Brief Summary of Group Theory

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1 Group Theory

These notes are based on an old summary of the group theory, especially Lie group theory, needed to understand the standard model of particle physics. This was a summary provided to students when I gave the Unification course some years ago.

The notes are too short for students to learn to from. However, they give a good guide to the group theory that is used in the Unification course. So they may help you to find which parts of a text are needed for background reading, and which can be ignored. Many text books start by focussing on finite groups (groups with few elements) not Lie groups but finite groups are not really needed for Unification. Some of the concepts used in finite groups are used in Lie group theory but you do not need the detail on finite groups provided in a typical group theory text.

The main topic missing from a typical group theory text and from these notes that is useful in the QFFF MSc is a discussion of the properties (mostly the representations) of the Poincaré group[1] of space-time symmetries. That is rarely covered in basic group theory texts but see Section 1.4 below for some suggestions.

1.1 Summary of Group Theory

A **group** is a set G of elements $g \in G$ which can be combined under **group multiplication law**, $*$, such that it obeys four axioms:

(i) **Closure**: $\forall a, b \in G$, $a * b \in G$

(ii) **Associativity**: $\forall a, b, c \in G$, $a * (b * c) = (a * b) * c$

(iii) **Identity**: $\exists e \in G$ s.t. $g * e = e * g = g \; \forall \; g \in G$

(iv) **Inverse**: $\forall g \in G \; \exists g^{-1} \in G$ s.t. $g * g^{-1} = g^{-1} * g = e$

Only certain special groups, called **Abelian groups**, have the additional property:

$$\text{(Abelian Groups)} \quad a * b = b * a \; \forall \; a, b \in G. \tag{1.1}$$

There are many distinct groups, even sets of the same number of elements can often be given different group multiplications rules which lead to distinct groups. By distinct we mean that there is no one-to-one and onto map which preserves the group structure. Such maps are called **isomorphisms**.

In practice groups are not found as the elements of some abstract set with some abstract multiplication law, but are found in terms of other objects with known properties. This is called a **representation** of the group to distinguish it from the fundamental and pure abstract

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[1]Technically, this is an example of a non-compact Lie group while these notes deal with the unitary representations associated with compact Lie groups found in the Standard Model.

group.[2] In practice we will use only **matrix representations** where each abstract group element g is represented by a matrix, denoted as $\mathsf{D}(g)$. That is,

$$\text{(abstract group)} \;\; c = a * b \quad \Rightarrow \quad \mathsf{D}(c) = \mathsf{D}(a).\mathsf{D}(b) \;\; \text{(matrix representation)} \tag{1.2}$$

Normal matrix multiplication is playing the role here of group multiplication. If we have a representation made from $d \times d$ matrices, then we say that the **dimension of the representation** is d. Matrix multiplication always satisfies associativity, the identity is always represented by the unit matrix $\mathsf{D}(e) = \mathbb{1}$ and the inverse group element must be the inverse matrix $\mathsf{D}(g^{-1}) = [\mathsf{D}(g)]^{-1}$.[3] If every abstract group element is represented by a unique matrix, then we say we have a **faithful representation**:

$$\text{(Faithful Representation)} \;\; \mathsf{D}(a) \neq \mathsf{D}(b) \text{ if } a \neq b \; \forall a, b \in G, \tag{1.3}$$

Only faithful representations can be used to define groups.

Many representations of groups are not faithful. All groups have a representation where all elements are represented by the number 1 and this is called the **trivial representation**:

$$\text{(Trivial Representation)} \;\; \mathsf{D}(g) = 1 \; \forall \; g \in G. \tag{1.4}$$

Many representations are essentially the same as others. For this you need to understand **similarity transformations** for matrices. Many representations we use are unitary representations using unitary matrices but these only exist for certain types of group such as finite groups and compact Lie groups used for internal symmetries in particle physics. The non-compact Lie groups, such as the space-time symmetry group, need not be unitary.

It is also useful to be aware that representations may be **reducible**, that is you can write any representation as a direct sum of smaller representations called **irreducible representations**. Put another way, all representations of a Lie group can be built out of combinations of the irreducible representations of that group. So we spend much time in group theory trying to find and list the irreducible representations of each group.

One important type of group are **product groups**. These are constructed from two groups, G and H, and are denoted $G \times H$. Each element of the abstract group can be thought as as a pair of elements (g, h) where $g \in G, h \in H$ and the multiplication law is

$$(g_1, h_1) * (g_2, h_2) = (g_1 * g_2, h_1 * h_2) \in G \times H, \quad g_1, g_2 \in G, \;\; h_1, h_2 \in H \tag{1.5}$$

Note the key property of product groups is that all elements can be split into a product of two terms, each involving one of the identities $e_G \in G$ and $e_H \in H$, and these two terms commute

$$(g, h) = (g, e_H) * (e_G, h) = (e_G, h) * (g, e_H) \tag{1.6}$$

In terms of a matrix representation, a product group is formed by taking a **direct product** of the matrices making up representations of the two groups.

The number of elements in a group is called the **order** of a group. Groups with a finite order called **finite groups**, and groups of infinite order are the **infinite groups**.

[2] Rather than define a group by labouriously specifying the result of every possible combination of these abstract group elements $a * b$ (usually in a **group multiplication table**, almost all groups are defined using **representations**.

[3] Thus closure is the only tricky group axiom. Thus matrices lend themselves to the representation of groups, though most sets of matrices do not form groups.

1.2 Lie Group Summary

A **Lie Group** is a special type of infinite group. Elements of these groups[4], D, can be parameterised by a finite number of *continuous* and *real* parameters ϵ, so that $\mathsf{D} \equiv \mathsf{D}(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$. We invariably choose the identity group elements to be $\mathbb{1} = \mathsf{D}(0, 0, \ldots, 0)$. The number of such parameters needed, n, is the **dimension of the Lie group**, not to be confused with the size of the matrices used, d — the dimension of the representation. For most representations, $d \neq n$. The **fundamental representation** is the representation of smallest dimension.

If the elements of a faithful representation of a Lie group are always finite $|D_{ij}| < \infty$ then we have a **compact Lie group**. These obey a useful theorem:

> All compact Lie groups have finite dimensional unitary representations,
> i.e. $\exists\{\mathsf{D}(g)\}$ s.t. $\mathsf{D}^\dagger(g)\mathsf{D}(g) = \mathbb{1}$.

Such unitary finite dimensional representation matrices can then be written in terms of hermitian d dimensional matrices A

$$\mathsf{D}(g) = \exp\{i\mathsf{A}(g)\}, \quad \mathsf{A}^\dagger = \mathsf{A} \tag{1.7}$$

Note that $\mathsf{A} = \mathsf{0}$ generates the identity element. One can then show that the matrices $\mathsf{A}(g)$ are elements of another type of algebraic object called a **Lie algebra** $\mathcal{A}$ which is defined through the following axioms:

(i) The Lie Algebra is a *vector space*, so that laws of multiplication by a real number and addition of elements are defined

   (a) $c_1 A_1 + c_2 A_2 \in \mathcal{A}, \quad \forall\, A_1, A_2 \in \mathcal{A}, \quad c_1, c_2 \in \mathbb{R}$

(ii) Elements of the algebra can also be multiplied together to give another algebra element (closure under multiplication) and this product is denoted as
$[A_1, A_2] \in \mathcal{A}, \quad \forall\, A_1, A_2 \in \mathcal{A},$

(iii) Elements of a Lie algebra also satisfy the **Jacobi identity**
$[[A_1, A_2], A_3] + [[A_2, A_3], A_1] + [[A_3, A_1], A_2] = 0 \quad \forall\, A_1, A_2, A_3 \in \mathcal{A},$

Every abstract Lie group is linked with an abstract Lie algebra. However, a small number of Lie groups, differing in the group elements a long way from the identity, can share the same Lie algebra, since the relationship between Lie group and algebra is only a simple one-to-one map near the identity.

Like the group elements, there are many representations of a Lie Algebra, and we see that for every d-dimensional unitary matrix representation of the Lie group, we have a d-dimensional matrix hermitian representation of the algebra, the matrices $\{\mathsf{A}(g)\}$ of (1.7). Finite dimensional square matrices automatically satisfy the Jacobi identity and the Lie Algebra product law is just the usual commutator of matrices

$$\text{(abstract) } [A_1, A_2] \in \mathcal{A} \quad \Rightarrow \quad [\mathsf{A}_1, \mathsf{A}_2] = \mathsf{A}_1\mathsf{A}_2 - \mathsf{A}_2\mathsf{A}_1 \in \mathcal{A} \text{ (matrix representation)} \tag{1.8}$$

Note that the usual laws of matrix multiplication and subtraction are used in this definition of the algebra product on the right-hand side. Thus closure of the algebra is the only problem for matrix representations.

As with all vector spaces, we can express all vectors in terms of sums of basis vectors. For a Lie algebra, the basis elements are called **generators** and are denoted as $\{\mathsf{T}^a\}$ in a matrix representation so

$$\mathsf{A} = \sum_a c^a \mathsf{T}^a, \ \forall \mathsf{A} \in \mathcal{A}, \ c^a \in \mathbb{R}, \quad \mathsf{T}^a = (\mathsf{T}^a)^\dagger. \tag{1.9}$$

[4] We will only use matrix representations here

Returning to the Lie group elements we see that we have[5]

$$\mathsf{D}(g) = \exp\{i\sum_{a=1}^{n}\epsilon_a \mathsf{T}^a\}, \quad \epsilon_a \equiv \epsilon_a(g) \in \mathbb{R}, \quad a = 1, 2, \dots n = \dim(G) \tag{1.10}$$

There are as many generators as ϵ_a coefficients i.e. the Lie group dimension. For $\epsilon_a = 0\ \forall a$ we get the identity element $\mathsf{D}(e) = \mathbb{1}\exp\{i0\}$. Depending on the group and the representation, only for a limited range of real values do the ϵ_a's give unique group elements. At least for group elements 'near' the identity element, i.e. 'small' ϵ_a, we can choose a fixed set T^a matrices, and let different values of ϵ_a take us through the different group elements[6]

$$D(g) \approx \mathbb{1} + i\sum_{a=1}^{n}\epsilon_a \mathsf{T}^a + \frac{i}{2}\sum_{a,b=1}^{n}\epsilon_a\epsilon_b[\mathsf{T}^a, T^b] + \dots . \tag{1.11}$$

The Lie algebra has a scalar product[7] which for matrix representations is given by the trace $\mathrm{Tr}\{\mathsf{A}_1\mathsf{A}_2\}$. This means we can define orthonormal generators, and in particle physics we *invariably* define

$$\mathrm{Tr}\{\mathsf{T}^a\mathsf{T}^b\} = \frac{1}{2}\delta_{ab}\,, \quad a, b = 1, 2, \dots, n\,. \tag{1.12}$$

The one exception is the generator of a U(1) group where normalisation is usually a matter of choice.

In terms of the generators, closure of multiplication in the Lie algebra has a simple form

$$[\mathsf{T}^a, \mathsf{T}^b] = if^{abc}\mathsf{T}^c\,, \quad a, b, c = 1, 2, \dots, n, \,, \quad f^{abc} = -f^{bac} = -f^{acb} \in \mathbb{R}\,. \tag{1.13}$$

where the f^{abc} are called the **structure constants**. They are real and completely antisymmetric and completely specify the algebra (and hence the Lie group close to the identity). However, many different values of these constants are possible for the same Lie algebra.

For a matrix representation of a product Lie group $G \times H$, we have that the associated Lie algebra is the *sum* of the algebras of the two parts where generators associated with each part commute, i.e.

$$\mathcal{A}_{G\times H} = \left\{\sum_{a=1}^{\dim G+\dim H} c_a\mathsf{T}^a\right\} = \{\mathsf{A}_G + \mathsf{A}_H\}, \tag{1.14}$$

$$[\mathsf{T}^b, \mathsf{T}^c] = 0, \qquad \mathsf{A}_G = \sum_{b=1}^{\dim G} c_b\mathsf{T}^b \in \mathcal{A}_G, \quad \mathsf{A}_H = \sum_{c=1+dimG}^{\dim G+\dim H} c_c\mathsf{T}^c \in \mathcal{A}_H \tag{1.15}$$

A Lie group which can *not* be expressed as a product of two smaller Lie groups, i.e. its algebra can not be split into two mutually commuting parts, is called a **simple Lie Group**. If a Lie group is a product group but none of thee parts is a pure U(1) group then it is called a **semi-simple Lie group**.

There are as many coefficients ϵ_a and matrices T^a as the dimension of the compact Lie group, $a = 1, 2, \dots n$. The real coefficients, $\{\epsilon_a\}$ vary with the group element chosen.

[5]There is no meaning attached to raised rather than lowered indices in the Lie group and algebra context.

[6]Technically, you may not be able to reach all group elements, in particular ones far from the identity. This depends on the *global* properties of the group.

[7]A vector space need not have a scalar product but most encountered in physics do.

1.3 Some SU(2) and SO(3) representations

The compact Lie group SU(2) is three dimensional. Thus we work with generators T^a_{ij} where $a, b, c = 1, 2, 3$ while i, j range over the dimension of the representation. In the following, we work with representations which satisfy the usual orthogonality

$$\mathrm{Tr}\{T^a T^b\} \;=\; \frac{1}{2}\delta^{ab} \tag{1.16}$$

The Cartan sub-algebra has only one generator in it, i.e. only one of the three generators can be diagonal.

Two-dimensional representation — SU(2) only

The matrices satisfy

$$[T^a, T^b] = i\epsilon^{abc}T^c. \tag{1.17}$$

where ϵ^{abc} is the totally anti-symmetric tensor with $\epsilon^{123} = +1$. A suitable choice for the generators

$$\mathsf{T}^a = \frac{1}{2}\boldsymbol{\tau}^a, \tag{1.18}$$

half the **Pauli matrices** which are given by

$$\boldsymbol{\tau}^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\tau}^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\tau}^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.19}$$

The Pauli matrices obey $\boldsymbol{\tau}_i\boldsymbol{\tau}_j = \delta_{ij} + \epsilon_{ijk}\boldsymbol{\tau}_k$.

Three-dimensional representations — SU(2) and SO(3)

The adjoint representation is the three-dimensional case. There are two common forms found, related to each other by a unitary transformation. In both the following cases, the matrices satisfy

$$[\mathsf{T}^a, \mathsf{T}^b] = \frac{i}{2}\epsilon^{abc}\mathsf{T}^c. \tag{1.20}$$

where ϵ^{abc} is the totally anti-symmetric tensor with $\epsilon^{123} = +1$. *Note* that the structure constants differ from those used in the two-dimensional case, (1.17), by a factor of 2.

The first three dimensional representation has T^3 in diagonal form[8].

$$\mathsf{T}^1 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \mathsf{T}^2 = \frac{1}{2\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \mathsf{T}^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{1.21}$$

A second way of writing the three dimensional representation has no generator in diagonal form[9]. It is also the natural way to write the three dimensional representation in the way which is common for the adjoint representation. The adjoint representation is present in *all* Lie Algebras, where one can write $T^a_{ij} = -if^{aij}$, $a, i, j = 1, \ldots, \dim(G)$. Here this gives

$$T^a_{bc} \;=\; -\frac{i}{2}\epsilon_{abc} \tag{1.22}$$

[8]See for example Cheng and Li (4.52). It can be used both for SU(2) and SO(3) but in the case of the latter the way to represent fields is less obvious.

[9]If we think in terms of rotations of real three-dimensional vectors, the definition of SO(3), this second representation is quickly found. It is therefore natural to use this when wanting to find the representation of SO(3) in terms of real fields.

which can be written out as

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & +i & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2}\begin{pmatrix} 0 & 0 & +i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad T^3 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ +i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{1.23}$$

1.4 Further Reading

Note that these are very old notes so there will be more modern texts available.

There are numerous texts with much more detail than given here. Jones [1] especially Ch. 1-4,6,8 starts from the beginning and covers all the relevant applications without going into too much detail, e.g. good on Poincaré group ch. 9. Covers Lie groups in just sufficient depth for the Unification course. Tung [2] also covers basics, and does simple particle physics cases, interesting advanced topics like Poincaré group of space-time symmetries too.

Many of the books on quantum field theory will contain brief summaries of the required group theory. Cottingham and Greenwood [3] is a good book for the Unification course and this has a summary similar to the one here in appendix B but which also includes the Lorentz group (part of the Poincaré group) left out from this summary. I also like the one in Cheng and Li [4], chapter 4. It is a very compact outline of key ideas about group theory required for particle physics and a brief but good introduction of main particle symmetry groups.

For more advanced treatments try the following. Hammermesh [5] is a standard classic text, mathematical and perhaps a bit old fashioned but comprehensive and accessible by physicists. Try chapters 1-3 and 8. Georgi [6], chapters Ch. 1-3,7, has advanced particle symmetry topics covered for physicists. I have also used Cornwall [7] (lots of physical examples) and Joshi (try chapters 1,2,4) in my time.

References

[1] H.F. Jones, *Groups, Representation and Physics* (Institute of Physics Publishing, Bristol, 2nd edition 1998, ISBN `0-7503-0504-5`).

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