# Solutions - PS4

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## 1

The decay of $\pi^0$ into two photons is given by

$$\mathcal{L}_{\pi^0 AA} = \frac{e^2}{32\pi^2 F_\pi} \pi^0 F_{\mu\nu}(A) F_{\rho\sigma}(A) \epsilon^{\mu\nu\rho\sigma} \, .$$

Under the nonlinear symmetry transformation $\pi^0 \mapsto \pi^0 + \omega F_\pi$, we have

$$\delta\mathcal{L}_{\pi^0 AA} = \frac{e^2 \omega}{32\pi^2} F_{\mu\nu}(A) F_{\rho\sigma}(A) \epsilon^{\mu\nu\rho\sigma} \, . \tag{1.1}$$

From problem sheet 1, this is a total derivative, and consequently has no effect at the classical level.

From lecture notes, the relevant anomaly coefficients can be obtained by recalling that the (anomalous) axial currents are such that

$$\partial^\mu j^a_\mu \propto F_{\mu\nu}(A) F_{\rho\sigma}(A) \epsilon^{\mu\nu\rho\sigma} \operatorname{tr}\left(T^a Q^2_{EM}\right),$$

where $T^a$ are the generators. The matrix $Q_{EM}$ embedded in $\mathrm{SU}(2)_A$ and $\mathrm{SU}(3)_A$ is

$$Q_{EM} = \operatorname{diag}(2/3, -1/3) \in \mathrm{SU}(2)_A \quad \text{for } \pi^0 \sim \frac{u\bar{u} + d\bar{d}}{\sqrt{2}}, \tag{1.2}$$

$$Q_{EM} = \operatorname{diag}(2/3, -1/3, -1/3) \in \mathrm{SU}(3)_A \quad \text{for } \eta \sim \frac{u\bar{u} + d\bar{d} - 2s\bar{s}}{\sqrt{6}} \tag{1.3}$$

For the $\pi^0$ decay, we are interested in the $\mathrm{SU}(2)_A$ generators, and only the $a = 3$ contribution is non-zero. Similarly, for the $\eta$ decay, we are in $\mathrm{SU}(3)_A$ generators, and only the $a = 8$ contribution is relevant. Thus the relevant generators are $T_{3A} = \gamma_5 \sigma^3/2$ and $T_{8A} = \gamma_5 \lambda^8/2$. The eighth Gell-Mann matrix is proportional to $\operatorname{diag}(1, 1, -2)$ and with the requirement that $\operatorname{tr}(\lambda^8\lambda^8) = 2\delta^{88}$ we find the proportionality constant to be $\lambda^8 = (1/\sqrt{3}) \operatorname{diag}(1, 1, -2)$.

The ratio of anomaly coefficients is then

$$\frac{A_{3AQ_{EM}Q_{EM}}}{A_{8AQ_{EM}Q_{EM}}} = \frac{\operatorname{tr}\left(T_{3A} Q^2_{EM}\right)}{\operatorname{tr}\left(T_{8A} Q^2_{EM}\right)} = \frac{(2/3)^2 - (1/3)^2}{\frac{1}{\sqrt{3}}((2/3)^2 + (1/3)^2 - 2(1/3)^2)} = \sqrt{3} \, ,$$

The decay rate $\Gamma(P \to \gamma\gamma)$ is proportional to the absolute square of the anomaly coefficient (as the anomaly coefficient is proportional to the 'coupling constant' in the decay Lagrangian). By dimensional analysis, it should be multiplied by $m_P^3$. Thus,

$$R = \frac{\Gamma(\pi^0 \to \gamma\gamma)}{\Gamma(\eta \to \gamma\gamma)} = \left|\frac{A_{3AQ_{EM}Q_{EM}}}{A_{8AQ_{EM}Q_{EM}}}\right|^2 \frac{m_{\pi^0}^3}{m_\eta^3} = 3\frac{m_{\pi^0}^3}{m_\eta^3} \simeq 0.045\,.$$

## 2

We will adopt the notation that $A(G_1, G_2, G_3)$ denote the anomaly coefficient corresponding to groups $G_1$, $G_2$ and $G_3$. Anomaly coefficient for a general representation $R$ has the properties

1. $A(\overline{R}) = -A(R)$
2. $A(R_1 \oplus R_2) = A(R_1) + A(R_2)$
3. $A(R_1 \otimes R_2) = A(R_1 \otimes \mathbb{I}_2 + \mathbb{I}_1 \otimes R_2) = \dim(R_1)A(R_2) + \dim(R_2)A(R_1)$

Now for the SM: $SU(3)_c \times SU(2)_L \times U(1)_Y$,

$$A(3,3,3) \propto \sum_{\alpha\beta\gamma} \mathrm{tr}\left(\lambda_\alpha\{\lambda_\beta, \lambda_\gamma\}\right) = 2\sum_{\alpha\beta\gamma} d_{\alpha\beta\gamma} = 0\,,$$

as the non-vanishing $d$ coefficients are $d_{118} = d_{228} = d_{338} = -d_{888} = 1/\sqrt{3}$, $d_{448} = d_{558} = d_{668} = d_{778} = -1/(2\sqrt{3})$, and $d_{344} = d_{355} = -d_{366} = -d_{377} = 1/2$. Then,

$$A(3,3,2) \propto \sum_{\alpha\beta a} \mathrm{tr}\left(\lambda_\alpha\{\lambda_\beta, \sigma_a\}\right) \propto \delta_{\alpha\beta}\,\mathrm{tr}\,\sigma_a = 0\,,$$

as the SU(3) and SU(2) generators commute. The $A(3,3,1)$ coefficient is more involved. Recalling that the $\mathrm{U}(1)_Y$ group is generated by the hypercharge operator $Y$, we have

$$A(3,3,1) \propto \sum_{\text{quarks}} Y = 3\big(2y_{Q_L} + y_{u_{Rc}} + y_{d_{Rc}}\big) = 0\,. \tag{2.1}$$

Moving on, we can easily see that $A(2,2,2)$ vanishes because $\{\sigma_a, \sigma_b\} \propto \delta_{ab}$ and $\sigma_a$ is traceless. As with the case with two 3's, $A(2,2,1)$ is more involved, and it is given by

$$A(2,2,1) \propto \sum_{\text{doublets}} Y = 3\big(3y_{Q_L} + y_{L_L}\big) = 0\,. \tag{2.2}$$

There are two more coefficients to check. $A(2,1,1)$ vanishes because it is proportional to the trace of a single Pauli matrix, and finally,

$$A(1,1,1) \propto \sum_{\text{fermions}} Y^3 \propto 2y_{L_L}^3 + y_{e_{Rc}}^3 + 6y_{Q_L}^3 + 3y_{u_{Rc}}^3 + 3y_{d_{Rc}}^3 = 0\,. \tag{2.3}$$

We have 3 non-trivial equations ((2.1)-(2.3)) and 5 hypercharges, so there are 2 hypercharges left undetermined.

Including the Higgs doublet $\phi$ with hypercharge $y_\phi$, we have the following Yukawa couplings:

$$\overline{Q}_L u_R \widetilde{\phi}\,, \quad \overline{Q}_L d_R \phi\,, \quad \overline{L}_L e_R \phi\,, \quad \text{complex conjugates.}$$

Classical invariance of hypercharge of these couplings require the following relations:

$$y_{u_R} - y_\phi - y_{Q_L} = 0\,, \quad y_{d_R} + y_\phi - y_{Q_L} = 0\,, \quad y_{e_R} + y_\phi - y_{L_L} = 0\,, \tag{2.4}$$

where $y_{u_R} = -y_{u_{Rc}}$, and so on. The negative of the sum of the first two equations give precisely (2.1), so there are only 2 extra independent relations. So, including the Higgs field, there are 6 hypercharges with 5 anomaly cancelling relations. The single, undetermined hypercharge, say $y_\phi$ can be set by convention; $y_\phi = 1/2$ in the Standard Model.

The generators of the SU(2) $\times$ U(1)$_Y$ sector are $T^i$ and $Y$ for $i \in \{1, 2, 3\}$. Recall that the unbroken generator $Q_{EM}$ is given by

$$Q_{EM}\phi = (\theta_i T^i + \eta Y)\phi = 0\,,$$

where $\theta_i \in \mathbb{R}$, and $\phi$ is the Higgs vacuum expectation value. For $\phi = (0, v/\sqrt{2})$, the solution is $\theta_1 = \theta_2 = 0$, and $\eta = \theta_3$. We can normalise with $\eta = 1$ giving $Q_{EM} = T^3 + Y$. The unique solution to the anomaly cancellation and hypercharge invariance equations given $y_\phi = 1/2$ is

$$y_{u_R} = \frac{2}{3}\,, \quad y_{d_R} = -\frac{1}{3}\,, \quad y_{Q_L} = \frac{1}{6}\,, \quad y_{L_L} = -\frac{1}{2}\,, \quad y_{e_R} = -1\,.$$

Their corresponding electric charges are:

$$\begin{aligned}
&u_R : \frac{2}{3}\,, \\
&d_R : -\frac{1}{3}\,, \\
&Q_L : Q_{EM} Q_L = \begin{pmatrix} \frac{1}{2} + \frac{1}{6} & 0 \\ 0 & \frac{1}{6} - \frac{1}{2} \end{pmatrix} \begin{pmatrix} u_L \\ d_L \end{pmatrix} \implies \text{charge 2/3 for } u_L \text{ and charge -1/3 for } d_L\,, \\
&L_L : Q_{EM} L_L = \begin{pmatrix} \frac{1}{2} - \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} - \frac{1}{2} \end{pmatrix} \begin{pmatrix} \nu \\ e_L \end{pmatrix} \implies \text{charge 0 for } \nu \text{ and charge -1 for } e_L\,, \\
&e_R : -1\,.
\end{aligned}$$

Now, let's suppose that the up quark has no current-quark mass, and $y_\phi$ and $y_{e_{Rc}} = 1 + \epsilon$ where $\epsilon$ is infinitesimal. Recall that $e = g_1 \cos\theta_W$, so $(1 + \epsilon)e = g_1 \cos\theta_W$, which for infinitesimal $\epsilon$ gives

$$e = g_1(1 - \epsilon)\cos\theta_W\,.$$

Since the up quark has no current-quark mass, the relations imposed on the hypercharges are (2.1)-(2.3) and the middle and right of (2.4). To lowest order in $\epsilon$, we have

$$y_{Q_L} = \frac{1}{6} + \frac{\epsilon}{3}\,, \quad y_{L_L} = -\frac{1}{2} - \epsilon\,, \quad y_{d_R} = -\frac{1}{3}(1 - \epsilon)\,, \quad y_{u_R} = \frac{2}{3} + \frac{\epsilon}{3}\,.$$

The electric charge of the neutron ($udd$) is then

$$Q_{EM;\text{neutron}} = y_{u_R} + 2y_{d_R} = \epsilon\,.$$

## 3

The intuitive way of finding the vev $\Phi$ that can give rise to a symmetry breaking $G = SU(5) \to H$ is one which leaves the $H = SU(3) \times SU(2) \times U(1)$ subgroup invariant, i.e. since the field is in the adjoint rep, commutes with the generators of $SU(3) \times SU(2) \times U(1)$. One can immediately see that a vev proportional to $Y \sim \mathrm{diag}(2, 2, 2, -3, -3)$ does the job. This is exactly the way we had found the vacuum orbits for $SU(3)$ symmetry breaking from Peter Higgs' paper in problem set 2. The required vev is $\langle\Phi\rangle = v_1\,\mathrm{diag}(2, 2, 2, -3, -3)$.

One can also argue for the required vev by looking for a simple enough specific parametrisation for $\Phi = \phi^a T^a$: Consider the terms $\mathrm{tr}\,\Phi^2$ and $\mathrm{tr}\,\Phi^4$ in the potential. It will be convenient if all of the terms generated by $\mathrm{tr}\,\Phi^2$ were quadratic, i.e., if all of the mixed second order terms vanish. That is, for $\Phi = v_a\phi_a$, $\mathrm{tr}(\phi_a\phi_b) \propto \delta_{ab}$. The existence of this parameterisation is guaranteed by group theory. It would also be convenient if all terms involving $v_1$ and $\phi_1$ in the $\mathrm{tr}\,\Phi^4$ terms to be at least quadratic in each other parameter. That is, $\mathrm{tr}(\phi_1^3\phi_b) = 0$ for $b \neq 1$. Fortunately, such a parameterisation exists. It is

$$\phi_1 = \mathrm{diag}(2, 2, 2, -3, -3)\,, \quad \phi_2 = \mathrm{diag}(1, -1, 0, 0, 0)\,, \quad \phi_3 = \mathrm{diag}(1, 1, -2, 0, 0)\,, \quad \phi_4 = \mathrm{diag}(0, 0, 0, 1, -1)\,.$$

We can now rewrite $V(\Phi, H = 0) = \mathcal{V}(v_i)$ in terms of these parameters. Recalling that our parameterisation is such that $\mathrm{tr}(\phi_1^3\phi_b) = 0$ for $b \neq 1$, every term in $\partial\mathcal{V}/\partial v_i$ will include at least one $v_1$. Thus, $v_1 \neq 0$ and $v_2 = v_3 = v_4 = 0$ is a solution to $\partial\mathcal{V}/\partial v_a = 0$.

With the vev $\langle\Phi\rangle = v_1\,\mathrm{diag}(2, 2, 2, -3, -3)$ the potential is

$$\mathcal{V}(v_1, v_2 = v_3 = v_4 = 0) = 900bv_1^4 + 210av_1^4 - 30m_1^2v_1^2$$

And so, the minimum of $\mathcal{V}$ is given by

$$v_1^2 = \frac{m_1^2}{14a + 60b}\,,$$

provided that $b > -7a/30$ so that the denominator is positive definite, and $a > 0$ so that the Lagrangian is bounded from below.

To find the mass term of the gauge fields after symmetry breaking, consider the kinetic term of the $\Phi$ field and expand it about $\overline{\Phi} + \delta\Phi$, focussing on terms independent of $\delta\Phi$,

$$\mathcal{L}_{\mathrm{kin}}(\Phi) = \frac{1}{2}\,\mathrm{tr}\left(D_\mu\Phi D^\mu\Phi\right)\,, \quad (D_\mu\Phi)_a = \partial_\mu v_a - ig_5 f_{abc}X_{\mu b}\Phi_c\,.$$

The $\delta v_a$ independent term in $\mathcal{L}_{\mathrm{kin}}(\overline{v} + \delta v)$ is then

$$-\frac{1}{4}g_5^2 f_{abc}f_{ade}\overline{v}_c\overline{v}_e X_{\mu b}X_d^\mu\,.$$

The conventional mass term is given by

$$-\frac{m^2}{2}X_{\mu a}X_a^\mu\,,$$

so

$$m^2 = \frac{1}{2}g_5^2 f_{aa1}f_{aa1}\overline{v}_1^2\,.$$

Calculating the commutators, we find that $f_{aa1} = 5$, and so

$$m = \sqrt{\frac{25}{2}} g_5 \overline{v}_1 \,.$$

For finding the transformation of the gauge fields under hypercharge, consider the same action under the adjoint commutator, but rescaled so instead of diag$(2,2,2,-3,-3)$, it is proportional to the SM hypercharge generator $y = \text{diag}(-1/3,-1/3,-1/3,1/2,1/2)$. This has a difference of a factor of $-1/6$, so the charge is $-5/6$ under the SM hypercharge.

Lastly, writing $H = (h_{t_1}, h_{t_2}, h_{t_3}, h_{d_1}, h_{d_2})^T$, then finding the mass term in $V(\overline{\Phi}, H)$ for $H$, we have

$$V(\overline{\Phi}, H) = -m_2^2(h^*_{t_i} h_{t_i} + h^*_{d_j} h_{d_j}) + 30\lambda_1 \overline{v}_1^2 (h^*_{t_i} h_{t_i} + h^*_{d_j} h_{d_j}) + \lambda_2 \overline{v}_1^2 (4 h^*_{t_i} h_{t_i} + 9 h^*_{d_j} h_{d_j}) + \cdots \,,$$

which gives

$$\begin{aligned} m_t^2 &= -m_2^2 + (30\lambda_1 + 4\lambda_2)\overline{v}_1^2 \,, \\ m_d^2 &= -m_2^2 + (30\lambda_1 + 9\lambda_2)\overline{v}_1^2 \,. \end{aligned}$$

## 4

From the lecture notes, the particles that make up the **5** rep of $SU(5)$ are diag$(d_R, d_R, d_R, e^+_L, \bar{\nu}_{eL})$ which yield the electromagnetic charge matrix embedded in the **5** rep as

$$\hat{Q} = c \, \text{diag}(-1/3, -1/3, -1/3, 1, 0) \tag{4.1}$$

where $c = \sqrt{3/2}$ is the constant of proportionality determined using the trace relation $\text{tr}(\hat{Q}\hat{Q}) = \frac{1}{2}$. We also know that the hypercharge matrix of the above particle content

$$\sqrt{\frac{5}{3}} \hat{Y} = Y = \text{diag}(-1/3, -1/3, -1/3, 1/2, 1/2) \tag{4.2}$$

when embedded in **5** rep of $SU(5)$. Lastly, the $T^3$ of $SU(2)$ can be embedded in $SU(5)$ as

$$T^3 = \frac{1}{2} \text{diag}(0, 0, 0, 1, -1) \tag{4.3}$$

where once again the proportionality constant is determined using the trace relation $\text{tr}(T^3 T^3) = \frac{1}{2}\delta^{33}$.

From which we immediately see that

$$\hat{Q} = \sqrt{\frac{3}{2}} (T^3 + \sqrt{\frac{5}{3}} \hat{Y}). \tag{4.4}$$

For the anomalies, first note that one can use the $\hat{Q}$ operator in a given representation to calculate the anomaly as it is just a particular linear combination of the generators of $SU(5)$ which keeps the anomaly coefficient of that representation invariant upto a factor of $d^{IJK}$. Furthermore it is easier to calculate the ratio $A(5^*)/A(10)$ because the factor $d^{IJK}$ drops out as we are the considering the same $\hat{Q}$ operator for both the $\mathbf{5}^*$ and **10** reps in the anomaly calculation.

The anti-fundamental rep **5**$^*$ has the charges switched from the fundamental **5** rep considered above, or specifically it has the particle content $\{(d_R)_C, L = (e_L, \nu_{eL})\}$ giving

$$tr_{5^*}(Q^3) = 3(1/3)^3 + (-1)^3 + 0 = -8/9, \tag{4.5}$$

while for anomaly of the **10** rep we need to consider its particle content, $\{(u_R)_C, Q_L = (u_L, d_L), (e_R)_C\}$ giving

$$\mathrm{tr}_{10}(Q^3) = 3(-2/3)^3 + 3(2/3)^3 + 3(-1/3)^3 + 1^3 = 8/9. \tag{4.6}$$

From which we see that

$$\frac{A(5^*)}{A(10)} = \frac{\mathrm{tr}_{5^*}(\hat{Q}^3)}{\mathrm{tr}_{10}(\hat{Q}^3)} = -1 \implies A(5^*) + A(10) = 0\,.$$

There are no changes to the gauge anomalies when one includes an SU(5) singlet right-handed fermion to generate neutrino masses via the see-saw mechanism, as the neutrino by definition does not transform under the gauge group.

# 5

For each gauge group, $n_f = 4n_g$, as they couple four particles/antiparticles per generation. Similarly, $C_A = 3, 2$ for SU(3) and SU(2), and $C_R = 1/2$, so

$$b_3 = \frac{1}{12\pi}\Big(11 \times 3 - 2 \times 4n_g \times \frac{1}{2}\Big) = \frac{1}{4\pi}\Big(11 - \frac{4n_g}{3}\Big)\,, \tag{5.1}$$

$$b_2 = \frac{1}{12\pi}\Big(11 \times 2 - 2 \times 4n_g \times \frac{1}{2} - n_H \times \frac{1}{2}\Big) = \frac{1}{4\pi}\Big(\frac{22}{3} - \frac{4n_g}{3} - \frac{n_H}{6}\Big)\,. \tag{5.2}$$

For $\hat{b}_1$, we need to compute $b_Y$, which from the notes is

$$b_Y = -\frac{1}{6\pi}\Big(\mathrm{tr}_{\text{fermions,left}}(Y^2) + \frac{1}{2}\,\mathrm{tr}_{\text{scalars}}(Y^2)\Big)\,,$$

and therefore,

$$\hat{b}_1 = \frac{1}{4\pi}\Big(-\frac{4n_g}{3} - \frac{n_H}{10}\Big)\,. \tag{5.3}$$

Next, the renormalisation group equation in the variable $z = \log\mu$ is

$$\frac{\partial g_R}{\partial z} = -\frac{b g_R^3}{4\pi}\,,$$

which can be solved to give

$$g_R = \pm\Big(\frac{b}{2\pi}z + \widetilde{c}\Big)^{-\frac{1}{2}}\,, \quad \widetilde{c} \in \mathbb{R}\,.$$

Hence, using the definition $\alpha_N = g_N^2/4\pi$, we have

$$\alpha_N(\mu) = \big(2b_N \log\mu + c\big)^{-1} \implies \log\mu = \big(2\alpha_N b_N\big)^{-1} - \frac{c}{2b_N}\,,$$

where $c = \sqrt{4\pi}\widetilde{c}$. Therefore,

$$\log\Big(\frac{Q^2}{\mu_0^2}\Big) = \frac{1}{b_N\alpha_N(Q)} - \frac{1}{b_N\alpha_N(\mu_0)}\,. \tag{5.4}$$

Assuming unification of the couplings at scale $Q$, and noticing that the RHS of (5.4) has no $N$ dependence, we have

$$\frac{1}{b_1\alpha_1(Q)} - \frac{1}{b_1\alpha_1(\mu_0)} = \frac{1}{b_2\alpha_2(Q)} - \frac{1}{b_2\alpha_2(\mu_0)} = \frac{1}{b_2\alpha_1(Q)} - \frac{1}{b_2\alpha_2(\mu_0)}\,,$$

which can be solved for $\alpha_1(Q)$ to give

$$\alpha_1(Q) = \frac{(b_2-b_1)\alpha_1(\mu_0)\alpha_2(\mu_0)}{b_2\alpha_2(\mu_0) - b_1\alpha_1(\mu_0)}\,.$$

Inserting this back into (5.4) yields the required result

$$\log\Big(\frac{Q^2}{\mu_0^2}\Big) = \frac{1}{b_1\alpha_1(Q)} - \frac{1}{b_1\alpha_1(\mu_0)} = \Big(\frac{1}{\alpha_1(\mu_0)} - \frac{1}{\alpha_2(\mu_0)}\Big)\frac{1}{b_2-b_1}\,.$$

Similarly, we have

$$\log\Big(\frac{Q^2}{\mu_0^2}\Big) = \Big(\frac{1}{\alpha_3(\mu_0)} - \frac{1}{\alpha_2(\mu_0)}\Big)\frac{1}{b_2-b_3}\,,$$

which implies that

$$\frac{1}{\alpha_3(\mu_0)} = \frac{1+B}{\alpha_2(\mu_0)} - \frac{B}{\alpha_1(\mu_0)}\,,\qquad B = \frac{b_3-b_2}{b_2-b_1}\,.$$

Substituting (5.1), (5.2) and (5.3), we find that

$$B = \frac{110+n_H}{2(110-n_H)} \simeq \frac{1}{2} + \frac{3n_H}{110} + \mathcal{O}(n_H^2)\,.$$

Lastly, in the MSSM, we must remember to include superpartners. If a vector transforms in the adjoint, then there is a spinor that transforms in the adjoint. And if there is a spinor transforming in the fundamental representation, there is a scalar transforming in the fundamental representation and vice versa. Therefore, $b_N$ becomes

$$b_N = \frac{1}{12\pi}\big(11C_A - 2n_f^AC_A - 8n_gC_F - 4n_gC_F\big)\,,$$

where $n_f^A = 1$ is the number of fermions in the adjoint. Using this, we find that

$$\begin{aligned} b_3 &= \frac{1}{4\pi}\big(9-2n_g\big)\,,\\ b_2 &= \frac{1}{4\pi}\Big(6-2n_g-\frac{n_H}{2}\Big)\,,\\ b_1 &= \frac{1}{4\pi}\Big(-2n_g-\frac{3}{10}n_H\Big)\,. \end{aligned}$$