1 a): Recall the definition of the field strength:

\[ F_{\mu\nu}^i = \partial_\mu A_\nu^i - \partial_\nu A_\mu^i + gf^{ijk} A_\mu^j A_\nu^k. \]  

(1.1)

Then,

\[ \delta F_{\mu\nu}^i = \partial_\mu \delta A_\nu^i - \partial_\nu \delta A_\mu^i + gf^{ijk} (\delta A_\mu^j) A_\nu^k + gf^{ijk} A_\mu^j \delta A_\nu^k, \]  

(1.2)

where we used the Leibniz property of the variational operator \( \delta \), and the fact that \( \delta \) commutes with \( \partial_\mu \). The quantity \( \delta A_\mu \) transforms under the adjoint representation of the group. To see this, recall that under a group action \( U \in G \), \( A_\mu \) transforms as

\[ A_\mu \rightarrow A'_\mu = UA_\mu U^{-1} - \frac{i}{g} (\partial_\mu U) U^{-1}. \]  

(1.3)

The final term on the left is independent of \( A_\mu \), and is known as the Maurer-Cartan form. With this, \( \delta A_\mu \), which is the difference of two gauge fields, transforms as

\[ \delta A_\mu \rightarrow \delta A'_\mu = U(\delta A_\mu)U^{-1}. \]  

(1.4)

The Maurer-Cartan form cancels, and we see that \( \delta A_\mu \) transforms in the adjoint representation. Recalling that the generators of the adjoint transformation are given by \( (T^i_{\text{adjoint}})^{jk} = -if^{ijk} \), we have

\[ D_\mu \delta A_\nu^i = \partial_\mu \delta A_\nu^i + gf^{ijk} A_\mu^j \delta A_\nu^k. \]  

(1.5)

Substituting this into (1.2), we obtain the Palatini identity:

\[ \delta F_{\mu\nu}^i = D_\mu \delta A_\nu^i - D_\nu \delta A_\mu^i. \]  

(1.6)

1 b): The operator form of \( F_{\mu\nu} \) is

\[ F_{\mu\nu} = \frac{i}{g} [D_\mu, D_\nu]. \]  

(1.7)
The Jacobi identity,

\[ [D_\mu, [D_\nu, D_\rho]] + [D_\nu, [D_\rho, D_\mu]] + [D_\rho, [D_\mu, D_\nu]] = 0, \]  

then implies that

\[ [D_\mu, F_{\nu\rho}] + [D_\nu, F_{\rho\mu}] + [D_\rho, F_{\mu\nu}] = 0. \]  

(1.9)

Now, the complete antisymmetrisation of a tensor \( B_{\mu\nu\rho} \) is given by

\[ B_{[\mu\nu\rho]} = \frac{1}{6} (B_{\mu\nu\rho} - B_{\mu\rho\nu} + B_{\rho\mu\nu} - B_{\rho\nu\mu} + B_{\nu\rho\mu} - B_{\nu\mu\rho}) . \]  

(1.10)

If, however, \( B_{\mu\nu\rho} = -B_{\mu\rho\nu} \) initially, then the above formula becomes

\[ B_{[\mu\nu\rho]} = \frac{1}{3} (B_{\mu\nu\rho} + B_{\rho\mu\nu} + B_{\nu\rho\mu}) . \]  

(1.11)

Let’s now define \( B_{\mu\nu\rho} = [D_\mu, F_{\nu\rho}] \). Clearly, \( B_{\mu\nu\rho} = -B_{\mu\rho\nu} \), and the Jacobi identity in (1.9) is simply the statement that

\[ B_{[\mu\nu\rho]} = 0. \]  

(1.12)

One way to ‘pick out’ the antisymmetric part of any tensor is to contract it with the epsilon symbol. Since we are in 4 dimensions (each index \( \mu \) runs over 4 values), the epsilon symbol has 4 indices. As such, we can rewrite (1.12) as

\[ \epsilon^{\mu\nu\rho\sigma} B_{\nu\rho\sigma} = 0 \implies [D_\nu, \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}] = 0 \implies [D_\nu, \tilde{F}^{\mu\nu}] = 0, \]  

(1.13)

where we used the definition

\[ \tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} . \]  

(1.14)

Finally, note that

\[ [\partial_\mu, X]\phi = (\partial_\mu X)\phi + X\partial_\mu\phi - X\partial_\mu\phi = (\partial_\mu X)\phi \implies [\partial_\mu, X] = \partial_\mu X \]  

(1.15)

for any operator \( X \) and field \( \phi \). Thus,

\[
0 = [D_\mu, \tilde{F}^{\mu\nu}] = [\partial_\mu - igA_\mu, \tilde{F}^{\mu\nu}]
= \partial_\mu \tilde{F}^{\mu\nu} - ig[A_\mu, \tilde{F}^{\mu\nu}]
= (\partial_\mu \tilde{F}^{\mu\nu}) T^i - igA_\mu^{i} \tilde{F}^{\mu\nu} k[T^j, T^k]
= (\partial_\mu \tilde{F}^{\mu\nu}) T^i + g f^{ijk} A_\mu^{i} \tilde{F}^{\mu\nu} k T^i
= (D_\mu \tilde{F}^{\mu\nu}) T^i ,
\]  

(1.16)
where we used the fact that $\tilde{F}^{\mu\nu}$ is in the adjoint representation. \[1.16\] implies that $D_\mu \tilde{F}^{\mu\nu} = 0$, as the generators $T^i$ are linearly independent.

1 c): We need to compute the variation of $F^i_{\mu\nu} \tilde{F}^{\mu\nu}$. First, note that

\[
F^i_{\mu\nu} \delta \tilde{F}^{\mu\nu} = \frac{1}{2} F^i_{\mu\nu} \epsilon^{\mu\nu\rho\sigma} \delta F^i_{\rho\sigma} = \frac{1}{2} F^i_{\rho\sigma} \epsilon^{\rho\sigma\mu\nu} \delta F^i_{\mu\nu} = \tilde{F}^{\mu\nu} \delta F^i_{\mu\nu}. \tag{1.17}
\]

Therefore,

\[
\delta(F^i_{\mu\nu} \tilde{F}^{\mu\nu}) = F^i_{\mu\nu} \delta \tilde{F}^{\mu\nu} + \tilde{F}^{\mu\nu} \delta F^i_{\mu\nu} = 2 \tilde{F}^{\mu\nu} \delta F^i_{\mu\nu}. \tag{1.18}
\]

Using the Palatini identity \[1.6\] in question 1 a), we have

\[
\delta(F^i_{\mu\nu} \tilde{F}^{\mu\nu}) = 2 \tilde{F}^{\mu\nu} \delta F^i_{\mu\nu}
= 4 \tilde{F}^{\mu\nu} D_\mu \delta A^i_{\nu}
= 4 \tilde{F}^{\mu\nu}(\partial_\mu \delta A^i_{\nu} + g f^{ijk} A^j_\mu \delta A^k_{\nu})
= 4 \partial_\mu(F^{\mu\nu} \delta A^i_{\nu}) - 4(\delta A^i_{\nu}) D_\mu \tilde{F}^{\mu\nu} + g f^{ijk} \tilde{F}^{\mu\nu} A^j_\mu \delta A^k_{\nu}
= 4 \partial_\mu(F^{\mu\nu} \delta A^i_{\nu}) - 4(\delta A^i_{\nu}) D_\mu \tilde{F}^{\mu\nu}
= 4 \partial_\mu(F^{\mu\nu} \delta A^i_{\nu}), \tag{1.19}
\]

where we used the Bianchi identity $D_\mu \tilde{F}^{\mu\nu} = 0$ we derived in question 1 b).

2

2 a): $T^i_a$ is Hermitian, so $(T^i_a)^* = T^i_b$. Using this, and the transformations of $\varphi_a$ and its conjugate (given in the problem sheet), we find that the fields

\[
\phi_{a1} = \frac{1}{2}(\varphi_a + \varphi^* a), \quad \phi_{a2} = -\frac{i}{2}(\varphi_a - \varphi^* a) \tag{2.1}
\]

transform as

\[
\delta \phi_{a1} = \frac{i}{2} \delta \theta^i (T^i_a \varphi_b - (T^i_a)^* \varphi^* b), \tag{2.2}
\]
\[
\delta \phi_{a2} = \frac{1}{2} \delta \theta^i (T^i_a \varphi_b + (T^i_a)^* \varphi^* b). \tag{2.3}
\]

We can invert \[2.1\] to give

\[
\varphi_a = \phi_{a1} + i \phi_{a2}, \quad \varphi^* a = \phi_{a1} - i \phi_{a2}. \tag{2.4}
\]
Substituting this into the transformations, we find

\[
\delta \phi_a^1 = -\delta \theta^i (\text{Im}(T_a^i b) \phi_{b1} + \text{Re}(T_a^i b) \phi_{b2}) ,
\]

\[
\delta \phi_a^2 = \delta \theta^i (\text{Re}(T_a^i b) \phi_{b1} - \text{Im}(T_a^i b) \phi_{b2}) .
\]

(2.5)

(2.6)

We can package \( \phi_a^1 \) and \( \phi_a^2 \) into a \( 2n \) component vector \( \phi_a^i \):

\[
\phi_a = \begin{pmatrix} \phi_a^1 \\ \phi_a^2 \end{pmatrix}
\]

(2.7)

which will transform under the real representation as

\[
\delta \phi_a = -\delta \theta^i \left( \text{Im}(T_a^i b) \phi_{b1} + \text{Re}(T_a^i b) \phi_{b2} \right) .
\]

(2.8)

We know, however, that

\[
\delta \phi_a = i \delta \theta^i (T_{\text{real}}^i \phi)_a .
\]

(2.9)

where \( T_{\text{real}}^i \) are the generators in the real representation. From (2.8), we can read off these generators to be

\[
T_{\text{real}}^i = i \begin{pmatrix} \text{Im}(T_a^i b) & \text{Re}(T_a^i b) \\ -\text{Re}(T_a^i b) & \text{Im}(T_a^i b) \end{pmatrix} ,
\]

(2.10)

which are purely imaginary, antisymmetric \( 2n \times 2n \) matrices.

2 b): Define \( S^{ij} = (T^i \overline{\phi}, T^j \overline{\phi}) \), where \( T^i \) is purely imaginary and antisymmetric, and \( \overline{\phi} \) is real.

**Symmetric:** Since \((u, v) = (v, u)\), \( S^{ij} = (T^i \overline{\phi}, T^j \overline{\phi}) = (T^j \overline{\phi}, T^i \overline{\phi}) = S^{ji} \).

**Real:** \((S^{ij})^* = (T^{is} \overline{\phi}, T^{js} \overline{\phi}) = (-T^i \overline{\phi}, -T^j \overline{\phi}) = S^{ij} \), as \( T^i \) is imaginary.

Since \( S^{ij} \) is symmetric and real, it is diagonalisable with

\[
S = \mathcal{O} D \mathcal{O}^T ,
\]

(2.11)

where \( \mathcal{O} \) is the orthogonal matrix constructed by the orthonormal eigenvectors of \( S \), and \( D \) is the diagonal matrix of the eigenvalues of \( S \). The mass squared matrix \( M^2 \) is proportional to \( S \), so if

\[
M^2(T^i \overline{\phi}) = 0 , \quad T^i \overline{\phi} \neq 0 ,
\]

(2.12)

for some values of \( i \), then \( \overline{\varphi}^i = T^i \overline{\phi} \) satisfies

\[
S \overline{\varphi}^i = 0 .
\]

(2.13)
This means that each $\varphi^i$ is a zero eigenvector of $S$, and consequently, each non-zero $T^i \varphi$ corresponds to a single zero eigenvalue in the mass squared matrix. The unique one-to-one correspondence between a non-zero $T^i \varphi$ and a zero eigenvalue is due to the diagonalisability of $S$.

3

3 a): The complex doublet is given by

$$\varphi = \begin{pmatrix} \phi_{4R3} + i\phi_{4R4} \\ \phi_{4R1} + i\phi_{4R2} \end{pmatrix}. \quad (3.1)$$

Using the results in question 2, we find that the real fields $\phi_1$ and $\phi_2$ are

$$\phi_1 = \begin{pmatrix} \phi_{4R3} \\ \phi_{4R1} \end{pmatrix}, \quad \phi_2 = \begin{pmatrix} \phi_{4R4} \\ \phi_{4R2} \end{pmatrix}, \quad (3.2)$$

and the 4-component real field $\phi$ is then

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} \phi_{4R3} \\ \phi_{4R1} \\ \phi_{4R4} \\ \phi_{4R2} \end{pmatrix}. \quad (3.3)$$

This transforms under the real representation with generators $T^i_{\text{real}}$ given in (2.10). What we want, however, are the generators for the field

$$\phi_{4R} = \begin{pmatrix} \phi_{4R1} \\ \phi_{4R2} \\ \phi_{4R3} \\ \phi_{4R4} \end{pmatrix}. \quad (3.4)$$

This can be obtained from $\phi$ by a transformation

$$\phi_{4R} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \phi := M\phi, \quad (3.5)$$

which means that the generators for $\phi_{4R}$, $T^i_{4R}$, can be written in terms of the generators for $\phi$, $T^i_{\text{real}}$, as

$$T^i_{4R} = MT^i_{\text{real}}M^{-1}. \quad (3.6)$$

We now list the generators $T^i$ in the complex representation:

$$T^1 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad T^3 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad T^4 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (3.7)$$

We can now obtain the desired $T^i_{4R}$ by applying (2.10) to obtain $T^i_{\text{real}}$ and then converting using (3.6).
As an explicit example, let’s consider $i = 1$. Here,

\[
\text{Im}(T^1) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{Re}(T^1) = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},
\]

so (2.10) gives

\[
T^1_{\text{real}} = \frac{i}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.
\] (3.8)

Finally, (3.6) tells us that

\[
T^1_{4R} = \frac{i}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \left( \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \right) = \frac{i}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.
\] (3.9)

(Mathematica is useful in computing these matrix products quickly.)

3 b): We will derive the general form of the vector mass matrix (given in the problem sheet) using group theory only, and assuming an unbroken symmetry given by $(T^3 + T^4)\overline{\phi} = 0$. From this, we have

\[
[T^3 + T^4, T^i T^j]_{\overline{\phi}} = (T^3 + T^4)T^i T^j \overline{\phi} + T^i T^j (T^3 + T^4)\overline{\phi} = (T^3 + T^4)T^i T^j \overline{\phi}.
\] (3.11)

Furthermore,

\[
[T^4, T^i T^j] = [T^4, T^i] T^j + T^i [T^4, T^j] = 0,
\] (3.12)

as $T^4$ generates the Abelian $U(1)_Y$ group, so $[T^3 + T^4, T^i T^j]_{\overline{\phi}} = [T^3, T^i T^j]_{\overline{\phi}}$. Using this, we have

\[
K^{ij} = g_i g_j (\overline{\phi}, [T^3, T^i T^j]_{\overline{\phi}})
\]

\[
= g_i g_j (\overline{\phi}, (T^3 + T^4)T^i T^j \overline{\phi})
\]

\[
= -g_i g_j ((T^3 + T^4)\overline{\phi}, T^i T^j \overline{\phi})
\]

\[
= 0,
\] (3.13)

where in the second line we used the fact that the generators are antisymmetric. Defining $P^{ij} = [T^3, T^i T^j] = [T^3, T^i] T^j + T^i [T^3, T^j]$, we obtain the identity

\[
K^{ij} = g_i g_j (\overline{\phi}, P^{ij} \overline{\phi}) = 0.
\] (3.14)

$i = j = 1$: $P^{11} = [T^3, T^1] T^1 + T^1 [T^3, T^1] = iT^2 \overline{T^1} + iT^1 \overline{T^2}$, so

\[
K^{11} = 2i g_2^2 (\overline{\phi}, T^2 T^1 \overline{\phi}) = 2(M^2)_{\text{gen}}^{12} = 0,
\] (3.15)

where we used the fact that $(\overline{\phi}, T^i T^j \overline{\phi}) = -(\overline{T^i \phi}, T^j \overline{\phi}) = -(\overline{T^j \phi}, T^i \overline{\phi}) = (\overline{\phi}, T^j T^i \overline{\phi})$, and the definition of
the vector matrix.

\[ i = 1, j = 2: \quad P^{12} = i(T^2T^2 - T^1T^1), \quad \text{so} \quad K^{12} = i((M^2)^{22}_{\text{gen}} - (M^2)^{11}_{\text{gen}}) = 0, \quad \text{meaning that} \quad (M^2)^{22}_{\text{gen}} = (M^2)^{11}_{\text{gen}}. \]

For \( i = 1, j = 3 \), we find that \( P^{13} = iT^2T^3 \), so \((M^2)^{23}_{\text{gen}} = 0. \) For \( i = 1, j = 4 \), we have \( P^{14} = iT^2T^4 \), so \((M^2)^{24}_{\text{gen}} = 0. \) For \( i = 2, j = 3 \), \( P^{23} = -iT^1T^3 \), so \((M^2)^{13}_{\text{gen}} = 0. \) Lastly, for \( i = 2, j = 4 \), \( P^{24} = -iT^1T^4 \), so \((M^2)^{14}_{\text{gen}} = 0. \)

Unfortunately, we cannot use the same trick to deal with the \((M^2)^{33}_{\text{gen}}, (M^2)^{34}_{\text{gen}}, \) and \((M^2)^{44}_{\text{gen}}\) components, as \( P^{33}, P^{34}, \) and \( P^{44} \) are trivially zero. What we know so far from our group theoretic discussion is then

\[
(M^2)_{\text{gen}} = \frac{1}{4} \begin{pmatrix} g_2^2 v^2 & 0 & 0 & 0 \\ 0 & g_2^2 v^2 & 0 & 0 \\ 0 & 0 & x & y \\ 0 & 0 & y & z \end{pmatrix}, \quad (3.16)
\]

where we parameterised \((M^2)^{11}_{\text{gen}}\) as \( g_2^2 v^2 / 4. \)

There is one last thing that we can do to constrain the bottom \( 2 \times 2 \) matrix. We know that there is an unbroken symmetry generated by \( T^3 + T^4 \), so Goldstone’s theorem and the Higgs mechanism tells us that there must be a single massless vector corresponding to this unbroken symmetry. This means that \((M^2)_{\text{gen}}\) must have a single zero eigenvalue that must occur in the bottom \( 2 \times 2 \) matrix. Another way of saying this is that the determinant of the bottom \( 2 \times 2 \) matrix must vanish:

\[
\text{det} \begin{pmatrix} x & y \\ y & z \end{pmatrix} = 0 \implies y = \pm xz. \quad (3.17)
\]

Choosing the negative root and parameterising the free parameters \((x, z)\) as \( x = g_2^2 u^2 \) and \( z = g_1^2 u^2 \), we find that the generic form of the vector mass matrix is

\[
(M^2)_{\text{gen}} = \frac{1}{4} \begin{pmatrix} g_2^2 v^2 & 0 & 0 & 0 \\ 0 & g_2^2 v^2 & 0 & 0 \\ 0 & 0 & g_2^2 u^2 & -g_1 g_2 u \\ 0 & 0 & -g_1 g_2 u & g_2^2 u^2 \end{pmatrix}. \quad (3.18)
\]

4

For this question, we need to recall Goldstone’s theorem, which states that each broken generator of a global symmetry corresponds to a massless mode. It is also useful to write out the generators of \( \text{SO}(3) \) in matrix form for clarity. We are given that \((T^i)^{jk} = -i \epsilon^{ijk}\), with \( i, j, k \in \{1, 2, 3\} \), and so

\[
T^1 = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad T^2 = i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad T^3 = i \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (4.1)
\]

4 a): \( \bar{\phi} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \).
Here, \( T^i \overline{\phi} = 0 \) trivially, so there are no broken generators, and the stability group is still \( \text{SO}(3) \). By Goldstone’s theorem, all 3 components of the fluctuations \( \varphi \) where \( \phi = \overline{\phi} + \varphi \) are massive.

**4 b):** \( \overline{\phi} = v \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \).

Here, \( T^1 \overline{\phi} = 0 \), but

\[
T^2 \overline{\phi} = -iv \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad T^3 \overline{\phi} = iv \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}.
\]

(4.2)

It is clear that there exists no non-zero \( a, b \in \mathbb{C} \) such that \((aT^2 + bT^3)\overline{\phi} = 0\), as \( T^2 \overline{\phi} \) and \( T^3 \overline{\phi} \) are linearly independent vectors in \( \mathbb{C}^3 \). Therefore, \( T^2 \) and \( T^3 \) are independently broken. The only unbroken generator is \( T^1 \). Exponentiating \( T^1 \), we find

\[
\exp(i\theta T^1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{pmatrix} \in \text{SO}(2).
\]

(4.3)

As such, the stability group generated by \( T^1 \) is simply \( \text{SO}(2) \). By Goldstone’s theorem, there will be 1 massive mode corresponding to the unbroken \( T^1 \) generator, and 2 massless modes corresponding to the broken \( T^2 \) and \( T^3 \) generators.

**4 c):** \( \overline{\phi} = \frac{v}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \).

Here, \((T^1 + T^2)\overline{\phi} = 0\), and the broken generators are \( T^1 - T^2 \) and \( T^3 \). Again, the stability group generated by \( T^1 + T^2 \) is \( \text{SO}(2) \). By Goldstone’s theorem, there will be 1 massive mode corresponding to \( T^1 + T^2 \), and 2 massless modes corresponding to \( T^1 - T^2 \) and \( T^3 \).

**4 d):** \( \overline{\phi} = \frac{v}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \).

The analysis is the same as 4 c) with the role of \( T^1 + T^2 \) exchanged with \( T^1 - T^2 \).

**4 e):** Consider the most general form of the vacuum expectation value

\[
\overline{\phi} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{C}^3.
\]

(4.4)

If \( x = y = z = 0 \), then there are no broken generators and the stability group is \( \text{SO}(3) \). If \( \overline{\phi} \neq 0 \), then for some \( a, b, c \in \mathbb{C} \), we have

\[
(aT^1 + bT^2 + cT^3)\overline{\phi} = i \begin{pmatrix} bz - cy \\ cx - az \\ ay - bx \end{pmatrix}.
\]

(4.5)
For the combination $aT^1 + bT^2 + cT^3$ to be an unbroken generator, we require the RHS of (4.5) to vanish. This requires solving the following 3 linear, simultaneous equations for the parameters $a, b, c$:

$$bz = cy, \quad cx = az, \quad ay = bx.$$  (4.6)

Without loss of generality, let’s assume $x \neq 0$ and $a \neq 0$. Then, the last two equations can be solved by

$$c = \frac{az}{x}, \quad b = \frac{ay}{x}.$$  (4.7)

With this expressions for $c$ and $b$, we find that the first equation $bz = cy$ is trivially satisfied:

$$bz - cy = \frac{ayz}{x} - \frac{azy}{x} = 0.$$  (4.8)

We have now found a unique solution for the unbroken generator, and it is given by

$$a\left(T^1 + \frac{y}{x}T^2 + \frac{z}{x}T^3\right)$$  (4.9)

Since $a \neq 0$, we can ignore the overall factor of $a$. The broken generators are formed by linearly independent combinations of $T^i$ that are orthogonal to the unbroken generator, and since there’s only one unbroken generator, there must be two broken one. After a bit of algebra, we find that one particular basis for the broken generators is

Unbroken: $\hat{T}^1 = T^1 + \frac{y}{x}T^2 + \frac{z}{x}T^3$  (4.10)

Broken: $\hat{T}^2 = \frac{y}{x}T^1 - T^2, \quad \hat{T}^3 = \frac{z}{x}T^1 + \frac{yz}{x^2}T^2 - \left(1 + \frac{y^2}{x^2}\right)T^3.$  (4.11)

Let’s just check that this works for the cases we talked about. For 4 b), we had $x = v, y = 0, z = 0$, so the unbroken generator is $\hat{T}^1 = T^1$, and the broken generators are $\hat{T}^2 = T^2$, and $\hat{T}^3 = T^3$. For 4 c), we had $x = v/\sqrt{2}, y = x, z = 0$, so $\hat{T}^1 = T^1 + T^2$, $\hat{T}^2 = T^1 - T^2$, and $\hat{T}^3 = -2T^3$. Finally, for 4 d), we had $x = v/\sqrt{2}, y = -x, z = 0$, so $\hat{T}^1 = T^1 - T^2$, $\hat{T}^2 = -(T^1 + T^2)$, and $\hat{T}^3 = -2T^3$.

5 a): Let $M \in \text{GL}(n, \mathbb{C})$ and define $K = M^\dagger M$. Clearly, $K^\dagger = (M^\dagger M)^\dagger = K$, so $K$ is Hermitian, which means that it has a set of non-zero eigenvectors $e_i, i \in \{1, \ldots, n\}$, with real eigenvalues $Ke_i = \lambda_i e_i$, $\lambda_i \in \mathbb{R}$. The eigenvectors are orthonormal with respect to the $\mathbb{C}^n$ inner product

$$(e_i, e_j) = \sum_{\alpha=1}^{n} (e_i^\alpha)^* e_j^\alpha = \delta_{ij},$$  (5.1)

where $e_i^\alpha$ is the $\alpha^{th}$ component of $e_i$. By orthonormality,

$$\lambda_i = (e_i, Ke_i) = (e_i, M^\dagger Me_i) = (Me_i, Me_i) = ||Me_i||^2 \geq 0,$$  (5.2)
where we used the definition of the Hermitian conjugate: \((u, Mv) = (M^\dagger u, v)\). This shows that the eigenvalues are strictly positive unless \(Me_i = 0\), which is a case we will ignore, and we shall write them as \(\lambda_i = \omega_i^2\), with \(\omega_i \in \mathbb{R} \setminus \{0\}\).

To diagonalise \(K\), construct the matrix \(U = (e_1, e_2, \ldots, e_n)\), where we treat \(e_i\) as column vectors. By orthonormality of the eigenvectors, we see that

\[ U^\dagger U = UU^\dagger = \mathbb{1}_n, \]  

so \(U\) is unitary. Therefore,

\[ K = K(UU^\dagger) = (KU)U^\dagger. \]  

By defining the diagonal matrix

\[ D^2 = \text{diag}(\omega_1^2, \omega_2^2, \ldots, \omega_n^2), \]  

we have

\[ Ku = K(e_1, e_2, \ldots, e_n) = (\omega_1^2e_1, \omega_2^2e_2, \ldots, \omega_n^2e_n) = UD^2. \]  

Thus,

\[ K = UD^2U^\dagger. \]  

5 b): Define \(H = UDU^\dagger\). To compute \(H^{-1}\), note that since \(\omega_i \neq 0\),

\[ D^m = \text{diag}(\omega_1^m, \omega_2^m, \ldots, \omega_n^m) \]  

for any \(m \in \mathbb{Z}\). This, combined with the unitarity of \(U\) then implies that

\[ H^{-1} = UD^{-1}U^\dagger. \]  

Now, define \(\tilde{U} = MH^{-1} = MUD^{-1}U^\dagger\). Then,

\[ \tilde{U}^\dagger \tilde{U} = UD^{-1}U^\dagger M^\dagger MUD^{-1}U^\dagger \]
\[ = UD^{-1}U^\dagger (UD^2U^\dagger) UD^{-1}U^\dagger \]
\[ = \mathbb{1}_n, \]

where we used the result in 5 a) that \(K = M^\dagger M = UD^2U^\dagger\). This means that \(\tilde{U}\) is unitary. With these definitions, we can now define the matrix \(V = \tilde{U}U = MUD^{-1}\), which is unitary because it is the product of unitary matrices. Then,

\[ V^\dagger MU = D^{-1}U^\dagger M^\dagger MU = D^{-1}U^\dagger (UD^2U^\dagger) U = D. \]
This can be rewritten as
\[ M = VDU^\dagger. \]  \hfill (5.12)

Note that if \( M \) is Hermitian, then
\[ M = M^\dagger \implies VDU^\dagger = UDV^\dagger \implies V = U, \]  \hfill (5.13)
giving the usual result for the diagonalisation of a Hermitian matrix.

6

Let \( A, B, C \in \mathbb{C} \) satisfy \( A + B + C = 0 \). It is useful to write \( A = ae^{i\alpha}, B = be^{i\beta} \) and \( C = ce^{i\gamma} \), where \( a, b, c \in \mathbb{R}^+ \) and \( \alpha, \beta, \gamma \in [0, 2\pi) \). Consider a triangle with vertices on \((0,0), A\) and \(A + B = -C\). The diagram will look like this:

![Diagram of a triangle with vertices at (0,0), A, and A+B=-C.

Clearly, \( \theta = \gamma - \alpha - \pi \), and the height \( h = a|\sin \theta| \). The area is then
\[ \text{Area} = \frac{1}{2} ac |\sin \theta|. \]  \hfill (6.1)

Now,
\[ |\text{Im}(AC^*)| = |\text{Im}(ace^{i(\alpha-\gamma)})| = ac |e^{i(\alpha-\gamma)} - e^{i(\alpha-\gamma+\pi)}| = ac |\sin \theta| \implies \text{Area} = \frac{1}{2} |\text{Im}(AC^*)|. \]  \hfill (6.2)

Since \( B = -(A + C) \), \( |\text{Im}(BC^*)| = |\text{Im}(AC^*) + \text{Im}(CC^*)| = |\text{Im}(AC^*)| \), as \( CC^* \in \mathbb{R} \). Similarly, \( |\text{Im}(AB^*)| = |\text{Im}(AA^*) + |\text{Im}(AC^*)| = |\text{Im}(AC^*)| \). Thus,
\[ \text{Area} = \frac{1}{2} |\text{Im}(AC^*)| = \frac{1}{2} |\text{Im}(BC^*)| = \frac{1}{2} |\text{Im}(AB^*)|. \]  \hfill (6.3)