Solutions - PS4

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1

The decay of π^0 into two photons is given by

$$\mathcal{L}_{\pi^0 AA} = \frac{e^2}{32\pi^2 F_\pi} \pi^0 F_{\mu\nu}(A) F_{\rho\sigma}(A) \epsilon^{\mu\nu\rho\sigma} \,.$$

Under the nonlinear symmetry transformation $\pi^0 \mapsto \pi^0 + \omega F_\pi$, we have

$$\delta\mathcal{L}_{\pi^0 AA} = \frac{e^2 \omega}{32\pi^2} F_{\mu\nu}(A) F_{\rho\sigma}(A) \epsilon^{\mu\nu\rho\sigma} \,. \tag{1.1}$$

From problem sheet 1, this is a total derivative, and consequently has no effect at the classical level.

The relevant anomaly coefficients can be obtained by recalling that the (anomalous) axial currents are such that

$$\partial^\mu j^a_\mu \propto F_{\mu\nu}(A) F_{\rho\sigma}(A) \epsilon^{\mu\nu\rho\sigma} \operatorname{tr}\left(T^a Q^2_{EM}\right) ,$$

where T^a are the generators. The matrix Q_{EM} embedded in $\mathrm{SU}(2)_V$ and $\mathrm{SU}(3)_V$ is

$$Q_{EM} = \operatorname{diag}(2/3, -1/3) \in \mathrm{SU}(2)_V \,, \quad Q_{EM} = \operatorname{diag}(2/3, -1/3, -1/3) \in \mathrm{SU}(3)_V \,.$$

For the π^0 decay, we are interested in the $\mathrm{SU}(2)_V$ generators, and only the $a = 3$ contribution is non-zero. Similarly, for the η decay, we are in $\mathrm{SU}(3)_V$ generators, and only the $a = 8$ contribution is relevant. Thus

$$\frac{A_{3AQ_{EM}Q_{EM}}}{A_{8AQ_{EM}Q_{EM}}} = \frac{\operatorname{tr}\left(T^3 Q^2_{EM}\right)}{\operatorname{tr}\left(T^8 Q^2_{EM}\right)} = \sqrt{3} \,,$$

where $T^3 = \sigma^3/2$ and $T^8 = \lambda^8/2$.

Finally, the decay rate $\Gamma(P \to \gamma\gamma)$ is proportional to the absolute square of the anomaly coefficient (as the anomaly coefficient is proportional to the 'coupling constant' in the decay Lagrangian) multiplied by m_P^3. Thus,

$$R = \frac{\Gamma(\pi^0 \to \gamma\gamma)}{\Gamma(\eta \to \gamma\gamma)} = \left| \frac{A_{3AQ_{EM}Q_{EM}}}{A_{8AQ_{EM}Q_{EM}}} \right|^2 \frac{m^3_{\pi^0}}{m^3_\eta} = 3 \frac{m^3_{\pi^0}}{m^3_\eta} \simeq 0.045 \,.$$

2

We will adopt the notation that $A(G_1, G_2, G_3)$ denote the anomaly coefficient corresponding to groups G_1, G_2 and G_3. Firstly,

$$A(3,3,3) \propto \sum_{\alpha\beta\gamma} \mathrm{tr}\left(\lambda_\alpha\{\lambda_\beta, \lambda_\gamma\}\right) = 2\sum_{\alpha\beta\gamma} d_{\alpha\beta\gamma} = 0\,,$$

as the non-vanishing d coefficients are $d_{118} = d_{228} = d_{338} = -d_{888} = 1/\sqrt{3}$, $d_{448} = d_{558} = d_{668} = d_{778} = -1/(2\sqrt{3})$, and $d_{344} = d_{355} = -d_{366} = -d_{377} = 1/2$. Then,

$$A(3,3,2) \propto \sum_{\alpha\beta a} \mathrm{tr}\left(\lambda_\alpha\{\lambda_\beta, \sigma_a\}\right) \propto \delta_{\alpha\beta}\,\mathrm{tr}\,\sigma_a = 0\,,$$

as the SU(3) and SU(2) generators commute. The $A(3,3,1)$ coefficient is more involved. Recalling that the $\mathrm{U}(1)_Y$ group is generated by the hypercharge operator Y, we have

$$A(3,3,1) \propto \sum_{\text{quarks}} Y = 3\big(2y_{Q_L} + y_{u_{Rc}} + y_{d_{Rc}}\big) = 0\,. \tag{2.1}$$

Moving on, we can easily see that $A(2,2,2)$ vanishes because $\{\sigma_a, \sigma_b\} \propto \delta_{ab}$ and σ_a is traceless. As with the case with two 3's, $A(2,2,1)$ is more involved, and it is given by

$$A(2,2,1) \propto \sum_{\text{doublets}} Y = 3\big(3y_{Q_L} + y_{L_L}\big) = 0\,. \tag{2.2}$$

There are two more coefficients to check. $A(2,1,1)$ vanishes because it is proportional to the trace of a single Pauli matrix, and finally,

$$A(1,1,1) \propto \sum_{\text{fermions}} Y^3 \propto 2y_{L_L}^3 + y_{e_{Rc}}^3 + 6y_{Q_L}^3 + 3y_{u_{Rc}}^3 + 3y_{d_{Rc}}^3 = 0\,. \tag{2.3}$$

We have 3 non-trivial equations ((2.1)-(2.3)) and 5 hypercharges, so there are 2 hypercharges left undetermined.

Including the Higgs doublet ϕ with hypercharge y_ϕ, we have the following Yukawa couplings:

$$\overline{Q}_L u_R \widetilde{\phi}\,, \quad \overline{Q}_L d_R \phi\,, \quad \overline{L}_L e_R \phi\,, \quad \text{complex conjugates.}$$

Classical invariance of hypercharge of these couplings require the following relations:

$$y_{u_R} - y_\phi - y_{Q_L} = 0\,, \quad y_{d_R} + y_\phi - y_{Q_L} = 0\,, \quad y_{e_R} + y_\phi - y_{L_L} = 0\,, \tag{2.4}$$

where $y_{u_R} = -y_{u_{Rc}}$, and so on. The negative of the sum of the first two equations give precisely (2.1), so there are only 2 extra independent relations. So, including the Higgs field, there are 6 hypercharges with 5 anomaly cancelling relations. The single, undetermined hypercharge, say y_ϕ can be set by convention; $y_\phi = 1/2$ in the Standard Model.

The generators of the SU(2) $\times$ U(1)$_Y$ sector are T^i and Y for $i \in \{1,2,3\}$. Recall that the unbroken generator Q_{EM} is given by

$$Q_{EM}\phi = (\theta_i T^i + \eta Y)\phi = 0\,,$$

where $\theta_i \in \mathbb{R}$, and ϕ is the Higgs vacuum expectation value. For $\phi = (0, v/\sqrt{2})$, the solution is $\theta_1 = \theta_2 = 0$, and $\eta = \theta_3$. We can normalise with $\eta = 1$ giving $Q_{EM} = T^3 + Y$. The unique solution to the anomaly cancellation and hypercharge invariance equations given $y_\phi = 1/2$ is

$$y_{u_R} = \frac{2}{3}\,, \quad y_{d_R} = -\frac{1}{3}\,, \quad y_{Q_L} = \frac{1}{6}\,, \quad y_{L_L} = -\frac{1}{2}\,, \quad y_{e_R} = -1\,.$$

Their corresponding electric charges are:

$$\begin{aligned}
&u_R : \frac{2}{3}\,,\\
&d_R : -\frac{1}{3}\,,\\
&Q_L : Q_{EM}Q_L = \begin{pmatrix} \frac{1}{2}+\frac{1}{6} & 0 \\ 0 & \frac{1}{6}-\frac{1}{2}\end{pmatrix}\begin{pmatrix} u_L \\ d_L\end{pmatrix} \implies \text{charge 2/3 for } u_L \text{ and charge -1/3 for } d_L\,,\\
&L_L : Q_{EM}L_L = \begin{pmatrix} \frac{1}{2}-\frac{1}{2} & 0 \\ 0 & -\frac{1}{2}-\frac{1}{2}\end{pmatrix}\begin{pmatrix} \nu \\ e_L\end{pmatrix} \implies \text{charge 0 for } \nu \text{ and charge -1 for } e_L\,,\\
&e_R : -1\,.
\end{aligned}$$

Now, let's suppose that the up quark has no current-quark mass, and y_ϕ and $y_{e_{Rc}} = 1+\epsilon$ where ϵ is infinitesimal. Recall that $e = g_1 \cos\theta_W$, so $(1+\epsilon)e = g_1\cos\theta_W$, which for infinitesimal ϵ gives

$$e = g_1(1-\epsilon)\cos\theta_W\,.$$

Since the up quark has no current-quark mass, the relations imposed on the hypercharges are (2.1)-(2.3) and the middle and right of (2.4). To lowest order in ϵ, we have

$$y_{Q_L} = \frac{1}{6}+\frac{\epsilon}{3}\,, \quad y_{L_L} = -\frac{1}{2}-\epsilon\,, \quad y_{d_R} = -\frac{1}{3}(1-\epsilon)\,, \quad y_{u_R} = \frac{2}{3}+\frac{\epsilon}{3}\,.$$

The electric charge of the neutron (udd) is then

$$Q_{EM;\text{neutron}} = y_{u_R} + 2y_{d_R} = \epsilon\,.$$

3

Consider the terms $\operatorname{tr}\Phi^2$ and $\operatorname{tr}\Phi^4$ in the potential. It will be convenient if all of the terms generated by $\operatorname{tr}\Phi^2$ were quadratic, i.e., if all of the mixed second order terms vanish. That is, for $\Phi = v_a\phi_a$, $\operatorname{tr}(\phi_a\phi_b) \propto \delta_{ab}$. The existence of this parameterisation is guaranteed by group theory. It would also be convenient if all terms involving v_1 and ϕ_1 in the $\operatorname{tr}\Phi^4$ terms to be at least quadratic in each other parameter. That is, $\operatorname{tr}(\phi_1^3\phi_b) = 0$ for $b \neq 1$. Fortunately, such a parameterisation exists. It is

$$\phi_1 = \operatorname{diag}(2,2,2,-3,-3)\,, \quad \phi_2 = \operatorname{diag}(1,-1,0,0,0)\,, \quad \phi_3 = \operatorname{diag}(1,1,-2,0,0)\,, \quad \phi_4 = \operatorname{diag}(0,0,0,1,-1)\,.$$

We can now rewrite $V(\Phi, H=0) = \mathcal{V}(v_i)$ in terms of these parameters. Recalling that our parameterisation is such that $\mathrm{tr}(\phi_1^3\phi_b) = 0$ for $b \neq 1$, every term in $\partial\mathcal{V}/\partial v_i$ will include at least one v_1. Thus, $v_1 \neq 0$ and $v_2 = v_3 = v_4 = 0$ is a solution to $\partial\mathcal{V}/\partial v_a = 0$. A brute force calculation (mathematica will be useful) gives

$$\mathcal{V}(v_1, v_2 = v_3 = v_4 = 0) = 900bv_1^4 + 210av_1^4 - 30m_1^2v_1^2\,.$$

And so, the minimum of $\mathcal{V}$ is given by

$$v_1^2 = \frac{m_1^2}{14a + 60b}\,,$$

provided that $b > -7a/30$ so that the denominator is positive definite, and $a > 0$ so that the Lagrangian is bounded from below.

To find the mass term of the gauge fields after symmetry breaking, consider the kinetic term of the Φ field and expand it about $\overline{\Phi} + \delta\Phi$, focussing on terms independent of $\delta\Phi$,

$$\mathcal{L}_{\mathrm{kin}}(\Phi) = \frac{1}{2}\,\mathrm{tr}\left(D_\mu\Phi D^\mu\Phi\right), \quad (D_\mu\Phi)_a = \partial_\mu v_a - ig_5 f_{abc}X_{\mu b}\Phi_c\,.$$

The δv_a independent term in $\mathcal{L}_{\mathrm{kin}}(\overline{v} + \delta v)$ is then

$$-\frac{1}{4}g_5^2 f_{abc}f_{ade}\overline{v}_c\overline{v}_e X_{\mu b}X_d^\mu\,.$$

The conventional mass term is given by

$$-\frac{m^2}{2}X_{\mu a}X_a^\mu\,,$$

so

$$m^2 = \frac{1}{2}g_5^2 f_{aa1}f_{aa1}\overline{v}_1^2\,.$$

Calculating the commutators, we find that $f_{aa1} = 5$, and so

$$m = \sqrt{\frac{25}{2}}g_5\overline{v}_1\,.$$

For finding the transformation of the gauge fields under hypercharge, consider the same action under the adjoint commutator, but rescaled so instead of $\mathrm{diag}(2,2,2,-3,-3)$, it is proportional to the SM hypercharge generator $y = \mathrm{diag}(-1/3,-1/3,-1/3,1/2,1/2)$. This has a difference of a factor of $-1/6$, so the charge is $-5/6$ under the SM hypercharge.

Lastly, writing $H = (h_{t_1}, h_{t_2}, h_{t_3}, h_{d_1}, h_{d_2})^T$, then finding the mass term in $V(\overline{\Phi}, H)$ for H, we have

$$V(\overline{\Phi}, H) = -m_2^2(h_{t_i}^*h_{t_i} + h_{d_j}^*h_{d_j}) + 30\lambda_1\overline{v}_1^2(h_{t_i}^*h_{t_i} + h_{d_j}^*h_{d_j}) + \lambda_2\overline{v}_1^2(4h_{t_i}^*h_{t_i} + 9h_{d_j}^*h_{d_j}) + \cdots\,,$$

which gives

$$m_t^2 = -m_2^2 + (30\lambda_1 + 4\lambda_2)\overline{v}_1^2\,,$$
$$m_d^2 = -m_2^2 + (30\lambda_1 + 9\lambda_2)\overline{v}_1^2\,.$$

4

The electromagnetic charge is

$$\hat{Q} = \mathrm{diag}(-1/3, -1/3, -1/3, 1, 0) = \sqrt{\frac{3}{2}}\big(T^3 + \sqrt{\frac{5}{3}}\hat{Y}\big)\,.$$

For the anomalies, it is easier to calculate the ratio $A(5^*)/A(10)$ due to the relative coefficient d^{IJK}. We have

$$\frac{A(5^*)}{A(10)} = \frac{\mathrm{tr}_{5^*}(\hat{Q}^3)}{\mathrm{tr}_{10}(\hat{Q}^3)} = \frac{3(1/3)^3 + (-1)^3 + 0}{3(-2/3)^3 + 3(2/3)^3 + 3(-1/3)^3 + 1^3} = -1 \implies A(5^*) + A(10) = 0\,.$$

There are no changes to the gauge anomalies when one includes an SU(5) singlet right-handed fermion to generate neutrino masses via the see-saw mechanism, as the neutrino by definition does not transform under the gauge group.

5

For each gauge group, $n_f = 4n_g$, as they couple four particles/antiparticles per generation. Similarly, $C_A = 3, 2$ for SU(3) and SU(2), and $C_R = 1/2$, so

$$b_3 = \frac{1}{12\pi}\Big(11 \times 3 - 2 \times 4n_g \times \frac{1}{2}\Big) = \frac{1}{4\pi}\Big(11 - \frac{4n_g}{3}\Big)\,, \tag{5.1}$$
$$b_2 = \frac{1}{12\pi}\Big(11 \times 2 - 2 \times 4n_g \times \frac{1}{2} - n_H \times \frac{1}{2}\Big) = \frac{1}{4\pi}\Big(\frac{22}{3} - \frac{4n_g}{3} - \frac{n_H}{6}\Big)\,. \tag{5.2}$$

For $\hat{b}_1$, we need to compute b_Y, which from the notes is

$$b_Y = -\frac{1}{6\pi}\Big(\mathrm{tr}_{\text{fermions,left}}(Y^2) + \frac{1}{2}\,\mathrm{tr}_{\text{scalars}}(Y^2)\Big)\,,$$

and therefore,

$$\hat{b}_1 = \frac{1}{4\pi}\Big(-\frac{4n_g}{3} - \frac{n_H}{10}\Big)\,. \tag{5.3}$$

Next, the renormalisation group equation in the variable $z = \log\mu$ is

$$\frac{\partial g_R}{\partial z} = -\frac{b g_R^3}{4\pi}\,,$$

which can be solved to give

$$g_R = \pm\Big(\frac{b}{2\pi}z + \widetilde{c}\Big)^{-\frac{1}{2}}\,, \quad \widetilde{c} \in \mathbb{R}\,.$$

Hence, using the definition $\alpha_N = g_N^2/4\pi$, we have

$$\alpha_N(\mu) = \big(2b_N \log\mu + c\big)^{-1} \implies \log\mu = \big(2\alpha_N b_N\big)^{-1} - \frac{c}{2b_N}\,,$$

where $c = \sqrt{4\pi}\widetilde{c}$. Therefore,

$$\log\Big(\frac{Q^2}{\mu_0^2}\Big) = \frac{1}{b_N\alpha_N(Q)} - \frac{1}{b_N\alpha_N(\mu_0)}\,. \tag{5.4}$$

Assuming unification of the couplings at scale Q, and noticing that the RHS of (5.4) has no N dependence, we have

$$\frac{1}{b_1\alpha_1(Q)} - \frac{1}{b_1\alpha_1(\mu_0)} = \frac{1}{b_2\alpha_2(Q)} - \frac{1}{b_2\alpha_2(\mu_0)} = \frac{1}{b_2\alpha_1(Q)} - \frac{1}{b_2\alpha_2(\mu_0)}\,,$$

which can be solved for $\alpha_1(Q)$ to give

$$\alpha_1(Q) = \frac{(b_2 - b_1)\alpha_1(\mu_0)\alpha_2(\mu_0)}{b_2\alpha_2(\mu_0) - b_1\alpha_1(\mu_0)}\,.$$

Inserting this back into (5.4) yields the required result

$$\log\Big(\frac{Q^2}{\mu_0^2}\Big) = \frac{1}{b_1\alpha_1(Q)} - \frac{1}{b_1\alpha_1(\mu_0)} = \Big(\frac{1}{\alpha_1(\mu_0)} - \frac{1}{\alpha_2(\mu_0)}\Big)\frac{1}{b_2 - b_1}\,.$$

Similarly, we have

$$\log\Big(\frac{Q^2}{\mu_0^2}\Big) = \Big(\frac{1}{\alpha_3(\mu_0)} - \frac{1}{\alpha_2(\mu_0)}\Big)\frac{1}{b_2 - b_3}\,,$$

which implies that

$$\frac{1}{\alpha_3(\mu_0)} = \frac{1+B}{\alpha_2(\mu_0)} - \frac{B}{\alpha_1(\mu_0)}\,, \quad B = \frac{b_3 - b_2}{b_2 - b_1}\,.$$

Substituting (5.1), (5.2) and (5.3), we find that

$$B = \frac{110 + n_H}{2(110 - n_H)} \simeq \frac{1}{2} + \frac{3n_H}{110} + \mathcal{O}(n_H^2)\,.$$

Lastly, in the MSSM, we must remember to include superpartners. If a vector transforms in the adjoint, then there is a spinor that transforms in the adjoint. And if there is a spinor transforming in the fundamental representation, there is a scalar transforming in the fundamental representation and vice versa. Therefore, b_N becomes

$$b_N = \frac{1}{12\pi}\big(11C_A - 2n_f^A C_A - 8n_g C_F - 4n_g C_F\big)\,,$$

where $n_f^A = 1$ is the number of fermions in the adjoint. Using this, we find that

$$
\begin{aligned}
b_3 &= \frac{1}{4\pi}\big(9 - 2n_g\big)\,, \\
b_2 &= \frac{1}{4\pi}\left(6 - 2n_g - \frac{n_H}{2}\right), \\
b_1 &= \frac{1}{4\pi}\left(-2n_g - \frac{3}{10}n_H\right).
\end{aligned}
$$