20 February 2024

The Standard Model and Beyond: Problem Set 3

1. * In this problem, we consider charged why charged pions decay preferentially into muons instead of electrons. The mechanism is known as "helicity suppression".

   (a) Transform the Dirac equation $(\gamma_\mu \partial^\mu + m)\psi = 0$ into momentum space with coordinates p^μ. Show that solutions to the momentum-space Dirac equation satisfy

$$(p^\mu p_\mu + m^2)\psi(p) = 0 \quad \text{and} \quad \psi(p) = \zeta \left(\frac{-p_i \gamma^0 \gamma^i + i m \gamma^0}{\sqrt{\mathbf{p}^2 + m^2}} \right) \psi(p)$$

   with $\zeta = 1$ if $p^0 > 0$ and $\zeta = -1$ if $p^0 < 0$, where the p_i are components of the spinor field's three-momentum, $i = 1, 2, 3$, and $|\mathbf{p}| = \sqrt{p^i p^i}$. The former solutions are called particle solutions; the latter are antiparticle solutions.

   The helicity Λ of a particle is the projection of its angular momentum 3-vector in the direction of the spatial momentum. Given that the Lorentz generators for spinors are $M_{\mu\nu} = -\frac{\mathrm{i}}{2}\gamma_{\mu\nu} = -\frac{\mathrm{i}}{4}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$ and the spatial angular-momentum 3-vector operator is $S_i = \frac{1}{2}\epsilon_{ijk} M_{jk}$, show that the helicity operator Λ can be written as

$$\Lambda = -\tfrac{1}{2}\gamma^5 \gamma^0 \gamma^i \frac{p_i}{|\mathbf{p}|} \tag{1}$$

   Hence, show that the helicity Λ of left- or right-chirality particles satisfies

$$\left.\begin{aligned} \Lambda\psi_L(p) &= \left(\frac{\zeta\sqrt{\mathbf{p}^2 + m^2}\,\psi_L(p) - i m \gamma^0 \psi_R(p)}{2|\mathbf{p}|} \right) \\ \Lambda\psi_R(p) &= -\left(\frac{\zeta\sqrt{\mathbf{p}^2 + m^2}\,\psi_R(p) - i m \gamma^0 \psi_L(p)}{2|\mathbf{p}|} \right) \end{aligned}\right\} \quad \zeta = \pm 1 \text{ for } p^0 \gtrless 0 \quad \text{where} \quad \gamma^5 \psi_{\substack{L\\R}} = \pm \psi_{\substack{L\\R}} \; . \tag{2}$$

   Hence, show that for massless or for ultrarelativistic spinor particles, chirality is related to helicity by

$$\begin{aligned} \Lambda\psi_L(p) &\sim \tfrac{\zeta}{2}\psi_L(p) \\ \Lambda\psi_R(p) &\sim -\tfrac{\zeta}{2}\psi_R(p) \; . \end{aligned} \tag{3}$$

   (b) Consider now a positively charged pion π^+ at rest, decaying into a positively charged lepton and the corresponding neutrino by exchange of a W^+_μ intermediate vector boson. An example is shown in Figure 1.

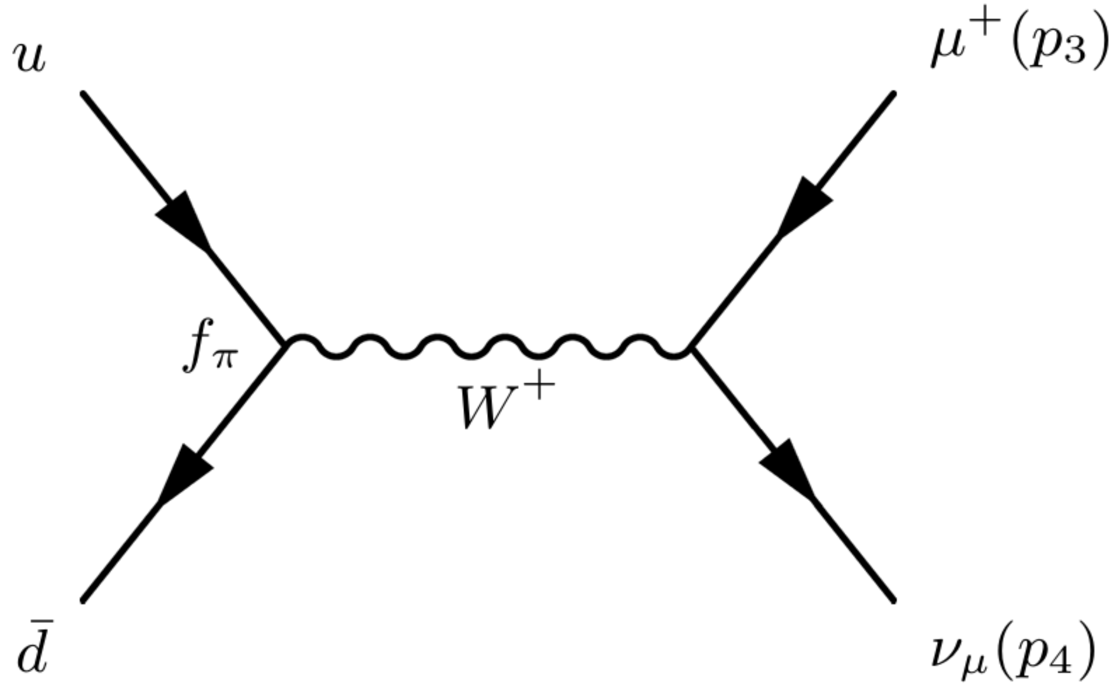

Figure 1: Pion decay into a muon

The problem is to explain the basic reason why the decay process shown in Figure **??** dominates over the analogous decay into a positron and its corresponding neutrino. In the minimal Standard Model, neutrinos are massless. Show that for the decay of a π^+ at rest, the charged lepton and the neutrino decay products would have to have the same helicity if the lepton were massless. Using the form of the Standard Model interaction between W_μ^+, the charged lepton and the neutrino, show that the decay of a pion via the process of Figure **??** would thus be impossible if the charged lepton were massless. For massive charged leptons, this helicity suppression is only partially effective, but it is stronger for lower mass charged leptons. Since the muon is about 200 times heavier than the electron, the helicity suppression for decay into muons is correspondingly weaker.

2. Now consider how to evaluate the relative rates of negatively charged pion decay into muons and electrons in the low-energy effective theory with a current-current interaction

$$\mathcal{L}_{\text{eff}} = 2\sqrt{2}G_F(J^{+\mu}J_\mu^- + J_Z^\mu J_{Z\mu})\,, \tag{4}$$

where only the $J_\mu^\pm$ charged currents participate in charged pion decay. In the effective theory, for decay of π^+ into a charged lepton ℓ and its corresponding neutrino ν_ℓ, these are

$$\begin{aligned} J^+_\mu &= \frac{1}{\sqrt{2}} c_1 f_\pi \partial_\mu \pi^+ \\ J^-_\mu &= -\frac{1}{2}\bar{\psi}_{\nu_\ell}\gamma_\mu(1+\gamma_5)\psi_\ell \end{aligned} \tag{5}$$

where f_π is the charged pion decay constant and c_1 is a constant that needs to be obtained experimentally.

The relevant term in the effective action is thus

$$\mathcal{L}_{\text{eff}} = -G_F c_1 f_\pi \partial^\mu \pi^+ \bar{\psi}_{\nu_\ell}\gamma_\mu(1+\gamma_5)\psi_\ell\,. \tag{6}$$

One obtains momentum-space Feynman rules for a quantum spinor field ψ by expanding in terms of $b_s(\mathbf{p})$, $d^\dagger_s(\mathbf{p})$ annihilation and creation operators, $\psi(x) = \sum_{\text{s}=\pm}\int \frac{d^3p}{2(2\pi)^3 p^0}[b_s(\mathbf{p})u_s(\mathbf{p})e^{ip\cdot x} + d^\dagger_s(\mathbf{p})v_s(\mathbf{p})e^{-ip\cdot x}]$, where $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ are c-number (*i.e.* ordinary commuting-number) spinor wavefunctions for incoming and outgoing particles respectively. The Dirac conjugates $\bar{u}_s(\mathbf{p})$ and $\bar{v}_s(\mathbf{p})$ are the wavefunctions for outgoing and incoming antiparticles. The tree level momentum-space decay amplitude for π^+ is then

$$\mathcal{T} = -iG_F c_1 f_\pi k^\mu \pi^+ \bar{u}_{\nu_\ell}\gamma_\mu(1+\gamma_5)v_\ell \tag{7}$$

where v_ℓ is the wavefunction of the outgoing charged lepton, $\bar{u}_{\nu_\ell}$ is the wavefunction of the outgoing neutrino and the four-momenta of the pion, charged lepton, and neutrino are k^μ, p_3^μ and p_4^μ as shown in Figure **??**. Use then $(-\not{p}_3 - im_\ell)v_\ell = 0$ and $-\not{p}_4 u_{\nu_\ell} = 0$ (remembering that the charged lepton and neutrino are outgoing) to show

$$\mathcal{T} = -G_F c_1 f_\pi \pi^+ m_\ell \bar{u}_{\nu_\ell}(1-\gamma_5)v_\ell \tag{8}$$

so $\mathcal{T}$ is proportional to the charged lepton mass m_ℓ. Squaring the modulus of $\mathcal{T}$ and ignoring normalisation factors in the π^+ wavefunction, one has

$$\begin{aligned} |\mathcal{T}|^2 &= -(G_F c_1 f_\pi m_\ell)^2 \bar{u}_{\nu_\ell}(1+\gamma_5)v_\ell \bar{v}_\ell(1-\gamma_5)u_{\nu_\ell} \\ &= -(G_F c_1 f_\pi m_\ell)^2\, \text{tr}\Big((1+\gamma_5)v_\ell\bar{v}_\ell(1-\gamma_5)u_{\nu_\ell}\bar{u}_{\nu_\ell}\Big)\,. \end{aligned} \tag{9}$$

Then perform a sum over spins of the final states using the spin-sum rules

$$\sum_{\text{spins}} v_\ell \bar{v}_\ell = -\not{p}_3 - im_\ell \tag{10}$$

$$\sum_{\text{spins}} u_{\nu_\ell}\bar{u}_{\nu_\ell} = -\not{p}_4 \tag{11}$$

to show that

$$\sum_{\text{spins}} |\mathcal{T}|^2 = (G_F c_1 f_\pi m_\ell)^2(-8p_3 \cdot p_4)$$
$$= 4(G_F c_1 f_\pi m_\ell)^2(m_\pi^2 - m_\ell^2) \tag{12}$$

The decay rate Γ is given by

$$\Gamma_\pi = \frac{1}{\tau_\pi} = \frac{|\mathbf{p}_3|}{8\pi m_\pi^2}\langle|\mathcal{T}|^2\rangle\,. \tag{13}$$

Show that the ratio of pion decay rates into electrons and into muons is consequently

$$R_\pi = \left(\frac{m_e}{m_\mu}\right)^2 \left(\frac{m_\pi^2 - m_e^2}{m_\pi^2 - m_\mu^2}\right)^2\,. \tag{14}$$

Inputting the experimental values of m_π, m_μ and m_e, one obtains $R_\pi = 1.283 \times 10^{-4}$.

3. The charge conjugation matrix corresponding to a mostly-plus Minkowski metric $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$ may be taken to be $C = -\gamma^0\gamma^2$ in notation where the Dirac conjugate of a spinor ψ is given by $\bar{\psi} := \psi^\dagger\gamma^0$.

   - Show the following:

$$
\begin{array}{ccc}
C^\dagger = C & C^{\mathrm{T}} = -C & C^2 = \mathbb{1} \\
\gamma^0\gamma^{\mu\,\dagger}\gamma^0 = \gamma^\mu & C\gamma^{\mu\,\mathrm{T}}C = -\gamma^\mu & C\gamma_5^{\mathrm{T}}C = \gamma_5 \\
\psi_{\mathrm{C}} := C(\bar{\psi})^{\mathrm{T}} = -\gamma^0 C\psi^* & \overline{(\gamma^\mu\psi)} = -\bar{\psi}\gamma^\mu & \overline{(\gamma_5\psi)} = -\bar{\psi}\gamma_5
\end{array}
$$

   - For generic Dirac spinor fields ψ and χ, derive the following transformation rules under charge conjugation for spinor bilinears, ignoring η_{C} phases [so just take $\psi \xrightarrow{\mathrm{C}} \psi_{\mathrm{C}} = C(\bar{\psi})^{\mathrm{T}}$]

$$
\begin{array}{cc}
\bar{\chi}\psi \xrightarrow{\mathrm{C}} \bar{\chi}_{\mathrm{C}}\psi_{\mathrm{C}} = \bar{\psi}\chi & \bar{\chi}\gamma_5\psi \xrightarrow{\mathrm{C}} \bar{\chi}_{\mathrm{C}}\gamma_5\psi_{\mathrm{C}} = \bar{\psi}\gamma_5\chi \\
\bar{\chi}\gamma^\mu\psi \xrightarrow{\mathrm{C}} \bar{\chi}_{\mathrm{C}}\gamma^\mu\psi_{\mathrm{C}} = -\bar{\psi}\gamma^\mu\chi & \bar{\chi}\gamma_5\gamma^\mu\psi \xrightarrow{\mathrm{C}} \bar{\chi}_{\mathrm{C}}\gamma_5\gamma^\mu\psi_{\mathrm{C}} = \bar{\psi}\gamma_5\gamma^\mu\chi \\
\bar{\chi}[\gamma^\mu,\gamma^\nu]\psi \xrightarrow{\mathrm{C}} \bar{\chi}_{\mathrm{C}}[\gamma^\mu,\gamma^\nu]\psi_{\mathrm{C}} = -\bar{\psi}[\gamma^\mu,\gamma^\nu]\chi &
\end{array}
$$

   Consider the Yukawa coupling $-ih_{mn}\bar{Q}^{Aa}_{\mathrm{L}m}u_{\mathrm{R}n}\tilde{\phi}_a + \mathrm{h.c}$ where m and n sum over the $1,2,3$ generations and $\tilde{\phi}_a = \epsilon_{ab}\phi^{*\,a}$. Show that this term is CP invariant provided h_{mn} is real. The charge conjugation transformation of ϕ_a is $\phi_a \xrightarrow{\mathrm{C}} \phi^{*a}$.

4. In order to preserve covariance under both gauge transformations and charge conjugation of the nonabelian covariant derivative $(D_\mu\psi)_A = \partial_\mu\psi_A - \mathrm{i}gA^i_\mu(T^i)_A{}^B\psi_B$ of a spinor field ψ_A, the gauge fields must transform $A^i_\mu \xrightarrow{\mathrm{C}} A'^i_\mu$ under charge conjugation as follows:

$$
\begin{array}{lll}
A'^i_\mu = -A^i_\mu & \text{for } (T^i)_A{}^B = (T^i)_B{}^A & \text{symmetric} \\
A'^i_\mu = A^i_\mu & \text{for } (T^i)_A{}^B = -(T^i)_B{}^A & \text{antisymmetric}
\end{array}
$$

   - Show that this charge conjugation transformation of the gauge fields A^i_μ leads to a covariant transformation of the field strength $F^i_{\mu\nu}$, *i.e.* that $F^i_{\mu\nu}$ flips sign when A^i_μ does but stays invariant if A^i_μ does.