1 Identities for Lorentz Group Representations

We will use undotted and dotted spinor indices ($\alpha \in \{1,2\}$, $\dot{\alpha} \in \{1,2\}$) as well as vector indices ($\mu \in \{0,1,2,3\}$). While the raising and lowering of vector indices is done with the mostly plus metric $\eta_{\mu\nu}$ (or it's inverse $\eta^{\mu\nu}$), the spinor indices are raised and lowered with the epsilon symbols $\varepsilon^{\alpha\beta}$, $\varepsilon_{\alpha\beta}$, $\varepsilon^{\dot{\alpha}\dot{\beta}}$, $\varepsilon_{\dot{\alpha}\dot{\beta}}$. Objects with an odd number of spinor indices are Grassmann odd, i.e. they anticommute with other Grassmann odd objects. We pick the convention

$$\varepsilon^{12} = -\varepsilon^{21} = 1 \qquad \text{and} \qquad \varepsilon_{12} = -\varepsilon_{21} = -1 \qquad \text{thus} \qquad \varepsilon^{\alpha\beta}\varepsilon_{\beta\gamma} = \delta^{\alpha}_{\gamma}\,, \ldots \tag{1}$$

(i) Show the completeness relation

$$\varepsilon_{\alpha\beta}\varepsilon^{\gamma\delta} = -\delta^{\gamma}_{\alpha}\delta^{\delta}_{\beta} + \delta^{\delta}_{\alpha}\delta^{\gamma}_{\beta} \tag{2}$$

The pulling up and down of indices is as follows

$$\psi^{\alpha} = \varepsilon^{\alpha\beta}\psi_{\beta}\,,\ \psi_{\alpha} = \varepsilon_{\alpha\beta}\psi^{\beta} \qquad \text{and} \qquad \bar{\psi}_{\dot{\alpha}} = \varepsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}\,,\ \bar{\psi}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}}\,. \tag{3}$$

Remember under $so(4)$ the following fields transform as

$$\psi_{\alpha} \text{ as } (1/2,0) \qquad \text{and} \qquad \bar{\psi}_{\dot{\alpha}} \text{ as } (0,1/2)\,. \tag{4}$$

Contracted spinor indices may be suppressed ~~with the convention~~. The dot on top of the index is really a representation of a balloon filled with helium. The index holds on to the balloon and flies up. Without such a balloon the index falls down, naturally...

$$\begin{aligned} \chi\psi = \chi^{\alpha}\psi_{\alpha}(= \chi^{\alpha}\varepsilon_{\alpha\beta}\psi^{\beta}) \qquad &\text{and} \qquad \bar{\chi}\bar{\psi} = \bar{\chi}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}(= \bar{\chi}_{\dot{\alpha}}\varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}}) \\ (0,0) = \Lambda^2(1/2,0) \qquad &\text{and} \qquad (0,0) = \Lambda^2(0,1/2)\,. \end{aligned} \tag{5}$$

Note that it's important to stick to this convention! Show

(ii)

$$\chi_{\alpha}\psi^{\alpha} = -\chi\psi \qquad \text{and} \qquad \bar{\chi}^{\dot{\alpha}}\bar{\psi}_{\dot{\alpha}} = -\bar{\chi}\bar{\psi} \tag{6}$$

(iii)

$$\chi\psi = \psi\chi \qquad \text{and} \qquad \bar{\chi}\bar{\psi} = \bar{\psi}\bar{\chi} \tag{7}$$

(iv)

$$\psi^2 = \psi\psi = 2\psi_2\psi_1 = 2\psi^2\psi^1 \qquad \text{and} \qquad \bar{\psi}^2 = \bar{\psi}\bar{\psi} = 2\bar{\psi}_1\bar{\psi}_2 = 2\bar{\psi}^1\bar{\psi}^2 \tag{8}$$

(v)

$$\psi^{\alpha}\psi^{\beta} = -\frac{1}{2}\psi^2\varepsilon^{\alpha\beta} \qquad \text{and} \qquad \bar{\psi}^{\dot{\alpha}}\bar{\psi}^{\dot{\beta}} = \frac{1}{2}\bar{\psi}^2\varepsilon^{\dot{\alpha}\dot{\beta}} \tag{9}$$

(vi)

$$(\chi\psi)^* = \bar{\chi}\bar{\psi} \tag{10}$$

We can now introduce the $\sigma^\mu_{\alpha\dot{\alpha}}$ constants that help us switch between spinor and vector indices (Remember that $(1/2,0)\otimes(0,1/2)=(1/2,1/2)$).

$$\sigma^0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{11}$$

And their conjugate

$$\bar{\sigma}^{\mu\dot{\alpha}\alpha} = \varepsilon^{\dot{\alpha}\dot{\beta}}\varepsilon^{\alpha\beta}\sigma^\mu_{\beta\dot{\beta}}\,. \tag{12}$$

Show the completeness relations

(vii)

$$Tr(\sigma^\mu\bar{\sigma}^\nu) = -2\eta^{\mu\nu} \qquad \text{and} \qquad \sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\dot{\beta}\beta}_\mu = -2\delta^\alpha_\beta\delta^{\dot{\alpha}}_{\dot{\beta}}\,. \tag{13}$$

We define

$$V_{\alpha\dot{\alpha}} = \sigma^\mu_{\alpha\dot{\alpha}}V_\mu\,. \tag{14}$$

Note, we are only changing the description, not the representation. The sigmas are just a bunch of constants after all (remember intertwiners?). Show that

(viii)

$$V^\mu = \frac{-1}{2}\bar{\sigma}^{\mu\dot{\alpha}\alpha}V_{\alpha\dot{\alpha}} \tag{15}$$

We can use sigma matrices to compute tensor products of spinors.

(ix) Show

$$(\psi\sigma^\mu\bar{\chi})^* = \chi\sigma^\mu\bar{\psi} \qquad \text{and} \qquad \psi\sigma^\mu\bar{\chi} = -\bar{\chi}\bar{\sigma}^\mu\psi\,. \tag{16}$$

What is the relevant tensor product?

(x) Show

$$(\psi\sigma^\mu\bar{\sigma}^\nu\chi)^* = \bar{\chi}\bar{\sigma}^\nu\sigma^\mu\bar{\psi} \qquad \text{and} \qquad \psi\sigma^\mu\bar{\sigma}^\nu\chi = \chi\sigma^\nu\bar{\sigma}^\mu\psi\,. \tag{17}$$

What is the relevant tensor product?

(xi) Write the metric tensor $g_{\mu\nu}$ in spinor notation

(xii) Write the 4-dimensional epsilon $\epsilon_{\mu\nu\sigma\rho}$ tensor in spinor notation

(xiii) Defining $(*F)_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$, show that

$$F^\pm_{\mu\nu} = F_{\mu\nu} \pm i(*F)_{\mu\nu}\,; \tag{18}$$

and express both parts in terms of **E** and **B**.

(xiv) Show that

$$\sigma^{\mu\nu\beta}_\alpha := \frac{1}{4}(\sigma^\mu_{\alpha\dot{\alpha}}\bar{\sigma}^{\nu\dot{\alpha}\beta} - \sigma^\nu_{\alpha\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\beta}) \tag{19}$$

are generators of the Lorentz group acting on the $(1/2,0)$ representation.

(xv) Show that

$$\psi\sigma^{\mu\nu}\psi = 0\,. \tag{20}$$

Prove the Fierz identities

(xvi) For 4 spinors $\psi_1, \ldots, \psi_4$

$$(\psi_1\psi_2)(\bar\psi_3\bar\psi_4) = \frac{1}{2}(\psi_1\sigma^\mu\bar\psi_4)(\bar\psi_3\bar\sigma_\mu\psi_2) \tag{21}$$

(xvii)

$$(\psi\sigma^\mu\bar\psi)(\psi\sigma^\nu\bar\psi) = -\frac{1}{2}\eta^{\mu\nu}(\psi\psi)(\bar\psi\bar\psi)\,. \tag{22}$$

For 4 spinors $\psi_1, \ldots, \psi_4$ show

(xviii)

$$(\psi_1\psi_2)(\psi_3\psi_4) + (\psi_1\psi_3)(\psi_2\psi_4) + (\psi_1\psi_4)(\psi_2\psi_3) = 0\,. \tag{23}$$

(xix)

$$(\psi_1\psi_2)(\psi_3\sigma^\mu\bar\psi_4) + (\psi_1\psi_3)(\psi_2\sigma^\mu\bar\psi_4) + (\psi_1\sigma^\mu\bar\psi_4)(\psi_2\psi_3) = 0\,. \tag{24}$$

(xx)

$$(\psi_1\sigma^\mu\bar\psi_2)(\psi_3\sigma^\nu\bar\psi_4) - (\psi_1\psi_3)(\bar\psi_2\bar\sigma^\mu\sigma^\nu\bar\psi_4) - (\psi_1\sigma^\nu\bar\psi_4)(\bar\psi_2\bar\sigma^\mu\psi_3) = 0. \tag{25}$$

Show

(xxi)

$$(\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu)_\beta^\alpha = -2\eta^{\mu\nu}\delta_\beta^\alpha \tag{26}$$

(xxii)

$$\sigma^\mu\bar\sigma^\nu = -\eta^{\mu\nu} + 2\sigma^{\mu\nu}\,. \tag{27}$$

(xxiii) We know that $(1/2, 0) \otimes (1/2, 0) = (0, 0) \oplus (1, 0)$. How can you decompose

$$\psi_\alpha\chi_\beta = ... + ...\ ? \tag{28}$$

(xxiv) Extra credit: Find any typos and report them!

2 Supermultiplets

(i) Write down the content of massless multiplets with highest helicity 1/2 and 1.

(ii) Write down the content of massive multiplets with highest spin 1/2 and 1.

(iii) Count the number of bosonic versus fermionic degrees of freedom for each multiplet above and confirm that they match.

(iv) Compare the degrees of freedom of the massless and the massive multiplets. How does this connect to the Higgs mechanism?

3 Superspace and Superfields for $4d$ $\mathcal{N}=1$

Superspace for $4d$ $\mathcal{N}=1$ has coordinates $(x^\mu, \theta_\alpha, \bar{\theta}_{\dot{\alpha}})$. The SUSY generators Q_α and $\bar{Q}_{\dot{\alpha}}$ can be expressed as differential operators on superspace

$$\begin{aligned} Q_\alpha &= +\partial_\alpha - i\bar{\theta}^{\dot{\alpha}}\partial_{\alpha\dot{\alpha}} = +\frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \\ \bar{Q}_{\dot{\alpha}} &= -\bar{\partial}_{\dot{\alpha}} + i\theta^\alpha\partial_{\alpha\dot{\alpha}} = -\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \end{aligned} \tag{29}$$

(i) Check that

$$\{Q_\alpha, Q_\beta\} = 0 \tag{30}$$

and

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2P_{\alpha\dot{\alpha}}\,. \tag{31}$$

The covariant derivatives on superspace are defined as

$$\begin{aligned} D_\alpha &= +\partial_\alpha + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \\ \bar{D}_{\dot{\alpha}} &= -\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \end{aligned} \tag{32}$$

(ii) Show that $\bar{D}_{\dot{\alpha}}$ annihilates both θ^α and $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$.

A superfield C is called chiral if

$$\bar{D}_{\dot{\alpha}}C = 0\,. \tag{33}$$

(iii) Show that any chiral field may be written as

$$C(x^\mu, \theta^\alpha, \bar{\theta}^{\dot{\alpha}}) = \phi(x) + \sqrt{2}\theta\psi(x) + \theta^2F(x) + i\theta\bar{\sigma}^\mu\bar{\theta}\partial_\mu\phi(x) - \frac{i}{\sqrt{2}}\theta^2\partial_\mu\psi(x)\sigma^\mu\bar{\theta} + \frac{1}{4}\theta^2\bar{\theta}^2\partial^\mu\partial_\mu\phi(x) \tag{34}$$

(iv) Show that if C is chiral, then

$$Q_\alpha C \quad \text{and} \quad \bar{Q}_{\dot{\alpha}}C \quad \text{are chiral.} \tag{35}$$

(v) Count the off-shell degrees of freedom of the chiral superfield. Do fermionic and bosonic degrees of freedom cancel? How about on-shell?

(vi) What is an anti-chiral superfield? Show that $\bar{C}$ is antichiral.

There is a second kind of common superfield. The vector superfield. A superfield V is called a vector superfield if

$$\bar{V} = V \tag{36}$$

(vii) Write the most general vector superfield.

(viii) Count the off shell degrees of freedom of the vector superfield. Do fermionic and bosonic degrees of freedom cancel? How about on-shell?

Let's turn to integration on superspace. Remember that we're dealing with Berezin integration (i.e. differentiation really).

(ix) Show that

$$\int d^4\theta K(C^i, \bar{C}^{\bar{i}}, X) = \int d^2\theta d^2\bar{\theta} K(C^i, \bar{C}^{\bar{i}}, X)\,, \tag{37}$$

where C^i are chiral and X is a collection of arbitary superfields, is invariant under Susy transformations. You can show that Q acting on a superfield has a $\theta^2\bar{\theta}^2$ term which is a total derivative $\partial(\dots)$.

(x) Show that

$$\int d^4\theta f(C) = \partial(\dots)\,. \tag{38}$$

4 Kähler Potentials, optional, you can hand it in

Kähler potentials are objects from differential geometry. If you open e.g. Nakahara you will find a chapter on Kähler geometry.
Let's consider a complex, Riemannian, symplectic manifold M with metric g, complex structure I and symplectic form ω. The manifold is said to be Kähler if the following relation holds

$$g(I(X), Y) = \omega(X, Y) \qquad \text{for} \quad X, Y \in \Gamma(TM)\,. \tag{39}$$

Kähler manifolds have nice properties. Locally (i.e. on an open patch U) The complex structure allows us to introduce holomorphic coordinates ϕ^i and $\bar{\phi}^{\bar{i}}$. We will from now on work on this coordinate chart in complex coordinates. Partial derivatives are

$$\partial_i = \frac{\partial}{\partial \phi^i} \qquad \text{and} \qquad \partial_{\bar{i}} = \frac{\partial}{\partial \bar{\phi}^{\bar{i}}} \tag{40}$$

It can be shown that the metric can be expressed as

$$g_{i\bar{i}}(\phi, \bar{\phi}) = \partial_i \partial_{\bar{i}} K(\phi, \bar{\phi})\,. \tag{41}$$

This is also called the Kähler metric. The symplectic form (Kähler form) can also be expressed in terms of derivatives of the Kähler potential. Show that K is not unique in the sense that $g_{i\bar{i}}$ is invariant under

$$K(\phi, \bar{\phi}) \mapsto K(\phi, \bar{\phi}) + (f(\phi) + \text{complex conjugate})\,. \tag{42}$$

The Christoffel symbols are

$$\Gamma^i_{jk} = g^{\bar{l}i} \partial_k g_{j\bar{l}} \qquad \text{and} \qquad \bar{\Gamma}^{\bar{i}}_{\bar{j}\bar{k}} = g^{\bar{l}i} \partial_{\bar{k}} g_{l\bar{j}}\,, \tag{43}$$

and the Riemann tensor is

$$R_{i\bar{j}k\bar{l}} = \partial_k \partial_{\bar{l}} g_{i\bar{j}} - \Gamma^m_{ik} g_{m\bar{m}} \Gamma^{\bar{m}}_{\bar{j}\bar{l}}\,. \tag{44}$$

Let's turn back to Supersymmetry! Interpreting the first components ϕ^i and $\bar{\phi}^i$ of our chiral superfields C^i and their aintichiral conjugates $\bar{C}^{\bar{i}}$ as coordinates on a manifold (also called the target space), show that we can express the supersymmetric Lagrangian

$$\begin{aligned}\mathcal{L} = \int d^4\theta K(C^i, \bar{C}^{\bar{i}}) =& g_{i\bar{i}} F^i \bar{F}^i - \frac{1}{2} F^i g_{\imath\bar{i}} \Gamma^{\bar{i}}_{\bar{j}\bar{k}} \bar{\psi}^{\bar{j}} \bar{\psi}^{\bar{k}} - \frac{1}{2} \bar{F}^{\bar{i}} g_{i\bar{i}} \Gamma^i_{jk} \psi^i \psi^k \\ & - g_{i\bar{i}} \partial_\mu \phi^i \partial^\mu \bar{\phi}^{\bar{i}} - i g_{i\bar{i}} \bar{\psi}^{\bar{i}} \bar{\sigma}^\mu \mathcal{D}_\mu \psi^i + \frac{1}{4} (\partial_k \partial_{\bar{l}} g_{i\bar{j}}) \psi^i \psi^k \bar{\psi}^{\bar{j}} \bar{\psi}^{\bar{l}} \,,\end{aligned} \tag{45}$$

where

$$\mathcal{D}_\mu \psi^i = \partial_\mu \psi^i + \Gamma^i_{jk} \partial_\mu \phi^k \psi^j \,. \tag{46}$$

Vary $\bar{F}$ and find the equations of motion

$$F^i = \frac{1}{2} \Gamma^i_{jk} \psi^j \psi^k \,, \tag{47}$$

assuming that $g_{i\bar{i}}$ is invertible. By substituting the e.o.m. back into the Lagrangian, show that

$$\mathcal{L} = -g_{i\bar{i}} \partial_\mu \phi^i \partial^\mu \bar{\phi}^{\bar{i}} - i g_{i\bar{i}} \bar{\psi}^{\bar{i}} \bar{\sigma}^\mu \mathcal{D}_\mu \psi^i + \frac{1}{4} R_{i\bar{j}k\bar{l}} \psi^i \psi^k \bar{\psi}^{\bar{j}} \bar{\psi}^{\bar{l}} \,. \tag{48}$$

What is the interpretation of the ψ's on the target space side?