The Standard Model and Beyond: Problem Set 4

1. A symmetric-space nonlinear realisation of a group \(G\) with linear realisation on a subgroup \(H\) is based on a Lie algebra

\[
[V_i, V_j] = i f_{ij}^k V_k \quad [V_i, A_\ell] = i f_{i\ell}^m A_m \quad [A_\ell, A_m] = i f_{\ell m}^k V_k
\]  

(1.1)

where the \(V_i\) are generators of the stability subgroup \(H\) and the \(A_\ell\) are generators of the \(G/H\) coset representatives. Note that this symmetric-space symmetry algebra admits an automorphism \(A_\ell \rightarrow -A_\ell\). For the nonlinear realisation, write \(g(x) = e^{i\xi(x)A_k} e^{i\theta(x)}V_i = u(\xi^k(x))h(\theta^i(x))\) where the \(\xi^k(x)\) are the nonlinearly transforming Goldstone fields. Upon transforming by an \(x^\mu\) independent element \(g_0\) of \(G\), one has

\[
g_0 g = u(\xi'^k(x))h(g_0, \xi'^j(x))
\]

(1.2)

by repolarizing into \(G/H\) and \(H\) factors.

For chiral symmetry derived from \(N\) underlying quark species, \(G = SU(N)_L \times SU(N)_R\) where \(H\) is the diagonal “vector” \(SU(N)_V\) subgroup, and the individual \(SU(N)_{L,R}\) factors act on the \(q_L\) left (\(\gamma_5\) eigenvalue +1) and the \(q_R\) right (\(\gamma_5\) eigenvalue -1) chiral components of the quarks independently. An arbitrary element \(g\) of \(G\) may be written \(g = L(\alpha_L)R(\alpha_R) = e^{i\alpha_L \cdot T_L} e^{i\alpha_R \cdot T_R} = e^{\frac{i}{2}(\alpha_L^1 + \alpha_R^1)T_1 + \frac{i}{2}(\alpha_L^2 - \alpha_R^2)\gamma_5 T_5}\) corresponding to \(V_i = 1_{4 \times 4} T_i\) for the diagonal \(SU(N)_V\) vector subgroup \(H\) generators and \(A_k = \gamma_5 T_k\) for the \(G/H\) axial coset generators, in which the \(T_k\) are generators of \(SU(N)\). The \(g_0\) transformation of equation (1.2) thus would have \(\xi_0^i = \frac{1}{2}(\alpha_L^1 - \alpha_R^1)\) and \(\theta_0^i = \frac{1}{2}(\alpha_L^2 + \alpha_R^2)\). The \(1_{4 \times 4}\) and \(\gamma_5\) matrices are inherited from the underlying spinorial quark structure; even though one is now dealing only with bosonic fields, these matrices play a key rôl in representing the \(SU(N)_L \times SU(N)_R\) algebra.

For original left and right chiral-projection quark transformations \(q \rightarrow q' = Lq_L + Rq_R\), one can define modified chiral quark fields \(q_L = u\tilde{q}_L\)\(, q_R = u^\dagger\tilde{q}_R\), where the form of the second definition is determined by the parity transform of the first, which sends \(\gamma_5 \rightarrow -\gamma_5\). This is the \(A_i \rightarrow -A_i\) automorphism for this \(G/H\) symmetric space. It is helpful to use a representation in which \(\gamma_5 = \text{diag}(1, -1)\) is diagonal (with \(1\) now a \(2 \times 2\) unit submatrix). Keeping the +1 eigenvalue associated to the upper-left \(2 \times 2\) subblock of this \(4 \times 4\) matrix, the effect of a left \(\leftrightarrow\) right parity transformation may then be viewed as interchanging the upper-left and lower-right \(2 \times 2\) subblocks of \(\gamma_5\) so \(u_P = \Lambda_P u P^{-1}\) with \(\Lambda_P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\).
• Show that the $G/H$ coset representative structure $u = e^{i\xi(x)\gamma_5 T_k}$ implies $u(x) = \text{diag}(w(x)1, w^\dagger(x)1)$ where $w(x) = e^{i\xi(x)T}$. The detailed form of the transformation $u \to u'$ is determined by the requirement of preserving this $G/H$ coset representative structure for $u' = e^{i\xi'(x)\gamma_5 T_k}$.

For a quark transformation $q \to q' = Lu\tilde{q}_l + Ru^\dagger\tilde{q}_r$ one accordingly has nonlinear transformations $Lu = u'h_1$, $Ru^\dagger = u''h_1$ and $\tilde{q}'_l = h_1\tilde{q}_l$, $\tilde{q}'_r = h_1\tilde{q}_r$, where $u' = u(\xi')$, $h_1 = h_1(g_0, u(x))$ and the forms of the $R$ transformation laws are determined by the parity transforms of the $L$ transformation laws, i.e. using $R = L_P = \Lambda_P L \Lambda_P^{-1}$.

• Show that for $SU(N)_L$ or $SU(N)_R$ transformations, i.e. $g_0 = L$ resp. $g_0 = R$, one thus has

$$u' = LH_1$$
$$u'' = h_1u^\dagger L^\dagger$$

(resp.)

$\text{Equivalently for} \quad u' = h_1uR^\dagger$ (1.3a)

where in each case $h_1 = h_1(g_0, u(x))$.

• Show that for a combined $g_0 = LR$ transformation $X = \text{diag}(\ell 1, r 1)$ that the induced $H$ transformation $h_1$ is determined by the nonlinear relation $h_1 w^\dagger \ell^\dagger = rw^\dagger h_1^\dagger$ equivalently for left-sided and right-sided versions of the transformation:

$$u' = Xu h_1^\dagger$$
$$u'' = h_1u^\dagger X^\dagger$$

(resp.)

$\text{Equivalently for} \quad u' = h_1uX_P^\dagger$ (1.4a)

where $X_P = \Lambda_P X \Lambda_P^{-1}$.

The standard left-sided transformation of the form (1.2) for a nonlinear realisation of $G$ (or the equivalent right-sided transformation from (1.4a)) requires finding $h_1(g_0, u(x))$ from the implicit relation given above. General $G$ invariants are then built by requiring local $h_1$ invariance for “matter” fields such as the $\tilde{q}$ quarks. Such constructions make use of the standard Maurer-Cartan form $udu^{-1} = udu^\dagger = Z \cdot A + M \cdot V$, where the $G/H$ coset projection $Z = Z_\mu dx^\mu$ transforms according to $Z \to h_1 Z h_1^\dagger$, while the $M = M_\mu dx^\mu$ stability subgroup $H$ projection transforms like a gauge field for $H$, i.e. $M \to h_1 dh_1^\dagger + h_1 M h_1^\dagger$. 

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In order to construct an invariant Lagrangian for the Goldstone fields \( u = e^{i \xi \cdot A} \) themselves, however, a more convenient form of the realisation is obtained by defining \( U = u^2 \).

- In order to construct a normalised Goldstone-field kinetic term, write \( \xi^k(x) = F_\pi \pi^k(x) \) so
  \[
  U(\pi(x)) = e^{\frac{2i\pi^k(x)}{F_\pi} \gamma_5 T_k}
  \]
  where \( T_k = \frac{1}{2} \lambda_k \) and for \( N = 2 \) the \( \lambda_k \) are the SU(2) Pauli matrices \( \sigma_k \) while for \( N = 3 \) the \( \lambda_k \) are the SU(3) Gell-Mann matrices. Show that the standard nonlinear-realisation kinetic term
  \[
  L_{\text{kin}} = \frac{F_\pi^2}{16} \text{tr}(Z_\mu Z^\mu) \]
  can then be written in the form
  \[
  L_{\text{kin}} = -\frac{F_\pi^2}{4} \text{tr}(\partial_\mu U \partial^\mu U^\dagger)
  \]
  where the trace is over both \( \gamma_5 \) and adjoint SU(\( N \)) indices.

2. In an SU(5) grand-unified model extending the SU(3) \( \times \) SU(2) \( \times \) U(1) Standard Model, the Higgs sector can be built using an adjoint 24 Higgs field \( \Phi \) and a fundamental 5 Higgs field \( H \). One then has a general renormalizable Higgs potential
  \[
  V(\Phi, H) = V(\Phi) + V(H) + \lambda_1 (\text{tr}\Phi^2)(H^\dagger H) + \lambda_2 (H^\dagger \Phi^2 H)
  \]
  \[
  V(\Phi) = -m_1^2 \text{tr}(\Phi^2) + a[\text{tr}(\Phi^4)]^2 + b[\text{tr}(\Phi^2)]^2
  \]
  \[
  V(H) = -m_2^2 (H^\dagger H) + \lambda (H^\dagger H)^2
  \]

- Show that one may arrange a first SU(5) \( \Phi \rightarrow \) SU(3) \( \times \) SU(2) \( \times \) U(1) stage of symmetry breaking producing \( \langle \Phi \rangle = v_1 \text{diag}(2, 2, 2, -3, -3) \) with \( v_1^2 = m_1^2/(16a + 60b) \) provided \( a > 0 \) and \( b > -7a/30 \).
- Show that the Higgs effect gives a mass \( \sqrt{25/2} g_5 v_1 \) to the \( X_{\mu A}^a \) transforming as \( (3, \bar{2}) \) under SU(3) \( \times \) SU(2).
- Show that \( X_{\mu A}^a \) transform with charge \( -5/6 \) under the conventionally normalized hypercharge \( Y \) symmetry by considering the commutation relations of the SU(5) generator that is proportional to the \( Y \) generator.

As a result of the first stage of symmetry breaking, the SU(5) Higgs field \( H \) breaks into a \( (3, 1) \) SU(3) colour triplet \( h_t \) and a \((1,2) \) SU(2) doublet \( h_d \).

- Show that the \( h_t \) and \( h_d \) fields acquire mass terms
  \[
  m_t^2 = -m_2^2 + (30\lambda_1 + 4\lambda_2)v_1^2
  \]
  \[
  m_d^2 = -m_2^2 + (30\lambda_1 + 9\lambda_2)v_1^2
  \]

In order for the second stage of symmetry breaking SU(3) \( \times \) SU(2) \( \times \) U(1) \( \rightarrow \) U(1)\(_{\text{EM}} \) to take place, one needs to have \( m_d^2 < 0 \). In order to have the correct hierarchy of symmetry breakings with the second stage taking place at energies \( v_2 \simeq 246 \text{ GeV}/c^2 \), one needs to have \( |m_d^2| << v_1^2 \).
3. The renormalization group equation describes the change in a renormalized coupling \( g_R \) occasioned by a change in renormalization reference scale \( \mu \) is

\[
\mu \frac{\partial g_R}{\partial \mu} = \beta(g_R).
\]

Writing \( \beta = -\frac{b g^3}{4 \pi} \), one finds the one-loop contribution to the \( b \) coefficient for an SU(\( N \)) gauge group coupling constant \( g_N \)

\[
b_N = \frac{1}{12\pi}(11C_A - 2n_f^i C_{R_i} - n_o^i C_{R_i})
\] (3.1)

where \( n_f^i \) is the number of left- or right-chiral fermions carrying the \( i^{th} \) irreducible representation of the gauge group. \( C_A = N \) is the adjoint representation Dynkin index and \( C_F = \frac{1}{2} \) is the Dynkin index for the fundamental representation. For a U_1 subgroup of a simple unified gauge group, one needs to be careful to rescale the Lie algebra generator and corresponding coupling constant with respect to the traditional Standard Model hypercharge \( Y \) generator: \( \hat{g}_1 = \sqrt{\frac{5}{3}} g_Y \), so \( \hat{b}_1 = \frac{3y_Y}{5} \).

- Show that the \( b_3, b_2 \) and \( \hat{b}_1 \) coefficients for the Standard Model are given by

\[
b_3 = \frac{1}{4\pi}(11 - 4n_g) \quad b_2 = \frac{1}{4\pi}\left( \frac{22}{3} - \frac{4n_g}{3} - \frac{n_H}{6} \right), \quad \hat{b}_1 = \frac{1}{4\pi}\left( -\frac{4n_g}{3} - \frac{n_H}{10} \right)
\] (3.2)

where \( n_g \) is the number of matter generations and \( n_H \) is the number of Higgs doublets.

- Define \( \alpha_N = \frac{g^2}{4\pi} \) and consider the evolution of \( \alpha_3, \alpha_2 \) and \( \alpha_1 \) between a scale \( \mu_0 \) and a scale \( \mu = Q \). Show that

\[
\ln(\frac{Q^2}{\mu_0^2}) = \left( \frac{1}{\alpha_1(\mu_0)} - \frac{1}{\alpha_2(\mu_0)} \right) / (b_2 - \hat{b}_1).
\] (3.3)

- Assuming unification of couplings at the \( Q \) scale, i.e. \( \alpha_3(Q) = \alpha_2(Q) = \alpha_1(Q) \), obtain the corresponding relation at scale \( \mu_0 \) that would have to be found,

\[
\frac{1}{\alpha_3(\mu_0)} = (1 + B) \frac{1}{\alpha_2(\mu_0)} - B \frac{1}{\alpha_1(\mu_0)}
\] (3.4)

where

\[
B = \frac{b_3 - b_2}{b_2 - \hat{b}_1}.
\] (3.5)

- Show that in a version of the Standard Model with \( n_H \) Higgs doublets one would have

\[
B^{th} = \frac{\frac{1 + \frac{n_H}{110} - \frac{3n_H}{110}}{2(1 - \frac{n_H}{110})}}{\approx} \frac{1}{2} + \frac{3}{110} n_H
\] (3.6)

- In the MSSM with \( n_g \) generations and \( n_H \) Higgs doublets, show that one has

\[
b_3 = \frac{1}{4\pi}(9 - 2n_g), \quad b_2 = \frac{1}{4\pi}(6 - 2n_g - \frac{n_H}{2}), \quad \hat{b}_1 = \frac{1}{4\pi}(-2n_g - \frac{3n_H}{10})
\] (3.7)

and consequently for \( n_H = 2, n_g = 3 \) one has \( B^{th}_{MSSM} = \frac{5}{7} \).