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The Standard Model and Beyond: Problem Set 2

1. This is a problem about vacuum orbits in an SU(3) symmetric model, suggested by the penultimate paragraph of Peter Higgs' famous 1966 paper. Consider a field transforming in the adjoint representation of SU(3). This may be written in terms of 3×3 hermitian traceless matrices $\Phi_i{}^j$, $i, j = 1, 2, 3$, transforming as $\Phi \to U \Phi U^\dagger$ where $U_i{}^j \in \mathrm{SU}(3)$. Let the potential for Φ be $V = \frac{1}{4}(\mathrm{tr}(\Phi^2) - k^2)^2$.

   a) It is a standard result in matrix algebra that a hermitian matrix M can be diagonalized by a unitary similarity transformation $M \to S M S^{-1}$, where S is unitary. Hence, show that Φ can be diagonalized by an SU(3) transformation. Also show, however, that the eigenvalues of Φ are never changed by such an SU(3) transformation.

   b) Find the condition determining the vacuum $\bar{\Phi}$ values in the above potential. Show that the vacuum orbits may be characterized by the eigenvalues of $\bar{\Phi}$, and hence that one may parametrize the vacuum orbits by the diagonalized $\bar{\Phi}$ matrix on a given orbit. How many free parameters determine a given vacuum orbit for this model?

   c) Show that for generic $\bar{\Phi}$ eigenvalues, the little group H is $\mathrm{U}(1) \times \mathrm{U}(1)$. Give an explicit form for such little-group matrices when $\bar{\Phi}$ is diagonal. How many Goldstone modes and how many Higgs modes are there for such a generic vacuum? In a gauged extension of this model, the Goldstone modes, but not the Higgs, will be absorbed into vector-field masses.

   d) Find the masses of the Higgs modes for a diagonal $\bar{\Phi}$ vacuum with eigenvalues of the form $(\lambda, 0, -\lambda)$. Are there Higgs modes with accidentally vanishing masses in this case, even though such vanishing is not required by Goldstone's theorem? Remember that masses are determined purely by the expansion in fluctuation fields to second order about the given vacuum – there may in addition be cubic and quartic interaction terms. The mass spectra of Higgs, *i.e.* non-Goldstone, modes varies significantly from one model to another.

   e) Find a special set of $\bar{\Phi}$ eigenvalues for which the little group becomes $\mathrm{U}(2) \cong \mathrm{SU}(2) \times \mathrm{U}(1)$. How many genuine Goldstone modes and how many Higgs modes are there for vacua on such an orbit? For a diagonal $\bar{\Phi}$ of this special type, identify the matrix elements that correspond to the Goldstone modes and the matrix elements that correspond to the Higgs modes. Limit the analysis to infinitesimal fluctuations away from the vacuum $\bar{\Phi}$.

   f) In what representations of H do the Higgs fields transform for vacua with $H = \mathrm{U}(2)$? How many of these modes are accidentally massless, even though this is not required by Goldstone's theorem? What would happen to vector-field masses as one approached this orbit from neighboring generic orbits in a gauged version of this model?

2. In a seesaw mechanism model for Majorana spinors N and n with mass terms $\mathcal{L}_{\text{mass}} = -i(M\bar{N}N + \frac{vp}{2\sqrt{2}}(\bar{n}N + \bar{N}n))$, show that one finds mass eigenstates $n_\pm$ with mass values $m_\pm = \frac{M}{2}(1 \pm \sqrt{1 + \frac{p^2v^2}{2M^2}})$. For $M \gg \frac{pv}{\sqrt{2}}$, one thus finds a very large eigenvalue $m_+ \simeq M$ and an eigenvalue $m_- \simeq -\frac{p^2v^2}{8M}$ with magnitude $|m_-| \ll \frac{pv}{\sqrt{2}}$. Find the corresponding low-mass Majorana spinor field n_- as a combination of N and n. Show that the negative sign in m_- is not physically meaningful by making a field redefinition of the corresponding spinor $n_- \to \tilde{n}_- = i\gamma_5 n_-$ which a) preserves the Majorana condition for $\tilde{n}_-$ and b) flips the sign of the $-im_-\bar{\tilde{n}}_-\tilde{n}_-$ mass term while leaving invariant the kinetic term $-i\bar{\tilde{n}}_-\gamma^\mu\partial_\mu\tilde{n}_-$.

3. For a complex symmetric matrix $\mathbf{S}$, consider the eigenvectors of $\mathbf{M} = \mathbf{S}\mathbf{S}^\dagger$ and first show that $\mathbf{M}$ may be diagonalized by a unitary matrix $\mathbf{U}$ constructed from the eigenvectors of $\mathbf{M}$, *i.e.* $\mathbf{U}^\dagger\mathbf{M}\mathbf{U} = \mathbf{D}$ where $\mathbf{D} = \text{diag}(h_1, h_2, \ldots, h_N)$ is a diagonal matrix with real nonnegative elements. Then let $\mathbf{T} = \mathbf{U}^\dagger\mathbf{S}(\mathbf{U}^\dagger)^{\text{T}}$ so $\mathbf{T}\mathbf{T}^\dagger = \mathbf{D}$. Show that $[\mathbf{D}, \mathbf{T}] = \mathbf{0}$ and consequently that $T_{i\ell}(h_i - h_\ell) = 0$, so that $\mathbf{T}$ must itself be diagonal provided the eigenvalues h_i are all distinct. Consequently, show that one may choose phases for the eigenvalues of $\mathbf{T}$ such that $\mathbf{T} = \text{diag}(\sqrt{h_1}, \sqrt{h_2}, \ldots, \sqrt{h_N})$.

   Writing $\mathbf{S} = \mathbf{U}\mathbf{D}\mathbf{U}^{\text{T}}$ for $\mathbf{U}$ unitary and $\mathbf{D}$ diagonal is known as Takagi factorization.

4. * In a system with N generations and PMNS unitary matrix U which diagonalizes the neutrino mass matrix, the relation between flavour eigenstates $|\nu_\alpha\rangle$ and mass eigenstates $|i\rangle$ is $|\nu_\alpha\rangle = \sum_i U^*_{\alpha i}|i\rangle$. Consider ultrarelativistic neutrinos with momentum $p = |\vec{p}| >> m_i$ for any of the mass eigenvalues m_i, whose energies can accordingly be approximated by $E_i = \sqrt{p_i^2 + m_i^2} \simeq p_i + \frac{m_i^2}{2p_i} \approx E + \frac{m_i^2}{2E}$. For ultrarelativistic neutrinos and $c = 1$, one has a time of flight T to distance travelled L relation $T \approx L$.

   Show that the probability for a neutrino originally of flavour α to be later observed with flavour β is then to leading order

$$P_{\alpha\to\beta} = |\langle\nu_\beta|\nu_\alpha(T)\rangle|^2 = |\sum_i U^*_{\alpha i}U_{\beta i}e^{-im_i^2L/2E}|^2 \quad .$$

   Show that this probablility may be rewritten as

$$P_{\alpha\to\beta} = \delta_{\alpha\beta} - 4\sum_{i>j}\text{Re}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin^2\left(\frac{\Delta m^2_{ij}L}{4E}\right) + 2\sum_{i>j}\text{Im}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin\left(\frac{\Delta m^2_{ij}L}{2E}\right)$$

   where $\Delta m^2_{ij} = m_i^2 - m_j^2$.

   The CP asymmetry is $A^{\alpha\beta}_{\text{CP}} = P_{\alpha\to\beta} - P_{\bar{\alpha}\to\bar{\beta}} = 4\sum_{i>j}\text{Im}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j})\sin\left(\frac{\Delta m^2_{ij}L}{2E}\right)$. In terms of the Jarlskog invariant J, determined in the $N = 3$ case by $\text{Im}(U^*_{\alpha i}U_{\beta i}U_{\alpha j}U^*_{\beta j}) = -J\sum_{\gamma,k}\epsilon_{\alpha\beta\gamma}\epsilon_{ijk}$, show that the CP asymmetry is given by

$$A^{\alpha\beta}_{\text{CP}} = 16J\sum_\gamma \epsilon_{\alpha\beta\gamma}\sin\left(\frac{\Delta m^2_{21}L}{4E}\right)\sin\left(\frac{\Delta m^2_{32}L}{4E}\right)\sin\left(\frac{\Delta m^2_{31}L}{4E}\right) \quad .$$