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GAUGE THEORIES OF GRAVITY AND SUPERGRAVITY*

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Introduction

The unification of gravity with the other fundamental interactions of elementary particles has been a major goal of theoretical physics for most of this century, and still remains elusive. One obvious connecting thread is the notion of gauge invariance. All the currently favoured theories of strong weak and elctromagnetic interactions are gauge theories, and it has long been known that gravity may also be so regarded. It is therefore very interesting to examine this relationship in more detail. One of the chief reasons for renewed interest in this subject is the advent of super-gravity, which may also very profitably be regarded as a gauge theory.

The earliest attempts at a unified theory of gravity and elctromagnetism, the only other fundamental interaction then known, were strongly geometrical. The object was to extend general relativity to a complete geometrical theory of all interactions, the holy grail of a unified field theory so long sought by Einstein and others. Though gauge theories have developed from the other end, starting from particle physics rather than gravity, they are clearly in this geometrical tradition.

Notable among early proposals for a unified theory was Weyl's (1918) theory of gravity and electromagnetism, where the gauge principle (Prinzip der Eichinvarianz) first appeared. The transformations he considered were in one respect very different from what we now understand by gauge transformations. Weyl argued that since physics is unchanged by a change of the length scale it should, in a relativisitic theory, be possible to make space-time-dependent scale changes. His gauge group was therefore not U(1), but the non-compact Abelian group R (the additive real line), and he tried to associate the electromagnetic field with these transformtions. It became clear, however that this theory is incompatible with quantum mechanics. In a later work (Weyl 1929a,b) Weyl introduced the crucial factor of i, and proposed for the first time to associate electromagnetism with phase transformations of the electron field. Remarkably enough, in these same papers he developed also the vierbein formalism which is needed in gauge theories of gravity. (He did not use the term Vierbein, i.e. four-leg, but rather Achsenkreuz , or axes-cross).

The crucial step in developing a gauge theory of gravity was taken by Utiyama (1958). The gauge principle had been applied for the first time to a non-Abelian gauge group by Yang and Mills (1954), specifically to the SU(2) isospin group. Utiyama considered gauge theories of a general group, and then showed how gravity might be regarded as a gauge theory of the Lorentz group SO(3,1). His formulation was not entirely satisfactory, because the vierbein had to be introduced in an essentially ad hoc manner. This defect was at least partly remedied by Kibble (1961) and Sciama (1962), who considered not the Lorentz group but its inhomogeneous counterpart, the Poincaré group.

The basic problem is that the analogy between gravity and any other gauge theory is necessarily less than perfect. This is because of the intimate connection between the "gauge" transformations and the space-time co-ordinates, which is of course peculiar to gravity. A considerable variety of approaches is possible, each with its own

particular virtues. Many authors have contributed to our present understanding. Their work has been very fully reviewed by Hehl et al. (1976), and we shall not attempt a complete coverage here.

General relativity need not be regarded, however, as a theory in a class by itself. Considerable benefit is obtained, even in regard to our understanding simply of general relativity itself, by considering it as a member of a wider class of theories, including in particular the super-gravity theories, of which the simplest may be regarded as the gauge theory of the graded Poincaré group. It is also helpful in some circumstances to regard general relativity itself as a gauge theory not of the Poincaré group but of the de Sitter group SO(3,2), with a similar extension for supergravity to the corresponding graded group.

In this article, we shall review the treatment of gravity as a gauge theory of the Poincaré group, and of the de Sitter group, and go on to discuss the analogous formulation of supergravity theories. It will be useful to begin with a brief review of the gauge principle, applied to electromagnetism and to more general non-Abelian gauge theories.

The Gauge Principle

The transformations considered by Weyl in his original formulation of the gauge principle (1918) were scale transformations, of the form

$$q(x) \rightarrow \Omega(x)q(x) \qquad (1)$$

where q(x) is some quantity with the dimensions of length, and x denotes space-time position.

Unfortunately, Weyl's theory in its original form turned out to be incompatible with quantum mechanics. In 1929 he put forward a new version of his idea, in which the transformations incorporated a crucial extra factor of i. This (Weyl 1929a,b) is the first statement of the gauge principle in a form we should recognize today.

Let us consider a theory described by a complex field variable $\phi(x)$. Supose that the theory is invariant under the phase transformations

$$\phi(x) \rightarrow e^{i\lambda}\phi(x), \qquad (2)$$

where λ is a constant real parameter. Often it is easier to consider the infinitesimal form of the transformation, namely

$$\delta\phi(x) = i\delta\lambda\phi(x), \qquad (3)$$

where $\delta\lambda$ is a constant infinitesimal parameter. If the theory is described a Lagrangian function L of ϕ and its space-time derivatives, that function is required to be invariant under the transformation (3). A simple example is provided by the Lagrangian function for the Dirac equation, namely

$$L_0 = \frac{1}{2} i(\bar{\phi}\gamma^\mu\partial_\mu\phi - \partial_\mu\bar{\phi}\ \gamma^\mu\phi) - im\bar{\phi}\ \phi. \qquad (4)$$

Now one can argue that in a relativistic theory invariance under the global transformations (2) or (3) is

unnatural. We expect that parts of the system with spacelike separation should be completely independent, and that if there is a transformation under which the equations are invariant, then it ought to be possible to make a similar transformation but with a parameter that varies from one region to another. Thus we are led to ask whether the system possesses an invariance under the local transformation corresponding to the global transformation (3), namely

$$\delta\phi(x) = i\delta\lambda(x)\phi(x), \qquad (5)$$

where now $\delta\lambda$ is an infinitesimal function of space-time position.

It is easily seen that in general the Lagrangain is not invariant under the local transformation (5). The trouble comes from the space-time derivatives of ϕ, which transform non-linearly :

$$\delta(\partial_\mu\phi) = \partial_\mu(\delta\phi) = i\delta\lambda\partial_\mu\phi + i(\partial_\mu\delta\lambda)\phi. \qquad (6)$$

Because of the term involving the derivatives of $\delta\lambda$, the Lagrangian is not invariant under this transformation. Indeed there is no way of constructing a non-trivial Lagrangian depending only on ϕ and its derivatives that is invariant. It can only be done if we are willing to introduce other field variables besides ϕ.

However, the nature of the problem suggests the mode of solution: what we have to do is to replace the derivatives $\partial_\mu\phi$ in L by new "covariant derivatives" $D_\mu\phi$ with a simple homogeneous transformation law, namely

$$\delta(D_\mu\phi) = i\,\delta\lambda D_\mu\phi, \qquad (7)$$

exactly as for ϕ itself. To construct the covariant derivative, we need to introduce a new field, with an inhomogeneous transformation. If we write

$$D_\mu\phi = \partial_\mu\phi + iA_\mu\phi, \qquad (8)$$

then we obtain the correct transformation law provided that the new field $A_\mu(x)$ transforms according to

$$\delta A_\mu = -\partial_\mu\delta\lambda. \qquad (9)$$

If A_μ is interpreted as the electromagnetic vector potential, then (9) is the familiar gauge transformation. Thus by seeking to make the theory invariant under the local transformations (5), we are led to introduce the electromagnetic field. This is the basic point of the gauge principle.

If in the Lagrangian (4) we substitute $D_\mu\phi$ for the ordinary derivative $\partial_\mu\phi$, we obtain a new Lagrangian that may be written

$$L = L_0 - A_\mu\bar{\phi}\gamma^\mu\phi. \qquad (10)$$

This is essentially the Dirac Lagrangian in the presence of a classical external electromagnetic field. Actually to make the analogy precise, we ought to identify A_μ not with the vector potential, but with e times the vector potential, where e is the electronic charge.

It is useful to note that the covariant derivatives, unlike ordinary derivatives, do not commute. Instead we find

$$D_\mu D_\nu\phi - D_\nu D_\mu\phi = iF_{\mu\nu}\phi, \qquad (11)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \qquad (12)$$

A very important property of $F_{\mu\nu}$ is that, unlike A_μ, it is invariant under the transformations (9). Moreover, it is also clear that any gauge-invariant, local function of A_μ must in fact be a function of $F_{\mu\nu}$ only.

Now in complete theory, A_μ as well as ϕ must be regarded as a dynamical variable. the Lagrangian ought to contain kinetic terms for it, but clearly in adding such terms we do not want to spoil the gauge invariance. If we impose on the Lagrangian the natural requirements of locality and gauge invariance, then we are restricted to fucntions of F alone. The simplest Lorentz-invariant function of this kind is of course

$$L = -\frac{1}{4e^2} F_{\mu\nu}(x)F^{\mu\nu}(x), \qquad (13)$$

where e is an arbitrary constant. By redefining the field A (and hence F), setting the original A equal to e times a new A, we can cast the Lagrangian composed of (10) and (13) into the form

$$L = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} i(\bar{\phi}\gamma^\mu\partial_\mu\phi - \partial_\mu\bar{\phi}\gamma^\mu\phi) - im\bar{\phi}\phi - eA_\mu\bar{\phi}\gamma^\mu\phi. \qquad (14)$$

which may be recognized as the standard Lagrangian of quantum electrodynamics.

Thus starting from a Lagrangian invariant under the global phase transformation (2) the gauge principle leads us to introduce the electromagnetic field.

Non-Abelian Gauge Fields

It is easy to extend the argument to a non-Abelian symmetry group, G. Although it is by now rather well known, it may be useful to review the formalism briefly, to provide a basis for the comparison with the special case of the gravitational field, discussed below.

Consider a field ϕ transforming according to

$$\delta\phi(x) = \delta\omega\phi(x) = \delta\omega^a T_a \phi(x), \tag{15}$$

where the matrices T_a form a representation of the Lie algebra of G, defined by the commutation relations

$$[T_a, T_b] = t_a{}^c{}_b T_c, \tag{16}$$

in which $t_a{}^c{}_b$ are the structure constants of the group.

As before, the ordinary derivatives of ϕ transform in an inhomogeneous manner. We can define a covariant derivative, $D_\mu\phi$, which transforms exactly as ϕ itself, by setting

$$D_\mu\phi = \partial_\mu\phi + A_\mu\phi, \tag{17}$$

where A_μ is now a matrix, belonging to the Lie algebra, of the form

$$A_\mu = A_\mu{}^a T_a. \tag{18}$$

The transformation law of A is now

$$\delta A_\mu = [\delta\omega, A_\mu] - \partial_\mu\delta\omega, \tag{19}$$

or, in terms of components,

$$\delta A_\mu{}^a = t_b{}^a{}_c \delta\omega^b A_\mu{}^c - \partial_\mu\delta\omega^a. \tag{20}$$

The commutator of two covariant derivatives is given by

$$D_\mu D_\nu\phi - D_\nu D_\mu\phi = F_{\mu\nu}\phi, \tag{21}$$

where now

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] = F_{\mu\nu}{}^a T_a, \tag{22}$$

with

$$F_{\mu\nu}{}^a = \partial_\mu A_\nu{}^a + t_b{}^a{}_c A_\mu{}^b A_\nu{}^c. \tag{23}$$

The fields F are no longer invariant under gauge transformations, but they do have a simple linear transformation,

$$\delta F_{\mu\nu} = [\delta\omega, F_{\mu\nu}], \tag{24}$$

or, equivalently,

$$\delta F_{\mu\nu}{}^{a} = t_{b}{}^{a}{}_{c}\delta\omega^{b}F_{\mu\nu}{}^{c}. \tag{25}$$

Any gauge-invariant local function of A and its derivatives must, as before, be expressible in terms of the field F alone. The simplest such Lagrangian is

$$L = -\frac{1}{4g^2}(F_{\mu\nu}, F^{\mu\nu}), \tag{26}$$

where g is a coupling constant, analogous to the charge e, and (., .) denotes the Lie algebra inner product,

$$(A, B) = -\frac{1}{2}\mathrm{tr}(AB) = g_{ab}A^{a}B^{b},$$

with

$$g_{ab} = -\frac{1}{2}t_{a}{}^{c}{}_{d}t_{b}{}^{d}{}_{c}.$$

It is again possible to rescale the fields so that g disappears from the quadratic terms in the Lagrangian. Of course, (26) is no longer quadratic, but includes cubic and quartic self-interaction terms for the gauge potentials A.

Gravity and the Poincaré Group

Let us now turn to the real subject of this review, the

incorporation into the framework of gauge theories of the gravitational field. In a qualitative, heuristic sense it is obvious that there is a close analogy between the transition from local phase transformations to global gauge transformations on the one hand and that from Lorentz or Poincaré transformations to general coordinate transformations on the other. However, it is far from trivial to formulate the analogy in a precise and useful way.

There are of course two complementary ways of looking at space-time transformations: the active point of view in which the coordinates are held fixed and the state of the system is rotated, translated, etc., and the passive view in which the same state is described in terms of a different coordinate system. To be specific, we shall adopt an active viewpoint.

Under a Poincaré transformation

$$x^{\mu} \to x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} + a^{\mu}$$

a field ϕ belonging to a representation S of the Lorentz group (or, more precisely of its two-fold covering group) transforms according to

$$\phi'(x') = S(\Lambda)\phi(x).$$

Under the infinitesimal transformation $x \to x + \delta x$, with

$$\delta x^{\mu} = \delta\epsilon^{\mu}{}_{\nu}x^{\nu} + \delta a^{\mu}, \tag{27}$$

we have $\phi \to \phi' = \phi + \delta\phi$, with

$$\delta\phi = \frac{1}{2}\delta\varepsilon^{ab}S_{ab}\phi - \delta x^{\mu}\partial_{\mu}\phi, \qquad (28)$$

where $S_{ab} = -S_{ba}$ are the generators of the representation $S(\Lambda)$. (We use Latin indices for what will be local Lorentz indices.) Comparing (28) with (15), we find the major difference between this and the previous case - namely the fact that the role of the generators T_a is played in part by the differential operators ∂_{μ}.

A closely related problem is apparent from (27). If we seek to generalize these transformations by allowing $\delta\varepsilon$ and δa to become functions of space-time, we lose the distinction between Lorentz transformations and translations. The transformations with $\delta\varepsilon = 0$ and arbitrary $\delta a(x)$ already include all general co-ordinate transformations, and nothing further is apparently gained by including $\delta\varepsilon$ also. On the other hand, if we choose an arbitrary general co-ordinate transformation, specified by $\delta x^{\mu}(x)$, the transformation of ϕ is not in general well-defined, until we specify what is meant by $\delta\varepsilon^{ab}$ in (27).

Let us nevertheless proceed to consider what happens if the parameters $\delta\varepsilon$ and δa are allowed to become functions of space-time position. Since the specification of δa depends on the choice of origin, it is more convenient to use instead as independent variables $\delta\varepsilon^{ab}$ and δx^{μ}. To keep a clear separation between the terms involving these two independent parameters, we shall, as in (28), use Latin indices for $\delta\varepsilon$, and Greek for δx.

We assume that the action integral is Poincaré-invariant. Then as before the main reason for non-invariance under the more general transformations is the behaviour of

the derivative of ϕ, namely

$$\partial_{\mu}\delta\phi = \frac{1}{2}\delta\varepsilon^{ab}S_{ab}\partial_{\mu}\phi - \delta x^{\lambda}\partial_{\lambda}\partial_{\mu}\phi$$
$$+ \frac{1}{2}(\partial_{\mu}\delta\varepsilon^{ab})S_{ab}\phi - (\partial_{\mu}\delta x^{\lambda})\partial_{\lambda}\phi. \qquad (29)$$

The third term on the right hand side is precisely of the same form as those we have eliminated previously by defining covariant derivatives. We can do the same thing here. If we write

$$\nabla_{\mu}\phi = \partial_{\mu}\phi + \frac{1}{2}\omega_{\mu}{}^{ab}S_{ab}\phi \qquad (30)$$

where the new field ω is ascribed a transformation law containing the inhomogeneous term, $-\partial_{\mu}\delta\varepsilon^{ab}$, we can arrange to cancel this third term. (The transformation law must also contain a homogeneous term containing the structure constants of the Lorentz group.)

The last term in (29), however, cannot be removed by exactly the same procedure. In it the role of the matrix generator T_a is played by the differential operator ∂_{λ}. Thus if we are to remove it by adding an extra term to the derivative, that term must be proportional not to ϕ but to its derivative $\partial_{\lambda}\phi$. In other words the modification to the derivative is not additive but multiplicative: the new covariant derivative may be obtained by multiplying (30) by a new field with a suitable transformation law. We write

$$D_a\phi = e^{\mu}{}_a\nabla_{\mu}\phi, \qquad (31)$$

with a suitable choice for the transformation low of $e^{\mu}_{\ a}$. We then arrive finally at a transformation law for the covariant derivative that is free of derivatives of the parameters, namely

$$\delta(D_a\phi) = \frac{1}{2}\delta\varepsilon^{bc}S_{bc}D_a\phi - \delta\varepsilon^{b}_{\ a}D_b\phi - \delta x^{\mu}\partial_{\mu}D_a\phi. \qquad (32)$$

Substituting (31) for the derivative of ϕ in the original Poincare-invariant action integral cures part of the problem, but there is one further complication. The Lagrangian is not in fact invariant - even under the original transformation (27) - but rather transforms as a scalar, $\delta L = -\delta x^{\mu}\partial_{\mu}L$. What is required, however, is that the action integral be invariant. The necessary and sufficient condition for this is that the Lagrangian itself transform as a scalar density, i.e.

$$\delta L = -\partial_{\mu}(\delta x^{\mu}L).$$

To achieve this, it is not necessary to introduce any new fields: all we have to do is to multiply L by

$$e = \det(e^{\mu}_{\ a})^{-1}. \qquad (33)$$

In particular, if we start with the free Dirac Lagrangian (4), we arrive in this way at

$$L_{\phi} = e[\tfrac{1}{2}ie^{\mu}_{\ a}(\bar{\phi}\gamma^{a}\nabla_{\mu}\phi - \nabla_{\mu}\bar{\phi}\gamma^{a}\phi) - im\bar{\phi}\phi], \qquad (34)$$

where

$$\nabla_{\mu}\phi = \partial_{\mu}\phi + \frac{1}{8}\omega_{\mu ab}[\gamma^{a},\gamma^{b}]\phi.$$

The Gravitational Lagrangian

As before, the covariant differential operators do not commute. From (30) we easily find

$$\nabla_{\mu}\nabla_{\nu}\phi - \nabla_{\nu}\nabla_{\mu}\phi = \frac{1}{2}R_{\mu\nu}^{\ \ ab}S_{ab}\phi, \qquad (35)$$

where $R_{\mu\nu} = \partial_{\mu}\omega_{\nu} - \partial_{\nu}\omega_{\mu} + [\omega_{\mu}, \omega_{\nu}]$, i.e.

$$R_{\mu\nu\ b}^{\ \ a} = \partial_{\mu}\omega_{\nu\ b}^{\ a} - \partial_{\nu}\omega_{\mu\ b}^{\ a} + \omega_{\mu\ c}^{\ a}\omega_{\nu\ b}^{\ c} - \omega_{\nu\ c}^{\ a}\omega_{\mu\ b}^{\ c}. \qquad (36)$$

It is not quite so straightforward to evaluate the commutator of the differential operators D_a. This is because, according to (32), $D_a\phi$ does not transform precisely in the same way as ϕ itself, but with an extra term reflecting the extra index, so that when it in turn is differentiated, an extra term is required in the definition of the second covariant derivative. However a simple calculation shows that

$$D_aD_b\phi - D_bD_a\phi = \frac{1}{2}R_{ab}^{\ \ cd}S_{cd}\phi + T_{ab}^{\ \ c}D_c\phi, \qquad (37)$$

with

$$R_{ab}^{\ \ cd} = e^{\mu}_{\ a}e^{\nu}_{\ b}R_{\mu\nu}^{\ \ cd} \qquad (38)$$

and

$$T_{ab}{}^{c} = (e^{\mu}{}_{a}\nabla_{\mu}e^{\nu}{}_{b} - e^{\mu}{}_{b}\nabla_{\mu}e^{\nu}{}_{a})e^{\nu}{}_{c}. \qquad (39)$$

Here $e_{\mu}{}^{a}$ is the inverse of $e^{\mu}{}_{a}$, i.e.

$$e_{\mu}{}^{a}e^{\mu}{}_{b} = \delta^{a}{}_{b}, \quad e_{\mu}{}^{a}e^{\nu}{}_{a} = \delta^{\nu}_{\mu}, \qquad (40)$$

and the covariant derivative ∇_{μ} of e is defined by

$$\nabla_{\mu}e^{\nu}{}_{a} = \partial_{\mu}e^{\nu}{}_{a} - \omega_{\mu}{}^{b}{}_{a}e^{\nu}{}_{b}.$$

We must now find a Lagrangian for the new fields ω and e. If it is to yield an action integral invariant under the local form of the Poincare group transformation, it can only involve ω through the covariant combination R. There is no such restriction on e, but its derivatives can appear only through T. Unlike the case of an ordinary gauge theory, it is in this case possible to construct a Lagrangian that is only of first order in the derivatives, namely

$$L_{g} = M^{2}eR_{ab}{}^{ab}, \qquad (41)$$

where M is a constant with the dimensions of mass related to Newton's constant G by $G = 1/8\pi M^{2}$. The next simplest terms are those in T^{2} and R^{2}.

It is of course possible to give a geometric interpretation to the resulting theory. The fields $e^{\mu}{}_{a}$ may

be interpreted as the components of a vierbein, defining four independent tangent vector fields,

$$e_{a} = e^{\mu}{}_{a}\partial_{\mu}.$$

The dual one-forms are

$$e^{a} = e_{\mu}{}^{a}dx^{\mu}.$$

They can be used to define a metric

$$ds^{2} = \eta_{ab}e^{a} \otimes e^{b} = g_{\mu\nu}\, dx^{\mu}\, dx^{\nu}, \qquad (42)$$

where $\eta = \mathrm{diag}\ (-1, +1, +1, +1)$ is the Minkowski metric tensor, which we use to raise and lower the indices a, b, ..., and

$$g_{\mu\nu} = e_{\mu}{}^{a}e_{\nu a} \qquad (43)$$

Note that in terms of this metric the vector fields e_{a} form an orthonormal tetrad.

The theory defined by the Lagrangian (41) together with (34) is very nearly the standard Einstein theory. The only difference is that it incorporates Cartan's torsion, T (Cartan 1922). The Euler-Lagrange equation obtained by varying e is essentially Einstein's equation, with the energy-momentum tensor of the matter field on the right; the equation for ω provides the relation between ω and e, and relates the torsion to the spin angular momentum of the matter. (See Kibble (1961) and Hehl et al (1976) for a full review.)

Spontaneously Broken SO(3,2) Gauge Theory

We have seen that it is possible to use the gauge principle, though in a somewhat modified form, to arrive at essentially the Einstein theory, starting from a theory invariant under the global Poincaré group. However the analogy with other gauge theories is less than perfect, because of the special way we had to handle the translations. So far as the Lorentz transformations are concerned, we have a theory with full local gauge symmetry, but this is not the case for the translations. It is possible, however, to formulate gravity in a way that treats the whole Poincaré group in a more unified way. The physical distinction between the Lorentz and translation generators arises, in this formulation, through spontaneous symmetry breaking, which leads to a nonlinear realization of the translational part of the group.

This model provides a particularly interesting example of the phenomenon of dynamical rearrangement of symmetry, first studied by Professor H. Umezawa, to whom this volume is dedicated (Umezawa 1965; see also Matsumoto, Papstamatiou and Umezawa 1975).

It is convenient, following Stelle and West (1980), to begin not with the Poincare group itself, but the de Sitter group SO(3,2). This has the advantage of being semi-simple, so that all the generators are initially on a manifestly equal footing. (It would be equally possible to use the group SO(4,1), but the choice of SO(3,2) will be useful later in connection with supersymmetry.) The SO(3,2) group is then

broken spontaneously down to its Lorentz subgroup SO(3,1). In the process the local gauge symmetry corresponding to the 'translation' generators is preserved, but becomes nonlinearly realized.

We begin with a pure SO(3,2) gauge theory, described by the gauge potentials $\omega_\mu{}^{AB} = -\omega_\mu{}^{BA}$, where A and B run over the indices 0,1,2,3,5. These indices are raised or lowered with the metric η_{AB}, with $\eta_{00} = \eta_{55} = -1$ and $\eta_{ij} = \delta_{ij}$ for i,j = 1,2,3. The corresponding gauge fields are

$$F_{\mu\nu}{}^A{}_B = \partial_\mu\omega_\nu{}^A{}_B - \partial_\nu\omega_\mu{}^A{}_B + \omega_\mu{}^A{}_C\omega_\nu{}^C{}_B - \omega_\nu{}^A{}_C\omega_\mu{}^C{}_B. \quad (44)$$

Note that at this stage the internal indices and the space-time indices are quite unrelated.

In constructing an invariant action integral, we may use, in addition to F, the invariant SO(3,2) tensor ϵ_{ABCDE}, and the invariant tensor density $\epsilon^{\mu\nu\rho\sigma}$. With these ingredients, the only polynomial Lagrangian one can construct is the integrand of the Pontryagin index, $\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}{}^A{}_B F_{\rho\sigma}{}^B{}_A$, which is of course a total divergence.

It is possible to construct an appropriate Lagrangian in terms of these variables, but only by going to a non-polynomial Lagrangian. A simpler form is obtained by introducing in addition to the gauge field an auxiliary field y^A, which satisfies a constraint equation. In either form the resulting theory exhibits a spontaneous breaking of the de Sitter symmetry down to the Lorentz subgroup SO(3,1). However, this is much clearer in terms of the auxiliary field y.

The Lagrangian we choose is

$$L = m\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}{}^{AB}F_{\rho\sigma}{}^{CD}\epsilon_{ABCDE}y^E - \lambda(y^A y_A + m^{-2}). \quad (45)$$

Here m is a constant mass, and λ is a variable that plays the role of a Lagrange multiplier, imposing the constraint

$$y^A y_A = -m^{-2}.$$

(The equivalent non-polynomial form is obtained by eliminating y and λ from (45).)

We can always choose a gauge in which

$$y = {}^0y = (0,0,0,0,m^{-1}), \quad (46)$$

so that the effective Lagrangian becomes

$$L = \epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}{}^{ab}F_{\rho\sigma}{}^{cd}\epsilon_{abcd5}. \quad (47)$$

(We use a, b, ... to denote internal indices restricted to the values 0,1,2,3). In this form it is clear that the nonzero value of y^5 corresponds to a breaking of the gauge symmetry group G = SO(3,2), down to the isotropy subgroup of ${}^0y$, the Lorentz subgroup H = SO(3,1).

In the special gauge (46) only the SO(3,1) symmetry is manifest. To see what has happened to the translational part of the symmetry group, one must relax this condition.

Let us denote the SO(3,2) generators by M_{AB}, and in particular define

$$P_a = mM_{a5}. \quad (48)$$

Note that these generators are defined here to be anti-hermitean. These 'translation' generators then satisfy the commutation relations

$$[P_a, P_b] = m^2 M_{ab}. \quad (49)$$

In the limit $m \to 0$, the group G = SO(2,3) goes over into its Wigner-Inonu contraction, the Poincare group. The little group H is generated by the M_{ab}. An element g in the neighbourhood of the identity in G may be parametrized in the form

$$g = \exp(\zeta^a P_a)h, \quad (50)$$

where $h \in H$ and the ζ^a parametrize the quotient space G/H. Now if σ_5 denotes the five-vector representation of SO(3,2), we may write

$$y = \sigma_5(g)\,{}^0y = {}^0y + \zeta + O(m\zeta^2), \quad (51)$$

thereby establishing a one-to-one correspondence between values of the field y and the coset label ζ. More explicitly,

$$y^a = m^{-1}(\zeta^a/\zeta)\sinh(m\zeta), \quad y^5 = m^{-1}\cosh(m\zeta), \quad (52)$$

with $\zeta = (\zeta^a\zeta_a)$.

The field ζ is the Goldstone field associated with the spontaneous breaking of G down to H. It transorms nonlinearly with respect the G: under the element $g_0 \in G$, $\zeta \to \zeta'$, where

$$g_0 \exp(\zeta^a P_a) = \exp[\zeta'^a(g_0,\zeta)P_a]\, h_1(g_0,\zeta). \qquad (53)$$

with $h_1 \in H$. In general, ζ' and h_1 are nonlinear functions of g_0 and ζ.

Now consider any other field ϕ, transforming according to some designated representation σ of SO(3,2). In general, ϕ will of course break up into several distinct SO(3,1) representations. Using the 'Goldstone' field ζ introduced above, we can define a corresponding field $\bar{\phi}$ carrying a nonlinear realization of SO(3,2) by

$$\bar{\phi}(x) = \sigma[\exp(-\zeta^a(x)P_a)]\, \phi(x). \qquad (54)$$

The transformation law of $\bar{\phi}$ is easily found. Under the transformation $g_0(x) \in G$,

$$\bar{\phi}(x) \to \bar{\phi}'(x) = \sigma[h_1(g_0(x),\zeta(x))]\, \bar{\phi}(x). \qquad (55)$$

Note that since the argument of σ is an element of H for all g_0, the parts of $\bar{\phi}$ belonging to distinct irreducible representations of H transform independently. Under a transformation g_0 restricted to the Lorentz subgroup H, the transformation is linear. But for transformations including nontrivial 'translation' parts, it is nonlinear. What we have done is to exchange the original fields ϕ transforming according to a single irreducible (linear) representation of G for a set of nonlinearly transforming fields in which the

components with different Lorentz indices transform quite separately.

In particular, from the SO(3,2) gauge potentials $\omega_\mu{}^{AB}$ we may construct two separate fields carrying nonlinear realizations of G – the spin connection $\bar{\omega}_\mu{}^{ab}$ and the (inverse) vierbein field

$$\bar{e}_\mu{}^a = m^{-1}\bar{\omega}_\mu{}^{a5}, \qquad (56)$$

which serves to relate the internal and space-time indices. Under an SO(3,2) transformation, $\bar{\omega}$ transforms as in (55), but with an extra inhomogeneous term $h_1\partial_\mu h_1{}^{-1}$, while $\bar{e}$ transforms exactly as in (55).

The curvature constructed from $\bar{\omega}$ splits in a similar way. From (44), we have

$$\bar{F}_{\mu\nu}{}^{ab} = \bar{R}_{\mu\nu}{}^{ab} - m^2(\bar{e}_\mu{}^a\bar{e}_\nu{}^b - \bar{e}^a{}_\nu\bar{e}_\mu{}^b), \qquad (57)$$

where $\bar{R}$ is the curvature tensor constructed from the gauge potentials $\bar{\omega}_\mu{}^a{}_b$, and

$$\bar{F}_{\mu\nu}{}^{a5} = m(\bar{\nabla}_\mu\bar{e}_\nu{}^a - \bar{\nabla}_\nu\bar{e}_\mu{}^a), \qquad (58)$$

with

$$\bar{\nabla}_\mu\bar{e}_\nu{}^a = \partial_\mu\bar{e}_\nu{}^a + \bar{\omega}_\mu{}^a{}_b\bar{e}_\nu{}^b. \qquad (59)$$

We can now see how the theory yields Einstein's general relativity. In a general gauge the Lagrangian takes the same form as (47) but with barred fields $\bar{F}$ in place of F. Substituting (57) into (47), we obtain three terms, symbolically $\bar{R}\bar{R}$, $m^2\bar{e}\bar{e}\bar{R}$ and $m^4\bar{e}\bar{e}\bar{e}\bar{e}$, respectively. The first of these is the integrand of the Gauss-Bonnet topological invariant, and so gives no contribution to the equations of motion. The second is the usual scalar curvature lagrangian of the Einstein theory. The third represents a cosmological constant, of order m^4. Because of this constant, the maximally symmetric solution of the field equations is an anti-de Sitter space with $\bar{F}_{\mu\nu}{}^a{}_b = 0$, or

$$\bar{R}_{\mu\nu}{}^a{}_b = m^2(\bar{e}_\mu{}^a\bar{e}_{\nu b} - \bar{e}_\nu{}^a\bar{e}_{\mu b}). \qquad (60)$$

We have seen therefore that one can construct a gauge theory of gravity in which not only the Lorentz transformations but also the translations appear as true local gauge symmetries, provided that the translation part of the group is spontaneously broken (and hence nonlinearly realized).

Local Translations & General Coordinate Transformations

In the above discussion of the vierbein as the gauge field of the translations, it was essential to recognise that the local translations are a spontaneously broken part of the Poincare or (anti) de Sitter gauge group. The familiar vierbein and spin connection are then nonlinear combinations of the original gauge fields and the Goldstone field. One can also ask whether there is any relation between the broken local translations and the general coordinate

transformations. Of course, the structures of the local Poincare or de Sitter groups and the diffeomorphism group are greatly different, so there can be no exact relation. Nonetheless, there is at least an infinitesimal relation that has proven heuristically useful.

It is suggestive to rewrite the effect of an active general coordinate transformation on the vierbein as follows:

$$\begin{aligned}\delta_{gc}(\zeta)e_\mu{}^a &= \zeta^\nu\partial_\nu e_\mu{}^a + \partial_\mu\zeta^\nu e_\nu{}^a \\ &= D_\mu(\zeta^\nu e_\nu{}^a) - \zeta^\nu\omega_\nu{}^a{}_b e_\mu{}^b - \zeta^\nu F_{\mu\nu}{}^a \\ &= [D_\mu(\zeta^\nu h_\nu{}^\Sigma)]^a - \zeta^\nu F_{\mu\nu}{}^a \end{aligned} \qquad (61)$$

where

$$D_\mu\rho^\Sigma = \partial_\mu\rho^\Sigma - h_\mu{}^\Pi\rho^\Lambda t_\Lambda{}^\Sigma{}_\Pi \qquad (62)$$

$$F_{\mu\nu}{}^\Sigma = \partial_\mu h_\nu{}^\Sigma - \partial_\mu h_\nu{}^\Sigma - h_\mu{}^\Lambda h_\nu{}^\Pi t_\Lambda{}^\Sigma{}_\Pi \qquad (63)$$

and Σ Π Λ run over all the indices of the Poincare/de Sitter group, with $h_\mu{}^a = e_\mu{}^a$, $h_\mu{}^{ab} = \omega_\mu{}^{ab}$. The Lorentz rotation term with field dependent parameter $\zeta^\nu\omega_\nu{}^a{}_b$ has been absorbed into the nonderivative term in $[D_\mu(\zeta^\nu\omega_\nu{}^{cd})]^a$. The identity (61) is due to Hehl et al. (1976). The second term in (61) vanishes if the translational curvature vanishes, but this is just the torsion, which is zero if $\omega_\mu{}^{ab}$ is the usual spin connection constructed from the vierbein; i.e.

$$F_{\mu\nu}{}^{a} = 0 \tag{64}$$

is solved by

$$\omega_{\mu}{}^{ab} = \frac{1}{2}\, e^{\nu a}(\partial_{\mu}e_{\nu}{}^{b} - \partial_{\nu}e_{\mu}{}^{b}) - \frac{1}{2}\, e^{\nu b}(\partial_{\mu}e_{\nu}{}^{a} - \partial_{\nu}e_{\mu}{}^{a})$$

$$- \frac{1}{2}\, e^{\rho a}e^{\sigma b}(\partial_{\rho}e_{\sigma c} - \partial_{\sigma}e_{\rho c})e_{\mu}{}^{c}. \tag{65}$$

Thus, if the torsion vanishes and the spin connection takes its standard form, there is an infinitesimal relation between a general coordinate transformation with parameter ζ^{μ} and a set of Poincare/de Sitter transformations with field-dependent parameters $\zeta^{\nu}e_{\nu}{}^{a}$, $\zeta^{\nu}\omega_{\nu}{}^{ab}$ (i.e. $\zeta^{\nu}h_{\nu}{}^{\Pi}$).

Writing a general coordinate transformation as $D_{\mu}\rho^{a}$ implies only an infinitesimal correspondence to a translation, as can be seen by taking the commutator of two such transformations, which yields a curvature instead of the group relation, as can be seen in eq. (37). Nonetheless, the notion of such an 'open gauge algebra', the softening of a Lie algebraic structure by allowing curvature terms in the commutators, has proven to be a useful generalization of the notion of a local symmetry (see Regge (1985) and Sohnius (1983) for reviews). The curvatures occuring in the commutators of gravitational covariant derivatives are just one example of such a softening. In supergravity, one encounters also commutators giving local transformations that are field dependent. Thus, in an open gauge algebra, instead of structure constants, one may have structure functions, depending on gauge fields and their curvatures.

Although only an infinitesimal correspondence, viewing the general coordinate transformations as local translations $D_{\mu}\rho^{a}$ has been heuristically useful in supergravity. The curvatures for gauged graded extensions of the Poincare [Chamseddine and West, 1977], anti de Sitter [MacDowell and Mansouri, 1977] or conformal groups [Kaku et al. 1978; Townsend and van Nieuwenhuizen, 1979] turn out to have useful structures for building actions, even though the translational and super-translational (i.e. supersymmetry) parts of these algebras are spontaneously broken. An example of such usefulness is the quadratic form of the Lagrangian for general relativity with a cosmological constant given in eq. (47) [MacDowell and Mansouri 1977]. There, the anti de Sitter curvature (57) contains bilinears in vierbeins that give the Hilbert action from cross terms and also the cosmological term, while the Lorentz curvature-squared terms integrate to zero by the Gauss-Bonnet identity.

The usefulness of the 'group' curvatures in building the actions for general relativity, ordinary and conformal supergravity theories derives from the infinitesimal relations between the true 'world' local symmetries (i.e. general coordinate transformations and supersymmetry transformations) and the gauged versions of 'group' transformations. Thus, although the translations are spontaneously broken and consequently nonlinearly realized, it can still be useful to retain the full groups's curvatures such as (57) without breaking them into parts covariant under the little group H.

If an infinitesimal local world transformation can be interpreted as a group gauge transformation subject to some constraint such as (64), then the full group curvatures will transform simply under the world transformations. The world transformations consist of two components: the full group transformation laws for curvatures plus certain corrections due to the noncovariance of the constraints imposed under the broken parts of the group. Thus, (64) is only covariant under local Lorentz transformations and not under the group translations. Consequently, the field $\omega_\mu{}^{ab}$ which can be solved for in (64) transforms now by the chain rule, starting from the transformation of the field $e_\mu{}^a$ in terms of which it is solved, and not by its original group transformation. The difference $\delta' = \delta_{\text{chain rule}} - \delta_{\text{gauge law}}$ enters into the transformation of the curvatures via a generalization of the Palatini identity

$$\delta' F_{\mu\nu}{}^{\Sigma} = D_\mu \delta' h_\nu{}^{\Sigma} - D_\nu \delta' h_\mu{}^{\Sigma}. \qquad (66)$$

A particularly simple situation arises when the constraints imposed to link world and group transformations can be viewed as some of the field equations of the action to be constructed. Thus, (64) can be viewed as the field equation for the field $\omega_\mu{}^{ab}$ following from the Lagrangian (47). Consequently, if one imposes (64) prior to variation of the action, all terms in $\delta'\omega_\mu{}^{ab}$ must cancel. This simple observation leads to an enormous simplification in verifying the supersymmetry invariance of the N=1 supergravity action [Chamseddine and West, 1977; Townsend and van Niewenhuizen, 1977].

The program for constructing locally invariant theories starting from some rigid symmetry group comprises then the following elements:

1. expression of the world symmetry transformations in terms of the broken group transformations by imposing appropriate constraints covariant under the little group H.
2. construction of an action that is invariant under the little group H and the world transformations, taking into account the group form of the world transformations resulting from 1) and the changes δ' in transformations of fields that have been solved for in the constraints.

Parts 1) and 2) are not completely independent, for until the construction is complete, some world transformations (e.g. supersymmetry) are not completely known. So one starts writing an action manifestly invariant under H built from curvatures and searches for H covariant constraints to impose in order to get the remaining world invariances.

It should be emphasised that the above programme proceeds by analogy only with the structure of Yang-Mills gauge theories and is not uniformly successful in all cases. It works well in the case of N=1 anti de Sitter (adS) and N=1 conformal supergravity. In N=2 adS supergravity the curvatures alone are not sufficient to build the action, and one must use the vierbein explicitly in order to write the spin-one graviphoton's Lagrangian, $eF_{\mu\nu}F^{\mu\nu}$; likewise, the spin 3/2 fields need terms involving $F_{\mu\nu}\varepsilon^{i}$ in their supersymmetry tranformations (with spinorial parameters ε^{i}) that do not follow from group theory alone [Townsend and van Niewenhuizen, 1977]. Extended conformal supergravities can be built using such techniques, but with yet further modifications, since in that case even the spectra of the theories must be completed by non-gauge fields [Bergshoeff et al. 1981]. Nonetheless, a range of impressively complicated theories has been constructed along these lines, so it is clear that the analogy is a powerful one. Here we shall focus on just the two most successful cases: N=1 adS supergravity and N=1 conformal supergravity.

N=1 Supergravity

In N=1 adS supergravity, the graded group from which one starts is the simplest supersymmetric extension of the anti de Sitter group, OSp(1/4). This group has an even (bosonic) sector Sp(4) ≈ SO(3,2) and an odd sector generated by a spinor under Sp(4), denoted Q_α. Rescaling the translational parts of Sp(4) by the inverse radius m of adS space as in (48), we have for the Q_α the anticommutation relation

$$\{Q_\alpha, Q_\beta\} = -\tfrac{i}{2}(\gamma_a C)_{\alpha\beta}\, P^a - im(\sigma_{ab}C)_{\alpha\beta}M^{ab}, \qquad (67)$$

where C is the charge congugation matrix.

Considering now local OSp(1/4) transformations, we introduce gauge fields $e_\mu{}^a$, $\omega_\mu{}^{ab}$, $\phi_\mu{}^\alpha$. From the commutators of covariant derivatives defined with these we obtain the

curvatures:

$$F_{\mu\nu}{}^{ab}(M) = R_{\mu\nu}{}^{ab}(M) - m^2(e_\mu{}^a e_\nu{}^b - e_\mu{}^b e_\nu{}^a) + im(\bar\phi_\mu\sigma^{ab}\phi_\nu) \qquad (68)$$

where $R_{\mu\nu}{}^{ab}(M)$ is the usual Lorentz curvature constructed from $\omega_\mu{}^{ab}$,

$$F_{\mu\nu}{}^{a}(P) = \partial_\mu e_\nu{}^a - \partial_\nu e_\mu{}^a + \omega_\mu{}^a{}_b\, e_\nu{}^b - \omega_\nu{}^a{}_b e_\mu{}^b + \tfrac{i}{2}\bar\phi_\mu\gamma^a\phi_\nu \qquad (69)$$

and

$$F_{\mu\nu}{}^{\alpha}(Q) = D_\mu\bar\phi_\nu{}^\alpha - D_\nu\bar\phi_\mu{}^\alpha + \tfrac{m}{2}(\bar\phi_\nu\gamma_\mu - \bar\phi_\nu\gamma_\mu) \qquad (70)$$

where

$$D_\mu\lambda = (\partial_\mu + \tfrac{1}{2}\omega_\mu{}^{ab}\sigma_{ab})\lambda. \qquad (71)$$

The action can be written as an expression of the Yang-Mills type, quadratic in curvatures

$$I = \int d^4x F_{\mu\nu}{}^{\Sigma} F_{\rho\sigma}{}^{\Lambda}\, O^{\mu\nu\rho\sigma}_{\Sigma\Lambda} \qquad (72)$$

where the $O^{\mu\nu\rho\sigma}_{\Sigma\Lambda}$ are in this case independent of fields, and are numerically invariant under the little group H = SL(2,C).

The selection of constants $O^{\mu\nu\rho\sigma}_{\Sigma\,\Lambda}$ and of constraints proceeds straightforwardly, as in the case of general relativity with a cosmological constant discussed before. In order to have the local translations infinitesimally equivalent to general coordinate transformations, the translational curvature must again be set to zero,

$$F_{\mu\nu}{}^{a}(P) = 0 \ . \qquad (73)$$

As in (65), this equation can be solved algebraically for the spin connection. However, the solution now involves bilinears in the spin 3/2 field $\phi_\mu{}^\alpha$ as well:

$$\omega_\mu{}^{ab} = \omega_\mu{}^{ab}(e) + \tfrac{1}{4}(\bar{\phi}_\mu\gamma^a\phi^b + \bar{\phi}^a\gamma_\mu\phi^b - \bar{\phi}_\mu\gamma^b\phi^a) \qquad (74)$$

where $\phi_b = e^\nu{}_b\phi_\nu$, $\gamma_\mu = e_\mu{}^a\gamma_a$. The action is a sum of terms quadratic in $F_{\mu\nu}{}^{ab}(M)$ and in $F_{\mu\nu}{}^{\alpha}(Q)$. The bosonic part must have the same structure as (47) (with ϵ_{abcd} instead of ϵ_{abcd5}), and the only available matrix with the right parity for the fermionic part is $\gamma_5 C$, so the action is

$$I = \int d^4x\,\epsilon^{\mu\nu\rho\sigma}\{F_{\mu\nu}{}^{ab}(M)F_{\rho\sigma}{}^{cd}(M)\ \epsilon_{abcd} + F_{\mu\nu}{}^{\alpha}(Q)F_{\rho\sigma}{}^{\beta}(Q)(\gamma_5 C)_{\alpha\beta}\} \qquad (75)$$

$$= \int d^4x\,\epsilon^{\mu\nu\rho\sigma}\{F_{\mu\nu}{}^{ab}(M)F_{\rho\sigma}{}^{cd}(M)\ \epsilon_{abcd} + \bar{F}_{\mu\nu}(Q)\gamma_5\ F_{\rho\sigma}(Q)\}$$

$$(76)$$

The apparent higher derivative terms in (76) are once again total derivatives, both in the bosonic sector as in (47), and also in the fermionic sector. The m dependent terms in F(Q) and F(P) give rise to the Hilbert and Rarita-Schwinger (spin 3/2) actions through cross terms and give rise directly to a cosmological term and a mass-like term $m\bar{\phi}_\mu\sigma^{\mu\nu}\phi_\nu$ for the spin 3/2 field. Once again, in (76) we have the simple situation where the spin connection $\omega_\mu{}^{ab}$ can be treated as an independent field, and variation of it produces the constraint (73) as a field equation. Thus, in checking the supersymmetry invariance of the action (76), no account need be taken of the difference $\delta'\omega_\mu{}^{ab}$ from the group transformation law for $\omega_\mu{}^{ab}$ that arises upon solving the constraint. Consequently, the transformations of the curvatures F(M) and F(Q) can be taken directly from the group theory. Supersymmetry invariance follows even though the indices on the curvatures are contracted only with respect to the Lorentz group. This is easily checked using the commutation relations of the group, which determine the transformations of the curvatures. A term that deserves special notice is that arising from the P^a term on the right hand side of (67); this indicates that under an OSP (1/4) supersymmetry transformation, F(Q) transforms into F(P). This term is not cancelled by the variation of the $[F(M)]^2$ term. Instead, it vanishes by the constraint (73). Thus, the constraint which is required to make an infinitessimal relation between general coordinate transformations and the local translations is also essential for the supersymmetry invariance of the action.

Conformal Supergravity

Conformal supergravity is derived from the graded group SU(2,2/1) (Kaku et al, 1978; Townsend and van Nieuwenhuizen, 1979). This group extends the 15 parameter conformal group (translations P_a, Lorentz rotations M_{ab}, dilatations D and conformal boosts K_a) with two types of spinorial generators, i.e. Q- and S- supersymmetries, plus an additional bosonic generator of chiral rotations A. The Q-supersymmetry is the square root of the translations P_a,

$$\{Q^\alpha, Q^\beta\} = -\frac{i}{2}(\gamma^a C)^{\alpha\beta} P_a \qquad (77)$$

and the S-supersymmetry is the square root of the conformal boosts:

$$\{S^\alpha, S^\beta\} = \frac{i}{2}(\gamma^a C)^{\alpha\beta} K_a . \qquad (78)$$

S- and Q- supersymmetries anticommute to give D, M_{ab} and A:

$$\{Q^\alpha, S^\beta\} = -\frac{i}{2} C^{\alpha\beta} D + i(\sigma^{ab}C)^{\alpha\beta} M_{ab} + i(\gamma_5 C)^{\alpha\beta} A. \qquad (79)$$

Under the dilatations, Q and S have opposite weights, i.e. $[Q,D] = \frac{1}{2}Q$, $[S,D] = -\frac{1}{2}S$, and similarly under the chiral rotations A, $[Q,A] = -\frac{3}{4}\gamma_5 Q$, $[S,A] = \frac{3}{4}\gamma_5 S$. There are

also the additional commutation relations $[S,P_a] = \gamma_a Q$, $[Q,K_a] = -\gamma_a S$, and the usual commutation relations of the conformal group, $[P_a,D] = P_a$, $[K_a,D] = -K_a$, $[K_a,P_b] = -2(\eta_{ab}D + M_{ab})$.

The construction of conformal supergravity starts as before, by introducing gauge fields for all generators of the group; the general Lie-algebra-valued gauge field is

$$h_\mu = e_\mu{}^a P_a + \frac{1}{2}\omega_\mu{}^{ab} M_{ab} + \bar{\phi}_\mu Q + f_\mu{}^a K_a + b_\mu D + \bar{\phi}_\mu S + A_\mu A . \qquad (80)$$

The corresponding curvatures are

$$F_{\mu\nu}{}^{ab}(M) = \partial_\mu \omega_\nu{}^{ab} + \omega_\mu{}^{ac}\,\omega_{\nu c}{}^b - 2(e_\mu{}^a f_\nu{}^b - e_\mu{}^b f_\nu{}^a)$$

$$- i\bar{\phi}_\mu \sigma^{ab} \phi_\nu - (a \leftrightarrow b)$$

$$F_{\mu\nu}{}^a(P) = \partial_\mu e_\nu{}^a + \omega_\mu{}^{ab} e_{\nu b} + \frac{i}{4}\bar{\phi}_\mu \gamma^a \phi_\nu + e_\mu{}^a b_\nu - (a \leftrightarrow b)$$

$$F_{\mu\nu}{}^a(K) = \partial_\mu f_\nu{}^a + \omega_\mu{}^{ab} f_{\nu b} - \frac{i}{4}\bar{\phi}_\mu \gamma^a \phi_\nu - f_\mu{}^a b_\nu - (a \leftrightarrow b)$$

$$F_{\mu\nu}(D) = \partial_\mu b_\nu + 2e_{\mu a} f^a{}_\nu + \frac{i}{2}\bar{\phi}_\mu \phi_\nu - (a \leftrightarrow b)$$

$$F_{\mu\nu}(Q) = D_\mu \bar{\phi}_\nu + \bar{\phi}_\mu \gamma_\nu - \frac{1}{2} b_\mu \bar{\phi}_\nu + \frac{3}{4} A_\mu \bar{\phi}_\nu \gamma_5 - (a \leftrightarrow b)$$

$$F_{\mu\nu}(S) = D_\mu\bar{\phi}_\nu - \bar{\phi}_\mu\gamma_a f_\nu{}^a + \frac{1}{2} b_\mu\bar{\phi}_\nu - \frac{3}{4} A_\mu\bar{\phi}_\nu\gamma_5 - (a-b)$$

$$F_{\mu\nu}(A) = \partial_\mu A_\nu - i\,\bar{\phi}_\mu\gamma_5\phi_\nu\,. \qquad (81)$$

The action of conformal supergrvity is once more quadratic in curvatures, only now we really do have a higher derivative action, for the theory to be constructed is the supersymmetric extension of the Weyl invariant theory with Lagrangian $e\,C_{\mu\nu\alpha\beta}\,C^{\mu\nu\alpha\beta}$. Written in terms of the curvatures (81), the action is simpoly

$$I_{Conf\ SG} = \int d^4x\,\varepsilon^{\mu\nu\rho\sigma}[F_{\mu\nu}{}^{ab}(M)\,F_{\rho\sigma}{}^{cd}(M)\,\varepsilon_{abcd}$$

$$-8F_{\mu\nu}(Q)\,\gamma_5\,C\,F_{\rho\sigma}(S) + 4i\,F_{\mu\nu}(A)\,F_{\rho\sigma}(D)] \qquad (82)$$

The particular coefficients in this action are determined by the invariances, as we shall see.

In order to have a full world superconformal invariance of (82), constraints are again needed. Once more, these have the significance of turning infinitesimal P-gauge transformations into general coordinate transformations. On a general member of the multiplet $h_\mu{}^\Sigma$, an infinitesimal general coordinate transformation with parameter ζ^μ is given by the generalization of the identity (61):

$$\delta_{gc}(\zeta)h_\mu{}^\Sigma = D_\mu(\zeta^\nu h_\nu{}^\Pi)^\Sigma - \zeta^\nu F_{\mu\nu}{}^\Sigma \qquad (83)$$

For the P gauge field $e_\mu{}^a$ itself, this takes the form of a local group transformation with parameters $\zeta^\nu h_\nu{}^\Pi$, by virtue of the generalization of (64), which can be solved to give $\omega_\mu{}^{ab}$ in terms of the other fields,

$$F_{\mu\nu}{}^a(P) = 0\,. \qquad (84)$$

For the other fields, however, the curvatures $F_{\mu\nu}{}^\Sigma$ are not all set to zero - if they were, there would be no remaining degrees of freedom in the theory. Instead, there are just two more constraints that need to be imposed, each allowing for the algebraic solution for one of the SU(2,2/1) gauge fields:

$$F_{\mu\nu}(Q)\gamma^\nu = 0 \qquad (85)$$

which can be solved to give ϕ_μ in terms of the other fields and

$$e^{\lambda a}e_\mu{}^b\,F_{\nu\lambda ab}(M,\hat{\omega}) = 0 \qquad (86)$$

where

$$\hat{\omega}_\mu{}^{ab} = \omega_\mu{}^{ab} + \frac{i}{2}\,e_\mu{}^c\,\varepsilon_{abcd}\,A^d, \qquad (87)$$

involving the chiral gauge field A^d. The constraint (86) allows for the K-gauge field $f_{\mu a}$ to be solved in terms of the remaining independent fields $e_\mu{}^a$, ϕ_μ, b_μ and A_μ.

The constraints (85) and (86) give rise to changes in the transformations of the fields ϕ_μ amd $f_{\mu a}$, as compared to their original gauge group transformations. Thus, there is now a $\delta'\phi_\mu$ and a $\delta' f_\mu{}^a$ for those transformations under which (85) and (86) are not covariant, viz. under P and Q transformations. The set of constraints (84,85,86) is fully covariant under K,M,D,S and A transformations. The changes $\delta'_Q\phi_\mu$ and $\delta'_Q f_\mu{}^a$ are needed to make the algebra of transformations upon the remaining independent fields close to give general coordinate transformations in the form (83) instead of the original local P transformations. Thus, eg. for the chiral gauge field A_μ, introducing anticommuting parameters ϵ_1 and ϵ_2 for Q transformations, one has the algebra

$$[\delta_Q(\epsilon_1),\ \delta_Q(\epsilon_2)]A_\mu = \delta_P(\tfrac{1}{2}\bar{\epsilon}_2\gamma^a\epsilon_1)A_\mu - (1\bar{\epsilon}_2\gamma_5\delta'_Q\phi_\mu - (1\leftrightarrow 2)) \qquad (88)$$

The correction $\delta'_Q\phi_\mu$ turns the second term in (88) into an expression with $F_{\mu\nu}(A)$, so that the right hand side involves not $\delta_P A_\mu$ but $\delta_{gc}A_\mu$ in the form (83). Thus, for all fields of the theory, and not just $e_\mu{}^a$, the constraints have the significance of turning local translations into general coordinate transformations.

The constraints also guarantee the invariance of the superconformal action under all the world symmetries. In checking this, the change in the $\omega_\mu{}^{ab}$ transformation $\delta'_Q\omega_\mu{}^{ab}$ must now be taken into account, for (84) is not an equation of motion of the action with $\omega_\mu{}^{ab}$ varied independently, unlike the cases of general relativity and AdS supergravity. On the other hand, the constraint (86), which

allows $f_\mu{}^a$ to be solved for, is obtained by varying the action (82) with respect to $f_\mu{}^a$, so $\delta'_Q f_\mu{}^a$ does not enter into the check of Q invariance. The other symmetries M,K,D,A and S leave the constraints (84,85,86) invariant, so invariance under these symmetries can be checked by the group theory transformations alone. The particular coefficients in the action (82) are uniquely determined by the requirement of invariance under S- supersymmetry. Invariance under K follows using the constraints (84) and (85), while the M,D and A invariances are manifest.

Invariance of the action under K transformations has the consequence that the dilatational gauge field b_μ drops out of the action, since it transforms inhomogeneously under K transformations,

$$\delta_k b_\mu = 2\zeta_\mu^{(k)}\ . \qquad (89)$$

Thus, the remaining propagating fields in the theory are $e_\mu{}^a$, $\phi_\mu{}^\alpha$ and the chiral gauge field A_μ.

Conformal supergravity provides a rich example of the interplay between ordinary gauge invariances and the local world symmetries of a theory incorporating the gravitational field $e_\mu{}^a$. In its derivation, the role of the constraints (84,85,86) in effecting the conversion of the local translations into general coordinate transformations is crucial. The final action (82) is quite simple in structure when written in terms of the SU(2,2/1) curvatures, but becomes impressively complicated when written out fully in terms of $e_\mu{}^a$, $\phi_\mu{}^\alpha$ and A_μ. This construction is thus one of the most successful applications of gauge theory ideas to a gravity theory.

Conclusion

The analogy between general relativity and gauge theories has been most useful in the attempt to formulate a quantum theory of gravity. Indeed, Feynman (1963a,b) explicitly took Yang-Mills theory as a kind of 'finger exercize' for the real problem of quantizing gravity. In the course of history, this finger exercize has developed into the core of our understanding of the strong, weak and electromagnetic interactions. The solution to the original problem of quantum gravity remains elusive, however.

The cross-fertilization that has taken place between the subjects of quantum gravity and gauge theories illustrates both the strengths and weaknesses of the analogy. All of the perturbative approach to quantizing gravity has been lifted from gauge theories. Moreover, gauge-theoretic ideas have played a central role in the modern understanding of all interactions of gravity with spinors. The vierbein formalism was elucidated through the gauge theory analogy. Later on came supergravity, which developed through a heavy reliance on gauge-theoretic ideas. Supergravity has provided at least a crack in the otherwise impregnable barrier of the terrible ultraviolet problem of quantum gravity. Although it is by now pretty clear that supergravity in itself does not lead to an acceptable quantum theory, the general expectation remains that local supersymmetry will play an essential role in future developments. Thus, it seems likely that space-time and internal symmetries must in the end be united

in a future 'super' grand unification.

The weaknesses of the gauge theory-gravity analogy are also made apparent by the history of quantum gravity. For, despite many useful insights, we are still left without an acceptable quantum theory of gravitation. The differences between gravity and gauge fields are clearest in the area we have explored in this article: in the natures of their respective local symmetries. The diffeomorphism group is not simply a finite dimensional Lie group given a local interpretation. The perturbative approach to quantum gravity brushes aside this essential difference, and the price to be paid is the nonrenormalizability of perturbative quantum gravity. In the end, the answer may entail revising our concepts both of space-time and of quantization of such a highly nonlinear theory.

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