1 Problem 1

1.1 Palatini Identity

Show, for infinitesimal variations of non-Abelian Yang–Mills gauge fields:

$$\delta F^i_{\mu\nu} = D_{\mu} \delta A^i_\nu - D_{\nu} \delta A^i_{\mu} \ .$$

(1.1)

Begin by considering the following form of the field–strength tensor for a non-Abelian Yang–Mills field:

$$F^i_{\mu\nu} = \partial^\mu A^i_\nu - \partial^\nu A^i_{\mu} + g f^{ijk} A^j_{\mu} A^k_{\nu} \ ,$$

(1.2)

take the infinitesimal variation of this (to first order), noting the use of the Leibniz rule to the last term:

$$\delta F^i_{\mu\nu} = \partial^\mu \delta A^i_\nu - \partial^\nu \delta A^i_{\mu} + g f^{ijk} \delta A^j_{\mu} A^k_{\nu} + g f^{ijk} A^j_{\mu} \delta A^k_{\nu} \ .$$

(1.3)

Next, use the following definition for the covariant derivative:

$$D^\mu A^i_\nu = \partial^\mu A^i_\nu - i g T^{ijk}_{\mu\nu} \ A^j_{\mu} A^k_{\nu} \ ,$$

(1.4)

where, the standard convention utilized in physics is applied:

$$T^{ijk}_{\mu\nu} = -i T^{ijk}_{\mu\nu} \ ,$$

(1.5)

and, the using antisymmetry of the structure constants:

$$f^{ijk} = - f^{ikj} \ \ \Rightarrow \ \ \ g f^{ijk} \delta A^j_{\mu} A^k_{\nu} = -g f^{ijk} A^j_{\mu} \delta A^k_{\nu} \ ,$$

(1.6)

Eq. (1.3) may be rearranged as:

$$\delta F^i_{\mu\nu} = \partial^\mu \delta A^i_\nu - i g T^{ijk}_{\mu\nu} \ A^j_{\mu} \delta A^k_{\nu} - \partial^\nu \delta A^i_{\mu} - i g T^{ijk}_{\mu\nu} A^j_{\mu} \delta A^k_{\nu} - D_{\mu} \delta A^i_\nu - D_{\nu} \delta A^i_{\mu} \ .$$

(1.7)

1.2 Non-Abelian Bianchi Identity

Show, for a non-Abelian Yang–Mills field–strength tensor:

$$D^\mu F^\nu_{\mu\nu} = 0 \ ,$$

(1.8)

where, $F^\nu_{\mu\nu} = \frac{1}{2} \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} \ .$

First, take a generic result about antisymmetric tensors:

$$T_{\mu\nu\rho} = T_{\rho[\nu\rho]} \Rightarrow T_{\rho[\rho\nu]} = \frac{1}{6} \ ( T_{\mu\nu\rho} - T_{\mu\rho\nu} + T_{\rho\mu\nu} - T_{\nu\rho\mu} - T_{\nu\mu\rho} ) = \frac{1}{3} \ ( T_{\mu\nu\rho} + T_{\rho\mu\nu} + T_{\nu\rho\mu} ) \ .$$

(1.9)

Here, since $\varepsilon^{\mu\nu\alpha\beta} = \varepsilon^{[\mu\nu\alpha\beta]}$ and $T^{[\mu\nu]} S_{\mu\nu} = T^{\mu\nu} S_{[\mu\nu]}$,

$$D^\mu F^\nu_{\mu\nu} = 0 \ \Leftrightarrow \ \ D^\mu [F_{\mu\nu}] = 0 \ \Leftrightarrow \ \ D^\mu F_{\nu\rho} + D^\nu F_{\rho\mu} + D^\rho F_{\mu\nu} = 0 \ .$$

(1.10)

This already looks much like the cyclic combination which appears in the Jacobi identity. Therefore consider:

$$[D^\mu, [D^\nu, D^\rho]] \ ,$$

(1.11)

These covariant derivatives are defined by their action on fields (for instance, a general field $\phi$), however:
\[
[\partial_\mu, [D_\nu, D_\rho]] [\phi] = \partial_\mu [[D_\nu, D_\rho]\phi] - [D_\nu, D_\rho] \partial_\mu [\phi]
= \partial_\mu [D_\nu, D_\rho] \phi + [D_\nu, D_\rho] \partial_\mu [\phi] - [D_\nu, D_\rho] \partial_\mu [\phi]
= \partial_\mu [[D_\nu, D_\rho]] \phi.
\] (1.12)

Here \([\cdot, \cdot]\) means “take the commutator with respect to the Lie algebra,” and \(O[\cdot, \cdot]\) means “apply the differential operator \(O\).” The second equality is due to the Leibniz rule, and the full statement means that \([\partial_\mu, [D_\nu, D_\rho]] [\phi]\) is independent of derivatives of \(\phi\), or acts only through multiplication. Similarly:

\[
[-ig A^i_\mu T^i, [D_\nu, D_\rho]] = -ig A^i_\mu [T^i, -ig F^i_{\nu\rho}] = -g^2 A^i_\mu F^i_{\nu\rho} [T^i, T^j] = -ig^2 A^i_\mu F^i_{\nu\rho} f^{ijk} T^k
= g^2 A^i_\mu T_{\alpha\beta\mu} F^i_{\nu\rho} T^\alpha T^\beta.
\] (1.13)

Putting Eqs. (1.12) and (1.13) together, find:

\[
[D_\mu, [D_\nu, D_\rho]] = [\partial_\mu - ig A_\mu, [D_\nu, D_\rho]] = [\partial_\mu, [D_\nu, D_\rho]] + [-ig A^i_\mu T^i, [D_\nu, D_\rho]] = -ig(\partial_\mu \delta^{ij} + ig T_{\alpha\beta\mu} A^i_\mu) F^i_{\nu\rho} T^\alpha T^\beta
= -ig D_{\nu\rho} F_{\nu\rho}.
\] (1.14)

Hence each of the left hand side terms in Eq. (1.10) is proportional to:

\[
[D_\mu, [D_\nu, D_\rho]] + [D_\rho, [D_\mu, D_\nu]] + [D_\nu, [D_\rho, D_\mu]] = 0.
\] (1.15)

### 1.3 \(F \wedge F\)

Show the infinitesimal variation of \(F_{\mu\nu}\tilde{F}^{\mu\nu}\) is a total derivative. Therefore such terms do not contribute to the classical field equations.

\[
\delta [F_{\mu\nu}(1/2 \varepsilon^{\mu\nu\alpha\beta} F_{\alpha\beta})] = \partial_\nu \delta [A_\nu] - D_\nu \delta [A_\nu] - D_\nu \delta [A_\mu] \tilde{F}^{\mu\nu} = 4D_\mu \left[ \delta [A_\nu] \right] \tilde{F}^{\mu\nu}.
\] (1.16)

Using the Palatini identity:

\[
\delta [F_{\mu\nu} \tilde{F}^{\mu\nu}] = 2(D_\nu \delta [A_\nu] - D_\nu \delta [A_\mu]) \tilde{F}^{\mu\nu} = 4D_\mu \left[ \delta [A_\nu] \right] \tilde{F}^{\mu\nu}.
\] (1.17)

Now applying Leibniz again:

\[
\delta [F_{\mu\nu} \tilde{F}^{\mu\nu}] = 4D_\mu \left[ \delta [A_\nu] \right] \tilde{F}^{\mu\nu} = 4D_\mu \left[ \delta [A_\nu] D_\mu \tilde{F}^{\mu\nu} \right].
\] (1.18)

The first term of the above is a gauge invariant. That is the term as it appears in the Lagrangian:

\[
\int \frac{1}{2} \delta \left[ \text{tr} \left\{ F_{\mu\nu} \tilde{F}^{\mu\nu} \right\} \right] d^4 x = \int \left[ \delta [A_\nu] \right] \tilde{F}^{\mu\nu} d^4 x,
\] (1.19)

is invariant (transforms trivially) under the action of the gauge group. Hence:

\[
\delta [F_{\mu\nu} \tilde{F}^{\mu\nu}] = 4D_\mu \left[ \delta [A_\nu] \right] \tilde{F}^{\mu\nu} = 4 \partial_\mu \left[ \delta [A_\nu] \right] \tilde{F}^{\mu\nu}.
\] (1.20)

Therefore this term is a total derivative and does not contribute to the classical equations of motion.
2 Problem 2

2.1 Converting Complex Representations to Real Representations

Defining real and imaginary parts of a scalar may be done through the use of the (non-holomorphic) \( \Re \) and \( \Im \) functions:

\[
\Re[z] = \frac{z + z^*}{2},
\]

\[
\Im[z] = \frac{z - z^*}{2i}.
\]

Defining \( \phi_{1,2} \) so that:

\[
\phi = \phi_1 + i\phi_2,
\]

that is:

\[
\phi_1 = \Re[\phi], \quad \phi_2 = \Im[\phi].
\]

Then the terms of the action of the Lie algebra generators on \( \phi \) is:

\[
iT^k \phi = i(\Re[T^k] + i\Im[T^k])(\phi_1 + i\phi_2)
\]

\[
= i\Re[T^k] \phi_1 - i\Im[T^k] \phi_2 - \Im[T^k] \phi_1 - \Re[T^k] \phi_2,
\]

(2.4)

Defining a new (real) scalar (\( \tilde{\phi} \)) that transforms under a real representation of the same algebra:

\[
\frac{\partial}{\partial \theta_k} \left[ \exp(i\theta_k T^k) \phi \right] = iT^k \phi,
\]

(2.5)

\[
\frac{\partial}{\partial \theta_k} \left[ \exp(i\theta_k \tilde{T}^k) \tilde{\phi} \right] = i\tilde{T}^k \tilde{\phi}.
\]

(2.6)

Note, \( \tilde{T} \) must be imaginary (cancel the \( i \)) and antisymmetric (i.e. Hermitian and Imaginary to cancel the \( i \)):

\[
\tilde{T}^{k\dagger} = -\left( \tilde{T}^k \right)^T
\]

(2.7)

for imaginary \( \tilde{T}^k \). \( \tilde{T}^k \) is a matrix,

\[
\tilde{T}^k = \begin{pmatrix} \tilde{T}_{UL}^k & \tilde{T}_{UR}^k \\ \tilde{T}_{LL}^k & \tilde{T}_{LR}^k \end{pmatrix},
\]

(2.8)

\[
\tilde{\phi} = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}.
\]

(2.9)

With this notation:

\[
i\tilde{T}^k \tilde{\phi} = \begin{pmatrix} i\tilde{T}_{UL}^k \phi_1 + i\tilde{T}_{UR}^k \phi_2 \\ i\tilde{T}_{LL}^k \phi_1 + i\tilde{T}_{LR}^k \phi_2 \end{pmatrix} \rightarrow \begin{pmatrix} -\Im[T^k] \phi_1 - \Re[T^k] \phi_2 \\ -i (\Re[T^k] \phi_1 - i\Im[T^k] \phi_2) \end{pmatrix},
\]

(2.10)

therefore taking Eqs. (2.4) and (2.10) together implies:

\[
i\tilde{T}_{UL}^k = -\Im[T^k],
\]

(2.11)

\[
i\tilde{T}_{UR}^k = -\Re[T^k],
\]

(2.12)

\[
i\tilde{T}_{LL}^k = \Re[T^k],
\]

(2.13)

\[
i\tilde{T}_{LR}^k = -\Im[T^k].
\]

(2.14)

The above may be written:

\[
\tilde{T}^k = -i \begin{pmatrix} -\Im[T^k] & -\Re[T^k] \\ \Re[T^k] & -\Im[T^k] \end{pmatrix}.
\]

(2.15)
2.2 Symmetry of Mass Matrices

The inner product of the vacuum after the action of Lie algebra generators is:

\[ S^{ij} = (T^i\bar{\phi}, T^j\bar{\phi}) \]  \hspace{1cm} (2.16)

Since these generators are Hermitian with respect to the \((\cdot, \cdot)\) inner product, and the operator is symmetric:

\[ = (T^iT^j\bar{\phi}, \bar{\phi}) = (T^jT^i\bar{\phi}, \bar{\phi}) = S^{ji}. \]  \hspace{1cm} (2.18)

Furthermore, considering any vector annihilated by the mass matrix:

\[ M^{2}_{AB}T^i_{BC}\bar{\phi}_C = 0. \]

\[ \phi^i_B = T^i_{BC}\bar{\phi}_C \neq 0 \]

\[ \Rightarrow M^{2}_{AB}\phi^i_B = 0 = 0 \cdot \phi^i. \]

That is if a matrix annihilates a vector, that vector is an eigenvector. In this case, a fluctuation about the vacuum with zero mass, i.e., a Goldstone boson. Additionally, \(S^{ij}\) is real, since it is the product of two purely imaginary vectors. Furthermore, this may be diagonalized using an orthogonal transformation:

\[ \tilde{S}^{ij} = (O^T SO)^{ij}, \]  \hspace{1cm} (2.19)

where \(O\) are orthogonal matrices. The nonzero diagonal components of \(\tilde{S}\) correspond to independent non-vanishing \(T^i\phi\). Therefore, all non-vanishing \(T^i\phi\) correspond to different eigenvectors of equation:

\[ M^{2}_{AB}T^i_{BC}\bar{\phi}_C = 0, \]  \hspace{1cm} (2.20)

with eigenvalue 0.

3 Problem 3

3.1 The Real Representation

Considering how the \(SU(2) \times U(1)\) term appears int the Lagrangian:

\[ L_{\text{eff}} = \left| \left( -\frac{i}{2}g_2W^i_\mu\sigma^i - \frac{i}{2}g_1B_\mu \right) \phi_{2C} \right|^2, \]  \hspace{1cm} (3.1)

the desired matrices transform as:

\[ iT^i_{2C}\bar{\phi}_{2C} \rightarrow iT^i_{4R}\phi_{4R}. \]  \hspace{1cm} (3.2)

Expanding out the effective term:

\[ \frac{i}{2} \begin{pmatrix} g_1B_\mu + g_2W^{i3}_\mu & g_2W^{i1}_\mu + iW^{i2}_\mu \\ g_2W^{i1}_\mu - ig_2W^{i2}_\mu & g_1B_\mu + g_2W^{i3}_\mu \end{pmatrix} \begin{pmatrix} \phi_{4R3} + i\phi_{4R4} \\ \phi_{4R1} + i\phi_{4R2} \end{pmatrix}. \]  \hspace{1cm} (3.3)

Then taking the above expression field by field:

\[ \frac{1}{2}g_2W^{i1}_\mu \begin{pmatrix} \frac{i}{4}\phi_{4R1} - \phi_{4R2} \\ \frac{i}{4}\phi_{4R3} - \phi_{4R4} \end{pmatrix} \rightarrow \frac{1}{2}g_2W^{i1}_\mu \begin{pmatrix} -\phi_{4R4} \\ \phi_{4R3} \\ -\phi_{4R1} \end{pmatrix} = ig_2W^{i1}_\mu T^1 \begin{pmatrix} \phi_{4R1} \\ \phi_{4R2} \\ \phi_{4R3} \end{pmatrix}. \]  \hspace{1cm} (3.4)

Therefore:
Similar calculations yield each $T^i_{4R}$:

\[
T^2_{4R} = \frac{i}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \\
T^3_{4R} = \frac{i}{2} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \\
T^4_{4R} = \frac{i}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.
\] (3.6)

### 3.2 Mass Squared Matrix

Next, calculating $(M^2)_{ij}$ is straightforward, for example:

\[
(M^2)^{34} = -\frac{g_1 g_2}{4} (v \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}) \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} v \\ 0 \\ 0 \\ 0 \end{pmatrix} = -\frac{1}{4} g_1 g_2 v^2.
\] (3.7)

This calculation may be sped through caching, either the general form of $S_{ij}$ as described in the problem sheet, or the general form of $T^i \bar{\phi}_{4R}$ which are:

\[
T^1 \bar{\phi}_{4R} = \frac{i}{2} \begin{pmatrix} 0 \\ 0 \\ 0 \\ -v \end{pmatrix}, \\
T^2 \bar{\phi}_{4R} = \frac{i}{2} \begin{pmatrix} 0 \\ 0 \\ -v \\ 0 \end{pmatrix}, \\
T^3 \bar{\phi}_{4R} = \frac{i}{2} \begin{pmatrix} v \\ 0 \\ 0 \\ 0 \end{pmatrix}, \\
T^4 \bar{\phi}_{4R} = \frac{i}{2} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.
\] (3.9)

Taking the possible products of these find the mass squared matrix is:

\[
(M^2) = \frac{1}{4} \begin{pmatrix} g_1^2 v^2 & 0 & 0 & 0 \\ 0 & g_1^2 v^2 & 0 & 0 \\ 0 & 0 & g_1^2 v^2 & -g_1 g_2 v^2 \\ 0 & 0 & -g_1 g_2 v^2 & g_2^2 v^2 \end{pmatrix}.
\] (3.10)

### 3.3 Custodial Symmetry

In general, however, simply requiring that there be only one uneaten mode means that some linear combination of generators annihilates the vacuum gives a different mass squared matrix. In the current representation with the current Lagrangian:

\[
g_i g_j \bar{\phi} T^i T^j \bar{\phi} =
\]

\[
\frac{1}{4} \begin{pmatrix} g_2^2 |\bar{\phi}|^2 & 0 & 0 & g_1 g_2 (2ac + 2bd) \\ 0 & g_2^2 |\bar{\phi}|^2 & 0 & g_1 g_2 (2bc - 2ad) \\ 0 & 0 & g_2^2 |\bar{\phi}|^2 & g_1 g_2 (c^2 + d^2 - a^2 - b^2) \\ g_1 g_2 (2ac + 2bd) & g_1 g_2 (2bc - 2ad) & g_1 g_2 (c^2 + d^2 - a^2 - b^2) & g_2^2 |\bar{\phi}|^2 \end{pmatrix}.
\] (3.11)

Here $\bar{\phi}^\dagger = (a \ b \ c \ d)$. Breaking custodial symmetry with this Lagrangian is impossible, a more complicated scalar sector (specifically a higher representation of $SU(2) \times U(1)$) is required.
However, insisting that an (arbitrary, for simplicity sake $T^3 + T^4$) linear combination of $SU(2)$ generators and $U(1)$ generators remains unbroken yields:

\[
(T^3 \bar{\phi}, (T^3 + T^4) \bar{\phi}) = 0 \Rightarrow (T^3 \bar{\phi}, T^3 \bar{\phi}) = -(T^3 \bar{\phi}, T^4 \bar{\phi})
\]  (3.13)

multiplying by the relevant coupling constants:

\[
g_1^2 S^{33} = -g_1 g_2 S^{34} \Rightarrow (M^2)^{33} = -g_1 g_2 (M^2)^{34}
\]  (3.14)

Furthermore this real symmetric matrix must have three non-zero eigenvalues, therefore, diagonalize the further terms to get:

\[
(M^2) = \frac{1}{4} \begin{pmatrix}
  a & 0 & 0 & b \\
  0 & a & 0 & c \\
  0 & 0 & d & -\frac{g_1}{g_2} d \\
  b & c & -\frac{g_2}{g_2} d & -\frac{g_1}{g_2} d
\end{pmatrix}.
\]  (3.15)

Given that no linear combination of the remaining generators may be unbroken, that is:

\[
(T^i \bar{\phi}, (T^i + k T^4) \bar{\phi}) \neq 0
\]  (3.16)

for $i = 1, 2$ and for all $k \neq 0$, therefore:

\[
S^{ii} + k S^{i4} \neq 0 \Rightarrow S^{i4} = 0,
\]  (3.17)

for $i = 1, 2$. Therefore choosing the correct normalizations of $a$, and $d$, get:

\[
(M^2) = \frac{1}{4} \begin{pmatrix}
  g_2^2 v^2 & 0 & 0 & 0 \\
  0 & g_2^2 u^2 & 0 & 0 \\
  0 & 0 & g_2^2 u^2 & -g_1 g_2 u^2 \\
  0 & 0 & -g_1 g_2 u^2 & g_1^2 u^2
\end{pmatrix}.
\]  (3.18)

With broken custodial symmetry.

## 4 Problem 4

### Setup

\[
T^{ijk} = -\epsilon^{ijk}
\]  (4.1)

are generators of $SO(3)$ algebra in the adjoint representation:

\[
T^1 = -i \begin{pmatrix}
  0 & 0 & 0 \\
  0 & 0 & 1 \\
  0 & -1 & 0
\end{pmatrix}, \quad T^2 = -i \begin{pmatrix}
  0 & 0 & -1 \\
  0 & 0 & 0 \\
  1 & 0 & 0
\end{pmatrix}, \quad T^2 = -i \begin{pmatrix}
  0 & 1 & 0 \\
  -1 & 0 & 0 \\
  0 & 0 & 0
\end{pmatrix}.
\]  (4.2)

### 4.1 VEV $a$

Let

\[
\bar{\phi} = \begin{pmatrix}
  0 \\
  0 \\
  0
\end{pmatrix},
\]  (4.3)

then

\[
T^k \bar{\phi} = \begin{pmatrix}
  0 \\
  0 \\
  0
\end{pmatrix},
\]  (4.4)

for all $k$. This means that no generator is broken and the stability group is $H = G = SO(3)$.

Expanding $\phi$ around $\bar{\phi}$:

\[
\phi = \bar{\phi} + \phi_1 \begin{pmatrix}
  1 \\
  0 \\
  0
\end{pmatrix} + \phi_2 \begin{pmatrix}
  0 \\
  1 \\
  0
\end{pmatrix} + \phi_3 \begin{pmatrix}
  0 \\
  0 \\
  1
\end{pmatrix},
\]  (4.5)

where $\phi_1, \phi_2, \phi_3$ are massive fields.
4.2 VEV b

\[ \bar{\phi} = v \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad T^1 \bar{\phi} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad (4.6) \]

so \( T^1 \) is unbroken. Let a generic linear combination of \( T^2, T^3 \) act on \( \bar{\phi} \):

\[ (aT^2 + bT^3)\bar{\phi} = ia \begin{pmatrix} 0 \\ 0 \\ v \end{pmatrix} + ib \begin{pmatrix} 0 \\ v \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \quad (4.7) \]

For this last equality to be true, we need \( a = b = 0 \), therefore no linear combination of \( T^2, T^3 \) annihilates \( \bar{\phi} \), and \( T^2, T^3 \) are both broken generators. Since there is only one unbroken generator, the stability group is \( H = SO(2) \). If we expand the field \( \phi \) around \( \bar{\phi} \):

\[ \phi = v \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \phi_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \phi_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \phi_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (4.8) \]

\( \phi_1 \) is massive since it is perpendicular to the \( SO(3) \) orbit of \( \bar{\phi} = v \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \), an \( S^2 \) with radius \( v \). \( \phi_2 \) and \( \phi_3 \) are massless since they go in the two tangent directions to the orbit.

4.3 VEV c

\[ \bar{\phi} = \frac{v}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}. \quad (4.9) \]

Let a linear combination of the generators act on \( \bar{\phi} \):

\[ (aT^1 + bT^2 + cT^3)\bar{\phi} = i \frac{v}{\sqrt{2}} \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \quad (4.10) \]

\[ = i \frac{v}{\sqrt{2}} \begin{pmatrix} -c \\ c \\ a \end{pmatrix} = 0. \quad (4.11) \]

This requires \( c = 0 \) and \( a = b \). Therefore, the generator \( T^1 + T^2 \) is unbroken and the generators \( T^1 - T^2 \) and \( T^3 \) are broken. The stability group is \( H = SO(2) \): 2d rotations in the plane perpendicular to \( \bar{\phi} = i \frac{v}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \). Let

\[ \phi = \bar{\phi} + \phi_1 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \phi_2 \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \phi_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad (4.12) \]

\( \phi_1 \) is massive, \( \phi_2 \) and \( \phi_3 \) are massless.

4.4 VEV d

\[ \bar{\phi} = \frac{v}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}. \quad (4.13) \]

Let

\[ (aT^1 + bT^2 + cT^3)\bar{\phi} = i \frac{v}{\sqrt{2}} \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \quad (4.14) \]

\[ = i \frac{v}{\sqrt{2}} \begin{pmatrix} -c \\ c \\ a \end{pmatrix} = 0. \quad (4.15) \]
This is satisfied for \( c = 0, a = -b \). Hence the generator \( T^1 - T^2 \) is unbroken and the generators \( T^1 + T^2 \) and \( T^3 \) are broken. The stability group is \( H = SO(2) \), the 2d rotations of the plane perpendicular to \( \tilde{\phi} \). Expanding the field \( \phi \):

\[
\phi = \phi_1 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + \phi_2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \phi_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} . \tag{4.16}
\]

\( \phi_1 \) is massive, \( \phi_2 \) and \( \phi_3 \) are massless.

### 4.5 Generic VEV

In the general case we have two options:

1. \( \tilde{\phi} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \), discussed in part a, three unbroken generators.

2. \( \tilde{\phi} = v\vec{n} \), where \( \vec{n} \) is a 3–vector of length 1,

\[
\vec{n} = \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} , \quad n_1^2 + n_2^2 + n_3^2 = 1 . \tag{4.17}
\]

The linear combination of generators \( a_i T^i \) that generates the \( SO(2) \) group of 2d rotations of the plane perpendicular to \( \vec{n} \) is unbroken. Two other linearly independent combinations \( b_i T^i \) and \( c_i T^i \) will be broken. Let \( \vec{a} \) denote the vector

\[
\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} , \tag{4.18}
\]

then

\[
a_i T^{ijk} \vec{n}^k = 0 \tag{4.19}
\]

\[
\Rightarrow -ia_i \epsilon^{ijk} \vec{n}^k = 0 \tag{4.20}
\]

\[
\Rightarrow -ia_i \epsilon^{ijk} v n^k = 0 \tag{4.21}
\]

\[
\Rightarrow -iv (\vec{a} \wedge \vec{n})^i = 0 . \tag{4.22}
\]

This is true when \( \vec{a} \) and \( \vec{n} \) are on the same direction.

### 5 problem 5

#### 5.1 Hermitian Diagonalization

\( M\dagger M \) is Hermitian since:

\[
(M\dagger M)^\dagger = m \dagger (M\dagger)^\dagger = M\dagger M . \tag{5.1}
\]

Therefore (prove by using the inner product) it must have real eigenvalues:

\[
(\psi, M\dagger M\psi) = \lambda(\psi, \psi) = (M\dagger M\psi, \psi) = \lambda^* (\psi, \psi) \Rightarrow \lambda^* = \lambda , \tag{5.2}
\]

its eigenvectors (with different eigenvalue) must be orthogonal:

\[
(\psi, M\dagger M\phi) = (M\dagger M\psi, \phi) = \lambda_\psi(\psi, \phi) = \lambda_\phi(\psi, \phi) \Rightarrow (\psi, \phi) = 0 \text{ when } \lambda_\psi \neq \lambda_\phi , \tag{5.3}
\]

therefore its eigenvectors (may be chosen to) form an orthonormal basis:

\[
\Sigma \psi_i \dagger \psi = \hat{1} , \tag{5.4}
\]
therefore $M^\dagger M$ is defined by its action on its eigenvectors:

$$M^\dagger M \phi = \hat{1} M^\dagger M 1 \phi = \sum \psi^\dagger \psi M^\dagger M \psi \phi = \sum \psi^\dagger \psi \psi \phi = \sum \lambda \psi \psi \phi = UD^2 U^\dagger \phi , \quad (5.5)$$

where $D^2$ is the diagonal with $(D^2)_i = \lambda_i$, the eigenvalue corresponding to the $i$th column of the matrix $(U^\dagger)_j = \psi_j$, which is composed of eigenvectors ($U$ is unitary since the basis was orthonormal). Alternately:

$$D^2 = U^\dagger M^\dagger M U . \quad (5.6)$$

To show that the entries of $D^2$ are real (guaranteed by Hermiticity) and positive consider eigenvector $\psi$ with eigenvalue $\lambda$:

$$D^2 \psi = \lambda \psi . \quad (5.7)$$

Multiplying with $\psi^\dagger$ from the left:

$$\psi^\dagger D^2 \psi = \lambda \psi \psi^\dagger . \quad (5.8)$$

Now define vector $\phi = MU \psi$. Hence:

$$\phi^\dagger \phi = \lambda \psi \psi^\dagger . \quad (5.9)$$

Since $\phi^\dagger \phi$ and $\psi \psi^\dagger$ are manifestly real and positive, $\lambda$ must be real and positive as well.

### 5.2 General Diagonalization

We define $H = UD U^\dagger$, hence:

$$H^{-1} = (UD U^\dagger)^{-1} = (U^\dagger)^{-1} D^{-1} U^{-1} . \quad (5.10)$$

$U$ is unitary, $U^{-1} = U^\dagger$, so

$$H^{-1} = UD^{-1} U^\dagger . \quad (5.11)$$

Furthermore $D$ is Hermitian and therefore:

$$H^\dagger = (U^\dagger)^{\dagger} D^\dagger U^\dagger = U D U^\dagger = H , \quad (5.12)$$

is Hermitian. Ergo:

$$\tilde{U} \tilde{U} = (H^{-1})^\dagger M^\dagger M H^{-1} = UD^{-1} U^\dagger U D^2 U^\dagger U D^{-1} U^\dagger = \hat{1} . \quad (5.13)$$

Finally:

$$V^\dagger MU = U^\dagger U^\dagger M U = U^\dagger U^{-1} M U = U^\dagger H M^{-1} M U = U^\dagger U D U^\dagger U = D . \quad (5.14)$$

Therefore $M$ is diagonalizable.

### 6 Problem 6

Given three complex numbers $A$, $B$, $C$ satisfying $A + B + C = 0$, show that the area of the triangle with vertices at the two–dimensional points $0$, $A$, $A + B$ is given by

$$\frac{1}{2} \Im (AB^*) = \frac{1}{2} \Im (AC^*) = \frac{1}{2} \Im (BC^*) . \quad (6.1)$$

Let:

$$A = \exp(i \alpha) a , \quad (6.2)$$

$$B = \exp(i \beta) b , \quad (6.3)$$

$$C = \exp(i \gamma) c , \quad (6.4)$$

where $\alpha, \beta, \gamma \in [0, 2\pi]$ and $a, b, c \in \mathbb{R}^+$. 


We can divide the triangle with base $C$ in two halves, then compute height $h$:

$$\sin(\theta) = \frac{h}{b} \quad \Rightarrow \quad h = b \sin(\theta). \quad (6.5)$$

From the figure we see

$$\theta = \beta - (\gamma - \pi) = \beta - \gamma + \pi. \quad (6.6)$$

And we know that

$$\text{Area} = \frac{1}{2} \text{(base)} \cdot \text{(height)} = \frac{1}{2} c h = \frac{1}{2} c b \sin(\beta - \gamma + \pi). \quad (6.7)$$

Therefore:

$$\frac{1}{2} |\Im(B^*)| = \frac{1}{2} |b c \exp(i\beta) \exp(-i\gamma)| = \frac{1}{2} b c |\exp(i(\beta - \gamma))|$$

$$= \frac{1}{2} b c |\sin(\beta - \gamma)| = \frac{1}{2} b c |\sin(\beta - \gamma + \pi)|. \quad (6.8)$$

If we arrange $A, B, C$ as in the figure, then $\theta$ is between 0 and $\pi$, so $|\sin(\theta)| = \sin(\theta)$. Hence:

$$\frac{1}{2} |\Im(B^*)| = \frac{1}{2} b c \sin(\beta - \gamma + \pi) = \text{Area}. \quad (6.10)$$

Similarly we can choose $A$ or $B$ to be the base and obtain:

$$\text{Area} = \frac{1}{2} |\Im(A^*)| = \frac{1}{2} |\Im(A^*)|. \quad (6.11)$$