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1 a): Fourier expanding $\psi(x)$ with

$$\psi(x) = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot x} \psi(p), \quad (1.1)$$

the Dirac equation $(\gamma^\mu \partial_\mu + m)\psi(x) = 0$ reads

$$(i\gamma^\mu p_\mu + m)\psi(p) = 0. \quad (1.2)$$

This implies that

$$0 = -(i\gamma^\mu p_\mu + m)(i\gamma^\nu p_\nu + m)\psi(p)$$

$$= (\gamma^\mu \gamma^\nu p_\mu p_\nu - 2mi\gamma^\mu p_\mu - m^2)\psi(p). \quad (1.3)$$

Since $p_\mu p_\nu$ is symmetric in $\mu$ and $\nu$, we have

$$\gamma^\mu \gamma^\nu p_\mu p_\nu = \frac{1}{2} \{\gamma^\mu, \gamma^\nu\} p_\mu p_\nu = \eta^{\mu\nu} p_\mu p_\nu = p^\mu p_\mu. \quad (1.4)$$

Also, using (1.2), we have

$$-2mi\gamma^\mu p_\mu \psi(p) = 2m^2 \psi(p). \quad (1.5)$$

Thus, (1.3) reads

$$(p^\mu p_\mu + m^2)\psi(p) = 0. \quad (1.6)$$

This means that on-shell,

$$p^\mu p_\mu + m^2 = 0 \implies p^0 = \zeta \sqrt{p^2 + m^2}, \quad (1.7)$$
where \( \zeta = \pm 1 \). Recalling that our metric convention is mostly plus, \( p_0 = -p^0 \), so that (1.2) is

\[
( -i \zeta \sqrt{p^2 + m^2} \gamma^0 + i \gamma^i p_i + m) \psi(p) = 0 .
\]

Using the fact that \( (i\gamma^0)^2 = +\mathbb{1}_4 \), we have

\[
( - \zeta \sqrt{p^2 + m^2} \mathbb{1}_4 - \gamma^0 \gamma^i p_i + i m \gamma^0) \psi(p) = 0 ,
\]

or equivalently,

\[
\psi(p) = -\frac{\zeta}{\sqrt{p^2 + m^2}} (\gamma^0 \gamma^i p_i - i m \gamma^0) \psi(p) .
\]

The helicity operator is

\[
\Lambda = \frac{S_i p_i}{|p|} = \frac{1}{2|p|} \epsilon_{ijk} p_k M_{jk} = \frac{1}{|p|} (p_1 M_{23} + p_2 M_{31} + p_3 M_{12}) .
\]

Recalling that \( M_{\mu \nu} = -i \gamma_{\mu \nu}/2 \), we have

\[
\Lambda = -\frac{i}{2|p|} (p_1 \gamma^2 \gamma^3 + p_2 \gamma^3 \gamma^1 + p_3 \gamma^1 \gamma^2) .
\]

Using \( (\gamma^0)^2 = -\mathbb{1}_4 \) and \( (\gamma^i)^2 = +\mathbb{1}_4 \), we can rewrite this as

\[
\Lambda = \frac{i}{2|p|} \gamma^0 \gamma^0 (p_1 \gamma^1) \gamma^1 \gamma^2 \gamma^3 + (p_2 \gamma^2) \gamma^2 \gamma^3 \gamma^1 + (p_3 \gamma^3) \gamma^3 \gamma^1 \gamma^2
\]

\[
= -\frac{i}{2|p|} \gamma^0 (p \gamma^i) \gamma^5
\]

\[
= -\frac{i}{2} \gamma^5 \gamma^0 \gamma^i \frac{p_i}{|p|} ,
\]

where we used the definition \( \gamma^5 = i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \), and the relations \( \{ \gamma^5, \gamma^\mu \} = 0 \).

The left and right-handed projections of \( \psi(p) \) are defined as

\[
\psi_{L/R}(p) = \frac{1}{2} (1 \pm \gamma^5) \psi(p) .
\]

Applying this to (1.10), we find that

\[
\psi_{L/R} = -\frac{\zeta}{\sqrt{p^2 + m^2}} (\gamma^0 \gamma^i p_i \psi_{L/R} - i m \gamma^0 \psi_{R/L}) ,
\]

as \( (1 \pm \gamma^5) \gamma^0 \gamma^i = \gamma^0 \gamma^i (1 \pm \gamma^5) \) and \( (1 \pm \gamma^5) \gamma^0 = \gamma^0 (1 \pm \gamma^5) \). Then, using the fact that \( \gamma^5 \psi_{L/R} = \pm \psi_{L/R} \),
we have

\[
\Lambda_{l/R} = \pm \frac{\zeta}{2|p|\sqrt{p^2 + m^2}} (\gamma^0 \gamma^j \gamma^j p_i p_j \psi_{l/R} - im \gamma^0 \gamma^j p_i \psi_{R/L})
\]

\[
= \pm \frac{\zeta}{2|p|\sqrt{p^2 + m^2}} (p^2 \psi_{l/R} - im \gamma^0 p_i \psi_{R/L}), \quad (1.16)
\]

where \(\gamma^0 \gamma^j \gamma^j p_i p_j = \gamma^i \gamma^j p_i p_j = \{\gamma^i, \gamma^j\} p_i p_j / 2 = p^2\). For the second term, note that (1.15) implies

\[
\gamma^j p_i \psi_{l/R} = \zeta \sqrt{p^2 + m^2} \gamma^0 \psi_{l/R} + im \psi_{R/L}.
\]

This means that

\[
\Lambda_{l/R} = \pm \left(\frac{\zeta \sqrt{p^2 + m^2} \psi_{l/R} - im \gamma^0 \psi_{R/L}}{2|p|}\right).
\]

In the massless or ultra-relativistic limit, we set \(m = 0\), and (1.18) becomes

\[
\Lambda_{l/R} \bigg|_{m=0} = \pm \frac{\zeta}{2} \psi_{l/R}.
\]

1 b): The charged current interaction in the minimal Standard Model couples \(W^+_\mu\) to a right-handed charged lepton and a corresponding (massless) left-handed neutrino. If the charged lepton is massless, then (1.19) implies that the lepton and neutrino must have opposite helicity.

In the \(\pi^+\) rest frame of the decay \(\pi^+ \rightarrow l^+ + \nu_l\), conservation of 3-momentum shows that \(P = p_{l^+} = -p_{\nu_l}\). Since \(\pi^+\) is spin 0, conservation of spin then says that \(S = s_{l^+} = -s_{\nu_l}\). Then,

\[
\Lambda_{l^+} = \frac{s_{l^+} \cdot p_{l^+}}{|p_{l^+}|} = \frac{S \cdot P}{|P|} = \frac{s_{\nu_l} \cdot p_{\nu_l}}{|p_{\nu_l}|} = \Lambda_{\nu_l}.
\]

This is a contradiction, as the charged current interaction prohibits the charged lepton and neutrino to have the same helicity. As such, the decay \(\pi^+ \rightarrow l^+ + \nu_l\) is prohibited unless \(l^+\) is massive.

2: The decay amplitude for \(\pi^+ \rightarrow l^+ + \nu_l\) is given to be

\[
\mathcal{T} = -i G_F c_1 f_{\pi} \pi^+ \bar{u}_{\nu_l} \not{k} (\not{1} + \gamma^5) \nu_l .
\]

where \(\not{k} = \gamma^\mu k_\mu\). 4-momentum conservation gives \(k^\mu = p_{l^+}^\mu + p_{\nu_l}^\mu\), where \(p_{l^+}^\mu\) and \(p_{\nu_l}^\mu\) are the 4-momenta of \(l^+\) and \(\nu_l\) respectively. Then

\[
u_l \not{k} (\not{1} + \gamma^5) \nu_l = u_{\nu_l} \not{p}_3 (\not{1} + \gamma^5) \nu_l + u_{\nu_l} \not{p}_4 (\not{1} + \gamma^5) \nu_l \quad (1.22)
\]

Since \(\gamma^\mu (\not{1} + \gamma^5) = (\not{1} - \gamma^5) \gamma^\mu\), we have \(\not{p} (\not{1} + \gamma^5) = (\not{1} - \gamma^5) \not{p}\). Then,

\[
u_l \not{p}_3 (\not{1} + \gamma^5) \nu_l + u_{\nu_l} \not{p}_4 (\not{1} + \gamma^5) \nu_l = u_{\nu_l} (\not{1} - \gamma^5) \not{p}_3 \nu_l + u_{\nu_l} \not{p}_4 (\not{1} + \gamma^5) \nu_l = -im_{\nu_l} u_{\nu_l} (\not{1} - \gamma^5) \nu_l , \quad (1.23)
\]
where we used the on-shell conditions \( p_4^\mu u_{\nu_4} = 0 \) and \( p_3^\mu v_l = -i m_l v_l \). Therefore,

\[
T = -G_F c_1 f_{\pi^+} m_{l} u_{\nu_4} (\mathbb{1}_4 - \gamma^5) v_l.
\] (1.24)

We are also given the absolute-square of this quantity:

\[
|T|^2 = - (G_F c_1 f_{\pi^+} m_{l})^2 \text{tr} \left( (\mathbb{1}_4 + \gamma^5) v_l \bar{\nu}_l (\mathbb{1}_4 - \gamma^5) u_{\nu_3} \bar{u}_{\nu_3} \right).
\] (1.25)

Using the sum of spins rules given, the sum of both the \( l^+ \) and \( \nu_l \) spins of \(|T|^2\) is

\[
\sum_{\text{spins}} |T|^2 = - (G_F c_1 f_{\pi^+} m_{l})^2 \text{tr} \left( (\mathbb{1}_4 + \gamma^5) (p_3 + im_l)(\mathbb{1}_4 - \gamma^5) p_4 \right).
\] (1.26)

Here,

\[
\text{tr} \left( (\mathbb{1}_4 + \gamma^5) (p_3 + im_l)(\mathbb{1}_4 - \gamma^5) p_4 \right) = \text{tr} \left( (\mathbb{1}_4 + \gamma^5) p_3 (\mathbb{1}_4 - \gamma^5) p_4 \right) + im_l \text{tr} \left( (\mathbb{1}_4 + \gamma^5) (\mathbb{1}_4 - \gamma^5) p_4 \right)
\]
\[
= \text{tr} \left( (\mathbb{1}_4 + \gamma^5)^2 p_3 p_4 \right)
\]
\[
= 2 \text{tr} \left( (\mathbb{1}_4 + \gamma^5) p_3 p_4 \right)
\]
\[
= 2 \text{tr} (p_3 p_4) + 2 \text{tr} (\gamma^5 p_3 p_4)
\]
\[
= 2 p_3^\mu p_4^\nu \text{tr} (\gamma^\mu \gamma^\nu) + 2 p_3^\mu p_4^\nu' \text{tr} (\gamma^5 \gamma^\mu \gamma^\nu)
\]
\[
= p_3^\mu p_4^\nu \text{tr} (\{\gamma^\mu, \gamma^\nu\}) + p_3^\mu p_4^\nu' \text{tr} (\gamma^5 \{\gamma^\mu, \gamma^\nu\})
\]
\[
= 8 p_3 \cdot p_4,
\] (1.27)

where \( p_3 \cdot p_4 = p_3^\mu p_4^\mu \), and we used the fact that \( \gamma^5 \) is traceless. Thus,

\[
\sum_{\text{spins}} |T|^2 = -8(G_F c_1 f_{\pi^+} m_{l})^2 p_3 \cdot p_4.
\] (1.28)

Working in the rest frame of \( \pi^+ \), 3-momentum and energy conservation shows that

\[
p_3^\mu = (E_3, p_3), \quad E_3 = \frac{m_\pi^2 + m_l^2}{2m_\pi}, \quad |p_3| = \frac{m_\pi^2 - m_l^2}{2m_\pi}
\]
\[
p_4^\mu = (|p_3|, -p_3).
\] (1.29)

Thus,

\[
p_3 \cdot p_4 = -E_3|p_3| - |p_3|^2 = -\frac{m_\pi^2 - m_l^2}{2},
\] (1.30)

which gives

\[
\sum_{\text{spins}} |T|^2 = 4(G_F c_1 f_{\pi^+} m_{l})^2 (m_\pi^2 - m_l^2).
\] (1.31)
The ratio of pion decay rates into electrons and muons is consequently

$$R_\pi = \frac{\Gamma_e}{\Gamma_\mu} = \frac{|p_{3;e}| \sum_{\text{spins}} |T_e|^2}{|p_{3;\mu}| \sum_{\text{spins}} |T_\mu|^2} = \left( \frac{m_e}{m_\mu} \right)^2 \left( \frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2} \right)^2.$$  (1.32)