Solutions - PS2

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1

1 a): Φ is Hermitian, so it admits 3 orthornormal eigenvectors v_1, v_2 and v_3, with real eigenvalues λ_1, λ_2 and λ_3. Φ is also traceless, so the eigenvalues must sum to zero:

$$\lambda_1 + \lambda_2 + \lambda_3 = 0\,. \tag{1.1}$$

By constructing the 3×3 unitary matrix $U = (v_1, v_2, v_3) \in \mathrm{SU}(3)$ (the unitarity comes from the orthonormality of the eigenvectors), where we treat v_i as column vectors, and the diagonal matrix $D = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, we have

$$\Phi U = (\lambda_1 v_1, \lambda_2 v_2, \lambda_3 v_3) = UD \implies \Phi = UDU^\dagger\,. \tag{1.2}$$

The process $\Phi \mapsto D = U^\dagger \Phi U$ is called diagonalisation, and it is clear by construction that the eigenvalues do not change. In accordance with the notation in the problem sheet, we identify $S = U^\dagger \in \mathrm{SU}(3)$.

1 b): The potential is given by

$$V(\Phi) = \frac{1}{4}\left(\,\mathrm{tr}(\Phi^2) - k^2\right)^2\,. \tag{1.3}$$

This is a continuous function in Φ that is semi-positive definite, so its minimum is given by $V(\overline{\Phi}) = 0$. As such, the vacuum is determined by the relation

$$\mathrm{tr}\left(\overline{\Phi}^2\right) = k^2\,. \tag{1.4}$$

From 1 a), since $\Phi = UDU^\dagger$, we have $\Phi^2 = UD^2U^\dagger$, so (1.4) becomes

$$\mathrm{tr}\left(D^2\right) = k^2 \implies \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = k^2\,. \tag{1.5}$$

Combining this with the traceless relation in (1.1), we find that there is only one free parameter (say, λ_1) that determines the vacuum. The vacuum orbit $\mathcal{O}_{\lambda_1}$, determined by λ_1, is defined as

$$\mathcal{O}_{\lambda_1} \coloneqq \{S\overline{\Phi}S^\dagger : S \in \mathrm{SU}(3)\}\,, \tag{1.6}$$

where it is clear that any $L \in \mathcal{O}_{\lambda_1}$ obeys $\operatorname{tr} L = 0$ and $\operatorname{tr}\left(L^2\right) = k^2$.

1 c): In order to find the little group, we need to find the unbroken generators. The transformation of the vacuum $\overline{\Phi}$ is given by

$$\overline{\Phi} \mapsto \overline{\Phi}' = S\overline{\Phi}S^\dagger\,, \quad S \in \mathrm{SU}(3)\,. \tag{1.7}$$

Writing $S = \mathbb{1}_3 + i\theta^a T^a$ and $\overline{\Phi} = \overline{\Phi}^a T^a$ (as Φ lives in the Lie algebra), where $a \in \{1, \ldots, 8\}$, we have

$$\overline{\Phi}' = \overline{\Phi} + i\theta^a \overline{\Phi}^b [T^a, T^b] + \mathcal{O}(\theta^2)\,. \tag{1.8}$$

The unbroken generators are then given by the solutions to

$$\theta^a \overline{\Phi}^b [T^a, T^b] = 0\,. \tag{1.9}$$

Another way of saying this is that we are trying to find generators in SU(3) that commutes with each other. For generic $\overline{\Phi}$ values, the answer to this problem is simply the Cartan sub-algebra of $\mathfrak{su}(3)$. Using the Gell-Mann basis, the Cartan sub-algebra is a 2-dimensional algebra generated by

$$T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}. \tag{1.10}$$

The exponentiation of this algebra gives the little group, and it is simply $H = \mathrm{U}(1) \times \mathrm{U}(1)$, with T^3 and T^8 generating each of the U(1) factors. A generic matrix $\mathcal{H} \in H$ is given by

$$\mathcal{H} = \exp\left(i\alpha T^3 + i\beta T^8\right) = \begin{pmatrix} e^{\frac{i}{2}\left(\alpha + \frac{1}{\sqrt{3}}\beta\right)} & 0 & 0 \\ 0 & e^{\frac{i}{2}\left(-\alpha + \frac{1}{\sqrt{3}}\beta\right)} & 0 \\ 0 & 0 & e^{\frac{i}{\sqrt{3}}\beta} \end{pmatrix}. \tag{1.11}$$

There are 2 unbroken generators out of a total of 8, so Goldstone's theorem states that there are 6 massless modes and 2 massive Higgs modes.

1 d): The mass-squared matrix is given by

$$(M^2)^{ik}_{jl} = \frac{\partial^2 V}{\partial \Phi^j_i \partial \Phi^l_k}\Big|_{\Phi = \overline{\Phi}}\,. \tag{1.12}$$

Explicitly,

$$\frac{\partial^2 V}{\partial \Phi^j_i \partial \Phi^l_k} = 2\Phi^i_j \Phi^k_l + \left(\operatorname{tr}(\Phi^2) - k^2\right)\delta^i_l \delta^k_j\,, \tag{1.13}$$

where we note that $\operatorname{tr}(\Phi^2) = \Phi^n_m \Phi^m_n$. Evaluating this at $\overline{\Phi}$, we notice that the second term on the right drops out due to (1.5), so

$$(M^2)^{ik}_{jl} = 2\overline{\Phi}^i_j \overline{\Phi}^k_l\,. \tag{1.14}$$

For any diagonal $\overline{\Phi}$ with $\overline{\Phi}_i^j = \lambda_{\underline{i}}\delta_i^j$, where there underline indicates no summing over i, we have

$$(M^2)^{ik}_{jl} = 2\lambda_{\underline{i}}\lambda_{\underline{k}}\delta^i_j\delta^k_l\,. \tag{1.15}$$

Perturbing about this vacuum with

$$\Phi(x) = \overline{\Phi} + \phi(x)\,, \tag{1.16}$$

we find that

$$V(\Phi) = \frac{1}{2}(M^2)^{ik}_{jl}\phi^j_i\phi^l_k = \lambda_{\underline{i}}\lambda_{\underline{k}}\delta^i_j\delta^k_l\phi^j_i\phi^l_k = \left(\lambda_1\phi^1_1 + \lambda_2\phi^2_2 + \lambda_3\phi^3_3\right)^2, \tag{1.17}$$

where the first two terms, $V(\overline{\Phi})$ and $\partial V/\partial\Phi^j_i|_{\Phi=\overline{\Phi}}$ vanish due to the properties of the vacuum. For the vacuum specified in the question, $\lambda_2 = 0$ and $\lambda_1 = -\lambda_3 = \lambda$, so

$$V(\Phi) = \lambda^2\left(\phi^1_1 - \phi^3_3\right)^2. \tag{1.18}$$

This means that the field $H = \phi^1_1 - \phi^3_3$ has a mass-squared $m^2 = 2\lambda^2$, but the second Higgs mode has an accidentally vanishing mass. Recalling that

$$\phi = \phi^a T^a\,, \tag{1.19}$$

where $a \in \{1, \ldots, 8\}$, we find that

$$\phi^1_1 - \phi^3_3 = \phi^3 + \sqrt{3}\,\phi^8\,, \tag{1.20}$$

where the LHS concerns the matrix components of ϕ and the RHS concerns the Lie algebra basis components of ϕ. This suggests that the appropriate basis to expand ϕ is in terms of the broken generators $\{T^1, T^2, T^4, T^5, T^6, T^7\}$ and

$$T = \frac{1}{2}\left(T^3 + \sqrt{3}T^8\right), \quad \widetilde{T} = \frac{1}{2}\left(T^8 - \sqrt{3}T^3\right), \tag{1.21}$$

with

$$\phi = \phi_{U1}T + \phi_{U2}\widetilde{T} + \phi^a T^a\,, \quad a \in \{1, 2, 4, 5, 6, 7\}\,. \tag{1.22}$$

Our analysis then shows that for the 2 Higgs modes ϕ_{U1} and ϕ_{U2}, only ϕ_{U1} has a non-zero mass ($m^2_{\mathrm{U1}} = 2\lambda^2$), where ϕ_{U2} has an accidentally vanishing mass.

1 e): The group $\mathrm{U}(2) \simeq \mathrm{SU}(2) \times \mathrm{U}(1)$ has 4 generators, so we are looking for a vacuum $\overline{\Phi}$ that admits 4 unbroken generators. Working again in the Gell-Mann basis for the SU(3) generators, we note that the generators T^1, T^2 and T^3 with

$$T^1 = \frac{1}{2}\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}, \quad T^2 = \frac{1}{2}\begin{pmatrix}0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix} \tag{1.23}$$

and T^3 given previously form an $\mathfrak{su}(2)$ sub-algebra. Furthermore, from 1 c), we can see that T^8 commutes with T^1, T^2 and T^3. This means that if we choose our vacuum to be

$$\overline{\Phi} = \lambda T^8 = \frac{\lambda}{2\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} , \tag{1.24}$$

then the unbroken generators determined by the relation in (1.9) are $\{T^1, T^2, T^3, T^8\}$. The fact that T^8 commutes with $\{T^1, T^2, T^3\}$ means that the algebra generated by $\{T^1, T^2, T^3, T^8\}$ is the direct product of the algebras generated by $\{T^1, T^2, T^3\}$ and T^8 respectively, which is just $\mathfrak{su}(2) \times \mathfrak{u}(1)$. Exponentiating and recalling that the resulting group must be a subgroup of SU(3), we find that the little group is $\mathrm{SU}(2) \times \mathrm{U}(1)$. By Goldstone's theorem, there will be 4 massless modes and 4 massive Higgs modes. The Goldstone modes will correspond to the generators $\{T^4, T^5, T^6, T^7\}$.

1 f): Since our vacuum in (1.24) is diagonal, we can again use our previous results to compute the Higgs mass spectrum, with

$$V(\Phi) = \left(\lambda_1 \phi_1^1 + \lambda_2 \phi_2^2 + \lambda_3 \phi_3^3\right)^2 = \frac{\lambda^2}{12}\left(\phi_1^1 + \phi_2^2 - 2\phi_3^3\right)^2 . \tag{1.25}$$

Repeating the analysis in 1 d), we find that by expanding

$$\phi = \phi^a T^a , \tag{1.26}$$

where $a \in \{1, \ldots 8\}$, we obtain

$$\phi_1^1 + \phi_2^2 - 2\phi_3^3 = 2\sqrt{3}\phi^8 . \tag{1.27}$$

As such, (1.25) shows that only ϕ^8 is massive with $m^2 = 2\lambda^2$, whereas the other Higgs modes, ϕ^1, ϕ^2 and ϕ^3 corresponding to the residual SU(2) generated by $\{T^1, T^2, T^3\}$ are accidentally massless.

In the gauged version of this model, a generic vacuum preserves a $\mathrm{U}(1) \times \mathrm{U}(1)$ symmetry, meaning that there will be $8 - 2 = 6$ massive vectors. For the specific vacuum considered here that preserves a U(2) symmetry, there will be $8 - 4 = 4$ massive vectors. This means that 2 vectors will become massless when one approaches this orbit from a neighbouring, generic orbit.

2

2 a): The mass term for the Majorana spinors N and n can be written as

$$\mathcal{L} = -i\left(\overline{n}, \overline{N}\right) \begin{pmatrix} 0 & \frac{vp}{2\sqrt{2}} \\ \frac{vp}{2\sqrt{2}} & M \end{pmatrix} \begin{pmatrix} n \\ N \end{pmatrix} . \tag{2.1}$$

To be more succinct, let's define

$$\psi = \begin{pmatrix} n \\ N \end{pmatrix}, \quad m = \begin{pmatrix} 0 & \frac{vp}{2\sqrt{2}} \\ \frac{vp}{2\sqrt{2}} & M \end{pmatrix}, \tag{2.2}$$

such that

$$\mathcal{L} = -i\overline{\psi} m \psi \,. \tag{2.3}$$

From this, it is clear that m is the mass matrix. The masses of the mass eigenstates are then given by the eigenvalues of m, which are obtained by the characteristic equation

$$\det \begin{pmatrix} -\lambda & \frac{vp}{2\sqrt{2}} \\ \frac{vp}{2\sqrt{2}} & M-\lambda \end{pmatrix} = \lambda(\lambda - M) - \frac{v^2p^2}{8} = 0 \,. \tag{2.4}$$

This has the solutions

$$\lambda = m_\pm = \frac{M}{2}\Big(1 \pm \sqrt{1 + \frac{v^2p^2}{2M^2}}\Big) \,. \tag{2.5}$$

To obtain the mass eigenstates n_+ and n_-, note that m is a real, symmetric matrix, so there exists a matrix $U \in \mathrm{SO}(2)$ such that

$$m = UDU^T \,, \tag{2.6}$$

where $D = \mathrm{diag}(m_+, m_-)$. It is a standard result that U is the matrix of orthonormal eigenvectors of m. Explicitly,

$$U = \begin{pmatrix} -\frac{\alpha_+}{\sqrt{1+\alpha_+^2}} & -\frac{\alpha_-}{\sqrt{1+\alpha_-^2}} \\ \frac{1}{\sqrt{1+\alpha_+^2}} & \frac{1}{\sqrt{1+\alpha_-^2}} \end{pmatrix}, \quad \alpha_\pm = \frac{2\sqrt{2}}{vp} m_\pm \,. \tag{2.7}$$

Using the diagonalisation of m, the mass Lagrangian becomes

$$\mathcal{L} = -i\overline{\psi} UDU^T \psi = -i\overline{\chi} D \chi \,, \quad \chi = U^T \psi \,, \tag{2.8}$$

where in the last step we used the fact that U is real. Since D is the diagonal matrix of mass eigenvalues, we see that the mass eigenstates $n_\pm$ are identified as

$$\chi = U^T \psi = \begin{pmatrix} n_+ \\ n_- \end{pmatrix} \,. \tag{2.9}$$

Using (2.7), we have

$$n_\pm = \frac{1}{\sqrt{1+\alpha_\pm^2}} \big(N - \alpha_\pm n\big) \,. \tag{2.10}$$

2 b): When $M >> 1$,

$$m_- = \frac{M}{2}\Big(1 - \sqrt{1 + \frac{v^2p^2}{2M^2}}\Big) = -\frac{v^2p^2}{8M} + \mathcal{O}\big(M^{-3}\big)\,. \tag{2.11}$$

To show that the negative sign in m_- is not physical meaningful, consider

$$\nu = i\gamma_5 n_-\,. \tag{2.12}$$

We will show: 1. ν is Majorana; 2. the mass term for ν has the correct sign (i.e. $-i|m_-|\overline{\nu}\nu$); 3. the kinetic term for ν also has the correct sign (i.e. $-i\overline{\nu}\gamma^\mu\partial_\mu\nu$).

1. The charge conjugate is given by $\psi_C = C(\overline{\psi})^T$, where C is the charge conjugation matrix. Then,

$$\nu_C = C(\overline{\nu})^T = C(-in_-^\dagger\gamma_5\gamma^0)^T = C(i\overline{n}_-\gamma_5)^T = i\gamma_5(\overline{n}_-)^T = i\gamma_5 n_- = \nu\,, \tag{2.13}$$

where we used the following properties: γ_5 is Hermitian, $\{\gamma_5, \gamma^\mu\} = 0$, $C\gamma_5^T = \gamma_5 C$ (c.f. question 5), and n_- is Majorana.

2. Starting with the mass term for n_- which has the wrong sign, we have

$$\mathcal{L}_{\text{mass}} = +i|m_-|\overline{n}_-\overline{n}_- = i|m_-|\overline{\big(-i\gamma_5\nu\big)}\big(-i\gamma_5\nu\big) = i|m_-|\big(i\nu^\dagger\gamma_5\gamma^0\big)\big(-i\gamma_5\nu\big) = -i|m_-|\overline{\nu}\nu\,, \tag{2.14}$$

where we used the following properties: γ_5 is Hermitian, $\{\gamma_5, \gamma^\mu\} = 0$, and $\gamma_5^2 = \mathbb{1}$.

3. Using all of the properties we used in 1. and 2., the kinetic term for n_- is

$$\mathcal{L}_{\text{kin}} = -i\overline{n}_-\gamma^\mu\partial_\mu n_- = -i(-i\overline{\nu}\gamma_5)\gamma^\mu\partial_\mu(-i\gamma_5\nu) = i\overline{\gamma}_5\gamma^\mu\gamma_5\partial_\mu\nu = -i\overline{\nu}\gamma^\mu\partial_\mu\nu\,. \tag{2.15}$$

3

Let S be a symmetric, complex $n\times n$ matrix; $S^T = S$. Consider $M = SS^\dagger$. M is Hermitian, as $M^\dagger = (SS^\dagger)^\dagger = SS^\dagger = M$. From Problem Sheet 1 Question 5, we know that M has an orthonormal set of eigenvectors $\{e_i\}$, with positive (semi)-definite eigenvalues h_i, and can be diagonalised as

$$M = UDU^\dagger\,, \tag{3.1}$$

where $U = (e_1, \dots, e_n)$ is the unitary matrix of the eigenvectors, and $D = \text{diag}(h_1, \dots, h_n)$ is the diagonal matrix of eigenvalues. Then, let $T = U^\dagger S(U^\dagger)^T$. Since S is symmetric, T is also symmetric. Now,

$$TT^\dagger = U^\dagger S(U^\dagger)^T U^T S^\dagger U = U^\dagger S(UU^\dagger)^T S^\dagger U = U^\dagger MU = D\,. \tag{3.2}$$

Using the fact that T is symmetric and that D is real, we see that

$$DT = TT^\dagger T = TT^*T^T = TD^* = TD \implies [D, T] = 0\,. \tag{3.3}$$

In component form,

$$0 = [D, T]_{ij} = \sum_k \left(h_{\underline{i}} \delta_{ik} T_{kj} - T_{ik} h_{\underline{k}} \delta_{kj} \right) = T_{ij} \left(h_{\underline{i}} - h_{\underline{j}} \right), \tag{3.4}$$

where the underline in the indices means that they are not summed over. Therefore, T must be diagonal provided that the eigenvalues are distinct. Consequently, we can write $T = \text{diag}(t_1, \ldots, t_n)$, and so

$$D = TT^\dagger \implies h_i = |t_i|^2 . \tag{3.5}$$

By choosing t_i to all be real, we have $t_i = \sqrt{h_i}$. Finally, recalling that $T = U^\dagger S (U^\dagger)^T$, we have $S = UTU^T$.

4

The mass eigenstates $|i\rangle$ obey $H|i\rangle = E_i|i\rangle \simeq (E + m_i^2/2E)|i\rangle$, therefore, the time evolution of the flavour eigenstate at time $T \simeq L$ is

$$|\nu_\alpha(T)\rangle = e^{-iHT}|\nu_\alpha\rangle = \sum_i U^*_{\alpha i} e^{-iHT}|i\rangle = e^{-iEL} \sum_i U^*_{\alpha i} e^{-im_i^2 L/2E}|i\rangle . \tag{4.1}$$

Note that there is an overall phase e^{-iEL} out in the front. Using the orthonormality of the mass eigenstates, $\langle j|i\rangle = \delta_{ij}$, we find that

$$\langle \nu_\beta | \nu_\alpha(T)\rangle = e^{-iEL} \sum_{ij} U^*_{\alpha i} U_{\beta j} e^{-im_i^2 L/2E} \langle j|i\rangle = \delta_{ij} = e^{-iEL} \sum_i U^*_{\alpha i} U_{\beta i} e^{-im_i^2 L/2E} . \tag{4.2}$$

The transition probability is then

$$P_{\alpha\to\beta} = |\langle \nu_\beta|\nu_\alpha(T)\rangle|^2 = \Big| \sum_i U^*_{\alpha i} U_{\beta i} e^{-im_i^2 L/2E} \Big|^2 . \tag{4.3}$$

Let's now expand the absolute value and rewrite the transition probability as

$$P_{\alpha\to\beta} = \sum_{ij} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} e^{-i\theta_{ij}} , \tag{4.4}$$

where $\theta_{ij} = \Delta m^2_{ij} L/2E$. We can rewrite this as

$$P_{\alpha\to\beta} = \sum_{ij} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} + \sum_{ij} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} (e^{-i\theta_{ij}} - 1) . \tag{4.5}$$

Using the fact that the PMNS matrix is unitary, the first term is

$$\sum_{ij} U^*_{\alpha i} U_{\beta i} U_{\alpha j} U^*_{\beta j} = \delta_{\alpha\beta}\delta_{\alpha\beta} = \delta_{\alpha\beta} . \tag{4.6}$$

For the second term, we note that

$$
\begin{aligned}
&U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}(e^{-i\theta_{ij}}-1)\\
&=\big(\mathrm{Re}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})+i\,\mathrm{Im}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\big)(\cos\theta_{ij}-1-i\sin\theta_{ij})\\
&=\mathrm{Re}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})(\cos\theta_{ij}-1)+\mathrm{Im}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin\theta_{ij}+\text{imaginary}\\
&=-2\,\mathrm{Re}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin^2\frac{\theta_{ij}}{2}+\mathrm{Im}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin\theta_{ij}+\text{imaginary}\,.
\end{aligned}
\tag{4.7}
$$

The transition probability is a real number, so the imaginary contributions in the equation above must evaluate to zero. Collecting the results, we have

$$
P_{\alpha\to\beta}=\delta_{\alpha\beta}-2\sum_{i\neq j}\mathrm{Re}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin^2\frac{\theta_{ij}}{2}+\sum_{i\neq j}\mathrm{Im}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin\theta_{ij}\,,
\tag{4.8}
$$

where we note that the summation is over all $i\neq j$ as $\theta_{ii}=0$. We can change the summation to all $i>j$ by simply multiplying a factor of 2 to each summation to account for the double counting. This gives

$$
P_{\alpha\to\beta}=\delta_{\alpha\beta}-4\sum_{i>j}\mathrm{Re}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin^2\frac{\theta_{ij}}{2}+2\sum_{i>j}\mathrm{Im}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin\theta_{ij}\,.
\tag{4.9}
$$

The CP asymmetry is given by

$$
A^{\alpha\beta}_{CP}=4\sum_{i>j}\mathrm{Im}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin\theta_{ij}\,.
\tag{4.10}
$$

For 3 generations,

$$
\mathrm{Im}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})=-J\sum_{\gamma,k}\epsilon_{\alpha\beta\gamma}\epsilon_{ijk}\,.
\tag{4.11}
$$

Therefore,

$$
A^{\alpha\beta}_{CP}=-4J\sum_{i>j}\sum_{\gamma,k}\epsilon_{\alpha\beta\gamma}\epsilon_{ijk}\sin\theta_{ij}=-4J\sum_{\gamma}\epsilon_{\alpha\beta\gamma}\big(\sin\theta_{12}+\sin\theta_{23}+\sin\theta_{31}\big)\,.
\tag{4.12}
$$

Let $A=\theta_{12}$, $B=\theta_{23}$ and $C=\theta_{31}$. It is clear that $A+B+C=0$, so

$$
\begin{aligned}
\sin B+\sin C&=2\sin\frac{B}{2}\cos\frac{B}{2}+2\sin\frac{C}{2}\cos\frac{C}{2}\\
&=2\sin\frac{B}{2}\cos\frac{A+C}{2}+2\sin\frac{C}{2}\cos\frac{A+B}{2}\\
&=2\sin\frac{B}{2}\Big(\cos\frac{A}{2}\cos\frac{C}{2}-\sin\frac{A}{2}\sin\frac{C}{2}\Big)+2\sin\frac{C}{2}\Big(\cos\frac{A}{2}\cos\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}\Big)\\
&=2\cos\frac{A}{2}\sin\frac{B+C}{2}-4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\\
&=-\sin A-4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\,.
\end{aligned}
\tag{4.13}
$$

Therefore,

$$
A^{\alpha\beta}_{CP}=16J\sum_{\gamma}\epsilon_{\alpha\beta\gamma}\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=16J\sum_{\gamma}\epsilon_{\alpha\beta\gamma}\sin\frac{\theta_{21}}{2}\sin\frac{\theta_{32}}{2}\sin\frac{\theta_{31}}{2}\,,
\tag{4.14}
$$

where we used $\sin x \sin y = \sin(-x)\sin(-y)$ in the last line.

5

Setup:

In the following, we will be using the Weyl representation:

$$\gamma_\mu = \begin{pmatrix} 0 & i\sigma_\mu \\ i\widetilde{\sigma}_\mu & 0 \end{pmatrix}, \quad \sigma_\mu = \left(\mathbb{1}_2, \sigma_i\right), \quad \widetilde{\sigma}_\mu = \left(\mathbb{1}_2, -\sigma_i\right), \tag{5.1}$$

where the σ_i are the Hermitian Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.2}$$

The charge conjugation and γ_5 matrices are Hermitian matrices defined as

$$C = -\gamma^0\gamma^2, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{5.3}$$

The metric signature is $(-,+,+,+)$, so $\gamma^0 = -\gamma_0$ and $\gamma^i = \gamma_i$, $i \in \{1,2,3\}$. From the Clifford algebra,

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}\mathbb{1}_4, \tag{5.4}$$

we deduce that $(\gamma^0)^2 = -\mathbb{1}_4$, $(\gamma^i)^2 = \mathbb{1}_4$, and $\{\gamma_5, \gamma_\mu\} = 0$.

Spinors ψ and χ are Grassmann odd quantities, so every time we exchange a pair of spinors, we have to add in a minus sign. For example, $\psi^T\chi = -\chi^T\psi$.

Computations 1:

- Straightforward computation gives $\gamma_0^\dagger = -\gamma_0$ and $\gamma_i^\dagger = \gamma_i$. We can write this compactly as

$$\gamma_\mu^\dagger = \gamma^0\gamma_\mu\gamma^0.$$

- $C^\dagger = -(\gamma^0\gamma^2)^\dagger = -(\gamma^2)^\dagger(\gamma^0)^\dagger = \gamma^2\gamma^0 = -\gamma^0\gamma^2 = C$.

- $C^2 = \gamma^0\gamma^2\gamma^0\gamma^2 = -\gamma^2(\gamma^0)^2\gamma^2 = \mathbb{1}_4$.

- Since γ_2 is real, $\gamma_2^T = (\gamma_2^*)^\dagger = \gamma_2^\dagger = \gamma_2$. The other gamma matrices γ_α, $\alpha = 0,1,3$, are purely imaginary, so $\gamma_\alpha^T = (\gamma_\alpha^*)^\dagger = -\gamma_\alpha^\dagger = -\gamma^0\gamma_\alpha\gamma^0$. This means that $\gamma_2^T = \gamma_2$, $\gamma_0^T = \gamma_0$, $\gamma_1^T = -\gamma_1$, and $\gamma_3^T = -\gamma_3$. We can write this compactly as

$$\gamma_\mu^T = \gamma^2\gamma^0\gamma_\mu\gamma^0\gamma^2 = -C\gamma_\mu C \implies C\gamma_\mu^T C = -\gamma_\mu.$$

- $C^T = -(\gamma^0\gamma^2)^T = -(\gamma^2)^T(\gamma^0)^T = -\gamma^2\gamma^0 = \gamma^0\gamma^2 = -C$.

- $C\gamma_5^T = iC(\gamma^3)^T(\gamma^2)^T(\gamma^1)^T(\gamma^0)^T = iC(C\gamma^3C)(C\gamma^2C)(C\gamma^1C)(C\gamma^0C) = i\gamma^0\gamma^1\gamma^2\gamma^3C = \gamma_5C$.

- $\psi_C = C(\overline{\psi})^T = C(\psi^\dagger\gamma^0)^T = C(\gamma^0)^T\psi^* = -C(C\gamma^0C)\psi^* = -\gamma^0C\psi^*$.

- $\overline{\gamma^\mu\psi} = (\gamma^\mu\psi)^\dagger\gamma^0 = \psi^\dagger(\gamma^\mu)^\dagger\gamma^0 = \psi^\dagger(\gamma^0\gamma^\mu\gamma^0)\gamma^0 = -\overline{\psi}\gamma^\mu$.

- $\overline{\gamma_5\psi} = \psi^\dagger\gamma_5^\dagger\gamma^0 = \psi^\dagger\gamma_5\gamma^0 = -\overline{\psi}\gamma_5$.

Computations 2:

- $\overline{\chi}_C = -\overline{\gamma^0C\psi^*} = -(\gamma^0C\psi^*)^\dagger\gamma^0 = -\psi^TC^\dagger(\gamma^0)^\dagger\gamma^0 = -\psi^TC$.

- $\overline{\chi}_C\psi_C = \chi^TC\gamma^0C\psi^* = -\chi^T(\gamma^0)^T\psi^* = +(\psi^*)^T\gamma^0\chi = \overline{\psi}\chi$.

- $\overline{\chi}_C\gamma_5\psi_C = \chi^TC\gamma_5\gamma^0C\psi^* = \chi^T(C\gamma_5C)(C\gamma^0C)\psi^* = -\chi^T\gamma_5^T(\gamma^0)^T\psi^* = +\psi^\dagger\gamma^0\gamma_5\chi = \overline{\psi}\gamma_5\chi$.

- $\overline{\chi}_C\gamma^\mu\psi_C = \chi^TC\gamma^\mu\gamma^0C\psi^* = \chi^T(C\gamma^\mu C)(C\gamma^0C)\psi^* = \chi^T(\gamma^\mu)^T(\gamma^0)^T\psi^* = -\psi^\dagger\gamma^0\gamma^\mu\chi = -\overline{\psi}\gamma^\mu\chi$.

- $\overline{\chi}_C\gamma_5\gamma^\mu\psi_C = \chi^T(C\gamma_5C)(C\gamma^\mu C)(C\gamma^0C)\psi^* = \chi^T\gamma_5^T(\gamma^\mu)^T(\gamma^0)^T\psi^* = -\overline{\psi}\gamma^\mu\gamma_5\chi = +\overline{\psi}\gamma_5\gamma^\mu\chi$.

- $\overline{\chi}_C\gamma^\mu\gamma^\nu\psi_C = \chi^T(C\gamma^\mu C)(C\gamma^\nu C)(C\gamma^0C)\psi^* = -\chi^T(\gamma^\mu)^T(\gamma^\nu)^T(\gamma^0)^T\psi^* = \overline{\psi}\gamma^\nu\gamma^\mu\chi$. Because the index positions have switched, we have

$$\overline{\chi}_C[\gamma^\mu,\gamma^\nu]\psi_C = \overline{\psi}[\gamma^\nu,\gamma^\mu]\chi = -\overline{\psi}[\gamma^\mu,\gamma^\nu]\chi\,.$$

Computation 3:

The term we would like to CP conjugate is

$$T = h_{mn}\overline{Q}_m(\mathbb{1}_4-\gamma_5)u_n\widetilde{\phi} + h^*_{mn}\overline{u}_m(\mathbb{1}_4+\gamma_5\gamma_5)Q_n\widetilde{\phi}^*\,. \tag{5.5}$$

Under C, we have

$$T \underbrace{\mapsto}_{C} h_{mn}\overline{u}_m(\mathbb{1}_4-\gamma_5)Q_n\widetilde{\phi}^* + h^*_{mn}\overline{Q}_m(\mathbb{1}_4+\gamma_5\gamma_5)u_n\widetilde{\phi}\,. \tag{5.6}$$

Under P, the projection operators $P_\pm = (\mathbb{1}\pm\gamma_5)/2 \to P_\mp$. As such,

$$T \underbrace{\mapsto}_{CP} h_{mn}\overline{u}_m(\mathbb{1}_4+\gamma_5)Q_n\widetilde{\phi}^* + h^*_{mn}\overline{Q}_m(\mathbb{1}_4-\gamma_5\gamma_5)u_n\widetilde{\phi}\,. \tag{5.7}$$

Comparing to (5.5), we find that h_{mn} must be real for CP invariance.

6

In terms of the Lie algebra valued gauge potential A_μ, charge conjugation acts as

$$A_\mu \underbrace{\mapsto}_{C} -A_\mu^T \,. \tag{6.1}$$

The field strength is defined as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \,. \tag{6.2}$$

Its transpose is

$$F_{\mu\nu}^T = \partial_\mu A_\nu^T - \partial_\nu A_\mu^T - ig[A_\nu^T, A_\mu^T] \,, \tag{6.3}$$

where we note that the order of indices is switched in the commutator. Therefore,

$$F_{\mu\nu} \underbrace{\mapsto}_{C} -\partial_\mu A_\nu^T + \partial_\nu A_\mu^T - ig[A_\mu^T, A_\nu^T] = -F_{\mu\nu}^T \,, \tag{6.4}$$

which is a covariant transformation.