The Standard Model and Beyond: Problem Set 1

1. a) Derive the Palatini identity \( \delta F_{\mu\nu} = D_\mu \delta A^i_{\nu} - D_\nu \delta A^i_{\mu} \) for infinitesimal variations of a nonabelian field strength arising from a variation \( \delta A^i_{\mu} \) of the Yang-Mills gauge field, in which \( D_\mu \) is a covariant derivative for the adjoint representation. Recall that \( T_{\text{adj}}^{kij} = -if^{kij} \) for the adjoint representation.

b) Derive the nonabelian identity \( D_\mu \tilde{F}^{\mu\nu} = 0 \) where \( \tilde{F}^{\mu\nu} := \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \) in which \( D_\mu \) is here a covariant derivative for the adjoint representation while \( (D_\mu)_{A}^{B} = \partial_\mu \delta_{A}^{B} - ig A_\mu \cdot T_{A}^{B} \) in a general representation \( T_{A}^{B} \). One then has \( [D_\mu, D_\nu]_{AB} = -ig F_{\mu\nu} \cdot T_{A}^{B}. \) It may be helpful to use the Jacobi identity \([D_\rho, [D_\mu, D_\nu]] + \text{cycle (} \rho, \mu, \nu \text{)} = 0.\)

c) Show that the infinitesimal variation of \( F^{\mu\nu} \tilde{F}^{\mu\nu} \) is a total derivative, so such terms in the field-theory Lagrangian do not contribute to the classical field equations.

2. a) Show that the complex \( n \)-dimensional fundamental representation \( \varphi_a \) of \( SU(n) \), with infinitesimal transformations generated by Hermitean Lie algebra generators \( T_i^a \)

\[
\delta \varphi_a = i \delta \theta^i T_i^a \varphi_b \\
\delta \varphi^*a = -i \delta \theta^i \varphi^*b T_i^a
\]

can be recast as a \( 2n \) dimensional real representation with Hermitean generators. One way to label this \( 2n \) dimensional real form is to use a \( 2n \) valued hybrid index notation \( A \leftrightarrow a \hat{z} \), in which one has \( \phi_A = \phi_{a\hat{z}} \) with \( a = 1, \ldots, n; \hat{z} = 1, 2 \), setting

\[
\phi_{a1} = \frac{1}{2}(\varphi_a + \varphi^{*a}), \quad \phi_{a2} = \frac{1}{2}(\varphi_a - \varphi^{*a})
\]

Demonstrate that the generators \( T_i^{bc} \) in this real form, with \( B \leftrightarrow b\hat{w}, \) etc., are Hermitean \( 2n \times 2n \) matrices. Since the \( 2n \) dimensional real form must have purely imaginary generators (so that \( iT_i \) is real), show that the generators of this real form are purely antisymmetric \( 2n \times 2n \) matrices.

b) Using the results of part a), show that for this real-form representation, the matrix

\[
S^{ij} = (T^i \bar{\phi}, T^j \bar{\phi})
\]

is real and symmetric, and thus diagonalizable, where \((v, w)\) is the usual \( 2n \) dimensional vector space inner product, \((v, w) = v_A w_A\) and \( \bar{\phi}_B \) is the vacuum value of \( \phi_B \). Hence, starting from the (mass)\(^2\) constraint \( M_{AB}^2 T_i^{bc} \bar{\phi}_c = 0 \), show that there is precisely one zero eigenvalue of \( M_{AB}^2 \) for each non-vanishing \( T_i \bar{\phi} \) vector. (Ignore the possibility of further accidentally vanishing eigenvalues about which one can’t learn anything from this discussion: if the above discussion does not require a (mass)\(^2\) eigenvalue to vanish, assume that it is positive.)
3. Consider the symmetry-breaking mechanism of the Standard Model from the point of view of Problem 2, with the scalar fields considered as a quartet of real fields $\phi_{4R}$ given in terms of the standard complex doublet $\varphi_{2c}$ by

$$\varphi_{2c} = \left( \phi_{4R3} + i\phi_{4R4}, \phi_{4R1} + i\phi_{4R2} \right).$$

Let the scalar vacuum be

$$\bar{\varphi}_{4R} = \begin{pmatrix} v \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

• Show that the SU(2)$_L$ generators $T_{4R}^{i=1,2,3}$ and the $U(1)_Y$ generator $T_{4R}^4$ in this scheme take the purely imaginary and antisymmetric forms

$$T_{4R}^1 = \frac{i}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \quad T_{4R}^2 = \frac{i}{2} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

$$T_{4R}^3 = \frac{i}{2} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, \quad T_{4R}^4 = \frac{i}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

The general expression for the vector mass (matrix)$^2$ in this real-field scheme is $(M_{ij}^2)_{\text{gen}} = g_i g_j (\bar{\varphi}_{4R}, T_{4R}^i T_{4R}^j \bar{\varphi}_{4R})$, in which $(v, w) = v_\lambda w_\lambda$ is the standard $\mathbb{R}_4$ inner product, $g_i$ is $g_T$ for all $T_{4R}^i$ in a simple $G$ group factor and the underlined $i, j$ do not provoke Einstein summation. Noting that $(T_{4R}^1)^2 = (T_{4R}^2)^2 = (T_{4R}^3)^2 = \frac{1}{4} \mathbb{I}_{4\times4}$ and $T_{4R}^3 T_{4R}^4 = T_{4R}^4 T_{4R}^3 = \frac{1}{4} \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, we found in the lectures that the SM vector mass (matrix)$^2$ takes the form

$$(M_{ij}^2)_{\text{SM}} = \frac{1}{4} \begin{pmatrix} g_2^2 v^2 & 0 & 0 & 0 \\ 0 & g_2^2 v^2 & 0 & 0 \\ 0 & 0 & g_2^2 v^2 & -g_2 g_1 v^2 \\ 0 & 0 & -g_2 g_1 v^2 & g_1^2 v^2 \end{pmatrix}.$$
where $g_2$ is the SU(2)$_L$ Yang-Mills coupling constant and $g_1$ is the U(1)$_Y$ hypercharge coupling constant.

- Then show that under an assumption only that the SM gauge group SU(2)$_L \times$ U(1)$_Y$ breaks spontaneously down to the little group U(1)$_{EM}$ generated by $T^3_{4r} + Y_{4r}$, the vector mass (matrix)$^2$ would generically have a structure

$$(M^2)_{ij}^{gen} = \frac{1}{4} \begin{pmatrix} g_2^2 v^2 & 0 & 0 & 0 \\ 0 & g_2^2 v^2 & 0 & 0 \\ 0 & 0 & g_2^2 u^2 & -g_2 g_1 u^2 \\ 0 & 0 & -g_2 g_1 u^2 & g_1^2 u^2 \end{pmatrix}$$

in which $u$ is not necessarily equal to $v$.

- In order to do this, it may be convenient to show firstly that the requirement that the U(1)$_{EM}$ generated by $T^3_{4r} + Y_{4r}$ be unbroken, i.e. that $(T^3_{4r} + Y_{4r})\bar{\phi}_{4r} = 0$, combined with use of the $G = SU(2)_L \times U(1)_Y$ Lie algebra implies $(M^2)_{11}^{gen} = (M^2)_{22}^{gen}$ and $(M^2)_{12}^{gen} = (M^2)_{21}^{gen} = 0$ and $(M^2)_{23}^{gen} = (M^2)_{13}^{gen} = 0$ and $(M^2)_{14}^{gen} = (M^2)_{24}^{gen} = 0$. Consequently, the upper-left $2 \times 2$ block of $(M^2)_{ij}^{gen}$ must be diagonal, while the upper-right and lower-left $2 \times 2$ blocks must vanish.

- Find that the remaining lower right-hand block is required to be symmetric, but is not otherwise constrained by the SU(2)$_L \times$ U(1)$_Y$ algebra since $Y_{4r}$ commutes with itself and with the $T^i_{4r}$. However, this remaining lower right-hand $2 \times 2$ block is still constrained by the requirement that $(M^2)_{ij}^{gen}$ have a zero eigenvalue, which thus requires the lower right-hand block’s determinant to vanish, yielding the above structure.

The special SM condition $u = v$ is the consequence of the “custodial SU(2)” symmetry.

4. Consider the different possibilities for symmetry breaking in a theory of a scalar field $\phi_i$ that is invariant under a rigid SO(3) symmetry, with the scalar field taken in the fundamental triplet representation of the group. The structure constants of SO(3) are $f^{ijk} = \epsilon^{ijk}$, so the adjoint $\simeq$ triplet representation generators are $T^i_{jk} = -i\epsilon^{ijk}$.

a) Consider the most general element of the SO(3) algebra acting on a constant vacuum value $\bar{\phi}_i$ and find

- which generators, or linear combinations of generators, are broken and which are unbroken
- the stability (aka little) group
- which combinations of the fields $\phi_i$ are massless and which will generally be massive, when expanded around $\bar{\phi}_i$ in the following cases, in which $v$ is a real constant:
a) $\bar{\phi}_i = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$,  

b) $\bar{\phi}_i = v \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$,  
c) $\bar{\phi}_i = \frac{v}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$,  
d) $\bar{\phi}_i = \frac{v}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$.

Find the most general form of the expectation value $\bar{\phi}_i$. Show that one always has 0 or 2 broken generators.

5. Show that a general complex but nonsingular (i.e. with nonvanishing determinant) matrix $M$ can be made diagonal with positive real entries by use of a biunitary transformation, $M \rightarrow V^\dagger M U$ where $V$ and $U$ are unitary matrices. Do this as follows:

a) State why $M^\dagger M$ is straightforward to diagonalize by a unitary transformation. Use a unitary transformation $U$ to do this and show that the resulting diagonal entries are real and positive. Call this diagonal matrix $D^2$. Define the matrix $D$ to be the diagonal matrix whose entries are the positive square roots of the corresponding entries in $D^2$.

b) Let $H = U D U^\dagger$. Show that $\tilde{U} = M H^{-1}$ is unitary. Thus, show that

$$V^\dagger M U = D$$

(1)

where $V = \tilde{U} U$ and $U$ are unitary.

This can be extended to the case of singular matrices by showing that they can be first be transformed to block-diagonal form with the zero eigenvalues on the diagonal in the upper left corner, then with a nonsingular sub-block to which the above argument can be applied.

6. Given three complex numbers $A$, $B$, $C$ satisfying $A + B + C = 0$, show that the area of the triangle with vertices at the two-dimensional points $0, A, A + B$ is given by $\frac{1}{2} |\text{Im}(AB^*)| = \frac{1}{2} |\text{Im}(AC^*)| = \frac{1}{2} |\text{Im}(BC^*)|$. 

4