partial^{\beta\dot{\alpha}} D_\beta \bar{D}^2 F \Big|_{\theta=\bar{\theta}=0} \qquad \text{total } \partial_m \text{ derivative}$$

Chiral Superspace integrals

If one can construct a chiral superfield from the basic superfield variables of a theory, then one can use it to construct a different type of invariant, which is integrated over just half the fermionic coordinates:

if $\bar{D}_{\dot\alpha} \mathcal{I} = 0$, then one has an invariant

$$I_{\text{chiral}} = \int d^4x \, d^2\theta \, \mathcal{I}(x,\theta,\bar\theta)$$

or, in chiral basis $I_{\text{chiral}} = \int d^4y \, d^2\theta \, \mathcal{I}(y,\theta)$.

One may check invariance as we did for the full superspace integral, first converting the $\int d^2\theta$ integrations into $\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta}\epsilon^{\alpha\beta}$ differentiations and adding total ∂_m derivatives to convert these to D^2:

$$I_{\text{chiral}} = \int d^4x \, D^2 \mathcal{I}$$

The check of invariance is the same as the check that all but the lowest component of $D^2\mathcal{I}$ are total ∂_m derivatives:

$$\delta\left(D^2\mathcal{I}\right)_| = \left(\left(\cancel{\epsilon^\alpha D_\alpha} + \bar\epsilon_{\dot\alpha}\bar D^{\dot\alpha}\right)D^2\mathcal{I}\right)_| \qquad D_\alpha D_\beta D_\gamma \equiv 0$$

$$= \bar\epsilon_{\dot\alpha}\left(\left[\bar D^{\dot\alpha}, D^2\right]\mathcal{I} + \cancel{D^2 \bar D^{\dot\alpha}\mathcal{I}}\right)_| \qquad \bar D_{\dot\alpha}\mathcal{I} = 0$$

$$= -4i\,\bar\epsilon_{\dot\alpha}\,\partial^{\beta\dot\alpha}\left(D_\beta \mathcal{I}\right)_{|\theta=\bar\theta=0} \quad \text{total } \partial_m \text{ derivative}$$

So, by the same token, one can write, ignoring $\int d^4x\,\partial_m$ terms,

$$I_{\text{chiral}} = \int d^4x \left(D^2\mathcal{I}\right)_{|\theta=\bar\theta=0}, \quad \text{"F term invariant"}$$

i.e. the integrand is the highest independent component ($\Leftrightarrow$ not a total ∂_m derivative) of the chiral superfield $\mathcal{I}$ (hence "F term"). Note that even though in standard superspace $\mathcal{I}$ does have [illegible] $\int d^4x\, d^2\bar\theta\, \mathcal{I}$. This becomes manifest in chiral basis,

Building superspace actions

For the Wess-Zumino model, the simplest real ($\Leftrightarrow$ hermitian) superfield that one can build that is quadratic in the chiral superfield Φ is $\bar\Phi\Phi$. So try this as a full superspace integrand:

$$I_{\text{kinetic}} = \frac{1}{16}\int d^4x \, d^2\theta \, d^2\bar\theta \, \bar\Phi\Phi \qquad \tfrac{1}{16} \text{ conventional normalization}$$

Although no derivatives are yet apparent, these can be revealed by exchanging $\int d\theta \to \frac{\partial}{\partial\theta} \to D$ and $\int d\bar\theta \to \frac{\partial}{\partial\bar\theta} \to \bar D$:

$$I_{\text{kinetic}} = \frac{1}{16}\int d^4x \, d^2\theta \left(\phi \bar D^2 \bar\Phi\right)_| \qquad \bar D_{\dot\alpha}\phi = 0$$

$$= \frac{1}{16}\int d^4x \left(D^2\phi \bar D^2\bar\Phi\right)_| + \frac{2}{16}\int d^4x\left(D^\alpha\phi \, D_\alpha \bar D^2\bar\Phi\right)_| + \frac{1}{16}\int d^4x\left(\phi D^2 \bar D^2 \bar\Phi\right)_|$$

Now, $\bar D^2 = (D^2)^* = \bar D_{\dot\alpha}\bar D^{\dot\alpha}$ $\qquad D^2 = D^\alpha D_\alpha$

So $\quad D_\alpha \bar D^2 \bar\Phi = \left[D_\alpha, \bar D^2\right]\bar\Phi + \cancel{\bar D^2 D_\alpha \bar\Phi} \qquad D_\alpha\bar\Phi = 0$

$$= -4i\,\partial_{\alpha\dot\alpha}\bar D^{\dot\alpha}\bar\Phi$$

Similarly, $\quad D^2\bar D^2\bar\Phi = -4i\,D^\alpha\partial_{\alpha\dot\alpha}\bar D^{\dot\alpha}\bar\Phi$

$$= (-4i)(-2i)\,\partial_{\alpha\dot\alpha}\partial^{\alpha\dot\alpha}\bar\Phi = -8\,\partial_{\alpha\dot\alpha}\partial^{\alpha\dot\alpha}\bar\Phi$$

and since $(\bar\sigma_m)^{\dot\alpha\beta}(\sigma_n)_{\beta\dot\alpha} = -2\eta_{mn}$

one finds

$$D^2\bar D^2\bar\Phi = 16\,\Box\bar\Phi$$

Thus, using the component-field definitions for the WZ chiral superfield Φ, one has

$$I_{\text{kinetic}} = \frac{1}{16}\int d^4x\left(hh^* - 8i\,\chi^\alpha\partial_{\alpha\dot\alpha}\bar\chi^{\dot\alpha} + 16\,a\Box a^*\right)$$

and inserting the rescalings $a = \frac{1}{\sqrt2}(A+iB)$, $\chi_\alpha = \sqrt2\,\psi_\alpha$, $h = \sqrt8\,(F-iG)$ one reobtains the WZ kinetic term

$$I_{\text{kinetic}} = \int d^4x\left(\tfrac12 A\Box A + \tfrac12 B\Box B - \tfrac{i}{2}\bar\Psi\partial\!\!\!/\Psi + \tfrac12 F^2 + \tfrac12 G^2\right)$$

since $-i\,\psi^\alpha\partial_{\alpha\dot\alpha}\bar\psi^{\dot\alpha} = -\frac{i}{2}\bar\Psi\partial\!\!\!/\Psi$

The superpotential

Mass and interaction terms are easy to build using chiral superspace integrals. We obtain these terms as specific instances of the chiral superspace integral of a superpotential $f(\phi)$, which is holomorphic in ϕ (i.e. a function of ϕ alone, not $\bar{\phi}$). Thus $\bar{D}_{\dot\alpha} f(\phi) = 0$, so $f(\phi)$ is chiral.

The invariant is

$$I_{\text{superpotential}} = \mathrm{Re} \int d^4x\, d^2\theta\, f(\phi) \qquad \text{Re for hermiticity}$$

Evaluate this in component fields as before: $\int d\theta \to \frac{\partial}{\partial\theta} \to D$

So
$$\int d^4x\, d^2\theta\, f(\phi) = \int d^4x \left(D^2 f(\phi)\right)_{|\theta=\bar\theta=0}$$

Suppose f is a function of a number of chiral superfields ϕ^a, $a = 1 \ldots k$. Then one has, for $f_{,a} = \frac{\partial f}{\partial \phi^a}$, etc.

$$D^2 f(\phi) = \sum_a D^2\phi^a f_{,a}(\phi) + \sum_{a,b} D^\alpha \phi^a D_\alpha \phi^b f_{,ab}(\phi)$$

and consequently, taking $\left(D^2 f(\phi)\right)_{|\theta=\bar\theta=0}$, one has

$$I_{\text{superpotential}} = \mathrm{Re} \int d^4x \left(\sum_a h^a f_{,a}(a) + \sum_{a,b} \chi^{\alpha a} \chi^b_\alpha f_{,ab}(a) \right)$$

The second term is a generalized Yukawa coupling. The first term generates the potential for ϕ^a when the auxiliary fields $h^a(x)$ are eliminated.

For auxiliary kinetic terms $\frac{1}{16}\int d^4x \sum_a h^a h^{a*}$, one has the auxiliary field equations

$$\frac{1}{16} h^a + \frac{1}{2} f^*_{,a} = 0 \qquad \frac{1}{16} h^{a*} + \frac{1}{2} f_{,a} = 0$$

where $f^*(a^*)$ is the complex conjugate of $f(a)$.

Note that complex conjugations come in the right places for supersymmetric covariance. Since $h^a = \left(D^2\phi^a\right)_{|\theta=\bar\theta=0}$ is the lowest component of an antichiral superfield, since $D_\alpha D^2 \phi^a = 0$. Thus it becomes equated to $-8 f^*_{,a}(a^*)$, lowest component of the antichiral superfield $-8\bar{f}_{,a}(\bar\phi)$.

Eliminating the auxiliary fields $h^a = -8 f^*_{,a}$, $h^{a*} = -8 f_{,a}$ and combining results from $\frac{1}{16}\int \sum_a h^a h^{a*}$ and $I_{\text{superpotential}}$, one obtains the potential term in the action

$$\int d^4x \sum_a \left(\frac{1}{16} h^a h^{a*} + \frac{1}{2}\left(h^a f_{,a} + h^{a*} f^*_{,a}\right)\right)$$
$$= -4 \int d^4x \sum_a f_{,a}(a) f^*_{,a}(a^*)$$

So the potential is given by the classic result

$$V(a, a^*) = 4 \sum_a |f_{,a}|^2$$

We can now trivially specialize to the WZ mass and interaction terms for a single WZ multiplet:

for $f(\phi) = \frac{m}{4}\phi^2$, one has

$$I_{\text{mass}} = \frac{m}{8}\int d^4x\, d^2\theta\, \phi^2 \quad + \text{c.c.}$$

expand in components:

$$\frac{m}{8}\int d^4x\, d^2\theta\, \phi^2 = \frac{m}{8}\int d^4x \left(D^2(\phi^2)\right)_|$$
$$= \frac{m}{4}\int d^4x \left(h a + \chi^\alpha \chi_\alpha\right)$$

Re-expressing in terms of A, B, ψ, F, G, one re-obtains the previously given component-field form of $I_{\text{mass}} = \int d^4x\, \mathcal{L}_{\text{mass}}$

Similarly, for $f(\phi) = \frac{g}{6\sqrt{2}}\phi^3$ one obtains the previously given invariant $I_{\phi^3} = \int d^4x\, \mathcal{L}_{\phi^3}$

$$I_{\phi^3} = \frac{g}{12\sqrt{2}}\int d^4x\, d^2\theta\, \phi^3 \quad + \text{c.c.}$$
$$= \int d^4x \left(\frac{g}{4\sqrt{2}} h a^2 + \frac{g}{2\sqrt{2}} \chi^\alpha \chi_\alpha a \right) \quad + \text{c.c.}$$

Positivity of the Energy

The manifestly non-negative form of the scalar field potential $V = 4\sum_a |f_{,a}|^2$ is symptomatic of a general feature of supersymmetric theories.

Return to the $N=1$ supersymmetry algebra written in the 4-component Majorana representation:

$$\{Q_\alpha, Q_\beta\} = 2(\gamma^m\gamma^0)_{\alpha\beta} P_m .$$

Now take the trace on the 4-component α,β indices:

$$2\, Q_\alpha Q_\alpha = 8 P^0 \qquad tr(\gamma^m\gamma^0) = 4\eta^{m0}$$

Thus $H = P^0 = \frac{1}{4}\sum_\alpha Q_\alpha^2 \qquad Q_\alpha^\dagger = Q_\alpha$ hermitean

This Hamiltonian is manifestly non-negative, and so must be its eigenvalues, the <u>energies</u>: for a positive-norm state $|\Psi_k\rangle$ with energy E_k, one has

$$\langle\Psi_k|H|\Psi_k\rangle = E_k\langle\Psi_k|\Psi_k\rangle = \tfrac{1}{4}\sum_\alpha\langle\Psi_k|Q_\alpha^2|\Psi_k\rangle$$
$$= \tfrac{1}{4}\sum_\alpha \|Q_\alpha|\Psi_k\rangle\|^2$$

since $Q_\alpha = Q_\alpha^\dagger$ is hermitean.

Thus, we learn directly that

$E_k \geq 0$ for all states

$E_k = 0$ only if $Q_\alpha|\Psi_k\rangle = 0$

i.e. if the state is annihilated by all four Q_α.

Since the vacuum state $|vac\rangle$ must be a state with the lowest possible energy, one learns that $E_{vac} = 0$ for <u>unbroken</u> supersymmetry $Q_\alpha|vac\rangle = 0$, and, conversely, that if supersymmetry is spontaneously <u>broken</u>, $Q_\alpha|vac\rangle \neq 0$, then $E_{vac} > 0$ and since E_{vac} must be the minimum energy, $E_k > 0$ for all states $|\Psi_k\rangle$ in a theory with spontaneously broken SUSY.

Gauge Theories in Superspace

The off-shell multiplet for super-Maxwell theory that we found contains $A_m(x)$, $\kappa(x)$, $D(x)$; we recall that since A_m is a gauge field, $\delta A_m = \partial_m\Lambda$, only 3 out of its four components count in the tally of non-gauge field components, so one has 3+1 bose vs. 4 fermi components.

The single gauge component $\Lambda(x)$ is clearly not adequate to form a superfield, so we must have it accompanied by some more fields if we are to find a superspace realization of the super-Maxwell theory.

The key is to generalize the gradient $\partial_m\Lambda$ to a superfield construction.

Clearly, a real general scalar superfield $V(x,\theta,\bar\theta)$, $V=\bar V$, contains candidates for the non-gauge components A_m, Ψ, D (reality needed since A_m + D are real for Super-Maxwell and there is a single Majorana spinor Ψ). But we need to have a gauge transformation δV to remove non-gauge component fields.

The simplest off-shell multiplet of $N=1$ supersymmetry is the chiral multiplet, described by a chiral superfield.

Let $\Lambda(x,\theta,\bar\theta)$ be a chiral superfield, $\bar D_{\dot\alpha}\Lambda = 0$, and consider the real scalar superfield that is its imaginary part:

$$X = i(\bar\Lambda - \Lambda)$$

Clearly, X cannot contain any more independent component fields than Λ, but they are clearly differently organised since X is real and not chiral, $D_\alpha X \neq 0$.

At this point, let us return to the expansion of a general scalar superfield, taken to be real so as to have a single vector component field: $V = V^*$

We have already given the first two levels, but let us change notation slightly:

$$V_{|\theta=\bar\theta=0} =: C(x)$$

$$(D_\alpha V)_| =: \tilde{\chi}_\alpha(x) \qquad (\bar{D}_{\dot\alpha} V)_| = \bar{\tilde{\chi}}_{\dot\alpha}(x) \qquad \text{since } V = V^*$$

then define

$$(D^2 V)_| =: H(x) + iK(x) \qquad (\bar{D}^2 V) = H(x) - iK(x)$$

Now we come to the vector, built from $D_\alpha \bar{D}_{\dot\beta} V$ and $\bar{D}_{\dot\beta} D_\alpha V$. Since we want to give component-field names only to new components and not to derivatives of lower components already named. Since $\left(\{D_\alpha, \bar{D}_{\dot\beta}\} V\right)_| = -2i\,\partial_{\alpha\dot\beta} C(x)$, we extract the vector component from the <u>commutator</u>:

$$\left([D_\alpha, \bar{D}_{\dot\beta}] V\right)_| =: A_{\alpha\dot\beta}(x) = \sigma^m_{\alpha\dot\beta} A_m(x)$$

Note that this definition of A_m is <u>real</u> $\longleftrightarrow$ $(A_{\alpha\dot\beta})^* = A_{\beta\dot\alpha}$:
$(D_\alpha \bar{D}_{\dot\beta} V)^* = -\bar{D}_{\dot\alpha} D_\beta V$, so $\left([D_\alpha, \bar{D}_{\dot\beta}] V\right)^* = [D_\beta, \bar{D}_{\dot\alpha}] V$
[Note on conjugating products of D's and $\bar{D}$'s: the conjugate of $\theta_\alpha \bar\theta_{\dot\beta}$ is $\theta_\beta \bar\theta_{\dot\alpha} = -\bar\theta_{\dot\alpha}\theta_\beta$; when we take conjugates of products of derivatives, we must include this minus sign but we do not invert the order of the operators after individually conjugating them, i.e. we do not obtain $D_\beta \bar{D}_{\dot\alpha} V$, which differs from the correct conjugate by $2i\,\partial_{\beta\dot\alpha} V$.]

Continuing further, we have:

$$(\bar{D}^2 D_\alpha V)_| =: \lambda_\alpha(x) \qquad (D^2 \bar{D}_{\dot\alpha} V)_| = \bar\lambda_{\dot\alpha}(x) \qquad \begin{matrix} \bar{D}^2 = \bar{D}_{\dot\beta}\bar{D}^{\dot\beta} \\ D^2 = D^\beta D_\beta \end{matrix}$$

and finally

$$(D^\alpha \bar{D}^2 D_\alpha V)_| =: D(x) = (\bar{D}_{\dot\alpha} D^2 \bar{D}^{\dot\alpha} V)_|$$

This last reality relation is important, so let's derive it in detail:

$$\begin{aligned} D^\alpha \bar{D}^2 D_\alpha V &= \bar{D}^2 D^2 V - 4i\,\partial^\alpha{}_{\dot\beta} \bar{D}^{\dot\beta} D_\alpha V && [D_\alpha, \bar{D}^2] = -4i\,\partial_{\alpha\dot\beta}\bar{D}^{\dot\beta} \\ &= \bar{D}_{\dot\beta} D^2 \bar{D}^{\dot\beta} V + 4i\,\bar{D}_{\dot\beta}\partial_\alpha{}^{\dot\beta} D^\alpha V - 4i\,\partial^\alpha{}_{\dot\beta}\bar{D}^{\dot\beta} D_\alpha V && [\bar{D}_{\dot\beta}, D^2] = 4i\,\partial_{\alpha\dot\beta} D^\alpha \\ &= \bar{D}_{\dot\beta} D^2 \bar{D}^{\dot\beta} V \end{aligned}$$

and $(D^\alpha \bar{D}^2 D_\alpha V)^* = \bar{D}_{\dot\alpha} D^2 \bar{D}^{\dot\alpha} V$, so $D(x)$ is real.

Now return to our proposed real scalar superfield constructed from the imaginary part of a chiral superfield: $X = i(\bar\Lambda - \Lambda)$. Apply the above rules for extracting components to this.

Let Λ have components $\frac{\hat{B}}{4}$, $\frac{\hat{C}}{2}$, $i\hat{\chi}_\alpha$, $-\hat{K}$, $\hat{H}$

(i.e. $\Lambda_| = \frac{1}{4}\hat{B} + \frac{i}{2}\hat{C}$, $(D_\alpha\Lambda)_| = i\hat\chi_\alpha$, $(D^2\Lambda)_| = -\hat{K} + i\hat{H}$)
and use these components in the expansion of X, recalling that $D_\alpha\bar\Lambda = 0$:

$$X_| = \hat{C}(x) \qquad (D_\alpha X)_| = \hat\chi_\alpha(x) \qquad (D^2 X)_| = \hat{H}(x) + i\hat{K}(x)$$

Next, we come to the vector component of X. From our expansion of the real general scalar superfield above, we know that this should be extracted as $([D_\alpha, \bar{D}_{\dot\beta}] X)_|$. Calculate this, again recalling that $\bar{D}_{\dot\beta}\Lambda = 0$, $D_\alpha\bar\Lambda = 0$:

$$\begin{aligned} D_\alpha \bar{D}_{\dot\beta} X &= i\, D_\alpha \bar{D}_{\dot\beta} \bar\Lambda = -i\,\bar{D}_{\dot\beta} D_\alpha \Lambda + i\{D_\alpha, \bar{D}_{\dot\beta}\}\bar\Lambda \\ &= 2\partial_{\alpha\dot\beta}\bar\Lambda \\ \bar{D}_{\dot\beta} D_\alpha X &= -i\,\bar{D}_{\dot\beta} D_\alpha \Lambda = -2\partial_{\alpha\dot\beta}\Lambda \end{aligned}$$

so

$$([D_\alpha, \bar{D}_{\dot\beta}] X)_| = 2\partial_{\alpha\dot\beta}(\Lambda + \bar\Lambda)_| = \partial_{\alpha\dot\beta}\hat{B}$$

Note that this is the first time we encounter $\hat{B}$ in the expansion of X, appearing under the $\partial_{\alpha\dot\beta}$ derivative only. So $i(\bar\Lambda - \Lambda)$ is a candidate for a <u>gauge transformation</u> of $V(x, \theta, \bar\theta)$.

Continuing on further in the expansion of X, we have

$$\bar{D}^2 D_\alpha X = -i\bar{D}^2 D_\alpha \Lambda = -i[\bar{D}^2, D_\alpha]\Lambda \quad \text{since } \bar{D}_{\dot\beta}\Lambda = 0$$
$$= 4\partial_{\alpha\dot\beta}\bar{D}^{\dot\beta}\Lambda = 0$$

and similarly $D^2\bar{D}_{\dot\alpha} X = i D^2\bar{D}_{\dot\alpha}\bar\Lambda = i[D^2,\bar{D}_{\dot\alpha}]\bar\Lambda = 4\partial_{\alpha\dot\alpha}D^\alpha\bar\Lambda = 0$ and consequently $D^\alpha\bar{D}^2 D_\alpha X = \bar{D}_{\dot\alpha}D^2\bar{D}^{\dot\alpha}X = 0$ as well.

Thus, lining up transformations with corresponding component fields of $V(x,\theta,\bar\theta)$, we have

$$V:\quad C(x),\quad \tilde\chi_\alpha(x),\quad H(x)+iK(x),\quad A_m(x),\quad \lambda_\alpha(x),\quad D(x)$$
$$X:\quad \hat{C}(x),\quad \hat\chi_\alpha(x),\quad \hat{H}(x)+i\hat{K}(x),\quad \partial_m\hat{B}(x),\quad 0,\quad 0$$

Thus, one can use the gauge transformation

$$\delta V = i(\bar\Lambda - \Lambda)$$

to completely gauge away the lower components C, $\tilde\chi_\alpha$, H, K of the scalar superfield V, and also to change the vector component A_m by a gradient $\partial_m B$. This is the supersymmetrically covariant expression of gauge symmetry.

One may use some of this gauge freedom to make a partial gauge choice: $C = \tilde\chi_\alpha = H = K = 0$, but leaving the A_m gauge unfixed. This is called <u>Wess-Zumino gauge</u>. Since it is not a supersymmetrically covariant gauge choice (a superfield equation would be required for that), the remaining components A_m, λ_α, D do not strictly form a proper linear realization of supersymmetry. But they almost do: only the unfixed longitudinal part of A_m is not a proper member of the multiplet. The price one pays for this is the occurrence of field-dependent A_m gauge transformations in the $[\delta_{\epsilon_1},\delta_{\epsilon_2}]A_m$ algebra realization, as we have seen.

Field strength superfield

Given the superspace gauge transformation $\delta V = i(\bar\Lambda - \Lambda)$, we need to identify the gauge-invariant content of the multiplet. From the line-up of components of V versus $X = i(\bar\Lambda - \Lambda)$ we see directly that the physical spinor field $\lambda_\alpha(x)$ and its supersymmetric variations are gauge invariant. Since λ_α was extracted by taking $(\bar{D}^2 D_\alpha V)|_{\theta=\bar\theta=0}$, we identify the gauge invariant "field strength" superfield as

$$W_\alpha = \bar{D}^2 D_\alpha V \qquad \text{chiral: } \bar{D}_{\dot\alpha}W_\alpha = 0$$
$$\text{conj:}\quad \bar{W}_{\dot\alpha} = D^2\bar{D}_{\dot\alpha}V \qquad \text{antichiral: } D_\alpha\bar{W}_{\dot\alpha} = 0$$

These conjugate superfields are linked by a kind of "Bianchi identity": $D^\alpha W_\alpha = \bar{D}_{\dot\beta}\bar{W}^{\dot\beta}$, since we have seen that $D^\alpha\bar{D}^2 D_\alpha V = \bar{D}_{\dot\beta}D^2\bar{D}^{\dot\beta}V$.

The lowest component of W_α is λ_α, as we have seen:

$$\lambda_\alpha = (W_\alpha)|$$

At the next level in this chiral superfield we have $D_{[\alpha}W_{\beta]} = \frac{1}{2}\epsilon_{\alpha\beta}D^\gamma W_\gamma$ and $D_{(\alpha}W_{\beta)}$. The antisymmetric part gives the auxiliary field D:

$$D = (D^\alpha W_\alpha)| \qquad (\text{also } D = (\bar{D}_{\dot\alpha}\bar{W}^{\dot\alpha})|\text{ by Bianchi id.})$$

To identify the symmetric part, we need to digress on the translation of 4-component antisymmetric tensors into 2-component notation.

For $F_{mn} = -F_{nm}$ $\quad m,n = 0,1,2,3$

we obtain $F_{\alpha\dot\beta\,\gamma\dot\delta} = -F_{\gamma\dot\delta\,\alpha\dot\beta}$ $\quad m \to \alpha\dot\beta \quad n \to \gamma\dot\delta$

Symmetrizing and antisymmetrizing, we find that $F_{\alpha\dot\beta\gamma\dot\delta}$ can have either $[\alpha\gamma](\dot\beta\dot\delta)$ structure or $(\alpha\gamma)[\dot\beta\dot\delta]$ only.

Thus, replacing the antisymmetric combinations by ϵ tensors times ϵ traces, we have

$$F_{\alpha\dot\beta\gamma\dot\delta} = \tfrac{1}{2}\left(\epsilon_{\alpha\gamma} F^{\lambda}{}_{\dot\beta\lambda\dot\delta} - \epsilon_{\dot\beta\dot\delta} F_{\alpha\dot\lambda\gamma}{}^{\dot\lambda}\right)$$

Thus, define $F_{\alpha\gamma} = -i F_{\alpha\dot\lambda\gamma}{}^{\dot\lambda}$ $\quad \bar F_{\dot\beta\dot\delta} = -i F^{\lambda}{}_{\dot\beta\lambda\dot\delta}$

and note that $F_{\alpha\gamma} = F_{\gamma\alpha}$, $\bar F_{\dot\beta\dot\delta} = \bar F_{\dot\delta\dot\beta}$ symmetric

So we can write

$$F_{\alpha\dot\beta\gamma\dot\delta} = \frac{i}{2}\left(\epsilon_{\alpha\gamma}\bar F_{\dot\beta\dot\delta} - \epsilon_{\dot\beta\dot\delta} F_{\alpha\gamma}\right).$$

If by F_{mn} we now mean the field strength for A_m, $F_{mn} = \partial_m A_n - \partial_n A_m$, we can translate this into 2-component notation as

$$F_{\alpha\gamma} = -2i\,\partial_{(\alpha\dot\lambda} A_{\gamma)}{}^{\dot\lambda}$$

$$\bar F_{\dot\beta\dot\delta} = -2i\,\partial^{\lambda}{}_{(\dot\beta} A_{\lambda\dot\delta)}$$

complex conjugates: $(F_{\alpha\gamma})^* = \bar F_{\dot\alpha\dot\gamma}$

We can now identify $F_{\alpha\beta}$ from the symmetric derivative:

$$F_{\alpha\beta} = \left(D_{\alpha} W_{\beta}\right)|$$

Check this using the expansion of V:

$$D_{(\alpha} W_{\beta)} = D_{(\alpha}\bar D^2 D_{\beta)} V = [D_{(\alpha}, \bar D^2] D_{\beta)} V \quad \text{since } D_{(\alpha}D_{\beta)} \equiv 0$$

$$= -4i\,\partial_{(\alpha\dot\lambda}\bar D^{\dot\lambda} D_{\beta)} V$$

now take $\frac{1}{2}[\bar D^{\dot\lambda}, D_\beta] + \frac{1}{2}\{\bar D^{\dot\lambda}, D_\beta\}$:

$$D_{(\alpha} W_{\beta)} = -4\,\partial_{(\alpha\dot\lambda}\partial_{\beta)}{}^{\dot\lambda} V - 2i\,\partial_{(\alpha\dot\lambda}[\bar D^{\dot\lambda}, D_{\beta)}] V$$

The first term is zero since $\partial_{[m}\partial_{n]} \equiv 0 \Rightarrow \partial_{(\alpha\dot\lambda}\partial_{\beta)}{}^{\dot\lambda} \equiv 0$

so we have

$$\left(D_{(\alpha} W_{\beta)}\right)| = -2i\,\partial_{(\alpha\dot\lambda} A_{\beta)}{}^{\dot\lambda} \quad \text{as promised}$$

Super Maxwell action

If we take the vector component A_m of V to have the canonical dimension, equal to 1, so that λ_α is of dimension $3/2$ and D of dimension 2, then the lowest dimensional ${}_{(3)}$ gauge-invariant superfields bilinear in W, $\bar W$ are $W^\alpha W_\alpha$ (naturally traced since $W_{(\alpha}W_{\beta)} \equiv 0$) and $W_\alpha \bar W_{\dot\beta}$. The latter isn't a Lorentz scalar, so isn't appropriate for a Lagrangian density, but the former is perfect for a chiral superspace action of dimension 4:

$$I_{S.Max} = -\,\mathrm{Re}\int d^4x\, \underbrace{d^2\theta\, W^\alpha W_\alpha}_{\dim 1+3=4}$$

invariant since W_α is chiral, $\bar D_{\dot\beta} W_\alpha = 0$

Let's evaluate this in components to see what we've constructed:

$$-\int d^4x\, d^2\theta\, W^\alpha W_\alpha = -\int d^4x \left(D^\beta D_\beta (W^\alpha W_\alpha)\right)|$$

$$= \underbrace{-2\int d^4x\,(W^\alpha D^2 W_\alpha)|}_{\text{term (1)}} + \underbrace{2\int d^4x\,(D^\beta W^\alpha D_\beta W_\alpha)|}_{\text{term (2)}}$$

Work on term (2): $\quad D_\beta W_\alpha = D_{(\beta} W_{\alpha)} + \tfrac{1}{2}\epsilon_{\beta\alpha} D^\gamma W_\gamma$

so

$$2\int d^4x \left(D^\beta W^\alpha D_\beta W_\alpha\right)| = \int d^4x \left((D^\gamma W_\gamma)^2\right)| + 2\int d^4x \left(D^{(\beta} W^{\alpha)} D_{(\beta} W_{\alpha)}\right)|$$

take the real part of this: $\int d^4x\, D^2 + 2\,\mathrm{Re}\int d^4x\, F^{\alpha\beta} F_{\alpha\beta}$

and $2\,\mathrm{Re}\int d^4x\, F^{\alpha\beta} F_{\alpha\beta} = -8\int d^4x\, F_{mn} F^{mn}$, Maxwell kinetic term

The remaining term ① gives the spinor kinetic term after some conversion:

$$D^2 W_\alpha = D^\beta D_\beta W_\alpha = \tfrac{1}{2} D^\beta (D_\beta W_\alpha + D_\alpha W_\beta) + \tfrac{1}{2} D^\beta \epsilon_{\beta\alpha} D^\gamma W_\gamma$$
$$= \tfrac{1}{2} D^2 W_\alpha - \tfrac{1}{2} D_\alpha D^\beta W_\beta - \tfrac{1}{2} D_\alpha D^\gamma W_\gamma$$

rearrange: $\tfrac{1}{2} D^2 W_\alpha = - D_\alpha D^\gamma W_\gamma$

so $\quad D^2 W_\alpha = -2 D_\alpha D^\gamma W_\gamma$

[Note: this manipulation was the 2-component equivalent of a Fierz transformation.]

Now, we also have the Bianchi identity $D^\alpha W_\alpha = \bar{D}_{\dot\alpha} \bar{W}^{\dot\alpha}$

so $\quad D^2 W_\alpha = -2 D_\alpha \bar{D}_{\dot\gamma} \bar{W}^{\dot\gamma}$

and $\bar{W}^{\dot\gamma}$ is antichiral, $D_\alpha \bar{W}^{\dot\gamma} = 0$

so $\quad D^2 W_\alpha = 4i\, \partial_{\alpha\dot\gamma} \bar{W}^{\dot\gamma}$

Thus, term ① is $-8i \int d^4x \left(W^\alpha \partial_{\alpha\dot\gamma} \bar{W}^{\dot\gamma} \right)\Big|$

So we have, translating back to 4-component notation,

$$I_{S.Max} = \int d^4x \left(-8 F_{mn} F^{mn} - 4i \bar\lambda \not{\partial} \lambda + D^2 \right)$$

which, up to rescalings, is the expected Super Maxwell action

[Aside: Strictly speaking, one could omit the injunction to take the real part of $\int d^4x\, d^2\theta\, W^\alpha W_\alpha$, for the imaginary part turns out to be a total derivative here. The spin 1 part of this total derivative is $\int d^4x\, F_{mn} {}^*F^{mn}$ where ${}^*F_{mn} = \tfrac{1}{2} \epsilon_{mnpq} F^{pq}$. This is a topological invariant known as the Pontriagin class.]

Superspace Field Equations

One can of course obtain the field equations for a supersymmetric theory by working out the action in component fields and then varying the result in the ordinary way. But one can also obtain the field equations from a direct superspace variation.

Consider $I_{S.Max}$, remembering that $W_\alpha = \bar{D}^2 D_\alpha V$. The unconstrained real scalar superfield V, known as the prepotential, is the independent quantity that should be varied: $V \to V + \delta V$.

Prior to doing this, let's re-write the action:

$$-\int d^4x\, d^2\theta\, W^\alpha W_\alpha = -\int d^4x\, d^2\theta\, \bar{D}^2 D^\alpha V \bar{D}^2 D_\alpha V$$

Since $\bar{D}_{\dot\beta} \bar{D}^2 \equiv 0$, we can pull to $\bar{D}$'s into the integration measure and rewrite this as a full superspace integral:

$$I_{S.Max} = -\int d^4x\, d^2\theta\, d^2\bar\theta\, D^\alpha V \bar{D}^2 D_\alpha V$$
$$= \int d^4x\, d^2\theta\, d^2\bar\theta\, V D^\alpha \bar{D}^2 D_\alpha V$$

[Note: in throwing away surface terms, we have a strictly real quantity since $D^\alpha \bar{D}^2 D_\alpha V$ is real]

This form is now easy to vary; the contributions from varying each of the two V's in the action simply double the contribution of one of them and we have

$$\delta I_{S.Max} = 2 \int d^4x\, d^2\theta\, d^2\bar\theta\, \delta V D^\alpha \bar{D}^2 D_\alpha V$$

Requiring $\delta I_{S.Max} = 0$ for arbitrary $\delta V(x,\theta,\bar\theta)$ gives the superspace field equation:

$$D^\alpha \bar{D}^2 D_\alpha V = 0$$

In terms of the field strength superfield W_α, one can write this as $D^\alpha W_\alpha = 0$.

The lowest component of this superfield equation is simply $D(x) = 0$, the auxiliary field equation.

So: the auxiliary field is the highest dimension field in the supermultiplet and also the lowest dimension component of the superfield containing the equations of motion. This dual rôle is a general characteristic of auxiliary fields.

The remaining component field equations occur at higher levels in $D^\alpha W_\alpha = 0$. Apply $\bar{D}_{\dot\beta}$, for example:

$$\bar{D}_{\dot\beta} D^\alpha W_\alpha = 0 \quad \Longrightarrow \quad -2i\,\partial_{\dot\beta}{}^{\alpha} W_\alpha = 0$$

since $\bar{D}_{\dot\beta} W_\alpha = 0$

The lowest component of this superfield equation is the Dirac equation $-2i\,\partial_{\dot\beta}{}^{\alpha}\lambda_\alpha = 0$.

Continue on to the next order:

$$D_\gamma \partial_{\dot\beta}{}^{\alpha} W_\alpha = 0 \quad \Longrightarrow \quad \partial_{\dot\beta}{}^{\alpha} D_{(\gamma} W_{\alpha)} = 0$$

re-using the original equation $D^\gamma W_\gamma = 0$

Rewriting this (and its complex conjugate) in 4-component notation, the lowest component gives $\partial_m F^{mn} = 0$, the Maxwell equation.

Wess-Zumino model equations

The WZ superspace action we found is, for a single chiral superfield Φ,

$$I_{WZ} = \frac{1}{16}\int d^4x\, d^2\theta\, d^2\bar\theta\, \bar\Phi\Phi + \mathrm{Re}\int d^4x\, d^2\theta\, f(\Phi)$$

In varying I_{WZ}, we must take care to preserve the chirality of Φ, $\bar{D}_{\dot\alpha}\Phi = 0$. At the same time, it is only straightforward to pick off field equations from the cofactors of <u>unrestricted</u> variations.

We may achieve this by taking

$$\delta\Phi = \bar{D}^2 \delta Z \qquad\qquad \delta\bar\Phi = D^2 \delta\bar{Z}$$

where $\delta Z(x,\theta,\bar\theta)$ is an infinitesimal complex general scalar superfield. Then we have

$$\delta I_{WZ} = \frac{1}{16}\int d^4x\, d^2\theta\, d^2\bar\theta\, \bar\Phi\, \bar{D}^2 \delta Z + \frac{1}{2}\int d^4x\, d^2\theta\, f'(\Phi)\, \bar{D}^2 \delta Z + \text{c.c.}$$

Integrating by parts in the 1st term and pulling $\bar{D}^2$ into the integration measure in the 2nd, possible since $\bar{D}_{\dot\alpha} f'(\Phi) = 0$, we have

$f'(\phi) = \dfrac{\partial f}{\partial\phi}$

$$\delta I_{WZ} = \frac{1}{16}\int d^4x\, d^2\theta\, d^2\bar\theta\, \bar{D}^2\bar\Phi\, \delta Z + \frac{1}{2}\int d^4x\, d^2\theta\, d^2\bar\theta\, f'(\Phi)\, \delta Z + \text{c.c.}$$

[Note the signs arising from integration by parts:

$$\begin{aligned}\int d^4x\, d^2\theta\, d^2\bar\theta\, Y\, \bar{D}_{\dot\beta}\bar{D}^{\dot\beta} X &= -\int d^4x\, d^2\theta\, d^2\bar\theta\, \bar{D}_{\dot\beta} Y\, \bar{D}^{\dot\beta} X \\ &= (-1)(-1)(-1)\int d^4x\, d^2\theta\, d^2\bar\theta\, \bar{D}^{\dot\beta}\bar{D}_{\dot\beta} Y\, X \\ &= +\int d^4x\, d^2\theta\, d^2\bar\theta\, \bar{D}^2 Y\, X\end{aligned}$$

]

Requiring $\delta I_{WZ} = 0$ for arbitrary infinitesimal $\delta Z(x,\theta,\bar\theta)$ (treated as independent from $\delta\bar{Z}$ in the usual fashion when varying complex fields), one reads off the superspace WZ field equation:

$$\frac{1}{16}\bar{D}^2\bar\Phi + \frac{1}{2} f'(\Phi) = 0$$

Similarly, the conjugate variation $\delta\bar{Z}(x,\theta,\bar\theta)$ gives

$$\frac{1}{16} D^2\Phi + \frac{1}{2}\bar{f}'(\bar\Phi) = 0$$

Evaluating this at $\theta = \bar\theta = 0$, the lowest component of this superfield equation is the WZ auxiliary field equation

$$\frac{2\sqrt{2}}{16}(F + iG) = -\frac{1}{2} f'\left(\frac{A+iB}{\sqrt{2}}\right)$$

So, for example, with $f(\phi) = \frac{m}{4}\phi^2$ we obtain as expected

Gauge-invariant couplings of chiral and vector multiplets

Consider a set of chiral superfields ϕ^a, with action

$$I_{ch} = \frac{1}{16}\int d^4x\, d^2\theta\, d^2\bar\theta\, \bar\Phi_a \Phi^a + \mathrm{Re}\int d^4x\, d^2\theta\, f(\phi)$$

$$\bar\Phi_a = (\Phi^a)^*$$

where now we use the Einstein summation convention, $\bar\Phi_a \Phi^a := \sum_a \bar\Phi_a \Phi^a$, and $f(\phi)$ is some holomorphic prepotential function of the ϕ^a.

Now suppose this model has rigid symmetries that commute with SUSY, i.e. that act on whole superfields,

$$\delta\phi^a = \xi^r T_r{}^a{}_b \phi^b$$

where the ξ^r denote a set of infinitesimal constant commuting parameters and the $T_r{}^a{}_b$ form a representation of some symmetry group G, so $r = 1, 2, \cdots, \dim(G)$.

In order for the chiral multiplet action I_{ch} to be invariant under these transformations, both the kinetic and superpotential terms must be invariant. For the kinetic term to be invariant, it is sufficient that G leave invariant the Kronecker $\delta^a{}_b$. [Anticipating that one might eventually want to consider $SU(N)$ groups, the index on $\bar\Phi_a$ has been taken to be lowered: $\bar\Phi_a$ and ϕ^a can transform in conjugate representations.]

In order for the superpotential term to be invariant, one must have $\delta f(\phi) = 0 \implies f_{,a} T_k{}^a{}_b \phi^b = 0 \quad \forall k$

where, as before, $f_{,a} = \dfrac{\partial f(\phi)}{\partial \phi^a}$

For example, one could have a renormalizable superpotential

$$f(\phi) = \frac{1}{2} m_{ab}\phi^a\phi^b + \frac{1}{6} g_{abc}\phi^a\phi^b\phi^c$$

where m_{ab} and g_{abc} are invariant tensors of the group G.

Let us begin with a $U(1)$ group G, so the ϕ^a transformations are $\phi^a \to \phi^{a\prime} = e^{it_a\xi}\phi^a$ (not summed) in which the t_a are constants denoting the charges of the ϕ^a. [Note that the constant $U(1)$ parameter ξ can here be allowed to be finite, as we have written the exact $U(1)$ transformation of ϕ^a.]

Invariance of the superpotential $f(\phi)$ requires
$m_{ab} = 0$ unless $t_a + t_b = 0$ and
$g_{abc} = 0$ unless $t_a + t_b + t_c = 0$.

Gauge symmetry in superspace

Now consider generalizing the above rigid $U(1)$ symmetry to a local gauge transformation in superspace:

$$\phi^{a\prime} = e^{it_a\Lambda(x,\theta,\bar\theta)}\phi^a$$

where in order to preserve chirality of ϕ^a, the superfield parameter Λ must also be chiral, $\bar D_{\dot\alpha}\Lambda = 0$.
Then $\bar\Phi_a$ must transform with an antichiral parameter:

$$\bar\Phi_a{}' = e^{-it_a\bar\Lambda}\bar\Phi_a \quad , \quad D_\alpha\bar\Lambda = 0 .$$

This works fine for the superpotential term, whose invariance derives from its being built from G-invariant tensors. However, we now have difficulties with the kinetic term, where replacing the constant ξ by a superfield $\Lambda(x,\theta,\bar\theta)$ ruins the invariance of $\bar\Phi_a\phi^a$, despite the fact that $\delta^a{}_b$ is an invariant tensor.

The kinetic term general scalar integrand transforms as

$$\bar\Phi_a\phi^a \longrightarrow \sum_a \bar\Phi_a\phi^a\, e^{-it_a(\bar\Lambda-\Lambda)}$$

This is ideally suited to be cancelled by the Maxwell gauge transformation of a vector multiplet: $V \to V' = V + i(\bar\Lambda - \Lambda)$.

This leads us to construct a gauge-invariant kinetic term with integrand $\sum_a \bar\Phi_a e^{t_a V}\phi^a$.

This kinetic term looks rather disturbing from the point of view of renormalizability owing to the exponential $e^{t_a V}$, but remember that we have a large gauge symmetry inherent in the $\Lambda(x,\theta,\bar\theta)$ chiral superfield parameter. If we choose Wess-Zumino gauge, $C = \tilde\chi_\alpha = H = K = 0$, for the prepotential superfield V, then V starts out with $\theta^\alpha\bar\theta^{\dot\beta}A_{\alpha\dot\beta} + \cdots$ so $V^3 = 0$ and the Taylor expansion of the exponential truncates.

Using the WZ gauge, one can evaluate the gauge-invariant kinetic term in component fields, simplifying the result by noting that in WZ gauge
$$V| = (D_\alpha V)| = (\bar D_{\dot\alpha} V)| = (D^2 V)| = (\bar D^2 V)| = 0.$$

The result is
$$\frac{1}{16}\int d^4x\, d^2\theta\, d^2\bar\theta \sum_a \bar\Phi_a e^{t_a V}\phi^a =$$
$$\frac{1}{16}\int d^4x \sum_a \Big(\bar h_a h^a - 8i\chi^{a\alpha}(\partial_{\alpha\dot\alpha} - \tfrac{i}{2}t_a A_{\alpha\dot\alpha})\bar\chi_a{}^{\dot\alpha}$$
$$+ 8(\partial_{\alpha\dot\alpha} - \tfrac{i}{2}t_a A_{\alpha\dot\alpha})a^a(\partial^{\alpha\dot\alpha} + \tfrac{i}{2}t_a A^{\alpha\dot\alpha})\bar a_a$$
$$+ 2t_a a^a \bar\lambda_{\dot\beta}\bar\chi_a{}^{\dot\beta} + 2t_a \bar a_a \lambda^\beta \chi^a{}_\beta$$
$$+ t_a D\bar a_a a^a\Big)$$

Note the appearance of $U(1)$ covariant derivatives $\partial_{\alpha\dot\alpha} - \frac{i}{2}t_a A_{\alpha\dot\alpha}$ as well as new couplings to the "photino" spinor field $\lambda_\alpha(x)$: $t_a a^a \bar\lambda_{\dot\beta}\bar\chi_a{}^{\dot\beta}$, and also to the vector multiplet's auxiliary field $D(x)$: $t_a D\bar a_a a^a$. In consequence of the latter, eliminating $D(x)$ by its (still) algebraic equation of motion produces a new potential term $\sum_a t_a^2(\bar a_a a^a)^2$.

Super QED

As an example of the above gauge-invariant construction, take two chiral "matter" superfields with opposite charges: $\phi_+' = e^{ie\Lambda}\phi_+$, $\phi_-' = e^{-ie\Lambda}\phi_-$. Note that although they transform oppositely, both ϕ_+ and ϕ_- are chiral, $\bar D_{\dot\alpha}\phi_+ = \bar D_{\dot\alpha}\phi_-$.

Coupling to the Super Maxwell prepotential V, one has the superspace action
$$I^{SQED} = \frac{1}{16}\int d^4x\, d^2\theta\, d^2\bar\theta\left(\bar\Phi_+ e^{eV}\phi_+ + \bar\Phi_- e^{-eV}\phi_-\right)$$
$$+ \frac{m}{2}\mathrm{Re}\int d^4x\, d^2\theta\, \phi_+\phi_- - \mathrm{Re}\int d^4x\, d^2\theta\, W^\alpha W_\alpha$$

This action contains kinetic terms for the 2-component spinor fields $\chi_{+\alpha}$ and $\chi_{-\alpha}$,
$$-\frac{i}{2}\left(\chi_+^\alpha\partial_{\alpha\dot\alpha}\bar\chi_+^{\dot\alpha} + \chi_-^\alpha\partial_{\alpha\dot\alpha}\bar\chi_-^{\dot\alpha}\right) = \mathcal{L}^{kin}_{elec}.$$

One can recover a more standard QED-type formulation by combining χ_+ and χ_- into a 4-component Dirac spinor
$$\Psi_D = \frac{1}{\sqrt2}\begin{pmatrix}\chi_{+\alpha}\\ \bar\chi_-^{\dot\alpha}\end{pmatrix}, \text{ so } \mathcal{L}^{kin}_{elec} = -i\bar\Psi_D \not\partial \Psi_D.$$

In this way, we may take Ψ as a model field for the electron, accompanied by complex "selectrons" $a_\pm$, coupled to a photon field A_m and its photino λ_α. The latter, like A_m, is neutral and so is described by a single 2-component complex spinor or by a Majorana 4-component spinor.

Super Yang - Mills Theories

First consider the problem of incorporating a nonabelian local symmetry at the most basic level, with propagating fields only.

Let $\mathfrak{g}$ be a semi-simple Lie algebra for a group G, with structure constants $f_{ab}{}^{c}$: $[T_a, T_b] = i f_{ab}{}^{c} T_c$, where the T_a are Lie algebra generators.

Let A_m and λ now be valued in the Lie algebra $\mathfrak{g}$:
$A_m = A_m{}^a T_a$ $\qquad \lambda = \lambda^a T_a$

The component field A_m transforms as a gauge field - while λ^a transforms in the adjoint representation of G. Accordingly, the YM field strength and the covariant derivative of λ^a are

$$F_{mn}{}^{a} = \partial_m A_n^a - \partial_n A_m^a + f_{bc}{}^{a} A_m{}^b A_n{}^c$$
$$(D_m\lambda)^a = \partial_m \lambda^a + f_{bc}{}^{a} A_m{}^b A_n{}^c$$

{ adjoint rep. generators are $(T_a)_b{}^c = -i f_{ab}{}^c$

For a semi-simple Lie algebra, the <u>Cartan - Killing metric</u> $g_{ab} = -\frac{1}{2} f_{ad}{}^{c} f_{bc}{}^{d}$ is nondegenerate ; if G is <u>compact</u>, then g_{ab} is <u>positive - definite</u>. One can use g_{ab} to lower the upper index on $f_{ab}{}^{c}$: $f_{abc} = g_{cd} f_{ab}{}^{d}$. For a semisimple group G, one finds that $f_{abc} = f_{[abc]}$, <u>totally antisymmetric</u>. Moreover, there is a basis for the Lie algebra of a compact semisimple group G such that $g_{ab} = \delta_{ab}$, so one may then ignore the distinction between upper and lower indices and consider $f_{ab}{}^{c}$ to be totally antisymmetric.

Consider now the action for A_m^a and λ^a :

$$I = \int d^4x \left[-\tfrac{1}{4} F_{mn}{}^{a} F^{mn\,a} - \tfrac{i}{2} \bar{\lambda}^a (\not{D}\lambda)^a \right]$$

{ The auxiliary field would enter simply as $\int d^4x\, D^a D^a$ }

The obvious gauge - covariant SUSY transformations are $\delta A_m{}^a = i\bar{\epsilon}\gamma_m\lambda^a$, $\delta\lambda^a = -\frac{1}{2}\gamma^{mn}F_{mn}{}^{a}\epsilon$.

In checking invariance, it is necessary to use two facts about covariant derivatives :

1) $\delta F_{mn}{}^{a} = 2\,(D_{[m}\delta A_{n]})^a$
2) One can integrate by parts

The variation of the action then gives

$$\delta_\epsilon I = \int d^4x \Big((D_m F^{mn})^a \delta_\epsilon A_n^a - i\bar{\lambda}^a\gamma^n D_n \delta_\epsilon \lambda^a$$
$$\qquad - \tfrac{i}{2} \bar{\lambda}^a \gamma^n f^{abc} \delta_\epsilon A_n{}^b \lambda^c \Big)$$

where the first line is a natural generalization of the variation of the super Maxwell action, and the second line arises from the variation of the gauge field A_n^b in the covariant derivative of λ^a in the spinor kinetic term $\bar{\lambda}^a \not{D} \lambda^a$.

Using the nonabelian Bianchi identity $(D_{[p}F_{mn]})^a \equiv 0$, the first line cancels in a way directly analogous to the Super Maxwell case, but the second line remains:

$$\delta_\epsilon I = \int d^4x \left(\tfrac{1}{2} f^{abc} \bar{\epsilon}\gamma_n\lambda^b \bar{\lambda}^a \gamma^n \lambda^c \right)$$

This term must vanish by itself, and it does so by virtue of a new identity:

$$\gamma_n \lambda^{[a} \bar{\lambda}^{b} \gamma^n \lambda^{c]} \equiv 0$$

where the [abc] antisymmetrization is enforced by f^{abc}. This identity can be proved either by use of the Fierz identity in 4-component notation, or by consideration of the included gamma-matrix part in 2-component notation. Write the above as $\lambda_\beta^{[a}\lambda_\gamma^{b}\lambda_\delta^{c]}(\gamma_n)_{\alpha\beta}(C\gamma^n)_{\gamma\delta}$ and note that [abc] antisymmetrization forces $(\beta\gamma\delta)$ symmetrization

Hence consider the expression $\gamma_{n\alpha(\beta}(\gamma^o\gamma^n)_{\gamma\delta)}$.
Test its properties by introducing a commuting Majorana spinor v_α and form $u = \gamma_n v(\bar{v}\gamma^n v)$. Now go to 2-component notation: $v \to v_\alpha, \bar{v}_{\dot\alpha}$. The quantity u is a Majorana spinor built from cubic terms in v_α and $\bar{v}_{\dot\alpha}$, which we can enumerate:
$v_\alpha v_\beta v_\gamma$, $v_\alpha v_\beta \bar{v}_{\dot\gamma}$, $v_\alpha \bar{v}_{\dot\beta}\bar{v}_{\dot\gamma}$, $\bar{v}_{\dot\alpha}\bar{v}_{\dot\beta}\bar{v}_{\dot\gamma}$
In order to form a Majorana spinor $\binom{u_\alpha}{\bar{u}^{\dot\alpha}}$ out of these, two indices of the same type (undotted or dotted) must be contracted with $\epsilon^{\alpha\beta}$ or $\bar{\epsilon}^{\dot\alpha\dot\beta}$. But the commuting components $v_\alpha, \bar{v}_{\dot\alpha}$ cannot support this: all possible terms $= 0$
Hence $u = \gamma_n v(\bar{v}\gamma^n v) = 0$ for arbitrary commuting v, and so $\gamma_{n\alpha(\beta}(\gamma^o\gamma^n)_{\gamma\delta)} \equiv 0$, implying $\delta_\epsilon I = 0$.

A directly similar discussion shows that the supersymmetry Noether current $J^m{}_\alpha = \frac{1}{2}(\gamma^{pq}F_{pq}{}^a\gamma^m\lambda^a)_\alpha$ is conserved, $\partial_m J^m_\alpha = 0$, by virtue of the SYM field equations
$(D^m F_{mn})^a - \frac{i}{2}\bar\lambda^b\gamma_n\lambda^c f^{abc} = 0$ and $(\not{D}\lambda)^a = 0$.

SYM in Superspace

Now return to superspace and consider how to generalize the super Maxwell case to nonabelian local symmetry. Chiral matter superfields will transform as $\phi' = e^{i\Lambda}\phi$, $\bar\phi' = \bar\phi e^{-i\bar\Lambda}$, where ϕ is now considered to transform in some representation T of a group G and Λ is a superfield matrix $[\Lambda_{ij}]$ where $\Lambda_{ij} = T_{ij}{}^a\Lambda^a$; $T_{ij}{}^a$ are the (hermitean) YM generators in the representation T
$\bar{D}_{\dot\alpha}\Lambda^a = D_\alpha\bar\Lambda^a = 0$ as in the Super Maxwell case.
Consider how to make an invariant kinetic term for ϕ:
$\bar\phi e^V\phi$ will be invariant provided the matrix superfield e^V transforms as $e^V \to e^{V'} = e^{i\bar\Lambda}e^V e^{-i\Lambda}$.
$V = [V_{ij}] = [T_{ij}{}^a V^a]$ has a non-polynomial transformation defined by the transformation of e^V. To lowest order in $\Lambda, \bar\Lambda$, one has
$V' = V + i(\bar\Lambda - \Lambda) + \cdots$ { agrees with S. Maxwell to lowest order

Given this gauge transformation, one can define a gauge covariant field strength:

$$W_\alpha = \bar{D}_{\dot\beta}\bar{D}^{\dot\beta}(e^{-V}D_\alpha e^V)$$

Let's see how the covariance works:
$$W_\alpha \to W_\alpha' = \bar{D}^2(e^{i\Lambda}e^{-V}D_\alpha(e^V e^{-i\Lambda})) \qquad \text{since } D_\alpha\bar\Lambda = 0$$
$$= \bar{D}^2(e^{i\Lambda}(e^{-V}D_\alpha e^V)e^{-i\Lambda}) + \bar{D}^2(e^{i\Lambda}D_\alpha e^{-i\Lambda})$$
$$= e^{i\Lambda}W_\alpha e^{-i\Lambda} \qquad \text{since } e^{i\Lambda}D_\alpha e^{-i\Lambda} \text{ is killed by } \bar{D}^2$$

Expansion of W_α into component matrix fields is best carried out using the gauge and supersymmetry covariant derivative $\nabla_\alpha = e^{-V}D_\alpha e^V$, satisfying $\{\nabla_\alpha, \bar\nabla_{\dot\alpha}\} = -2i\sigma^m_{\alpha\dot\alpha}\nabla_m$ and $\{\nabla_\alpha, \nabla_\beta\} = 0 = \{\bar\nabla_{\dot\alpha}, \bar\nabla_{\dot\beta}\}$ but now the nonabelian nature gives $-i[\nabla_m, \bar\nabla_{\dot\beta}] = \sigma_m{}^\beta{}_{\dot\beta}W_\beta$, $-i[\nabla_m, \nabla_\beta] = \sigma_{m\beta\dot\beta}\bar{W}^{\dot\beta}$

and $-i[\nabla_m, \nabla_n] = F_{mn}(x,\theta,\bar\theta)$, the superfield whose $\theta = \bar\theta = 0$ component is $F_{mn} = \partial_m A_n - \partial_n A_m + i[A_m, A_n]$.

The component field expansion of W_α gives thus the matrix component fields
λ_α, F_{mn}, D and higher components that are total D_m covariant derivatives, such as $\sigma^m_{\alpha\dot\alpha}D_m\bar\lambda^{\dot\alpha}$, etc.

One may thus write down the most general action for supersymmetric renormalizable interactions of chiral and gauge supermultiplets:

$$I_{\text{sym+matter}} = \frac{1}{2}\mathrm{Re}\int d^4x\, d^2\theta\, \mathrm{Tr}(W^\alpha W_\alpha) \qquad \mathrm{Tr} \text{ on YM indices}$$
$$+ \frac{1}{16}\int d^4x\, d^2\theta\, d^2\bar\theta\, \bar\phi e^V\phi + \mathrm{Re}\int d^4x\, d^2\theta\left(\tfrac{1}{2}m_{ij}\phi^i\phi^j + \tfrac{1}{6}g_{ijk}\phi^i\phi^j\phi^k\right)$$

where gauge invariance requires that m_{ij} and g_{ijk} be invariant tensors with respect to the YM symmetry group G.
The ϕ^i are the component superfields of ϕ in the YM representation T.

Spontaneous Supersymmetry Breaking

We have already noted that for an arbitrary superpotential $f(\phi^a)$, the potential $V = 4(\sum_a |f_{,a}|^2)|_{\theta=\bar\theta=0}$ is manifestly non-negative. Moreover, the total energy $H = P^0 = \frac{1}{4}\sum_\alpha Q_\alpha^{(2)}$ is a manifestly non-negative operator provided a theory has no ghosts, i.e. that it can be quantized in a normal Hilbert space without negative norm states.

From the above considerations, we learn that if SUSY is unbroken in a certain vacuum $|vac\rangle$, i.e. $Q_\alpha|vac\rangle = 0 \ \forall\alpha$, then $E_{vac} = 0$ since $H|vac\rangle = \frac{1}{4}\sum_\alpha Q_\alpha^2|vac\rangle = 0$.

Conversely, if SUSY is spontaneously broken, then $E_\psi > 0$ for all physical states $|\psi\rangle$, since one must have $E_\psi \geq E_{vac}$ and for $Q_\alpha|vac\rangle \neq 0$ in a theory with a positive definite norm, $E_{vac} = \frac{1}{4}\sum_\alpha \| Q_\alpha|vac\rangle\|^2 / \||vac\rangle\|^2 > 0$.

We wish to consider situations where SUSY is spontaneously broken, but Lorentz invariance is unbroken. Accordingly, vector fields and gradients cannot have vevs; only scalar fields can have nonvanishing vevs.

For a Wess-Zumino model without gauge coupling, one has $V = \frac{1}{16}h^a h^{a*}$ where $h^a = (D^2\phi^a)|_{\theta=\bar\theta=0} = -8(\bar f_{,a})|$, giving the above $V = 4(f_{,a}\bar f_{,a})|$ result.
Accordingly, in order to achive spontaneous SUSY breaking, one must arrange to have the vev of $h^a \neq 0$. This at first appears tricky. Consider the standard renormalizable superpotential $f(\phi) = \frac{1}{2}m_{ab}\phi^a\phi^b + \frac{1}{6}g_{abc}\phi^a\phi^b\phi^c$, giving $h^{a*} = -8(f_{,a})| = -8\left(m_{ab}\phi^b + \frac{1}{2}g_{abc}\phi^b\phi^c\right)|$ and $h^{a*} = h^a = 0$ for $a^a = 0$.

Of course, the potential may have a more complicated structure, with multiple minima. But one has to arrange things carefully to ensure that no state $|\psi\rangle$ has $E_\psi = V_\psi = 0$.

A key trick is to include *linear* terms in the superpotential: $f(\phi^a) = \eta_a\phi^a + \frac{1}{2}m_{ab}\phi^a\phi^b + \frac{1}{6}g_{abc}\phi^a\phi^b\phi^c$ so that $f_{,a} = \eta_a + m_{ab}\phi^b + \frac{1}{2}g_{abc}\phi^b\phi^c$ and by careful choice of η_a, m_{ab} and g_{abc} one can try to avoid ever having a solution to the equations $f_{,a} = 0 \ \forall a$.

The first model to achieve this was the O'Raifertaigh model (1975), which employed 3 chiral multiplets:

$$f = -\eta\phi_1 + \mu\phi_2\phi_3 + \lambda\phi_1\phi_3^2 \qquad \begin{array}{l}\dim\eta = 2\\ \dim\mu = 1\\ \dim\lambda = 0\end{array}$$

This gives the 3 auxiliary field equations

① $-\frac{1}{8}h_1^* = (\partial f/\partial\phi_1)| = \lambda a_3^2 - \eta$
② $-\frac{1}{8}h_2^* = (\partial f/\partial\phi_2)| = \mu a_3$
③ $-\frac{1}{8}h_3^* = (\partial f/\partial\phi_3)| = 2\lambda a_1 a_3 + \mu a_2$

Look for solutions to $h_1^* = h_2^* = h_3^* = 0$:
② $\Rightarrow a_3 = 0$ then ③ $\Rightarrow a_2 = 0$
but then ① gives $-\frac{1}{8}h_1^* = \eta \neq 0$, so $h_1^* = h_2^* = h_3^* = 0$ is impossible, and SUSY is spontaneously broken.

Note that if one had included a mass term $\frac{\mu'}{2}\phi_1^2$ for the ϕ_1 multiplet, SUSY would have remained unbroken since then ① would contain a term $\mu'\phi_1$, so one would have $h_1^* = 0$ for $a_1 = \frac{\eta}{\mu'}$.
Thus, the terms that are omitted from $f(\phi^a)$ are as important as the terms that are included.

The absence of a mass term for the supermultiplet containing the auxiliary field with a nonzero vev is a <u>characteristic feature</u> of Spontaneous supersymmetry breaking. Since fermion mass terms are read off directly from the second derivative of the superpotential, this implies the presence of a massless fermion, called the <u>Goldstone fermion</u>, somewhere in the theory.

In the O'Raifeartaigh model, consider the SUSY transformation of the fermion χ_1 belonging to the same multiplet as $h_1 \neq 0$.

$$\delta\chi_{1\alpha} = -\tfrac{1}{2} h_1 \epsilon_\alpha + 2i\,\partial_{\alpha\dot\beta} A_1 \bar\epsilon^{\dot\beta} \qquad \text{2-cpt. notation}$$

For $h_1 = 8\eta^*$, the "on-shell" SUSY transformations after elimination of auxiliary fields will include the transformation
$$\delta\chi_{1\alpha} = -4\eta^* \epsilon_\alpha + 2i\,\partial_{\alpha\dot\beta} A_1 \bar\epsilon^{\dot\beta} + \cdots$$
where the $\cdots$ indicate further terms coming from eliminating h_1. Thus, the SUSY transformations contain an inhomogeneous <u>shift</u> term for $\chi_{1\alpha}$.

Although SUSY may be spontaneously broken by the vacuum state $|vac\rangle$, the essence of <u>spontaneous</u> symmetry breaking is that the <u>theory</u> remains fully invariant under the symmetry. The symmetry is realized, however, in a particularly <u>nonlinear</u> fashion. This nonlinear transformation of the Goldstone fermion nonetheless manages to be an invariance of the kinetic part of the action — the inhomogeneous shift is by a constant spinor, so the lowest order term in the variation is killed by ∂. A mass term for χ_1 would clearly not be invariant, however, as there would be nothing to cancel the resulting variation linear in a spinor field. Thus the nonlinearly realized supersymmetry rules out a mass term for χ_1.

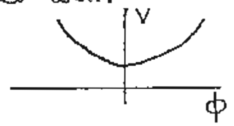
Note that the vacuum value of the potential $V_{min} = 4\eta^2$ is determined by the same parameter η that governs the Goldstone behaviour of χ_1.

Let's look at the mass and Yukawa interactions for the O'Raifeartaigh model in more detail. These are read off directly from our general treatment of the WZ model:
$$\mathrm{Re}\int d^4x\; \chi_a^{\alpha} \chi_{\alpha b} f_{,ab}(A)$$
In the present case, we have $f_{,13} = 2\lambda A_3$, $f_{,23} = \mu$ and $f_{,33} = 2\lambda A_1$; of these only $f_{,23}$ gives a mass term — $f_{,13}$ and $f_{,33}$ give Yukawa couplings:

<u>Fermion</u> mass + Yukawa terms
$$\tfrac{1}{2}\int d^4x\,(\mu\chi_2^{\alpha}\chi_{3\alpha} + 2\lambda A_3 \chi_1^{\alpha}\chi_{3\alpha} + 2\lambda A_1 \chi_3^{\alpha}\chi_{3\alpha}) + \text{c.c.}$$

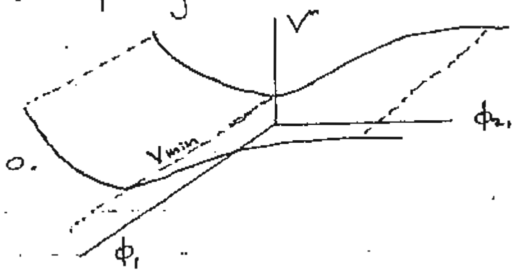
As expected, χ_1 is massless, while χ_2 and χ_3 have the same mass from their common off-diagonal mass term.

Now consider the masses of the bosons. Associated to the masslessness of χ_1, there tends to be a massless scalar as well. Although one frequently sees sketches like



to illustrate the potential of a theory with spontaneously broken supersymmetry, the full picture is better portrayed in a multidimensional sketch like

with a flat "valley" at $V_{min} > 0$.



Such flat valleys in the scalar field potential are not automatically present, but are typical. The mode corresponding to the valley floor is massless

Now consider in detail the bosonic mass terms in the O'Raifeartaigh model. Inserting the auxiliary field equations ①,②,③ into the potential $V = \frac{1}{16} h_a h^{\kappa}_a = 4(f_{,a} f_{,a})$, to obtain

$$V = 4\left(|\eta - \lambda A_3^2|^2 + \mu^2 |A_3|^2 + |\mu A_2 + 2\lambda A_1 A_3|^2\right).$$

Expand this out to quadratic order to find the masses:

$$V = 4\eta^2 + 4\left(\mu^2 A_2 A_2^* + \mu^2 A_3 A_3^* - \lambda\eta\left(A_3^2 + A_3^{*2}\right)\right) + \cdots$$

In order to see more clearly the masses, split $A_2 = a_2 + i b_2$ and $A_3 = a_3 + i b_3$ into real + imaginary parts, giving

$$V = 4\eta^2 + 4\left(\mu^2(a_2^2 + b_2^2) + (\mu^2 - 2\lambda\eta) a_3^2 + (\mu^2 + 2\lambda\eta) b_3^2\right) + \cdots$$

As anticipated, there is a flat valley in this potential: For $\mu^2 > |2\lambda\eta|$, both real and imaginary parts of A_1 are massless, while the masses of a_3 and b_3 are split (with the fermion mass μ for χ_3 lying in the middle). Thus, in theories with spontaneous supersymmetry breaking, component fields of the same supermultiplet can have different masses. [For $\mu^2 < |2\lambda\eta|$ one finds that the field a_3 takes a nonvanishing VeV $\left(\lambda^{-1}(\eta - \mu^2/2)\right)^{1/2}$ and the story is more complicated. However, there still are different masses within the same supermultiplet, and the potential still has a flat valley.]

Spontaneous supersymmetry breaking opens the possibility of having the superpartners of the known particles sufficiently massive to not yet be observable in accelerator experiments.

Supersymmetry breaking with vector supermultiplets: Fayet-Iliopoulos mechanism

Return to the super QED model, but include now the term $\kappa \int d^4x\, d^2\theta\, d^2\bar\theta\, V(x,\theta,\bar\theta)$. (Fayet-Iliopoulos term)

This is clearly supersymmetric. Check it for gauge invariance under $\delta V = i(\bar\Lambda - \Lambda)$. The variation gives

$$i\kappa \int d^4x\, d^2\theta\, d^2\bar\theta\, (\bar\Lambda - \Lambda) = i\kappa\left(\int d^4x\, d^2\bar\theta\, D^2\bar\Lambda - \int d^4x\, d^2\theta\, \bar D^2 \Lambda\right) = 0$$

since $\bar D_{\dot\alpha}\Lambda = D_\alpha \bar\Lambda = 0$. This clearly works only for a Maxwell $U(1)$ gauge symmetry, however, for which the adjoint rep is the singlet.

In components, the Fayet-Iliopoulos term is simply

$$\kappa \int d^4x \left(D^\alpha \bar D^2 D_\alpha V\right)\Big| = \kappa \int d^4x\, D(x)$$ linear in D auxiliary field.

We take this term together with the super QED action:

$$I^{FI} = -\tfrac{1}{2}\,\mathrm{Re}\int d^4x\, d^2\theta\, W^\alpha W_\alpha + \kappa \int d^4x\, d^2\theta\, d^2\bar\theta\, V$$
$$+ \tfrac{1}{16}\int d^4x\, d^2\theta\, d^2\bar\theta \left(\bar\Phi_+ e^{eV}\Phi_+ + \bar\Phi_- e^{-eV}\Phi_-\right) + \tfrac{m}{2}\,\mathrm{Re}\int d^4x\, d^2\theta\, \Phi_+\Phi_-$$

The scalar field potential $V = \frac{1}{2}D^2 + \frac{1}{16}h_+ h_+^* + \frac{1}{16} h_- h_-^*$ and the auxiliary field equations now give

$$D + \kappa + \tfrac{e}{16}\left(A_+^* A_+ - A_-^* A_-\right) = 0$$
$$\tfrac{1}{8} h_+^* + \tfrac{m}{2} A_- = 0$$
$$\tfrac{1}{8} h_-^* + \tfrac{m}{2} A_+ = 0$$

From these, one sees directly that there is no solution to $h_+ = h_- = D = 0$, so $V > 0$ and supersymmetry is spontaneously broken in this model.

Now substitute for the auxiliary fields in the potential:

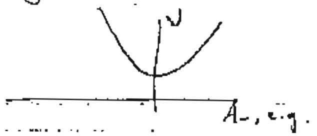
$$V = \tfrac{1}{2}k^2 + \left(m^2 + \tfrac{ke}{16}\right) a_+ a_+^* + \left(m^2 - \tfrac{ke}{16}\right) a_- a_-^* + \frac{e^2}{512}\left(a_+ a_+^* - a_- a_-^*\right)^2$$

There are two cases to consider: $m^2 > \frac{ke}{16}$ and $m^2 < \frac{ke}{16}$.

Case 1): $m^2 > \frac{ke}{16}$

In this case, we have two complex fields $a_\pm = \frac{A_\pm + iB_\pm}{\sqrt{2}}$ with $(\text{mass})^2$ $m^2 + \frac{ke}{16}$ for (A_+, B_+) and $(\text{mass})^2$ $m^2 - \frac{ke}{16}$ for (A_-, B_-)

Taking any one of these fields, the potential dependence is like



Note that, as in the O'Raifeartaigh model, the fermion masses are unchanged, so the $\pm\frac{ke}{16}$ terms give different masses for bosons and fermions in the same supermultiplet, characteristic of supersymmetry breaking.

— The massless fermion expected to accompany supersymmetry breaking is the spinor field λ of the $U(1)$ gauge multiplet. This may be verified by considering its supersymmetry transformation:

$$\delta\lambda = D\gamma_5\epsilon - \tfrac{1}{2}F_{mn}\gamma^{mn}\epsilon$$

and here $D = -k + \dots$ so $\delta\lambda$ starts off with an inhomogeneous shift $\delta\lambda = -k\gamma_5\epsilon + \dots$ characteristic of a Goldstone fermion

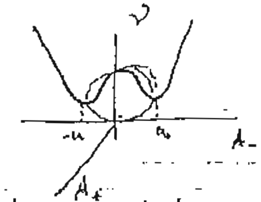
Now consider

Case 2): $m^2 < \frac{ke}{16}$

Recalling that $V = \frac{1}{2}D^2 + \frac{1}{16}|h_+|^2 + \frac{1}{16}|h_-|^2 > 0$ in all cases, we still have spontaneous supersymmetry breaking. But now we need to be more careful, because there is gauge symmetry breaking as well: the apparent $(\text{mass})^2$ for a_- is negative, so a_- will take a non-trivial vev: $\langle a_- \rangle = u$.

By a $U(1)$ gauge transformation, one can arrange for u to be real. One finds $u^2 = \frac{128}{e^2}\left(\frac{ke}{16} - m^2\right)$

graphing V versus A_-, we now have



Expanding the potential about the vev u for A_-, one finds that the $U(1)$ symmetry is also spontaneously broken.

There is again a massless Goldstone fermion field in this case, but it is a mixture of the λ from the $U(1)$ vector supermultiplet and the χ_- of the Φ_- supermultiplet. This mixture is apparent from the vev's of the auxiliary fields:

$$D = -\left(\frac{k}{2} + \frac{8}{e}m^2\right) \qquad h_- = -4mu \quad \text{, both nonzero}$$

— The non-vanishing vev of A_-, however gives rise to another phenomenon: the bosonic Goldstone mode (rolling mode in the minimum of V sketched above) would have been a massless scalar field, except it gets eaten by an ordinary $U(1)$ Higgs effect. As a consequence, the vector field A_m becomes massive. This can be seen in the gauge coupling between the vector and a_-:

$$\left(\partial_\mu - \tfrac{ie}{2}A_\mu\right) a_- \left(\partial^\mu + \tfrac{ie}{2}A^\mu\right) a_-^* \quad \text{contributes a}$$

mass term $\frac{u^2 e^2}{4} A_\mu A^\mu$.

The Standard Model and its Supersymmetric extension

Refs: M.E. Peskin "Beyond the Standard Model" hep-ph/9705479

D. Bailin + S. Love "Supersymmetric Gauge Field Theory and String Theory" (IOP Publishing, Bristol, 1994)

S. Weinberg "The Quantum Theory of Fields III" (C.U. Press, 2000)

The basic principle underlying the standard model is that of gauge-invariant interactions. The gauge group is a direct product $SU(3) \times SU(2) \times U(1)$, with three à priori unrelated coupling constants g_s, g, g'. The gauge bosons have multiplicities corresponding to their Lie algebras:

8 G_μ^A	$A = 1 \dots 8$	$SU(3)_c$	colour
3 $A_\mu^{\prime a}$	$a = 1,2,3$	$SU(2)_L$	isospin
1 B_μ		$U(1)_Y$	weak hypercharge

The fermions (leptons and quarks) carry representations of $SU(3) \times SU(2) \times U(1)$ that are chiral, ie. left and right-handed fermions $q_L = \frac{(1-i\gamma_5)}{2} q_L$ (or $\begin{pmatrix} q_\alpha \\ 0 \end{pmatrix}$ in Weyl basis), $q_R = \frac{(1+i\gamma_5)}{2} q_R$ (or $\begin{pmatrix} 0 \\ \bar{q}^{\dot\alpha} \end{pmatrix}$ in Weyl basis).

In a normalization where the unbroken electromagnetic $U(1)$ charge is $Q_{em} = I_3 + Y$, the left-handed quarks and leptons have $(SU(3), SU(2), U(1)_Y)$ representations; charges

$$q_{iL} = \begin{pmatrix} u_i \\ d_i \end{pmatrix}_L \qquad \overset{SU(3)\;\; SU(2)\;\; U(1)_Y}{(3,\ 2\ ;\ 1/6)} \qquad \left(\begin{matrix} SU(2) \\ \text{doublet} \leftrightarrow \\ I = \tfrac{1}{2} \end{matrix}\right)$$

$$e_{iL} = \begin{pmatrix} \nu_i \\ e_i \end{pmatrix}_L \qquad (1,\ 2\ ;\ -1/2)$$

while the right-handed quarks and leptons have

$$u_{iR} \qquad (3, 1; \tfrac{2}{3}) \qquad \left(\begin{matrix} SU(2) \text{ singlet} \\ \leftrightarrow I = 0 \end{matrix}\right)$$

$$d_{iR} \qquad (3, 1; -\tfrac{1}{3})$$

$$e_{iR} \qquad (1, 1; -1)$$

The index "i" here refers to a further multiplicity: there are 3 families

$i = 1,2,3$:

$$\begin{pmatrix} u \\ d \end{pmatrix} \qquad \begin{pmatrix} c \\ s \end{pmatrix} \qquad \begin{pmatrix} b \\ t \end{pmatrix}$$

$$\begin{pmatrix} \nu_e \\ e \end{pmatrix} \qquad \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} \qquad \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}$$

Since the antiparticle of a right-handed spinor is left-handed, one often classifies all species by their left-handed versions. Accordingly, instead of the right-handed particles we have their left-handed conjugates

$\bar{u}_i^R$	$(\bar{3}, 1; -\frac{2}{3})$	LH anti up, etc.
$\bar{d}_i^R$	$(\bar{3}, 1; 1/3)$	LH anti down, etc.
$\bar{e}_i^R$	$(1, 1; 1)$	LH anti electron, etc.

The final sector of the standard model is the Higgs sector. In the minimal version of the standard model, one has a single Higgs doublet

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \qquad (1, 2; \tfrac{1}{2})$$

The Standard model action is then of the form

$$① \quad I = \int d^4x \Big\{ -\tfrac{1}{4} \left(F^K_{mn}\right)^2 + i\, \bar{q}_i \not{D} q_i + i\, \bar{u}_i^R \not{D} u_i^R + i\, \bar{d}_i^R \not{D} d_i^R + i\, \bar{e}_i^R \not{D} e_i^R$$

$$② \quad + |D_\mu \phi|^2 - V(\phi)$$

$$③ \quad - \left(\lambda_u^{ij} \bar{u}_i^R \phi \cdot q_{jL} + \lambda_d^{ij} \bar{d}_i^R \phi^* \cdot q_{jL} + \lambda_e^{ij} \bar{e}_i^R \phi^* \cdot e_{jL} + \text{h.c.}\right) \Big\}$$

$K: A, a, Y$ — Note how charge assignments mesh.

①: Gauge kinetic terms + gauge couplings to fermions

②: Higgs sector

③: terms giving quark + lepton masses and Yukawa couplings.

For the minimal model Higgs potential, one takes

$$V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$$

which is minimized when $\phi^\dagger \phi = \frac{\mu^2}{2\lambda}$

One may choose the Higgs vacuum thus:

$$\langle\phi\rangle = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} v \end{pmatrix} \qquad \text{where } v^2 = \mu^2/\lambda$$

In the same notation, the most general ϕ configuration can be written $\phi = e^{i\alpha(x)\cdot\tau}\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + h(x)) \end{pmatrix}$ $\alpha(x)$: gauge parameters

where $\tau^a = \frac{\sigma^a}{2}$ are $SU(2)$ generators. The field $h(x)$ is the fluctuation orthogonal to a gauge transformation, i.e. it is gauge invariant, and represents the physical Higgs field. Its mass is given by $m_h^2 = 2\mu^2 = 2\lambda v^2$.

Inserting the Higgs vacuum value into the kinetic term for ϕ, the gauge couplings give masses for the W (charged) and Z (neutral) vector bosons: $m_W = g\frac{v}{2}$ $m_Z = \frac{v}{2}\sqrt{g^2 + (g')^2}$.

Since these particles have been discovered and their couplings and masses measured, one has the value $v = 246$ GeV.

The neutral Z^0_μ vector boson is a combination of A'^3_μ and B_μ:

$$Z^0_\mu = A'^3_\mu \cos\Theta_W + B_\mu \sin\Theta_W,$$

while the orthogonal combination $-A'^3_\mu \sin\Theta_W + B_\mu \cos\Theta_W = A_\mu$ has zero mass, and gives the unbroken electromagnetic gauge field, the photon. Θ_W is the weak mixing angle $\cos\Theta_W = \frac{m_W}{m_Z} = \frac{g}{\sqrt{g^2 + (g')^2}}$

The gauge & fermion sector of the standard model is quite firmly established. For example the value of $\sin^2\Theta_W$ is known from LEP and SLC experiments to about one part in 10^3. Moreover, important parts of this picture are fairly robust, in the sense that they do not depend strongly on the electroweak symmetry breaking mechanism, eg. $\sin^2\Theta_W$. However, the Higgs sector is much less established: the Higgs has not yet been found. So one can certainly contemplate modifications in that sector

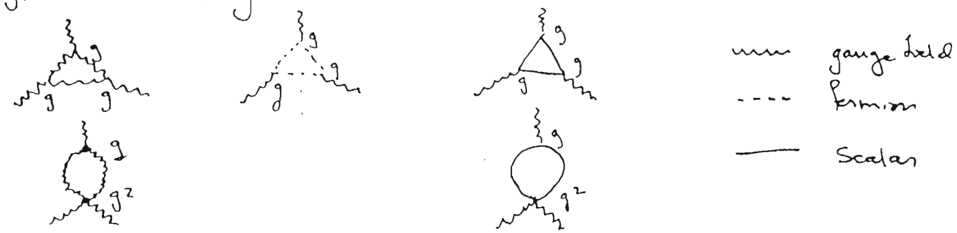
The evolution of couplings

Renormalization to remove ultraviolet divergences in a quantum field theory always introduces a scale dependence in the values of coupling constants. For example, consider the mass term for the Higgs field, $V_{mass} = -\mu^2 \phi^\dagger\phi$. The $(mass)^2 = -\mu^2$ has
N.B. "m^2" $= -\mu^2 < 0$ for sym. breaking
a radiative correction from the diagram  : if one cuts off the integral arising here at a scale Λ where the minimal Standard model breaks down, one finds a mass correction $\delta m^2 = \lambda \int^{\Lambda} \frac{d^4k}{(2\pi)^4} \frac{i}{k^2} = \frac{\lambda}{16\pi^2}\Lambda^2$. This is divergent as $\Lambda \to \infty$ and positive. The positivity is actually the worst feature of this correction, because having a negative $(mass)^2$ is essential for the symmetry breaking to work. In the minimal standard model, this correction must be cancelled by the Higgs bare mass. As the cutoff Λ is taken to be larger and larger, this cancellation must be made more and more finely, so the minimal standard model does not explain at all why $(mass)^2 \sim -v^2$ for v of the required (246 GeV) scale. This is called the "gauge hierarchy problem".

How seriously one takes the gauge hierarchy problem is a matter of taste — at the least, it is an indication of the incompleteness of the minimal standard model.

Another problem with the minimal S.M. is its inconsistency with unification — with the idea that the three different coupling constants g_3, g, g' should themselves derive from some high-energy unified gauge theory with a single coupling strength.

Consider a gauge coupling constant g_i. The renormalization of g_i involves diagrams like



After renormalization to remove the divergences, with subtractions performed at a low-energy scale of physical interest, e.g. v, then the sum of the tree-level diagram plus the above has an effective coupling given by
$$\frac{g_{iR}^2}{1+\frac{b_i}{8\pi^2}\, g_{iR}^2 \log\frac{Q}{v}} = g_i^2(Q)$$
where Q is the scale of momentum transfer.

g_{iR} : renormalized g_i; b_i = renormalization group coefficient.

This may also be expressed via the renormalization group equation
$$\frac{d}{d\log Q}\, g_i(Q) = -\frac{b_i}{(4\pi)^2}\, g_i^3(Q)\,.$$

Detailed calculation shows that the renormalization group coefficient for $SU(N)$ is
$$b_N = \left(\frac{11N}{3} - \frac{1}{3} n_f - \frac{1}{6} n_s\right) = \begin{bmatrix} SU(3) \\ 11 - \frac{4}{3} n_g \end{bmatrix}$$
$SU(2)$: $= \left[\frac{22}{3} - \frac{4}{3} n_g - \frac{1}{6} n_h\right]$, n_h = no. of Higgs doublets.

with n_g = no. of quark/lepton generations

n_f = no. of l-handed chiral fermions

n_s = no. of complex scalars coupled to the $SU(N)$ gauge field.

while for a $U(1)$ gauge theory with fermions carrying charges t_f and scalars carrying charges t_s one finds
$$b_1 = -\frac{2}{3}\sum_f t_f^2 - \frac{1}{3}\sum_s t_s^2 = -\frac{4}{3} n_g - \frac{1}{10} n_h$$

Note the celebrated fact that for $SU(N)$ with n_f and n_s sufficiently small, $b_N > 0$ ($N \neq 1$), leading to a decrease of $g_N(Q)$ as Q increases: this is called asymptotic freedom, and it underlies the entire basis of the perturbative approach to the strong interactions at high energies.

Now consider the hypothesis that at some energy scale $Q = m_u$ the three couplings g_s, g, g' come together at a common value $g(m_u)$. It is convenient to reformulate the problem in terms of quantities $\alpha_i = \frac{g_i^2}{4\pi}$, $i = 3,2,1$ and to perform the renormalizations at the scale of the Z^0 mass m_Z, so from the previous expression for $g_i^2(Q)$ one has
$$\alpha_i^{-1}(Q) = \alpha_i^{-1}(m_Z) + \frac{b_i}{2\pi}\log\frac{Q}{m_Z} \qquad i = 3,2,1$$

Setting $\alpha_3^{-1}(m_u) = \alpha_2^{-1}(m_u) = \alpha_1^{-1}(m_u) = \alpha^{-1}(m_u)$ at the hypothesised unification scale and then substituting into the three equations for $\alpha_i^{-1}(Q)$ at $Q = m_u$ one can eliminate the unknowns $\alpha^{-1}(m_u)$ and $\log\left(\frac{m_u}{m_Z}\right)$.

In consequence, at the scale m_Z one obtains the relation
$$\alpha_3^{-1} = (1+B)\,\alpha_2^{-1} - B\,\alpha_1^{-1} \qquad \text{where } B = \frac{b_3 - b_2}{b_2 - b_1}\,.$$

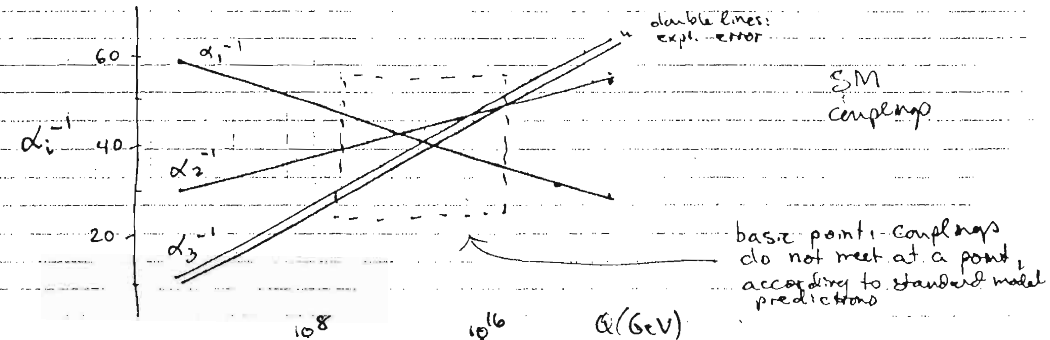
Now compare theory and experiment: for the Standard model with n_h Higgs doublets one has $B_{th} = \frac{1}{2} + \frac{3}{110} n_h$.

Experiment, however, gives the following ($\overline{MS}$ renormalized) couplings at $Q = m_Z$: $\alpha_3^{-1} = 8.47 \pm .22$ $\alpha_2^{-1} = 29.60 \pm .04$ $\alpha_1^{-1} = 58.98 \pm .08$

giving $B_{exp} = 0.719 \pm .008 \pm .03$
expt. error; omission of higher order corrections.

Clearly, B_{th} misses B_{exp} by a wide margin, for any plausible number of Higgs doublets n_h.

One frequently sees this situation presented as follows:



The Minimal Supersymmetric Standard Model (MSSM)

When one sets out to make a supersymmetric extension of the standard model, an immediate question is whether any of the known or currently anticipated (i.e. Higgs sector) particles can already be considered in superpartner pairs. Since the supersymmetry generator is a scalar under $SU(3) \times SU(2) \times U(1)$, superpartners must have the same standard-model charge assignments (i.e. SUSY and the standard-model symmetries must be in a direct product relation). Mostly, the possibilities of known/anticipated superpairings are ruled out, by the differing charge assignments under $SU(3) \times SU(2) \times U(1)$. There is one exception in the minimal standard model: the conjugate Higgs doublet ϕ^* has $(1,2;-\frac{1}{2})$, the same as e_{iL}. Although it might seem to be attractive to try to make superpartner relations using this, it turns out to lead to unwanted phenomenology, with certain cross sections that should be suppressed becoming too large.

Thus, one becomes resigned to adding as yet undiscovered superpartners for all the known/anticipated particles, denoted by a tilde over the usual notation:

quarks	$q_{iL} = \begin{pmatrix} u_i \\ d_i \end{pmatrix}$	$\longleftrightarrow$	squarks	$\tilde{q}_{iL} = \begin{pmatrix} \tilde{u}_i \\ \tilde{d}_i \end{pmatrix}$	scalar superpartners
	$\bar{u}^R_i, \bar{d}^R_i$	$\longleftrightarrow$		$\tilde{u}^R_i, \tilde{d}^R_i$	
electrons, neutrinos	$e_{iL} = \begin{pmatrix} \nu_{iL} \\ e_{iL} \end{pmatrix}$	$\longleftrightarrow$	selectrons, sneutrinos	$\tilde{e}_{iL} = \begin{pmatrix} \tilde{\nu}_{iL} \\ \tilde{e}_{iL} \end{pmatrix}$	
	$\bar{e}_i^R$	$\longleftrightarrow$		$\tilde{e}_i^R$	
gauge fields		$\longleftrightarrow$	gauginos		
gluons	A_μ^A	$\longleftrightarrow$	gluinos	$\tilde{A}_\mu^A$	fermionic superpartners.
	A_μ^{1a}	$\longleftrightarrow$		$\tilde{A}_\mu^{1a}$	
	B_μ	$\longleftrightarrow$		$\tilde{B}_\mu$	

After θ_W mixing to produce $W^\pm, Z^0$ have

	$W^\pm, Z^0$	$\longleftrightarrow$	$\tilde{W}^\pm, \tilde{Z}^0$	winos, zino
photon	A_μ	$\longleftrightarrow$	$\tilde{A}_\mu$	photino

Focus now on the superpotential, key to all the interactions with the Higgs sector. Given a superpotential W, the Yukawa couplings can be read off directly from the two-fermi terms in $D^\alpha D_\alpha W$. The SM Yukawa couplings $\lambda_u^{ij} \bar{u}_i^R \phi \cdot q_{jL} + \lambda_d^{ij} \bar{d}_i^R \phi^* \cdot q_{jL} + \lambda_\ell^{ij} \bar{e}_i^R \phi^* \cdot e_{jL}$ involve both ϕ and ϕ^*.

Now, all fields will be incorporated into LH chiral supermultiplets, $\bar{D}_{\dot\alpha} \Phi^K = 0$, and the superpotential W must be holomorphic in chiral superfields. Thus, one cannot use a conjugate superfield $\bar{\Phi}$ in W: the simplest supersymmetric model involves two Higgs doublet supermultiplets, H_1 with rep/charges $(1,2;-\frac{1}{2})$ and H_2 with rep/charges $(1,2;+\frac{1}{2})$.

Letting the chiral supermultiplets be denoted by capitalized versions of their known names (whether bosonic or fermionic), the MSSM superpotential is

$$W = \lambda_u^{ij} \bar{U}_i H_2 \cdot Q_j + \lambda_d^{ij} \bar{D}_i H_1 \cdot Q_j + \lambda_e^{ij} \bar{E}_i H_1 \cdot L_j$$

plus a Higgs mass term $\Delta W = \mu H_1 H_2$. At this stage, μ is the only new parameter beyond those present in the standard model. The μ mass term is necessary to prevent the occurrence of zero or small mass "neutralino" states that are ruled out by existing supersymmetry searches.

Now return to the question of coupling unification at some unknown higher energy. The gauginos present in the supersymmetric model give new contributions to the b_N renormalization group coefficients. For an $SU(N)$ gauge theory with n_f chiral supermultiplets, one finds

$$b_N = 3N - \tfrac{1}{2} n_f$$

and for a $U(1)$ gauge theory one has

$$b_1 = -\sum_f t_f^2$$

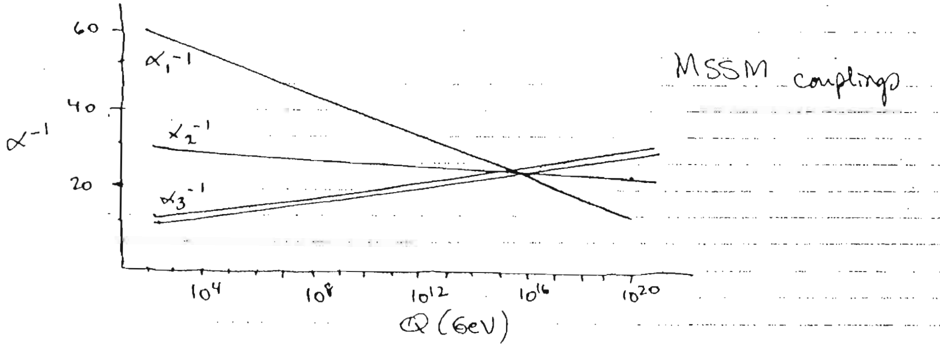
where t_f is the $U(1)$ charge of a chiral supermultiplet.

Evaluating these for $SU(3) \times SU(2) \times U(1)$ with n_g quark/lepton generations and n_h Higgs doublet supermultiplets, one has

$$b_3 = 9 - 2n_g$$
$$b_2 = 6 - 2n_g - \tfrac{1}{2} n_h$$
$$b_1 = -2n_g - \tfrac{3}{10} n_h$$

Inserting these into $B = \frac{b_3 - b_2}{b_2 - b_1}$ for $n_h = 2$, $n_g = 3$ one has $B = \frac{5}{7} = 0.714\ldots$

which agrees very well with the experimental value $0.719 \pm .008 \pm .03$. Sketching the evolution of coupling constants, one sees that they nicely meet now:



So one anticipates gauge coupling unification, and the arrival of yet more new physics at about 10^{16} GeV.

R-parity

With the large number of new superpartners in the MSSM as compared with the SM, there clearly is leeway for modifications within the above general scheme. However, there is often a useful discrete symmetry, called R-parity that restricts choices if one decides to be limited to R-parity-symmetric models.

The R-parity quantum number is defined for a superfield (e.g chiral) by $\Phi'(x,\theta) = \eta\,\phi(x,-\theta)$, so in terms of components one has

$$\phi'(x) = \eta\,\phi(x), \quad \psi'(x) = -\eta\,\psi(x), \quad F'(x) = \eta F(x).$$
$$= \Pi_R(\phi)\,\phi(x) \qquad = \Pi_R(\psi)\,\psi, \text{ etc}$$

One can choose the η's such that $\Pi_R = +1$ for conventional gauge fields, quark, lepton, Higgs fields, while $\Pi_R = -1$ for superpartners

Farrar and Weinberg showed that R-parity is equivalent to $\Pi_R = (-1)^{3(\mathcal{B}-\mathcal{L})}(-1)^F$, where $\mathcal{B}$ is baryon number (⅓ for a quark), $\mathcal{L}$ is lepton number, F is fermion number, so that if $\mathcal{B}-\mathcal{L}$ and F are conserved, then Π_R is conserved. Experimental searches for supersymmetry to date have assumed R-parity conservation.

Models with conserved R-parity have an important phenomenological consequence: the lightest superpartner is stable. Models without a conserved R-parity, in which the lightest superpartner can decay, are possible, but one must be careful to avoid unwanted effects such as rapid proton decay.

Supersymmetry breaking

Neither the O'Raifeartaigh nor the Fayet-Iliopoulos mechanism give desirable phenomenology, at least when they are considered in the purely flat-space context we have dealt with. Coupling to supergravity, however, gives important new possibilities, and also gives a way to cancel the cosmological constant that would otherwise result from a non-vanishing vacuum energy. The net effects of spontaneous supersymmetry breaking via supergravity may be summarized in a set of "soft breaking terms" that can be added to a supersymmetric action without disturbing the most important ultraviolet cancellations, in particular maintaining the cancellation of quadratic divergences in scalar mass terms.

The important* soft breaking terms for SM physics are the superrenormalizable ones (dimensions ≤ 3), and in particular the mass terms

$$\mathcal{L}_{soft} = -\sum_j m_j^2 |\phi_j|^2 - \sum_k m_k \underbrace{\bar{\tilde{\lambda}}^k \tilde{\lambda}^k}_{\text{gauginos}} + \underbrace{B\mu h_1 \cdot h_2}_{\Delta_\mu W} + A \underbrace{W(\phi)}_{\text{superpotential}}$$

dim 3, 2, only

Note: no terms like $\phi^* \phi\phi$ or $\bar{\psi}\psi$ chiral multiplet mass terms – these could regenerate unwanted quadratic divergences

– * higher order soft terms should be suppressed by inverse heavy masses

There are two basic scenarios for the origin of the soft terms discussed in the literature: supergravity-mediated supersymmetry breaking and gauge-mediated supersymmetry breaking. Both involve SUSY breaking originally at some higher scale M_S, which is then transmitted to the MSSM fields by <u>messenger fields</u>. For some auxiliary field VeV $\langle N \rangle \sim M_S^2$ causing the original SUSY breaking, subsequently transmitted to the MSSM fields by messengers of mass $\mathcal{M}_{mes.}$, the scale of the soft breaking masses will be $m_{soft} = \frac{M_S^2}{\mathcal{M}_{mes.}}$. For supergravity-mediated SUSY breaking, $\mathcal{M}_{mes.} \sim M_{Planck} \simeq 10^{19}\,\text{GeV}$ and then $M_S \sim 10^{11}$ GeV in order to have $m_{soft} \sim 1\,\text{TeV}$, as needed for electroweak symmetry breaking. In gauge-mediated SUSY breaking, it is assumed that the messengers carry $SU(3)\times SU(2)\times U(1)$ representations, with $M_S \sim 10^2 - 10^3$ TeV in order to have $m_{soft} \sim 1$ TeV.

Now consider gauge-mediated SUSY breaking in some more detail, and consider how one might achieve electroweak symmetry breaking. In the tree approximation, the total scalar potential takes the form $V = V_D + V_\mu + V_{soft}$ where V_D comes from the gauge multiplet D_K^2 terms and V_μ are the supersymmetric mass terms deriving from $\Delta_\mu W = \mu H_1 \cdot H_2$. Taken together, we have

$$V = \frac{g^2}{2}\left| h_1 \cdot h_2 \right|^2 + \frac{g^2 + (g')^2}{8}\left[h_1^* \cdot h_1 - h_2^* \cdot h_2 \right]^2$$

$$+ (\underline{m_1^2} + |\mu|^2)(h_1^* \cdot h_1) + (\underline{m_2^2} + |\mu|^2)(h_2^* \cdot h_2)$$

$$- \underline{B\mu}\, \mathrm{Re}(h_1 \cdot h_2)$$

underlined terms are soft SUSY breaking

$B\mu$ real + positive by overall phase adjustment

Now, for electroweak symmetry breaking, one needs to have a negative eigenvalue in the $(\text{mass})^2$ matrix. However, there is also a danger of instability resulting from this in directions where the quartic potential terms vanish. Specifically, one needs to ensure stability in directions $h_1 = \begin{pmatrix} \phi \\ 0 \end{pmatrix}$ $h_2 = \begin{pmatrix} 0 \\ \phi \end{pmatrix}$ for which the quartic terms vanish.

This stability requirement thus restricts the potential's parameters by $2|\mu|^2 + m_1^2 + m_2^2 - B\mu \geq 0$. At the same time, we require for electroweak symmetry breaking a negative eigenvalue for the $(\text{mass})^2$ matrix $\begin{pmatrix} m_1^2 + |\mu|^2 & -\frac{B\mu}{2} \\ -\frac{B\mu}{2} & m_2^2 + |\mu|^2 \end{pmatrix}$. This condition gives the restriction
$4(m_1^2 + |\mu|^2)(m_2^2 + |\mu|^2) < (B\mu)^2$.

The above two restrictions on the tree-level potential are not mathematically inconsistent, but they are awkward for tree-level electroweak symmetry breaking. Gauge-mediated supersymmetry breaking naturally gives the same soft-breaking mass terms to fields carrying the same $SU(3)\times SU(2)\times U(1)$ reps/charges, and this is the case, up to the sign of the Y hypercharge, for h_1 and h_2. $(1,2,-\frac{1}{2})$ $(1,2,\frac{1}{2})$ So gauge-mediated SUSY breaking naturally gives $m_1^2 = m_2^2$ at tree level. In that case, the above two inequalities become $0 < B\mu \leq 2(|\mu|^2 + m_1^2)$ and $(B\mu)^2 > 4(|\mu|^2 + m_1^2)^2$, which are clearly inconsistent.

The resolution of this problem involves radiative corrections and the high mass scale of the top quark. Letting $\langle h_1 \rangle = v_1$ and $\langle h_2 \rangle = v_2$ be the Higgs vevs, of which one may adjust the phase of v_1 to make it real, and then the vacuum conditions make v_2 turn out to be real too. Letting $\tan\beta = \frac{v_2}{v_1}$ and $M_Z^2 = \frac{1}{2}(g^2 + (g')^2)(v_1^2 + v_2^2)$ (the Z vector boson $(\text{mass})^2$) and $m_A^2 = 2|\mu|^2 + m_1^2 + m_2^2$ (the $(\text{mass})^2$ of a physical scalar), one finds $B\mu = m_A^2 \sin 2\beta$ and
$m_1^2 + |\mu|^2 = \frac{1}{2} m_A^2 - \frac{1}{2}(m_A^2 + M_Z^2)\cos 2\beta$, $m_2^2 + |\mu|^2 = \frac{1}{2} m_A^2 + \frac{1}{2}(m_A^2 + M_Z^2)\cos 2\beta$.
Since m_A^2 is the $(\text{mass})^2$ of a physical scalar, one must have $m_A^2 > 0$. And since $B\mu > 0$, this limits β to the range $0 < \beta < \pi/2$.

Now, in the 3rd generation, the ratio of top and bottom quark masses $\left(\frac{175\text{ GeV}}{4.5\text{ GeV}} \sim 39\right)$ is governed by $\tan\beta = v_2/v_1$. So, unless this large ratio is due to a large ratio of Yukawa couplings (unnatural), one should have $\tan\beta$ large, so $\beta \sim \pi/2$. But then $m_1^2 + |\mu|^2 \sim m_A^2 + \frac{1}{2} m_Z^2$ while $m_2^2 + |\mu|^2 \sim -\frac{1}{2} m_Z^2$, hardly equal and in apparent contradiction with the mechanism of gauge-mediated Susy breaking.

The resolution of the problem naturally involves the large scale of the top quark mass itself. Consider the renormalization group equations for the masses $M_{h_2}^2$, $M_{\tilde{t}_L}^2$ and $M_{\tilde{t}_R}^2$, where $\tilde{t}_L$ and $\tilde{t}_R$ are the L, R stops (scalar partners of the L, R top quarks)

$$\frac{d}{d\log Q} M_{h_2}^2 = \frac{1}{(4\pi)^2}\left\{\lambda_t^2\left(M_{h_2}^2 + M_{\tilde{t}_L}^2 + M_{\tilde{t}_R}^2\right) + \cdots\right\}$$

$$\frac{d}{d\log Q} M_{\tilde{t}_L}^2 = \frac{1}{(4\pi)^2}\left\{2\lambda_t^2\left(M_{h_2}^2 + M_{\tilde{t}_L}^2 + M_{\tilde{t}_R}^2\right) - \frac{32}{3} g_3^2 m_3^2 + \cdots\right\}$$

$$\frac{d}{d\log Q} M_{\tilde{t}_R}^2 = \frac{1}{(4\pi)^2}\left\{3\lambda_t^2\left(M_{h_2}^2 + M_{\tilde{t}_L}^2 + M_{\tilde{t}_R}^2\right) - \frac{32}{3} g_3^2 m_3^2 + \cdots\right\}$$

where λ_t is the top quark Yukawa coupling, g_3 is the SU(3) strong interaction coupling and m_3 is the mass of the gluino. Omitted terms are of electroweak strength.

The main point is that the effect of the strong interaction couplings $\sim g_3^2$ is to push $M_{\tilde{t}_L}^2$ and $M_{\tilde{t}_R}^2$ <u>up</u> at low energies (cf. down at high energies) while the effect of λ_t^2 is to push all three masses <u>down</u> at low energies. Consequently, in order for $M_{\tilde{t}_L}^2$ and $M_{\tilde{t}_R}^2$ to remain large at low energies, $M_{h_2}^2$ must go down through zero and become negative.

Accordingly, starting at high energies above the top scale with all Higgs $(\text{mass})^2$ parameters initially positive, RG evolution down to low energies <u>naturally</u> gives eventually a negative $(\text{mass})^2$ to h_2, causing electroweak symmetry breaking