Solutions - PS3

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1 a): Fourier expanding $\psi(x)$ with

$$\psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{ip_\mu x^\mu} \psi(p)\,, \tag{1.1}$$

the Dirac equation $(\gamma^\mu \partial_\mu + m)\psi(x) = 0$ reads

$$\big(i\gamma^\mu p_\mu + m\big)\psi(p) = 0\,. \tag{1.2}$$

This implies that

$$\begin{aligned} 0 &= -\big(i\gamma^\mu p_\mu + m\big)\big(i\gamma^\nu p_\nu + m\big)\psi(p) \\ &= \big(\gamma^\mu\gamma^\nu p_\mu p_\nu - 2mi\gamma^\mu p_\mu - m^2\big)\psi(p)\,. \end{aligned} \tag{1.3}$$

Since $p_\mu p_\nu$ is symmetric in μ and ν, we have

$$\gamma^\mu\gamma^\nu p_\mu p_\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} p_\mu p_\nu = \eta^{\mu\nu} p_\mu p_\nu = p^\mu p_\mu\,. \tag{1.4}$$

Also, using (1.2), we have

$$-2mi\gamma^\mu p_\mu \psi(p) = 2m^2\psi(p)\,. \tag{1.5}$$

Thus, (1.3) reads

$$\big(p^\mu p_\mu + m^2\big)\psi(p) = 0\,. \tag{1.6}$$

This means that on-shell,

$$p^\mu p_\mu + m^2 = 0 \implies p^0 = \zeta\sqrt{\boldsymbol{p}^2 + m^2}\,, \tag{1.7}$$

where $\zeta = \pm 1$. Recalling that our metric convention is mostly plus, $p_0 = -p^0$, so that (1.2) is

$$\big(- i\zeta \sqrt{\boldsymbol{p}^2 + m^2}\, \gamma^0 + i\gamma^i p_i + m \big) \psi(p) = 0\,. \tag{1.8}$$

Using the fact that $(i\gamma^0)^2 = +\mathbb{1}_4$, we have

$$\big(- \zeta \sqrt{\boldsymbol{p}^2 + m^2}\, \mathbb{1}_4 - \gamma^0\gamma^i p_i + im\gamma^0 \big) \psi(p) = 0\,, \tag{1.9}$$

or equivalently,

$$\psi(p) = -\frac{\zeta}{\sqrt{\boldsymbol{p}^2 + m^2}} \big(\gamma^0\gamma^i p_i - im\gamma^0 \big) \psi(p)\,. \tag{1.10}$$

The helicity operator is

$$\Lambda = \frac{S_i p_i}{|\boldsymbol{p}|} = \frac{1}{2|\boldsymbol{p}|} \epsilon_{ijk} p_i M_{jk} = \frac{1}{|\boldsymbol{p}|} \big(p_1 M_{23} + p_2 M_{31} + p_3 M_{12} \big)\,. \tag{1.11}$$

Recalling that $M_{\mu\nu} = -i\gamma_{\mu\nu}/2$, we have

$$\Lambda = -\frac{i}{2|\boldsymbol{p}|} \big(p_1 \gamma^2\gamma^3 + p_2 \gamma^3\gamma^1 + p_3 \gamma^1\gamma^2 \big)\,. \tag{1.12}$$

Using $(\gamma^0)^2 = -\mathbb{1}_4$ and $(\gamma^i)^2 = +\mathbb{1}_4$, we can rewrite this as

$$\begin{aligned}
\Lambda &= \frac{i}{2|\boldsymbol{p}|} \gamma^0\gamma^0 \big((p_1\gamma^1)\gamma^1\gamma^2\gamma^3 + (p_2\gamma^2)\gamma^2\gamma^3\gamma^1 + (p_3\gamma^3)\gamma^3\gamma^1\gamma^2 \big) \\
&= -\frac{i}{2|\boldsymbol{p}|} \gamma^0 \big((p_1\gamma^1)\gamma^0\gamma^1\gamma^2\gamma^3 + (p_2\gamma^2)\gamma^0\gamma^2\gamma^3\gamma^1 + (p_3\gamma^3)\gamma^0\gamma^3\gamma^1\gamma^2 \big) \\
&= -\frac{1}{2|\boldsymbol{p}|} \gamma^0 (p_i\gamma^i)\gamma^5 \\
&= -\frac{1}{2} \gamma^5\gamma^0\gamma^i \frac{p_i}{|\boldsymbol{p}|}\,,
\end{aligned} \tag{1.13}$$

where we used the definition $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, and the relations $\{\gamma^5, \gamma^\mu\} = 0$.

The left and right-handed projections of $\psi(p)$ are defined as

$$\psi_{L/R}(p) = \frac{1}{2} \big(1 \pm \gamma^5 \big) \psi(p)\,. \tag{1.14}$$

Applying this to (1.10), we find that

$$\psi_{L/R} = -\frac{\zeta}{\sqrt{\boldsymbol{p}^2 + m^2}} \big(\gamma^0\gamma^i p_i \psi_{L/R} - im\gamma^0 \psi_{R/L} \big)\,, \tag{1.15}$$

as $(1 \pm \gamma^5)\gamma^0\gamma^i = \gamma^0\gamma^i(1 \pm \gamma^5)$ and $(1 \pm \gamma^5)\gamma^0 = \gamma^0(1 \mp \gamma^5)$. Then, using the fact that $\gamma^5\psi_{L/R} = \pm\psi_{L/R}$, we have

$$\Lambda\psi_{L/R} = \pm\frac{\zeta}{2|\boldsymbol{p}|\sqrt{\boldsymbol{p}^2+m^2}}\left(\gamma^0\gamma^i\gamma^0\gamma^j p_i p_j \psi_{L/R} - im\gamma^0\gamma^i\gamma^0 p_i\psi_{R/L}\right)$$
$$= \pm\frac{\zeta}{2|\boldsymbol{p}|\sqrt{\boldsymbol{p}^2+m^2}}\left(\boldsymbol{p}^2\psi_{L/R} - im\gamma^i p_i\psi_{R/L}\right), \tag{1.16}$$

where $\gamma^0\gamma^i\gamma^0\gamma^j p_i p_j = \gamma^i\gamma^j p_i p_j = \{\gamma^i,\gamma^j\}p_i p_j/2 = \boldsymbol{p}^2$. For the second term, note that (1.15) implies

$$\gamma^i p_i \psi_{L/R} = \zeta\sqrt{\boldsymbol{p}^2+m^2}\,\gamma^0\psi_{L/R} + im\psi_{R/L}\,. \tag{1.17}$$

This means that

$$\Lambda\psi_{L/R} = \pm\Big(\frac{\zeta\sqrt{\boldsymbol{p}^2+m^2}\,\psi_{L/R} - im\gamma^0\psi_{R/L}}{2|\boldsymbol{p}|}\Big)\,. \tag{1.18}$$

In the massless or ultra-relativistic limit, we set $m = 0$, and (1.18) becomes

$$\Lambda\psi_{L/R}\Big|_{m=0} = \pm\frac{\zeta}{2}\psi_{L/R}\,. \tag{1.19}$$

1 b): The charged current interaction in the minimal Standard Model couples W^+_μ to a right-handed charged lepton and a corresponding (massless) left-handed neutrino. If the charged lepton is massless, then (1.19) implies that the lepton and neutrino must have opposite helicity.

In the π^+ rest frame of the decay $\pi^+ \to l^+ + \nu_l$, conservation of 3-momentum shows that $\boldsymbol{P} = \boldsymbol{p}_{l^+} = -\boldsymbol{p}_{\nu_l}$. Since π^+ is spin 0, conservation of spin then says that $\boldsymbol{S} = \boldsymbol{s}_{l^+} = -\boldsymbol{s}_{\nu_l}$. Then,

$$\Lambda_{l^+} = \frac{\boldsymbol{s}_{l^+}\cdot\boldsymbol{p}_{l^+}}{|\boldsymbol{p}_{l^+}|} = \frac{\boldsymbol{S}\cdot\boldsymbol{P}}{|\boldsymbol{P}|} = \frac{\boldsymbol{s}_{\nu_l}\cdot\boldsymbol{p}_{\nu_l}}{|\boldsymbol{p}_{\nu_l}|} = \Lambda_{\nu_l}\,. \tag{1.20}$$

This is a contradiction, as the charged current interaction prohibits the charged lepton and neutrino to have the same helicity. As such, the decay $\pi^+ \to l^+ + \nu_l$ is prohibited unless l^+ is massive.

2: The decay amplitude for $\pi^+ \to l^+ + \nu_l$ is given to be

$$\mathcal{T} = -iG_F c_1 f_\pi \pi^+ \overline{u}_{\nu_l}\not{k}(\mathbb{1}_4+\gamma^5)v_l\,. \tag{1.21}$$

where $\not{k} = \gamma^\mu k_\mu$. 4-momentum conservation gives $k^\mu = p_3^\mu + p_4^\mu$, where $p_{3,4}^\mu$ are the 4-momenta of l^+ and ν_l respectively. Then

$$u_{\nu_l}\not{k}(\mathbb{1}_4+\gamma^5)v_l = u_{\nu_l}\not{p}_3(\mathbb{1}_4+\gamma^5)v_l + u_{\nu_l}\not{p}_4(\mathbb{1}_4+\gamma^5)v_l \tag{1.22}$$

Since $\gamma^\mu(\mathbb{1}_4+\gamma^5) = (\mathbb{1}_4-\gamma^5)\gamma^\mu$, we have $\not{p}(\mathbb{1}_4+\gamma^5) = (\mathbb{1}_4-\gamma^5)\not{p}$. Then,

$$u_{\nu_l}\not{p}_3(\mathbb{1}_4+\gamma^5)v_l + u_{\nu_l}\not{p}_4(\mathbb{1}_4+\gamma^5)v_l = u_{\nu_l}(\mathbb{1}_4-\gamma^5)\not{p}_3 v_l + u_{\nu_l}\not{p}_4(\mathbb{1}_4+\gamma^5)v_l = -im_l u_{\nu_l}(\mathbb{1}_4-\gamma^5)v_l\,, \tag{1.23}$$

where we used the on-shell conditions $\not{p}_4 u_{\nu_l} = 0$ and $\not{p}_3 v_l = -im_l v_l$. Therefore,

$$\mathcal{T} = -G_F c_1 f_\pi \pi^+ m_l u_{\nu_l} (\mathbb{1}_4 - \gamma^5) v_l \,. \tag{1.24}$$

We are also given the absolute-square of this quantity:

$$|\mathcal{T}|^2 = -(G_F c_1 f_\pi m_l)^2 \,\mathrm{tr}\left((\mathbb{1}_4 + \gamma^5) v_l \overline{v}_l (\mathbb{1}_4 - \gamma^5) u_{\nu_l} \overline{u}_{\nu_l} \right) . \tag{1.25}$$

Using the sum of spins rules given, the sum of both the l^+ and ν_l spins of $|\mathcal{T}|^2$ is

$$\sum_{\text{spins}} |\mathcal{T}|^2 = -(G_F c_1 f_\pi m_l)^2 \,\mathrm{tr}\left((\mathbb{1}_4 + \gamma^5)(\not{p}_3 + im_l)(\mathbb{1}_4 - \gamma^5)\not{p}_4 \right) . \tag{1.26}$$

Here,

$$\begin{aligned}
\mathrm{tr}\left((\mathbb{1}_4 + \gamma^5)(\not{p}_3 + im_l)(\mathbb{1}_4 - \gamma^5)\not{p}_4 \right) &= \mathrm{tr}\left((\mathbb{1}_4 + \gamma^5)\not{p}_3(\mathbb{1}_4 - \gamma^5)\not{p}_4 \right) + im_l \,\mathrm{tr}\left((\mathbb{1}_4 + \gamma^5)(\mathbb{1}_4 - \gamma^5)\not{p}_4 \right) \\
&= \mathrm{tr}\left((\mathbb{1}_4 + \gamma^5)^2 \not{p}_3 \not{p}_4 \right) \\
&= 2\,\mathrm{tr}\left((\mathbb{1}_4 + \gamma^5) \not{p}_3 \not{p}_4 \right) \\
&= 2\,\mathrm{tr}\left(\not{p}_3 \not{p}_4 \right) + 2\,\mathrm{tr}\left(\gamma^5 \not{p}_3 \not{p}_4 \right) \\
&= 2 p_3^\mu p_4^\nu \,\mathrm{tr}\left(\gamma_\mu \gamma_\nu \right) + 2 p_3^\mu p_4^\nu \,\mathrm{tr}\left(\gamma^5 \gamma_\mu \gamma_\nu \right) \\
&= p_3^\mu p_4^\nu \,\mathrm{tr}\left(\{\gamma_\mu, \gamma_\nu\} \right) + p_3^\mu p_4^\nu \,\mathrm{tr}\left(\gamma^5 \{\gamma_\mu, \gamma_\nu\} \right) \\
&= 8 p_3 \cdot p_4 \,,
\end{aligned} \tag{1.27}$$

where $p_3 \cdot p_4 = p_3^\mu p_{\mu 4}$, and we used the fact that γ^5 is traceless. Thus,

$$\sum_{\text{spins}} |\mathcal{T}|^2 = -8 (G_F c_1 f_\pi m_l)^2 p_3 \cdot p_4 \,. \tag{1.28}$$

Working in the rest frame of π^+, 3-momentum and energy conservation shows that

$$\begin{aligned}
&p_3^\mu = (E_3, \boldsymbol{p}_3)\,, \quad E_3 = \frac{m_\pi^2 + m_l^2}{2m_\pi}\,, \quad |\boldsymbol{p}_3| = \frac{m_\pi^2 - m_l^2}{2m_\pi} \\
&p_4^\mu = (|\boldsymbol{p}_3|, -\boldsymbol{p}_3)\,.
\end{aligned} \tag{1.29}$$

Thus,

$$p_3 \cdot p_4 = -E_3 |\boldsymbol{p}_3| - |\boldsymbol{p}_3|^2 = -\frac{m_\pi^2 - m_l^2}{2}\,, \tag{1.30}$$

which gives

$$\sum_{\text{spins}} |\mathcal{T}|^2 = 4 (G_F c_1 f_\pi m_l)^2 (m_\pi^2 - m_l^2)\,. \tag{1.31}$$

The ratio of pion decay rates into electrons and muons is consequently

$$R_\pi = \frac{\Gamma_e}{\Gamma_\mu} = \frac{|\boldsymbol{p}_{3;e}| \sum_{\text{spins}} |\mathcal{T}_e|^2}{|\boldsymbol{p}_{3;\mu}| \sum_{\text{spins}} |\mathcal{T}_\mu|^2} = \left(\frac{m_e}{m_\mu}\right)^2 \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2 . \tag{1.32}$$