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The Standard Model and Beyond: Problem Set 4

1. We saw that the decay of a π^0 particle into two photons is well described by the interaction term

$$\mathcal{L}_{\pi^0 AA} = \frac{e^2}{32\pi^2 F_\pi} \pi^0 \epsilon_{\mu\nu\rho\sigma} F^{\mu\nu}(A) F^{\rho\sigma}(A)$$

which follows from the chiral symmetry anomaly that gives a violation of the $\pi^0 \to \pi^0 + \omega F_\pi$ nonlinear symmetry present at the classical level.

When mesons containing strange quarks are included, the $SU_{\mathrm{L}}(2) \times SU_{\mathrm{R}}(2)$ nonlinearly realized effective low-energy theory extends to a nonlinearly realized $SU_{\mathrm{L}}(3) \times SU_{\mathrm{R}}(3)$ effective theory, and the $SU(2)$ pion triplet extends to an $SU(3)$ octet. The τ^8 member of this octet is the eta meson, which has quark content $\eta \sim \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s})$. The eta meson also can decay into two photons, similarly to the π^0, and this decay also originates from a chiral symmetry anomaly.

- Show that, at the classical level, electromagnetic coupling does not disturb the nonlinear symmetries of the π^0 or the η fields.
- At the quantum level, the π^0 and η nonlinear symmetries are violated by anomalies. Show that the corresponding anomaly coefficients $A_{3AQ_{EM}Q_{EM}}$ and $A_{8AQ_{EM}Q_{EM}}$ are related by

$$\frac{A_{3AQ_{EM}Q_{EM}}}{A_{8AQ_{EM}Q_{EM}}} = \sqrt{3}$$

recalling that the $SU(3)$ generators are $\tau^a = \frac{1}{2}\lambda^a$, where the Gell-Mann matrices λ^a are normalized such that $\mathrm{tr}(\lambda^a \lambda^b) = 2\delta^{ab}$.

- Note that the amplitudes generated by these anomalies contain a $1/F_\pi$ factor and that decay rates are obtained from corresponding amplitudes-squared while decay rates should have dimensions of $[\text{time}]^{-1} \sim [\text{mass}]$. Consequently the decay rate $\Gamma(P \to \gamma\gamma)$ where P represents either a π^0 or an η must contain a factor of m_P^3. Accordingly, show that the decay rates of π^0 and η mesons into two gammas are related in leading approximation by

$$\frac{\Gamma(\pi^0 \to \gamma\gamma)}{\Gamma(\eta \to \gamma\gamma)} = 3 \frac{m_{\pi^0}^3}{m_\eta^3} \ .$$

Given that $m_{\pi^0} \sim 135\mathrm{MeV}/c^2$ and $m_\eta \sim 548\mathrm{MeV}/c^2$, find the ratio of decay rates expected from the chiral symmetry effective theory.

2. Considering just the chiral fermions of the minimal Standard Model (*i.e.* not including neutrino masses), assign general unknown hypercharges to the five chiral fermion species. Treat the right-handed fermions as left-handed after charge conjugation.

   - By demanding cancelation of the $SU(3) \times SU(2) \times U(1)$ gauge anomalies, derive three conditions that must be satisfied by the hypercharges y_L, $y_{(e_r)_C}$, y_{Q_L}, $y_{(u_R)_C}$ and $y_{(d_R)_C}$. These three conditions then leave two hypercharges undetermined.

   - Now include the Higgs sector with the standard single Higgs doublet ϕ_a, with initially undetermined hypercharge y_ϕ. Show that the requirements of classical hypercharge invariance for the possible Yukawa couplings yield three more relations on the various hypercharges. Show that one combination of these three relations is redundant with a combination of the fermionic anomaly-canceling conditions.

   - Including the Higgs doublet field, there are then six undetermined hypercharges overall, which must thus satisfy five relations. One of these six hypercharges may be set by definition, *e.g.* the standard Higgs value $y_\phi = \frac{1}{2}$. The $SU(2) \times U(1)$ symmetry breaking causes the Higgs field ϕ_a to take a nonvanishing vacuum value $\bar{\phi}_a$. Find the generator of the unbroken generator Q_{EM} and show that for the standard Higgs VEV in unitary gauge $\bar{\phi} = (0, \frac{1}{\sqrt{2}}v)$ one has $Q_{EM} = T^3 + Y$. Show then that the remaining relations on the five fermionic hypercharges fix their values to the established values for the Standard Model.

   - Suppose there were no $\bar{Q}_L u_R \tilde{\phi}$ Yukawa term in the Lagrangian (where $\tilde{\phi}_a = \varepsilon_{ab}\phi^{*b}$), so that the up quark would have vanishing current-quark mass. This would remove one relation imposed on the hypercharges. Suppose then that, instead of the SM value $y_{(e_r)_C} = 1$, some additional effect gave $y_{(e_r)_C} = 1 + \epsilon$. What would the relation between g_1, θ_W and the electric charge e then be? Treating ϵ as a infinitesimal quantity, find the corrections to lowest order in ϵ for y_L, y_{Q_L}, $y_{(u_R)_C}$ and $y_{(d_R)_C}$. What would the electric charge of the left-handed neutrino and of the neutron then be in units of e?

3. In an SU(5) grand-unified model extending the SU(3) × SU(2) × U(1) Standard Model, the Higgs sector can be built using an adjoint **24** Higgs field Φ and a fundamental **5** Higgs field H. One then has a general renormalizable Higgs potential

$$\begin{aligned} V(\Phi, H) &= V(\Phi) + V(H) + \lambda_1(\mathrm{tr}\Phi^2)(H^\dagger H) + \lambda_2(H^\dagger \Phi^2 H) \\ V(\Phi) &= -m_1^2 \mathrm{tr}(\Phi^2) + a\mathrm{tr}(\Phi^4) + b[\mathrm{tr}(\Phi^2)]^2 \\ V(H) &= -m_2^2(H^\dagger H) + \lambda(H^\dagger H)^2 \end{aligned} \tag{3.1}$$

- Show that one may arrange a first $\mathrm{SU}(5) \xrightarrow{\Phi} \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ stage of symmetry breaking producing $\langle\Phi\rangle = v_1 \mathrm{diag}(2, 2, 2, -3, -3)$ with $v_1^2 = m_1^2/(14a + 60b)$ provided $a > 0$ and $b > -7a/30$.
- Show that the Higgs effect gives a mass $\sqrt{25/2} g_5 v_1$ to the $X_{\mu A}{}^a$ transforming as $(3, \bar{2})$ under SU(3) × SU(2).
- Show that $X_{\mu A}{}^a$ transform with charge $-5/6$ under the conventionally normalized hypercharge Y symmetry by considering the commutation relations of the SU(5) generator that is proportional to the Y generator.

As a result of the first stage of symmetry breaking, the SU(5) Higgs field H breaks into a $(3, 1)$ SU(3) colour triplet h_t and a (1,2) SU(2) doublet h_d.

- Show that the h_t and h_d fields acquire mass terms

$$\begin{aligned} m_t^2 &= -m_2^2 + (30\lambda_1 + 4\lambda_2)v_1^2 \\ m_d^2 &= -m_2^2 + (30\lambda_1 + 9\lambda_2)v_1^2 \end{aligned} \tag{3.2}$$

In order for the second stage of symmetry breaking $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \xrightarrow{H} \mathrm{U}(1)_{\mathrm{EM}}$ to take place, one needs to have $m_d^2 < 0$. In order to have the correct hierarchy of symmetry breakings with the second stage taking place at energies $v_2 \simeq 246\ \mathrm{GeV}/c^2$, one needs to have $|m_d^2| << v_1^2$.

4. * The standard Dynkin-index convention $C_F = \frac{1}{2}$ for the fundamental representation of a Lie group G corresponds to the fundamental representation trace relation $\mathrm{tr}(T^I T^J) = \frac{1}{2}\delta^{IJ}$. For the group $SU(N)$, the corresponding symmetrized product relation for the fundamental-representation T^I generators is

$$\{T^I, T^J\} = \frac{1}{N}\delta^{IJ}\mathbb{1} + d^{IJK}T^K$$

which defines the $SU(N)$ anomaly symbol d^{IJK}; d^{IJK} is totally symmetric in I, J, K. Then for a general representation R, one defines the anomaly coefficient $A(R)$ by the relation

$$\mathrm{tr}(T^I\{T^J, T^K\}) = \frac{1}{2}A(R)d^{IJK} \ .$$

The anomaly coefficient $A(R)$ is independent of the particular choice of generators T^I, T^J, T^k and it is normalized to one for the fundamental representation: $A(F) = 1$. One can therefore make a simple choice of generators or combination of generators in a given representation R in order to calculate the anomaly coefficient $A(R)$ for that representation and the result will be the same for any choice of three generators in that representation, when multiplied by d^{IJK}.

- First show that, in terms of $SU(5)$ generators presented in the standard $SU(5)$ normalization, the electromagnetic charge becomes $\hat{Q} = \sqrt{\frac{3}{2}}(T^3 + \sqrt{\frac{5}{3}}\hat{Y})$, where T^3 is the diagonal $i = 3$ generator of the $SU(2)$ weak interaction subgroup and $\hat{Y}$ is the hypercharge generator in the standard $SU(5)$ normalization.

- In order to calculate the anomaly coefficients for the left-handed $\mathbf{5}^*$ and $\mathbf{10}$ representations of the $SU(5)$ grand-unified theory, calculate $\mathrm{tr}\,\hat{Q}^3$ for each of these representations. Hence show that

$$A(\mathbf{5}^*) + A(\mathbf{10}) = 0$$

so that the combination $\mathbf{5}^* + \mathbf{10}$ of left-handed fermion representations in the $SU(5)$ grand-unified theory is free from gauge symmetry anomalies.

- What happens to the gauge anomalies when one includes an $SU(5)$ singlet right-handed fermion in order to generate neutrino masses by a see-saw mechanism?

5. The renormalization group equation describes the change in a renormalized coupling g_R occasioned by a change in renormalization reference scale μ is $\mu\frac{\partial g_R}{\partial\mu} = \beta(g_R)$. Writing $\beta = -\frac{bg_R^3}{4\pi}$, one finds the one-loop contribution to the b coefficient for an SU(N) gauge group coupling constant g_N

$$b_N = \frac{1}{12\pi}(11C_A - 2n_f^iC_{R^i} - n_\phi^iC_{R^i}) \tag{5.1}$$

where n_f^i is the number of left- or right-chiral fermions carrying the i^{th} irreducible representation of the gauge group. $C_A = N$ is the adjoint representation Dynkin index and $C_F = \frac{1}{2}$ is the Dynkin index for the fundamental representation. For a U_1 subgroup of a simple unified gauge group, one needs to be careful to rescale the Lie algebra generator and corresponding coupling constant with respect to the traditional Standard Model hypercharge Y generator: $\hat{g}_1 = \sqrt{\frac{5}{3}}g_Y$, so $\hat{b}_1 = \frac{3b_Y}{5}$.

- Show that the b_3, b_2 and $\hat{b}_1$ coefficients for the Standard Model are given by

$$b_3 = \frac{1}{4\pi}(11 - \frac{4n_g}{3}) \qquad b_2 = \frac{1}{4\pi}(\frac{22}{3} - \frac{4n_g}{3} - \frac{n_H}{6}) , \qquad \hat{b}_1 = \frac{1}{4\pi}(-\frac{4n_g}{3} - \frac{n_H}{10}) \tag{5.2}$$

  where n_g is the number of matter generations and n_H is the number of Higgs doublets.

- Define $\alpha_N = \frac{g_n^2}{4\pi}$ and consider the evolution of α_3, α_2 and α_1 between a scale μ_0 and a scale $\mu = Q$. Show that

$$\ln(\frac{Q^2}{\mu_0^2}) = \left(\frac{1}{\hat{\alpha}_1(\mu_0)} - \frac{1}{\alpha_2(\mu_0)}\right)/(b_2 - \hat{b}_1) . \tag{5.3}$$

- Assuming unification of couplings at the Q scale, *i.e.* $\alpha_3(Q) = \alpha_2(Q) = \hat{\alpha}_1(Q)$, obtain the corresponding relation at scale μ_0 that would have to be found,

$$\frac{1}{\alpha_3(\mu_0)} = (1 + B)\frac{1}{\alpha_2(\mu_0)} - B\frac{1}{\hat{\alpha}_1(\mu_0)} \tag{5.4}$$

  where

$$B = \frac{b_3 - b_2}{b_2 - \hat{b}_1} . \tag{5.5}$$

- Show that in a version of the Standard Model with n_H Higgs doublets one would have

$$B^{\text{th}} = \frac{1 + \frac{n_H}{22}}{2(1 - \frac{n_H}{110})} \simeq \frac{1}{2} + \frac{3}{110}n_H \tag{5.6}$$

- In the MSSM with n_g generations and n_H Higgs doublets, show that one has

$$b_3 = \frac{1}{4\pi}(9 - 2n_g) , \qquad b_2 = \frac{1}{4\pi}(6 - 2n_g - \frac{n_H}{2}) , \qquad \hat{b}_1 = \frac{1}{4\pi}(-2n_g - \frac{3n_H}{10}) \tag{5.7}$$

  and consequently for $n_H = 2$, $n_g = 3$ one has $B^{\text{th}}_{\text{MSSM}} = \frac{5}{7}$.