1 Supersymmetric Quantum Mechanics

Consider a quantum mechanical system with the wave function

$$\Psi = \begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix} \tag{1}$$

where the first entry is bosonic and the second entry is fermionic and hence $(-1)^F = \sigma_3$. We can construct a Susy QM by defining the supercharges

$$Q^\dagger = \sigma_+(P + iW'(x)) \qquad \text{and} \qquad Q = \sigma_-(P - iW'(x)) \tag{2}$$

where $W(x)$ is a real function in x and its derivative $|W'(x)| \to \infty$ for $x \to \pm\infty$ and

$$P = -i\hbar\partial_x, \qquad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

(i) Use the Susy algebra to show that

(a)

$$2H = P^2 + (W'(x))^2 - \hbar W''(x)\sigma_3 \tag{4}$$

(b) for $E > 0$:

$$(-1)^F = \frac{1}{2E}[Q^\dagger, Q]\,. \tag{5}$$

(ii) Let $W(x) = \lambda x^2$. Diagonalise H and find the spectrum

$$E_{n,\pm} = \hbar\lambda(n + \frac{1}{2} \mp \frac{1}{2}) \tag{6}$$

(iii) The Witten index is defined as

$$\mathrm{Tr}[(-1)^F \exp(-\beta H)] \tag{7}$$

using the results form the lecture, determine whether it is non-zero for the following examples of W:

(a) $W = x^4 +$ lower orders

(b) $W = x^3 +$ lower orders

(c) $W = x^2$

(d) W has no critical points.

What does this have to do with Susy breaking?

2 Superspace and Superfields for QM

Supersymmetric quantum mechanics may be formulated using superspace with coordinates $(t, \theta, \bar{\theta})$. Superfields are fields defined on superspace. θ and $\bar{\theta}$ are Grassmann variables and obey the following equations

$$\begin{aligned} \theta^2 = \bar{\theta}^2 = 0, \qquad \theta\bar{\theta} = -\bar{\theta}\theta \\ \partial_\theta := \frac{\partial}{\partial\theta}, \qquad \partial_{\bar{\theta}} := \frac{\partial}{\partial\bar{\theta}}, \qquad \bar{\partial}_\theta = -\partial_{\bar{\theta}} \\ \partial_\theta 1 = 0, \qquad \partial_\theta\bar{\theta} = 0, \qquad \partial_{\bar{\theta}} 1 = 0, \qquad \partial_{\bar{\theta}}\theta = 0\,. \end{aligned} \tag{8}$$

any superfield may be expanded in the form

$$X(t, \theta, \bar{\theta}) = x(t) + \theta\psi(t) - \bar{\theta}\bar{\psi}(t) + \theta\bar{\theta}F(t) \tag{9}$$

(i) check that for x and F real

$$\bar{X} = X. \tag{10}$$

(ii) We can define the covariant derivative on superspace

$$D = \partial_\theta - i\bar{\theta}\partial_t \tag{11}$$

and

$$\bar{D} = -\partial_{\bar{\theta}} + i\theta\partial_t\,. \tag{12}$$

Compute

$$DX \qquad \text{and} \qquad \bar{D}X\,, \tag{13}$$

and show that

$$\{D, \bar{D}\} = 2i\partial_t\,. \tag{14}$$

(iii) Consider the supertranslations

$$(t, \theta, \bar{\theta}) \overset{\delta_\epsilon}{\mapsto} (t + i\epsilon\bar{\theta}, \theta + \epsilon, \bar{\theta}) \tag{15}$$

and

$$(t, \theta, \bar{\theta}) \overset{\delta_{\bar{\epsilon}}}{\mapsto} (t - i\theta\bar{\epsilon}, \theta, \bar{\theta} + \bar{\epsilon})\,. \tag{16}$$

By acting on a superfield, show that

$$[\delta_\epsilon, \delta_{\bar{\epsilon}}] = -\epsilon\bar{\epsilon}2i\partial_t = \epsilon\bar{\epsilon}2H\,. \tag{17}$$

and deduce that

$$\{\bar{Q}, Q\} = 2H\,. \tag{18}$$

(iv) Superfield $\mathcal{C}$ is said to be chiral, if it satisfies

$$\bar{D}\mathcal{C} = 0. \tag{19}$$

Show that

(a) products of chiral superfields are chiral

(b) $\bar{D}X$ is chiral for any X.

(c) $\int dt d\theta \mathcal{C}$ is SUSY invariant

3 Fermionic Zero Modes

We will now consider regular quantum mechanics with fermions and study them in a path integral. Let a_i be grassmann variables. The rules for Berezin integration are

$$\int da_i a_j = \delta_{ij}, \qquad \int da_i 1 = 0 \tag{20}$$

(i) Let us start with a discrete system. Consider the action

$$S = \sum_{i\alpha} \bar{\psi}_i A_{i\alpha} \psi_\alpha \tag{21}$$

where A has a single ψ zero mode, and there is one more ψ than $\bar{\psi}$ ($i = 1, \ldots, N$ and $\alpha = 1, \ldots, N+1$). Hence after a suitable transformation

$$A = \begin{pmatrix} \lambda_1 & 0 & \ldots & & 0 \\ 0 & \lambda_2 & & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \ldots & & \lambda_N & 0 \end{pmatrix}. \tag{22}$$

Compute

(a)

$$\int \prod_i d\bar{\psi}_i \prod_\alpha d\psi_\alpha \exp(-S) \tag{23}$$

(b)

$$\int \prod_i d\bar{\psi}_i \prod_\alpha d\psi_\alpha \exp(-S) \psi_{N+1} \tag{24}$$

(ii) We can now do a similar computation for Quantum Mechanics, i.e. 1-dimensional QFT. We will work with the euclidean path integral. Consider the action

$$S = \int dt\bar{\psi}(t)A\psi(t) \tag{25}$$

where A has a single ψ zero-mode. What does this mean?

Using an expansion of ψ and $\bar{\psi}$ in terms of eigenmodes of A, compute

(a)
$$\int [\mathcal{D}\bar{\psi}][\mathcal{D}\psi] \exp(-S)\psi(t) \tag{26}$$

(b)
$$\int [\mathcal{D}\bar{\psi}][\mathcal{D}\psi] \exp(-S)\bar{\psi}(t) \tag{27}$$

(c)
$$\int [\mathcal{D}\bar{\psi}][\mathcal{D}\psi] \exp(-S)\bar{\psi}(t_1)\psi(t_2) \tag{28}$$

(d)
$$\int [\mathcal{D}\bar{\psi}][\mathcal{D}\psi] \exp(-S)\bar{\psi}(t_1)\psi(t_2)\psi(t_3) \tag{29}$$

4 Representation Theory

Remind yourselves of the Lie-algebra representations of $su(2)$ and $so(4) = su(2) \times su(2)$. The $(n+1)$-dimensional representation of $su(2)$ may be labelled by its highest weight $[n]$, or by its spin $(\frac{n}{2})$. Hence we can label the representations of $so(4)$ by using a set of two spins (j_1, j_2).

(i) Compute the following tensor products for $su(2)$

- (a) $(j) \otimes (k)$
- (b) $\bigwedge^2(j)$
- (c) $Sym^2(j)$

(ii) Compute the following tensor products for $so(4)$

- (a) $(j_1, j_2) \otimes (k_1, k_2)$
- (b) $\bigwedge^2(j_1, j_2)$
- (c) $Sym^2(j_1, j_2)$

where $\bigwedge^2$ denotes the antisymmetric product, and Sym^2 denotes the symmetric product.