1 Supersymmetric Quantum Mechanics

Consider a quantum mechanical system with the wave function

\[ \Psi = \begin{pmatrix} \psi_+(x) \\ \psi_-(x) \end{pmatrix} \]  

where the first entry is bosonic and the second entry is fermionic and hence \((-1)^F = \sigma_3\). We can construct a Susy QM by defining the supercharges

\[ Q^\dagger = \sigma_+ (P + iW'(x)) \quad \text{and} \quad Q = \sigma_- (P - iW'(x)) \]  

where \(W(x)\) is a real function in \(x\) and its derivative \(|W'(x)| \to \infty\) for \(x \to \pm\infty\) and

\[ P = -i\hbar \partial_x, \quad \sigma_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]  

(i) Use the Susy algebra to show that

(a) \[ 2H = P^2 + (W'(x))^2 - \hbar W''(x)\sigma_3 \]  

(b) for \(E > 0\):

\[ (-1)^F = \frac{1}{2E}[Q^\dagger, Q]. \]  

(ii) Let \(W(x) = \lambda x^2\). Diagonalise \(H\) and find the spectrum

\[ E_{n,\pm} = \hbar \lambda (n + \frac{1}{2} \mp \frac{1}{2}) \]  

(iii) The Witten index is defined as

\[ \text{Tr}[(-1)^F \exp(-\beta H)] \]  

using the results from the lecture, determine whether it is non-zero for the following examples of \(W\):

(a) \(W = x^4 + \text{lower orders}\)
(b) \(W = x^3 + \text{lower orders}\)
(c) \(W = x^2\)
(d) \(W\) has no critical points.

What does this have to do with Susy breaking?
2 Superspace and Superfields for QM

Supersymmetric quantum mechanics may be formulated using superspace with coordinates \((t, \theta, \bar{\theta})\). Superfields are fields defined on superspace. \(\theta\) and \(\bar{\theta}\) are Grassmann variables and obey the following equations

\[
\theta^2 = \bar{\theta}^2 = 0, \quad \theta \bar{\theta} = -\bar{\theta} \theta
\]

\[
\partial_\theta := \frac{\partial}{\partial \theta}, \quad \partial_{\bar{\theta}} := \frac{\partial}{\partial \bar{\theta}}, \quad \bar{\partial}_\theta = -\partial_{\bar{\theta}}, \quad \bar{\partial}_{\bar{\theta}} = \partial_\theta
\]

\[\partial_\theta 1 = 0, \quad \partial_{\bar{\theta}} \bar{\theta} = 0, \quad \partial_\theta \bar{\theta} = 0, \quad \partial_{\bar{\theta}} \theta = 0.
\] (8)

any superfield may be expanded in the form

\[
X(t, \theta, \bar{\theta}) = x(t) + \theta \psi(t) - \bar{\theta} \bar{\psi}(t) + \theta \bar{\theta} F(t)
\]

(9)

(i) check that for \(x\) and \(F\) real

\[
\bar{X} = X.
\]

(10)

(ii) We can define the covariant derivative on superspace

\[
D = \partial_\theta - i \bar{\theta} \partial_t
\]

and

\[
\bar{D} = -\partial_{\bar{\theta}} + i \theta \partial_t.
\]

(11)

(12)

Compute

\[
DX \quad \text{and} \quad DX,
\]

(13)

and show that

\[
\{D, \bar{D}\} = 2i \partial_t.
\]

(14)

(iii) Consider the supertranslations

\[
(t, \theta, \bar{\theta}) \overset{\delta_\epsilon}{\rightarrow} (t + \epsilon e \bar{\theta}, \theta + \epsilon, \bar{\theta})
\]

(15)

and

\[
(t, \theta, \bar{\theta}) \overset{\delta_{\bar{\epsilon}}}{\rightarrow} (t - i \theta \bar{\epsilon}, \theta, \bar{\theta} + \bar{\epsilon}).
\]

(16)

By acting on a superfield, show that

\[
[\delta_\epsilon, \delta_{\bar{\epsilon}}] = -e \bar{e} 2i \partial_t = e \bar{e} 2H.
\]

(17)

and deduce that

\[
\{\bar{Q}, Q\} = 2H.
\]

(18)
Superfield $\mathcal{C}$ is said to be chiral, if it satisfies

$$\bar{D} \mathcal{C} = 0.$$  \hfill (19)

Show that

(a) products of chiral superfields are chiral
(b) $\bar{D} \mathbf{X}$ is chiral for any $\mathbf{X}$.
(c) $\int dt d\theta \mathcal{C}$ is SUSY invariant

3 Fermionic Zero Modes

We will now consider regular quantum mechanics with fermions and study them in a path integral. Let $a_i$ be grassmann variables. The rules for Berezin integration are

$$\int da_i a_j = \delta_{ij}, \quad \int da_i 1 = 0$$ \hfill (20)

(i) Let us start with a discrete system. Consider the action

$$S = \sum_{i\alpha} \bar{\psi}_i A_{i\alpha} \psi_\alpha$$ \hfill (21)

where $A$ has a single $\psi$ zero mode, and there is one more $\psi$ than $\bar{\psi}$ ($i = 1, \ldots, N$ and $\alpha = 1, \ldots, N + 1$). Hence after a suitable transformation

$$A = \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \lambda_N & 0 \end{pmatrix}.$$ \hfill (22)

Compute

(a) $$\int \prod_i d\bar{\psi}_i \prod_\alpha d\psi_\alpha \exp(-S)$$ \hfill (23)

(b) $$\int \prod_i d\bar{\psi}_i \prod_\alpha d\psi_\alpha \exp(-S) \psi_{N+1}$$ \hfill (24)
We can now do a similar computation for Quantum Mechanics, i.e. 1-dimensional QFT. We will work with the euclidean path integral. Consider the action

\[ S = \int dt \bar{\psi}(t) A \psi(t) \]  

where \( A \) has a single \( \psi \) zero-mode. What does this mean?

Using an expansion of \( \psi \) and \( \bar{\psi} \) in terms of eigenmodes of \( A \), compute

(a) \[ \int [D\bar{\psi}][D\psi] \exp (-S) \psi(t) \]  

(b) \[ \int [D\bar{\psi}][D\psi] \exp (-S) \bar{\psi}(t) \]  

(c) \[ \int [D\bar{\psi}][D\psi] \exp (-S) \bar{\psi}(t_1) \psi(t_2) \]  

(d) \[ \int [D\bar{\psi}][D\psi] \exp (-S) \bar{\psi}(t_1) \psi(t_2) \psi(t_3) \]  

4 Representation Theory

Remind yourselves of the Lie-algebra representations of \( su(2) \) and \( so(4) = su(2) \times su(2) \). The \((n + 1)\)-dimensional representation of \( su(2) \) may be labelled by its highest weight \([n]\), or by its spin \( n/2 \). Hence we can label the representations of \( so(4) \) by using a set of two spins \((j_1, j_2)\).

(i) Compute the following tensor products for \( su(2) \)

(a) \((j) \otimes (k)\)

(b) \(\wedge^2(j)\)

(c) \(\text{Sym}^2(j)\)

(ii) Compute the following tensor products for \( so(4) \)

(a) \((j_1, j_2) \otimes (k_1, k_2)\)

(b) \(\wedge^2(j_1, j_2)\)

(c) \(\text{Sym}^2(j_1, j_2)\)

where \(\wedge^2\) denotes the antisymmetric product, and \(\text{Sym}^2\) denotes the symmetric product.