Examination Paper

MSc IN QUANTUM FIELDS AND FUNDAMENTAL FORCES

TP.7 -- Supersymmetry

Wednesday, May 31st 2000

14:00 - 17:00

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Answer **Three** of the following **Five** questions

Use a separate booklet for each question. Make sure that each booklet carries your name, the course title and the number of the question attempted.

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1. Consider a particle of mass m in its Lorentz rest frame.

   a) Start from the $d=4$, N-extended supersymmetry algebra in 4-component notation,

$$\{Q^i, \bar{Q}^j\} = 2\delta^{ij}\gamma^m P_m \qquad\qquad i = 1, \dots, N$$

   where $\bar{Q}^j = (Q^j)^\dagger \gamma^0$. Specialise this algebra to the subspace of mass m states in the rest frame and show that, within this subspace, the supersymmetry algebra in 2-component notation takes the form

$$\{Q^i_\alpha, (Q^j_\beta)^*\} = 2m\delta_{\alpha\beta}\delta^i_j \qquad\qquad \alpha, \beta = 1, 2.$$

   To do this, recall that in the Weyl representation, the γ-matrices take the form

$$\gamma_m = \begin{pmatrix} 0 & \mathrm{i}\sigma_{m\,\alpha\dot{\beta}} \\ \mathrm{i}\bar{\sigma}_m^{\dot{\alpha}\beta} & 0 \end{pmatrix} \qquad\qquad m = 0, \dots, 3$$

   where $\sigma_{m\,\alpha\dot{\beta}} = (\mathbf{1}, \vec{\sigma})$, $\bar{\sigma}_m^{\dot{\alpha}\beta} = (\mathbf{1}, -\vec{\sigma})$ (where $\vec{\sigma}$ are the Pauli matrices), and a Majorana spinor λ^i is given by

$$\lambda^i = \begin{pmatrix} \lambda^i_\alpha \\ \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\lambda}_{\dot{\beta}\,i} \end{pmatrix} \qquad\qquad \text{where } \bar{\lambda}_{\dot{\beta}\,i} = (\lambda^i_\beta)^*.$$

   b) Give the explicit form of the representation matrices for the Lorentz little group acting on the Q^i_α and give their commutators with the Q^i_α.

   c) Use the Clifford vacuum construction and the above relations to construct the simplest $N=1$ and $N=2$ massive supermultiplets (in the latter case assume there are no central charges active).

2. a) Solve the chiral superfield constraint $\bar{D}_{\dot{\alpha}}\Phi = 0$, where $\bar{D}_{\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} - \mathrm{i}\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\partial_m$, showing thus that $\Phi(x,\theta,\bar{\theta})$ may be represented as

$$\Phi(y,\theta) = \varphi(y) + \sqrt{2}\theta^\alpha\psi_\alpha(y) + \theta^\alpha\theta_\alpha F(y), \qquad y^m = x^m + \mathrm{i}\theta^\alpha\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\ ,$$

   Show how this solution corresponds to a definition of the component fields of the supermultiplet obtained by taking lowest components of a sequence of superfields obtained by repeatedly differentiating Φ using $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \mathrm{i}\sigma^m_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m$.

   b) Starting from the chiral superfield action

$$I_{\mathrm{WZ}} = \int d^4x d^2\theta d^2\bar{\theta}\Phi^\dagger\Phi + \left(\int d^4x d^2\theta \mathcal{W}(\Phi) + \text{h.c.}\right)\ ,$$

   introduce the unconstrained superfield X such that $\Phi = \bar{D}_{\dot{\alpha}}\bar{D}^{\dot{\alpha}}X$, and find the field equation for Φ.

   c) Consider the system of three chiral superfields with a superpotential

$$\mathcal{W}(\Phi) = \lambda\Phi_1 + m\Phi_2\Phi_3 + g\Phi_1\Phi_2^2\ .$$

   Eliminate the auxiliary fields from the action, thus determining the scalar potential. Show that supersymmetry is broken at its minimum.

3. Consider the free field action in $d = 4$ for a real scalar field A, a real pseudoscalar field B and a Majorana spinor field ψ:

$$I = -\tfrac{1}{2}\int d^4x[\partial_m A\partial^m A + \partial_m B\partial^m B + i\bar{\psi}\gamma^m\partial_m\psi].$$

a) Assuming that the supersymmetry transformations for this action are linear, write the most general form of the transformations for the pseudoscalar B and the spinor ψ that is consistent by dimensional analysis with

$$\delta A = i\bar{\epsilon}\psi,$$

leaving three numerical coefficients to be determined.

b) Determine the restrictions on these numerical coefficients required to make the action invariant.

c) Determine the further restrictions on the coefficients needed to ensure that the commutators of two supersymmetry transformations, when acting on A and on B (with anticommuting parameters ϵ_1 and ϵ_2), yield spacetime translations according to

$$[\delta_{\epsilon_2}, \delta_{\epsilon_1}] = 2i\bar{\epsilon}_1\gamma^m\epsilon_2\partial_m.$$

d) Verify that the commutation relation of part c) also holds for the transformation of ψ, up to terms proportional to the ψ field equation.

Fierz identity for anticommuting α, β, γ: $\beta\bar{\alpha}\psi = -\frac{1}{4}(\bar{\alpha}\gamma_A\beta)\gamma^A\psi$, where

$$\gamma_A = \mathbf{1},\quad \gamma_5, \gamma_m, \gamma_5\gamma_m, \quad \frac{1}{\sqrt{2}}\gamma_{mn}$$
$$\gamma^A = \mathbf{1}, -\gamma_5, \gamma^m, \gamma_5\gamma^m, -\frac{1}{\sqrt{2}}\gamma^{mn}.$$

Basic relations: $\{\gamma^m, \gamma^n\} = 2\eta^{mn}$; $\gamma_{mn} = \frac{1}{2}(\gamma_m\gamma_n - \gamma_n\gamma_m)$; $\gamma_5^2 = -1$.
For Majorana α & β, the following hold:
$\bar{\alpha}\beta = \bar{\beta}\alpha$ $\qquad$ $\bar{\alpha}\gamma^m\beta = -\bar{\beta}\gamma^m\alpha$ $\qquad$ $\bar{\gamma}_5\gamma^m\beta = \bar{\beta}\gamma_5\gamma^m\alpha$ $\qquad$ $\bar{\alpha}\gamma_{mn}\beta = -\bar{\beta}\gamma_{mn}\alpha.$

4. The superspace action for supersymmetric QED, involving two chiral superfields ϕ_+ and ϕ_- ($\bar{D}_{\dot\alpha}\phi_\pm = 0$) and the real gauge superfield V is

$$I = -\int d^4x d^2\theta W^\alpha W_\alpha - \int d^4x d^2\bar\theta \bar{W}_{\dot\alpha} W^{\dot\alpha} + \int d^4x d^2\theta d^2\bar\theta (\bar\phi_+ e^{qV}\phi_+ + \bar\phi_- e^{-qV}\phi_- + 2\zeta V) + m(\int d^4x d^2\theta \phi_+\phi_- + \int d^4x d^2\bar\theta \bar\phi_+\bar\phi_-),$$

where $W_\alpha = \bar{D}^2 D_\alpha V$. The parameter ζ is known as the Fayet-Iliopoulos coefficient and q is the charge. Recall that $\{D_\alpha, \bar{D}_{\dot\beta}\} = -2\mathrm{i}\partial_{\alpha\dot\beta}$.

a) Show that the action is invariant under the generalised gauge transformation $\phi_+ \to e^{-\mathrm{i}\Lambda}\phi_+$, $\phi_- \to e^{\mathrm{i}\Lambda}\phi_-$, $V \to V + \mathrm{i}q^{-1}(\Lambda - \bar\Lambda)$, where Λ is a chiral superfield.

b) The $U(1)$ electromagnetic current occurs at the $\theta\bar\theta$ level of the current superfield

$$J = \bar\phi_+\phi_+ - \bar\phi_-\phi_-.$$

Defining the component expansion of this superfield by taking repeated superspace covariant derivatives and then setting $\theta = \bar\theta = 0$ to extract the lowest components of the resulting sequence of superfields, find the component electromagnetic vector current by applying $[D_\alpha, \bar{D}_{\dot\beta}]$.

c) The *Wess-Zumino* gauge for the gauge superfield V is one in which all the components of dimension lower than that of the component vector gauge field $A_m(x)$, as well as a complex scalar of the same dimension, are set to zero. Use this gauge to write out the equations of motion for the auxiliary fields of the theory and hence find the effective potential for the scalars.

The auxiliary fields are extracted from V and $\phi_\pm$ by the definitions $D = (D^\alpha\bar{D}^2 D_\alpha V)\Big|_{\theta=\bar\theta=0}$, $f_\pm = (D^2\phi_\pm)\Big|_{\theta=\bar\theta=0}$. You may proceed either by working out the component auxiliary field terms in the action and then varying to obtain the auxiliary field equations, or by varying the superfield action and extracting the auxiliary field parts of the superfield equations.

d) Show that supersymmetry is spontaneously broken when $0 < \frac{1}{2}q\zeta < m^2$. What happens when $m^2 < \frac{1}{2}a\zeta$?

5. Write an essay describing the principal features of the minimal supersymmetric standard model. Your discussion should cover the following: the supersymmetric representations used in combining known particles with superpartners, the structure of the mass-generation mechanism, renormalization properties, supersymmetry breaking, problems with tree-level gauge symmetry breaking and how these are resolved, and the naturalness of gauge coupling unification.