Jan. 12, 2024

The Standard Model and Beyond: Problem Set 1

1. a) Derive the Palatini identity $\delta F^i_{\mu\nu} = D_\mu \delta A^i_\nu - D_\nu \delta A^i_\nu$ for infinitesimal variations of a nonabelian field strength arising from a variation δA^i_μ of the Yang-Mills gauge field, in which D_μ is a covariant derivative for the adjoint representation. Recall that $T^{kij}_{\text{adj}} = -if^{kij}$ for the adjoint representation.
b) Derive the nonabelian identity $D_\mu \tilde{F}^{i\,\mu\nu} \equiv 0$ where $\tilde{F}^{i\,\mu\nu} := \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F^i_{\rho\sigma}$ in which D_μ is here a covariant derivative for the adjoint representation while $(D_\mu)_A{}^B = \partial_\mu \delta_A{}^B - igA_\mu \cdot T_A{}^B$ in a general representation $T^i{}_A{}^B$. One then has $[D_\mu, D_\nu]_A{}^B = -igF_{\mu\nu} \cdot T_A{}^B$. It may be helpful to use the Jacobi identity $[D_\rho, [D_\mu, D_\nu]] + \text{cycle}\ (\rho, \mu, \nu) = 0$.
c) Show that the infinitesimal variation of $F^i_{\mu\nu} \tilde{F}^{i\,\mu\nu}$ is a total derivative, so such terms in the field-theory Lagrangian do not contribute to the classical field equations.

2. a) Show that the *complex* n-dimensional fundamental representation φ_a of $\mathrm{S}U(n)$, with infinitesimal transformations generated by Hermitean Lie algebra generators $T^i{}_a{}^b$

$$\begin{aligned}\delta\varphi_a &= \mathrm{i}\delta\theta^i T^i{}_a{}^b \varphi_b \\ \delta\varphi^{*\,a} &= -\mathrm{i}\delta\theta^i \varphi^{*\,b} T^i{}_b{}^a\end{aligned}$$

can be recast as a $2n$ dimensional *real* representation with Hermitean generators. One way to label this $2n$ dimensional real form is to use a $2n$ valued hybrid index notation $A \leftrightarrow a\hat{z}$, in which one has $\phi_A = \phi_{a\hat{z}}$ with $a = 1, \ldots, n;\ \hat{z} = 1, 2$, setting

$$\phi_{a1} = \tfrac{1}{2}(\varphi_a + \varphi^{*\,a})\ , \quad \phi_{a2} = \tfrac{-\mathrm{i}}{2}(\varphi_a - \varphi^{*\,a})$$

Demonstrate that the generators $T^i{}_{BC}$ in this real form, with $B \leftrightarrow b\hat{w}$, *etc.*, are Hermitean $2n \times 2n$ matrices. Since the $2n$ dimensional real form must have purely imaginary generators (so that $\mathrm{i}T^i$ is real), show that the generators of this real form are purely antisymmetric $2n \times 2n$ matrices.
b) Using the results of part a), show that for this real-form representation, the matrix

$$S^{ij} = (T^i\bar{\phi}, T^j\bar{\phi})$$

is real and symmetric, and thus diagonalizable, where (v, w) is the usual $2n$ dimensional vector space inner product, $(v, w) = v_A w_A$ and $\bar{\phi}_B$ is the vacuum value of ϕ_B. Hence, starting from the $(\text{mass})^2$ constraint $M^2_{AB} T^i_{BC} \bar{\phi}_C = 0$, show that there is precisely one zero eigenvalue of M^2_{AB} for each non-vanishing $T^i\bar{\phi}$ vector. (Ignore the possibility of further accidentally vanishing eigenvalues about which one can't learn anything from this discussion: if the above discussion does not require a $(\text{mass})^2$ eigenvalue to vanish, assume that it is positive.)

3. Consider the symmetry-breaking mechanism of the Standard Model from the point of view of Problem 2, with the scalar fields considered as a quartet of real fields $\phi_{4\mathrm{R}i}$ given in terms of the standard complex doublet $\varphi_{2\mathrm{C}a}$ by

$$\varphi_{2\mathrm{C}} = \begin{pmatrix} \phi_{4\mathrm{R}3} + \mathrm{i}\phi_{4\mathrm{R}4} \\ \phi_{4\mathrm{R}1} + \mathrm{i}\phi_{4\mathrm{R}2} \end{pmatrix} .$$

Let the scalar vacuum be

$$\bar{\phi}_{4\mathrm{R}} = \begin{pmatrix} v \\ 0 \\ 0 \\ 0 \end{pmatrix} .$$

• Show that the $\mathrm{SU}(2)_\mathrm{L}$ generators $T^{i=1,2,3}_{4\mathrm{R}}$ and the $U(1)_Y$ generator $T^4_{4\mathrm{R}}$ in this scheme take the purely imaginary and antisymmetric forms

$$T^1_{4\mathrm{R}} = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad T^2_{4\mathrm{R}} = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

$$T^3_{4\mathrm{R}} = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \qquad T^4_{4\mathrm{R}} = \frac{\mathrm{i}}{2}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} .$$

The general expression for the vector mass (matrix)^2 in this real-field scheme is $(M^2)^{ij}_{\mathrm{gen}} = g_{\underline{i}} g_{\underline{j}} (\bar{\phi}_{4\mathrm{R}}, T^i_{4\mathrm{R}} T^j_{4\mathrm{R}} \bar{\phi}_{4\mathrm{R}})$, in which $(v, w) = v_A w_A$ is the standard $\mathbb{R}_4$ inner product, g_i is g_T for all $T^i_{4\mathrm{R}}$ in a simple G group factor and the underlined $\underline{i}, \underline{j}$ do not provoke Einstein summation. Noting that $(T^1_{4\mathrm{R}})^2 = (T^2_{4\mathrm{R}})^2 = (T^3_{4\mathrm{R}})^2 = \frac{1}{4}\mathbb{1}_{4\times 4}$ and $T^3_{4\mathrm{R}} T^4_{4\mathrm{R}} = T^4_{4\mathrm{R}} T^3_{4\mathrm{R}} = \frac{1}{4}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, we found in the lectures that the SM vector mass (matrix)^2 takes the form

$$(M^2)^{ij}_{\mathrm{SM}} = \frac{1}{4}\begin{pmatrix} g_2^2 v^2 & 0 & 0 & 0 \\ 0 & g_2^2 v^2 & 0 & 0 \\ 0 & 0 & g_2^2 v^2 & -g_2 g_1 v^2 \\ 0 & 0 & -g_2 g_1 v^2 & g_1^2 v^2 \end{pmatrix}$$

where g_2 is the $\mathrm{SU}(2)_\mathrm{L}$ Yang-Mills coupling constant and g_1 is the $\mathrm{U}(1)_\mathrm{Y}$ hypercharge coupling constant.

• Then show that under an assumption only that the SM gauge group $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$ breaks spontaneously down to the little group $\mathrm{U}(1)_\mathrm{EM}$ generated by $T^3_{4\mathrm{R}} + Y_{4\mathrm{R}}$, the vector mass (matrix)^2 would generically have a structure

$$(M^2)^{ij}_\mathrm{gen} = \frac{1}{4}\begin{pmatrix} g_2^2 v^2 & 0 & 0 & 0 \\ 0 & g_2^2 v^2 & 0 & 0 \\ 0 & 0 & g_2^2 u^2 & -g_2 g_1 u^2 \\ 0 & 0 & -g_2 g_1 u^2 & g_1^2 u^2 \end{pmatrix}$$

in which u is not neccessarily equal to v.

• In order to do this, it may be convenient to show firstly that the requirement that the $\mathrm{U}(1)_\mathrm{EM}$ generated by $T^3_{4\mathrm{R}} + Y_{4\mathrm{R}}$ be unbroken, *i.e.* that $(T^3_{4\mathrm{R}} + Y_{4\mathrm{R}})\bar{\phi}_{4\mathrm{R}} = 0$, combined with use of the $G = \mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$ Lie algebra implies $(M^2)^{11}_\mathrm{gen} = (M^2)^{22}_\mathrm{gen}$ and $(M^2)^{12}_\mathrm{gen} = (M^2)^{21}_\mathrm{gen} = 0$ and $(M^2)^{23}_\mathrm{gen} = (M^2)^{13}_\mathrm{gen} = 0$ and $(M^2)^{14}_\mathrm{gen} = (M^2)^{24}_\mathrm{gen} = 0$. Consequently, the upper-left 2×2 block of $(M^2)^{ij}_\mathrm{gen}$ must be diagonal, while the upper-right and lower-left 2×2 blocks must vanish.

• Find that the remaining lower right-hand block is required to be symmetric, but is not otherwise constrained by the $\mathrm{SU}(2)_\mathrm{L} \times \mathrm{U}(1)_\mathrm{Y}$ algebra since $Y_{4\mathrm{R}}$ commutes with itself and with the $T^i_{4\mathrm{R}}$. However, this remaining lower right-hand 2×2 block is still constrained by the requirement that $(M^2)^{ij}_\mathrm{gen}$ have a zero eigenvalue, which thus requires the lower right-hand block's determinant to vanish, yielding the above structure.

The special SM condition $u = v$ is the consequence of the "custodial SU(2)" symmetry.

4. * Consider the different possibilities for symmetry breaking in a theory of a scalar field ϕ_i that is invariant under a rigid $SO(3)$ symmetry, with the scalar field taken in the fundamental triplet representation of the group. The structure constants of $SO(3)$ are $f^{ijk} = \epsilon^{ijk}$, so the adjoint $\simeq$ triplet representation generators are $T^i{}_{jk} = -\mathrm{i}\epsilon^{ijk}$.
a) Consider the most general element of the SO(3) algebra acting on a constant vacuum value $\bar{\phi}_i$ and find

- which generators, or linear combinations of generators, are broken and which are unbroken
- the stability (aka little) group
- which combinations of the fields ϕ_i are massless and which will generally be massive, when expanded around $\bar{\phi}_i$ in the following cases, in which v is a real constant:

$$\text{a)}\ \bar{\phi}_i = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{b)}\ \bar{\phi}_i = v \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \text{c)}\ \bar{\phi}_i = \frac{v}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad \text{d)}\ \bar{\phi}_i = \frac{v}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

Find the most general form of the expectation value $\bar{\phi}_i$. Show that one always has 0 or 2 broken generators.

5. Show that a general complex but nonsingular (*i.e.* with nonvanishing determinant) matrix $\mathbf{M}$ can be made diagonal with positive real entries by use of a biunitary transformation, $\mathbf{M} \to \mathbf{V}^\dagger \mathbf{M} \mathbf{U}$ where $\mathbf{V}$ and $\mathbf{U}$ are unitary matrices. Do this as follows:

   a) State why $\mathbf{M}^\dagger \mathbf{M}$ is straightforward to diagonalize by a unitary transformation. Use a unitary transformation $\mathbf{U}$ to do this and show that the resulting diagonal entries are real and positive. Call this diagonal matrix $\mathbf{D}^2$ Define the matrix $\mathbf{D}$ to be the diagonal matrix whose entries are the positive square roots of the corresponding entries in $\mathbf{D}^2$.

   b) Let $\mathbf{H} = \mathbf{U}\mathbf{D}\mathbf{U}^\dagger$. Show that $\tilde{\mathbf{U}} = \mathbf{M}\mathbf{H}^{-1}$ is unitary. Thus, show that

$$\mathbf{V}^\dagger \mathbf{M} \mathbf{U} = \mathbf{D} \tag{1}$$

   where $\mathbf{V} = \tilde{\mathbf{U}}\mathbf{U}$ and $\mathbf{U}$ are unitary.

   This can be extended to the case of singular matrices by showing that they can be first be transformed to block-diagonal form with the zero eigenvalues on the diagonal in the upper left corner, then with a nonsingular sub-block to which the above argument can be applied.

6. Given three complex numbers $A,\ B,\ C$ satisfying $A+B+C=0$, show that the area of the triangle with vertices at the two-dimensional points 0, A, $A+B$ is given by $\frac{1}{2}|\mathfrak{Im}(AB^*)| = \frac{1}{2}|\mathfrak{Im}(AC^*)| = \frac{1}{2}|\mathfrak{Im}(BC^*)|$.