Generalised Geometry and Superstring Theory

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Ai miei genitori

To my parents

Abstract

In this dissertation, we describe various applications of generalised geometry in the context of superstring theory. We provide a review of the mathematical pre-requisites, i.e. complex geometry and generalised complex geometry. The relative background from physics is also discussed in detail: we introduce bosonic string theory and type II superstring theory, and then focus on toroidal compactifications of both theories with associated T-dualities. Then, we combine the mathematical results with the ones from physics to perform fluxless Calabi-Yau compactification of type II supergravity. As a final step before applying generalised geometry to strings, we develop the geometrical link between the bosonic fields appearing in superstring theory (together with their symmetries) and generalised geometry. Lastly, we focus on some examples of generalised geometry constructions in the context of string theory: we give a detailed review on how to formulate the NSNS sector of the type II supergravity action in the language of generalised geometry; we introduce Buscher's duality written in terms of generalised objects; we briefly present flux compactification of type II supergravity with reference to the generalised structures that emerge; we provide an overview on double field theory.

Contents

Introduction

Aim, Structure and Content

The aim of this dissertation is to describe generalised geometry and some of its recent applications to superstring theory. More specifically, its aim is to present them inside a self-contained work, which emphasizes the importance of T-duality and compactification in string theory. We introduce the above research topic starting from the "basics", in a detailed and consistent fashion. We assume that the reader has a background in quantum field theory and differential geometry. We also assume familiarity with bosonic string theory and the key concepts from representation theory and supersymmetry. In other words, this work is aimed at a graduate or advanced undergraduate student in theoretical/mathematical physics. Ideally, they would read the dissertation without much need to consult other sources, since almost all the derivations of the results (that do not belong to the "basics" mentioned above) are provided and discussed.[1]

This dissertation is divided into three parts. Part I is purely mathematical, part II is purely physical and part III merges the first two. Each part has a different style, tailored to its content. Part I consists solely of theorems, definitions, remarks and examples. In part II, the presentation of the content is more narrative and less neatly organised. Part III is half-way between the two.

Each part contains two chapters.

Chapter 1 (part I) presents the main results in complex geometry, from almost complex manifolds to Calabi-Yau manifolds. Differently from other sources, we do not provide many examples. Section 1 of chapter 5 can be thought as a condensate of examples of the structures presented in the first chapter.

[1] Footnotes will be used to flag the few cases where this does not happen, and references on where to find detailed proofs will be given.

In Chapter 2 (part I) we introduce generalised geometry. We put a lot of emphasis in justifying the interpretation of generalised objects in the light of ordinary objects from differential geometry. Spinors do not appear in this chapter, as they are (partially) dealt with in section 1 in chapter 6.
Chapter 3 (part II) consists of a brief introduction of bosonic string theory and superstring theory, focusing on the results needed for the subsequent chapters. We present the classical and quantum version of both theories, and derive their massless spectrum. We do not provide all calculations explicitly, but we show how to derive all the central results needed for the remainder of the dissertation.
In chapter 4 (part II), we treat compactifications. We start with field theories, then move to bosonic string theory and finally superstring theory. For the latter two, we study S^1 compactification first, and then general toroidal ones. In both cases, we give a detailed study the T-duality symmetry that emerges. Finally, we give a very short introduction of supergravity in 11 dimensions and type IIA/B supergravity in 10 dimensions.
Chapter 5 (part III) describes some applications of real differential geometry and complex geometry to (super)string theory. As examples of the former, we study the geometric nature of the fields in the NSNS sector of type II supergravity and their symmetries. We also introduce the conditions for Buscher's duality from a differential geometry prospective. Regarding complex geometry, we perform fluxless compactifications of the low energy limit of type II superstring theory.
In chapter 6 (part III), we study a few applications of generalised geometry to superstring theory. First, we derive the (generalised) geometry of flux compactifications. Secondly, we present the geometric formulation of type II supergravity via generalised geometry. Thirdly, we review how T-duality is naturally expressed in the generalised formalism. Finally, we provide a very short introduction to double field theory.

Introduction

The mismatch between logical order and chronological order has always been an extra challenge for scientists. Things are almost never discovered or developed in the same sequence as they are presented 100 years later. This work is no exception. We organised our presentation of the material based on pedagogical considerations and consistency. Our aim was to build all the necessary mathematical knowledge and give all the required background in physics at first, in order to be able to appreciate the latest applications of recent mathematical techniques to unsolved problems in physics. The topics presented, as one can imagine, did not appear historically in this order. We devote this introduction to explaining how the content of this dissertation fits into the historical development of physics and mathematics.

Differential geometry was pioneered by Gauss and Riemann, among others, in the 19th century.[2] Clearly, it did not have the rigorous and neat formulation that we are lucky enough to study today, known as modern differential geometry.[2] Such framework was developed in the first half of the 20th century, as part of the Hilbert's program to formalise mathematics, and builds on the work of Poincaré on the foundations of topology.[3] Among the many contributors, we mention Hausdorff, Cartan, Koszul, Chern, Hodge, De Rham.[4] A very brief review of the main results and constructions in real differential geometry is provided in the appendix, where almost all of the names just mentioned appear. The formal development of differential geometry spanned more than 50 years, ending with some results in complex geometry that are crucial for their application to physics, such as Calabi conjecture (1954) proved by Yau in 1978.[5] Chapter 1 is entirely devoted to presenting complex geometry, and it ends precisely with Calabi-Yau manifolds.

Differential geometry showed its strong link with physics (or viceversa) since Einstein's theory of general relativity. This merger reached a peak in the second half of the century, with the development of string theory; to the point that a physicist, Edward Witten, was awarded the Fields medal for mathematics for the first time in history. Let us now review the history of this theory, which is at the core of this dissertation.

String theory originated from the attempts of theoretical physics to model the behaviour of hadronic particles. At the end of the fifties, Regge found that quantum mechanical particles could be organised in Regge trajectories, a discovery that was confirmed experimentally a few years later with mesons. Roughly ten years later, Veneziano developed a model consistent with such feature, which was extended and generalised by Virasoro, among the others. Until then, there had been no mention to strings, but everything was understood in terms of dual resonance models. The idea of introducing a vibrating string in the model first appeared at the very end of the sixties, as a possible interpretation of the Veneziano amplitude, in works by Nambu, Nielsen and Susskind. Ramond, Naveu and Schwarz worked shortly after that on incorporating fermions. But a few years later, interest was lost for dual resonance models and the associated research, since chromodynamics stole the spotlight.[6] [7]

It is from the mid seventies that string theory started to be thought as a fundamental theory of on nature unifying all forces, comprising gravity. Bosonic string theory, with its mod-

[2] We can trace back the history of the subject even further. It is possible to find very early ideas from the 17th century (see [1]).

ern interpretation, appeared in 1974 independently from Schwarz and Scherk, and Yoneya. Clearly, this theory could not be but a preliminary draft of a theory of everything. It contained only bosons, had a tachyon instability and required 26 space-time dimensions. However, in the second half of the seventies, the study of supersymmetry in field theories became very popular (the first supersymmetric Yang-Mills and supergravity theories appeared), and its application to string theory led to the GSO projection mechanism. This introduced spacetime fermions (and spacetime supersymmetry) in a consistent way in the theory, eliminating the tachyon and reducing the number of spacetime dimensions to 10. We are now at the beginning of the eighties. Extra dimensions and Kaluza-Klein reduction[3] of 11 dimensional supergravity became topics of growing interest, and studies on the recently proposed supersymmetric string theory continued by Green and Schwarz, among the others. It is in this period that type I, IIA and IIB theories were classified, but there was not much excitement from the scientific community. In 1984 things radically changed. They changed so much that such year marks the beginning of the so-called *first superstring revolution*. In 1984 Green and Schwarz found that, quite miraculously, type I superstring theory is anomaly free. Less than a year later, Gross, Harvey, Martinec, and Rohm (the string quartet) discovered an alternative consistent superstring theory: the *heterotic* one. By the end of 1985, there were five candidate superstring theories for the description of the universe: Type I, Type IIA, Type IIB, Heterotic $SO(32)$ and Heterotic $E_8 \times E_8$. Suddenly, the interest in string theory increased among theoretical physicists. In the same year, Calabi-Yau manifolds made their way into string theory: they were found to be the suitable internal space for the compactification of superstring theory. The rest of the decade consisted of years of intense study and interest towards string theory. [7] [9] [10]
Chapters 3 and 4 partially revisit the the above series of events. In chapter 3, we present bosonic string theory, deriving its low energy spectrum with the tachyon ground state and the number of dimensions. We then introduce superstring theory, again deriving its spectrum (using the GSO projection) and finding the number of space-time dimensions. Then, in chapter 4, we describe compactification as a route for obtaining the correct phenomenology from string theory. In this analysis, we pay particular attention to the emerging T-duality symmetry. We present Kaluza-Klein dimensional reduction in QFT, then bosonic string theory with only one dimension compactified on a circle and finally bosonic string theory with an arbitrary number of dimensions compactified on a torus. This adds a realistic feature to the theory: describing a 4-dimensional world. We also see the appearance of $O(n, n)$ as a

[3] This mechanism was introduced 60 years before.[8]

T-duality symmetry group, and review Buscher's approach to T-duality. To add a second realistic feature, fermions, we consider the compactification of superstring theory, again with circles. For the final step towards a realistic phenomenology, we introduce type II supergravity in 10 dimensions, and then, in chapter 5, we present Calabi-Yau compactification mentioned above. Let us now go back to the history.
After roughly a decade of important developments of the theories, a second revolution happened. If the first one consisted of a proliferation of theories, the second one had the opposite nature. As conjectured by Witten and others at the beginning of the 90's, a few years later the five string theories are found to be manifestation of a single 11-dimensional theory: M-theory. A complicated web of dualities and equivalences between the five underlying theories was developed and formalised, including S-duality, T-duality, U-duality, mirror symmetry and conifold transitions. The importance of D-branes was discovered by Polchinski.[7] [11]
Past is not history. Usually, historians give themselves a boundary, of roughly 20/50 years, which distinguishes past from history. We are now about to cross such boundary for the final segment of this section, where we present one of the latest developments of complex geometry and string theory: generalised geometry and its applications.
Generalised geometry was first introduced in 2002 by Hitchin, and later developed by its students Gualtieri and Cavalcanti, as a natural extension of complex geometry.[12] [13] [14] Chapter 2 is dedicated to describing its basic features. It did not take long for theoretical physicists to find a number of applications of this framework to string theory. One of them, presented by Waldram and others, consists of reformulating supergravity theories in the language of generalised geometry. [15] [16] It is not the first that appeared, but the one that is most extensively described in this dissertation (in chapter 6). Two other, very early, applications are the following: the use of generalised structures to classify flux compactifications and the development of double field theory. The first one being pursued by Grana and collaborators, while the second one was developed by Hull.[17] [18] Both are discussed in chapter 6.

Part I

The Mathematics

Ordinary Complex Geometry
1

In this chapter, we give an overview of complex geometry. The aim is to develop the necessary tools in order to define Calabi-Yau manifolds and appreciate their properties. We assume that the reader is already familiar with (real) differential geometry.
Section 1 develops the tools necessary to go from real manifolds to compelx ones.
In section 2, we introduce Hermitian manifolds, describing their geometry.
Section 3 is dedicated to the study of Kähler manifolds.
In section 4, we present many key definitions in complex geometry, which are relevant for the discussion of Calabi-Yau manifolds.
Section 5 presents Calabi-Yau manifolds and their main properties.
For a detailed introduction to complex geometry please see [19] and [20]. This chapter mainly follows chapter 8 of the former and sections V-VIII of the latter. A brief review of some standard results and formulae from differential geometry is also given in the appendix A1.
We conclude with a final technical note on our style for referencing previous equations. "See 1.2.3" stands for "see equation 1.2.3". When, instead, we refer to theorems, definitions or examples, we clearly state it.

1.1 (Almost) Complex Manifolds

In this section, we study the basic features of almost complex manifolds and complex manifolds. We also develop the language of complex forms, which will be used in the next sections. [19] is the main reference for this section.

Definition 1.1.1. Let f be a function on $\mathbb{C}^n$:

$$f : \mathbb{C}^n \to \mathbb{C} \tag{1.1.1}$$
$$(z_1, ..., z_n) \mapsto f(z_1, ..., z_n) = f_R(z_1, ..., z_n) + i f_I(z_1, ..., z_n), \tag{1.1.2}$$

with $z_j = x_j + iy_j$. Then, f is *holomorphic* (or *analytic*) if it satisfies the Cauchy-Riemann conditions:

$$\frac{\partial f_R}{\partial x_j} = \frac{\partial f_I}{\partial y_j}, \qquad \frac{\partial f_I}{\partial x_j} = -\frac{\partial f_R}{\partial y_j}, \qquad j = 1, ..., n. \tag{1.1.3}$$

Definition 1.1.2. A more general map of the form

$$f : \mathbb{C}^n \to \mathbb{C}^m \tag{1.1.4}$$
$$(z_1, ..., z_n) \mapsto (f_1(z_1, ..., z_n), ..., f_m(z_1, ..., z_n)), \tag{1.1.5}$$

is *holomorphic* (or *analytic*) if all the f_j's are holomorphic.

Definition 1.1.3. M is an m-dimensional *complex manifold* if it satisfies the following axioms:

(i) M is a Hausdorff topological space with open sets J.

(ii) M is equipped with a set of charts (U_i, ψ_i), the Atlas, where $U_i \in J$, such that $\cup_i U_i = M$ and ψ_i are homeomorphisms from U_i into open sets U_i' of $\mathbb{C}^m$.

(iii) Given $U_i \cap U_j \neq \emptyset$, the map $\phi_{ij} = \psi_i \circ \psi_j^{-1}$ (that is $\mathbb{C}^m \to \mathbb{C}^m$) is holomorphic.

Remark. To clarify our notation for complex and real dimensions, for the manifold in the above definition the following hold: $dim_{\mathbb{C}}(M) = m$ and $dim_{\mathbb{R}}(M) = 2m$.

Definition 1.1.4. Consider a m-dimensional complex manifold M and a n-dimensional complex manifold N. Let f be a map between them, $f : M \to N$. Take p to be a point in a chart (U, ψ), and let (V, ϕ) be the chart on N that contains $f(p)$. If the map $\phi \circ f \circ \psi^{-1} : \mathbb{C}^m \to \mathbb{C}^n$ is holomorphic, then f is a *holomorphic map* from M to N.

Theorem 1.1.1. The above definition is independent of the coordinate choice on M.

Proof. Let the chart (U, ϕ) have coordinates $\{z^\mu\}$ (with $z^\mu = x^\mu + iy^\mu$), and $w^\mu(z^\nu) = u^\mu(z^\nu) + iv^\mu(z^\nu)$ be holomorphic functions. Consider another chart (U', ϕ') with coordinates $\{z'^\mu\}$ ($z'^\mu = x'^\mu + iy'^\mu$) s.t. $U \cap U' \neq \emptyset$. Then, for a point $p \in U \cap U'$,

$$\frac{\partial u^\mu}{\partial x'^\nu} = \frac{\partial u^\mu}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial x'^\nu} + \frac{\partial u^\mu}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial x'^\nu} = \frac{\partial v^\mu}{\partial y^\alpha}\frac{\partial y^\alpha}{\partial y'^\nu} + \frac{\partial v^\mu}{\partial x^\alpha}\frac{\partial x^\alpha}{\partial y'^\nu} = \frac{\partial v^\mu}{\partial y'^\nu}, \tag{1.1.6}$$

where we used the fact that w^μ are holomorphic wrt z^μ, and z^μ are holomorphic wrt z'^μ. With an analogous calculation, one finds $\partial u^\mu/\partial y'^\nu = -\partial v^\mu/\partial x'^\nu$. Hence, w^μ are also holomorphic wrt z'^μ. □

Remark. Independence from the choice of chart in N can be shown with a similar argument. This is the first encounter with a kind of calculation (chain rule combined with analyticity) that will appear a number of times in the following pages.

Definition 1.1.5. Consider a differentiable manifold M with $\dim_{\mathbb{R}}(M) = m$. We recall that the set of functions on M, denoted by $\mathcal{F}(M)$, is the set of maps $f : M \to \mathbb{R}$. The *complexification* of $\mathcal{F}(M)$, $\mathcal{F}(M)^{\mathbb{C}}$, is given by the maps of the form $f : M \to \mathbb{C}$, where $f = g + ih$ with $g, h \in \mathcal{F}(M)$.

Remark. The usual notation applies: if $f = g + ih$, then $\bar{f} = g - ih$ and f is real if and only if $\bar{f} = f$.

Definition 1.1.6. Let V be a vector space with $\dim_{\mathbb{R}}(V) = n$. The *complexification* of V, $V^{\mathbb{C}}$, is a complex vector space with $\dim_{\mathbb{C}}(V^{\mathbb{C}}) = n$ whose elements are of the form $(X + iY) \in V^{\mathbb{C}}$, $X, Y \in V$. Addition and multiplication by a complex number are defined as follows:

$$\begin{aligned}(X_1 + iY_1) + (X_2 + iY_2) &= (X_1 + X_2) + i(Y_1 + Y_2),\\ (a + ib)(X_1 + iY_1) &= (aX_1 - bY_1) + i(bX_1 + aY_1),\end{aligned} \tag{1.1.7}$$

for $X_1, Y_1, X_2, Y_2 \in V$ and $a, b \in \mathbb{R}$.

Remark. V is a vector subspace of $V^{\mathbb{C}}$ defined by elements of the form $(X + i0) \in V^{\mathbb{C}}$. Vectors in V are called real. The complex conjugate of $Z = X + iY$ is, as usual, $\bar{Z} = X - iY$. A vector is real if $\bar{Z} = Z$.

Remark. If $\{e_i\}$ forms a basis for a vector space V, then the same set of vectors, viewed as complex, also provide a basis for $V^{\mathbb{C}}$, by letting the components be complex.

Definition 1.1.7. A linear operator A that acts on V can be *extended* to a new operator $A_{\mathbb{C}}$ that acts on the the complexification of V, $V^{\mathbb{C}}$. $A_{\mathbb{C}}$ is defined by its action on $(X+iY) \in V^{\mathbb{C}}$:

$$A_{\mathbb{C}}(X+iY) = A(X) + iA(Y). \tag{1.1.8}$$

Remark. We will always declare when we extend operators to act on a complexified vector space, but we will omit the subscript $\mathbb{C}$ to ease the notation.

Example 1.1.1. Consider a (1,2) tensor acting on V and V^*, $T : V \otimes V \otimes V^* \to \mathbb{R}$. Since it is a linear operator, it can be extended to act on $V^{\mathbb{C}}$ and $(V^*)^{\mathbb{C}}$ as $T : V^{\mathbb{C}} \otimes V^{\mathbb{C}} \otimes (V^*)^{\mathbb{C}} \to \mathbb{C}$. In addition to being a linear operator, T is also an element of a vector space (the space of (1,2) tensors $\mathscr{T}_2^1$), and hence it can be complexified. The result is $T = T_1 + iT_2$, with $T_1, T_2 \in \mathscr{T}_2^1$. We have that $\bar{T} = T_1 - iT_2$, and if $\bar{T} = T$ then T is a real (1,2) tensor.

Definition 1.1.8. Given a differentiable manifold M, its tangent space at a point, T_pM, admits the *complexification* $(T_pM)^{\mathbb{C}}$. By definition, its elements are of the form $Z = (X + iY) \in (T_pM)^{\mathbb{C}}$ with $X, Y \in T_pM$. They act on a complexified function $f = (g+ih) \in \mathcal{F}(M)^{\mathbb{C}}$ as:

$$Z[f] = X[g+ih] + iY[g+ih] = X[g] - Y[h] + i(X[h] + Y[g]). \tag{1.1.9}$$

Definition 1.1.9. The co-tangent space T_p^*M also admits a *complexification*, $(T_p^*M)^{\mathbb{C}}$, in an analogous way. It is the set of elements of the form $\omega = (\sigma + i\rho)$ with $\sigma, \rho \in T_p^*M$.

Remark. Tensors at specific point of the manifold can be extended and complexified, just as in example 1.1.1.

For tensor fields defined over the whole manifold, the complexification is again the same procedure: $Z = (X + iY) \in \mathscr{T}_q^p(M)^{\mathbb{C}}$ with $X, Y \in \mathscr{T}_q^p(M)$. We will denote the space of $(1,0)$ tensor fields, i.e. vector fields, equivalently as $\mathscr{T}_0^1(M)$, or TM for short.

Definition 1.1.10. Let M be a differentiable manifold and J a (real) tensor field of type (1,1) defined on the manifold. J provides a map of the form $J : T_pM \to T_pM$, and we denote J at the point $p \in M$ as J_p. Then M is an *almost complex manifold* if

$$J_p^2 = -\mathbb{1}_{T_pM} \qquad \forall p \in M., \tag{1.1.10}$$

where $\mathbb{1}_{T_pM}$ is the unity operator of T_pM. J takes the name of *almost complex structure*.

Theorem 1.1.2. For M to be an almost complex manifold, it must have even (real) dimensions.

Proof. Let J be the almost complex structure on M. It immediately follows from 1.1.10 that J_p has eigenvalues $\pm i$. Let $\dim_{\mathbb{R}}(M) = n$, so that the components of J_p form a $n \times n$ matrix. Suppose that p eigenvalues are $+i$ and q of them are $-i$ $(p + q = n)$. Then taking the determinant of 1.1.10 yields: $(i)^p(-i)^q = (-1)^q(i)^n = -1$, which implies that n must be even. $\square$

Theorem 1.1.3. The complexified tangent space of an almost complex manifold splits at each point into a direct sum of two disjoint vector subspaces, according to the eigenvalue of J_p.

Proof. Let J be an almost complex structure defined over a differentiable manifold M with $\dim_{\mathbb{R}}(M) = 2m$, so that $J_p^2 = -\mathbb{1}_{2m}$ is true for each $p \in M$. Let $T_pM^{\mathbb{C}}$ be the complexified tangent space, and let us extend J_p to act on it as $J_p : T_pM^{\mathbb{C}} \to T_pM^{\mathbb{C}}$. By definition, $J_p^2 = -\mathbb{1}_{2m}$ is still a property of the extended operator. Here, $\mathbb{1}_{2m}$ stands for the $2m \times 2m$ unit matrix. Then we can define two operators $\mathcal{P}^{\pm}$ pointwise as:

$$\mathcal{P}_p^{\pm} = \frac{1}{2}(\mathbb{1}_{2m} \mp iJ_p). \tag{1.1.11}$$

It is easy to check that they satisfy $(\mathcal{P}_p^{\pm})^2 = \mathcal{P}_p^{\pm}$, $\mathcal{P}_p^{+} + \mathcal{P}_p^{-} = \mathbb{1}$, $\mathcal{P}_p^{+}\mathcal{P}_p^{-} = \mathcal{P}_p^{-}\mathcal{P}_p^{+} = 0$. We also have that $J_p\mathcal{P}_p^{\pm}Z = \pm i\mathcal{P}_p^{\pm}Z$. Hence, any vector belonging to $T_pM^{\mathbb{C}}$ can be split into two components: both eigenvectors of J_p, one with eigenvalue $+i$ and the other with eigenvalue $-i$. The projection operators provide explicit maps from the complexified tangent space to two disjoint vector subspaces of the complexified tangent space (at each point). Thus,

$$(T_pM)^{\mathbb{C}} = T_pM^{+} \oplus T_pM^{-}, \tag{1.1.12}$$

with $T_pM^{\pm} = \{Z \in (T_pM)^{\mathbb{C}} |\ J_pZ = \pm iZ\}$. $\square$

Remark. Vectors belonging to T_pM^{+} are called holomorphic, vectors belonging to T_pM^{-} are called anti-holomorphic.
Clearly, if $Z \in T_pM^{+}$, then $\bar{Z} \in T_pM^{-}$. Hence: $T_pM^{-} = \overline{T_pM^{+}} = \{\bar{Z}|Z \in T_pM^{+}\}$.

Remark. The same decomposition as above applies to vector fields. Let $X \in \mathscr{T}_0^1(M)^{\mathbb{C}}$ be a complexified vector field. Then, we define $JZ \in (\mathscr{T}_0^1(M))^{\mathbb{C}}$ pointwise by $JZ|_p = J_pZ|_p$. We can construct projection operators as above, and we have that:

$$Z|_p = (\mathcal{P}_p^{+} + \mathcal{P}_p^{-})Z|_p = Z^{+}|_p + Z^{-}|_p, \tag{1.1.13}$$

where $Z^{\pm} = \mathcal{P}^{\pm} Z$ are again defined pointwise. We call Z^+ a holomorphic vector field and Z^- an anti-holomorphic vector field, and we have the same decomposition into the corresponding subspaces:

$$\begin{aligned} &\mathscr{T}_0^1(M)^{\mathbb{C}} = \mathscr{T}_0^1(M)^+ \oplus \mathscr{T}_0^1(M)^-, \\ \text{with} \quad &\mathscr{T}_0^1(M)^{\pm} = \{Z \in \mathscr{T}_0^1(M)^{\mathbb{C}} |\ J_p Z|_p = \pm i Z|_p \ \ \forall p\}. \end{aligned} \tag{1.1.14}$$

Theorem 1.1.4. A complex manifold is an almost complex manifold.

Proof. A complex manifold M is automatically differentiable, by definition. The existence of an almost complex structure J on M suffices to show that M is an almost complex manifold. We will construct J explicitly. Let $\dim_{\mathbb{C}} = m$. Given a point p in some chart, with coordinates $z^\mu = x^\mu + iy^\mu$ ($\mu = 1, ..., m$), the tangent space at p has the natural set of basis vectors: $\{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^m}, \frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^m}\}$. Similarly, $T_p^* M$ is spanned by $\{dx^1, ..., dx^m, dy^1, ..., dy^m\}$. This is called the *real basis*. It is the most natural one to introduce at first, but not the most convenient to work with (see next Remark).
We define:

$$J_p = dx^\mu \otimes \frac{\partial}{\partial y^\mu} - dy^\mu \otimes \frac{\partial}{\partial x^\mu}, \tag{1.1.15}$$

which in components reads:

$$J_p = \begin{pmatrix} 0 & -\mathbb{1}_m \\ \mathbb{1}_m & 0 \end{pmatrix}. \tag{1.1.16}$$

We can view this (1,1) tensor as the endomorphism $J_p : T_p M \to T_p M$ given by:

$$J_p(\frac{\partial}{\partial x^\mu}) = \frac{\partial}{\partial y^\mu}, \qquad J_p(\frac{\partial}{\partial y^\mu}) = -\frac{\partial}{\partial x^\mu}. \tag{1.1.17}$$

J_p takes the same constant form for any p and its action is independent of the chart. To see this, let the point p lie in the overlap between U and another chart V, with coordinates $z'^\mu = x'^\mu + iy'^\mu$. Then, since M is a complex manifold, in the overlap $z^\mu(z'^\nu) = x^\mu(z'^\nu) + iy^\mu(z'^\nu)$ are holomorphic functions. Hence, according to definition 1.1.1, the Cauchy-Riemann conditions apply, and we have:

$$J_p(\frac{\partial}{\partial x'^\mu}) = J_p\Big(\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial}{\partial x^\alpha} + \frac{\partial y^\alpha}{\partial x'^\mu}\frac{\partial}{\partial y^\alpha}\Big) = \frac{\partial y^\alpha}{\partial y'^\mu}\frac{\partial}{\partial y^\alpha} + \frac{\partial x^\alpha}{\partial y'^\mu}\frac{\partial}{\partial x^\alpha} = \frac{\partial}{\partial y'^\mu}. \tag{1.1.18}$$

Analogously, $J_p(\frac{\partial}{\partial y'^\mu}) = -\frac{\partial}{\partial x'^\mu}$. Therefore J it is a tensor field defined globally over the manifold, which clearly squares to minus identity. Hence, we have constructed an almost complex structure, and this proves that a complex manifold is an almost complex manifold. $\square$

Remark. We can define the set of vectors $\frac{\partial}{\partial z^\mu} = \frac{1}{2}(\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial y^\mu})$ and $\frac{\partial}{\partial \bar{z}^\mu} = \frac{\partial}{\partial x^\mu} + i\frac{\partial}{\partial y^\mu}$ to be the basis for $T_pM^{\mathbb{C}}$. Then, $dz^\mu = dx^\mu + idy^\mu$ and $d\bar{z}^\mu = dx^\mu - idy^\mu$ are the dual bases that span $(T_p^*M)^{\mathbb{C}}$. This is called the *complex basis*, and it will be the one we will use to make calculations later in this chapter.

We can now extend J_p, defined above, to act on the complexified tangent space $J_p : T_pM^{\mathbb{C}} \to T_pM^{\mathbb{C}}$, so that $J_p(\frac{\partial}{\partial z^\mu}) = i\frac{\partial}{\partial z^\mu}$ and $J_p(\frac{\partial}{\partial \bar{z}^\mu}) = -i\frac{\partial}{\partial \bar{z}^\mu}$. Equivalently, the (1,1) tensor expression is

$$J_p = idz^\mu \otimes \frac{\partial}{\partial z^\mu} - id\bar{z}^\mu \otimes \frac{\partial}{\partial \bar{z}^\mu}, \tag{1.1.19}$$

with components

$$J_p = \begin{pmatrix} i\mathbb{1}_m & 0 \\ 0 & -i\mathbb{1}_m \end{pmatrix}. \tag{1.1.20}$$

In index notation,[1]

$$(J_p)_\mu{}^\nu = i\delta_\mu{}^\nu, \qquad (J_p)_{\bar{\mu}}{}^{\bar{\nu}} = -i\delta_{\bar{\mu}}{}^{\bar{\nu}}. \tag{1.1.21}$$

This extended operator can be used to construct the projection operators explicitly and perform the splitting into holomorphic and anti-holomorphic components. In the matrix form, they are given by:

$$\mathcal{P}_p^+ = \begin{pmatrix} \mathbb{1}_m & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{P}_p^- = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{1}_m \end{pmatrix}. \tag{1.1.22}$$

This is the form of the complex structure that we will use in the rest of this dissertation.

Theorem 1.1.5. On a complex manifold, a holomorphic vector is such in any chart.

Proof. Let M be a complex manifold and U be a chart with coordinates $\{z^\mu\}$. Consider a generic holomorphic vector X in U: $X = X^\mu \frac{\partial}{\partial z^\mu}$. Then, let X lie in the overlap between U and another chart V, with coordinates $\{z'^\mu\}$. As before, we have that $z'^\mu(z^\nu) = x'^\mu(z^\nu) +$

[1] For the index convention, we follow [19], [20] and [21], while in [22] the position of the indices is inverted.

$iy'^{\mu}(z^{\nu})$ are analytic. Hence:

$$\frac{\partial}{\partial z^{\mu}} = \frac{1}{2}(\frac{\partial}{\partial x^{\mu}} - i\frac{\partial}{\partial y^{\mu}}) = \frac{1}{2}(\frac{\partial x'^{\alpha}}{\partial x^{\mu}}\frac{\partial}{\partial x'^{\alpha}} + \frac{\partial y'^{\alpha}}{\partial x^{\mu}}\frac{\partial}{\partial y'^{\alpha}}) - \frac{i}{2}(\frac{\partial x'^{\alpha}}{\partial y^{\mu}}\frac{\partial}{\partial x'^{\alpha}} + \frac{\partial y'^{\alpha}}{\partial y^{\mu}}\frac{\partial}{\partial y'^{\alpha}}) =$$
$$\frac{1}{2}\frac{\partial x'^{\alpha}}{\partial x^{\mu}}(\frac{\partial}{\partial x'^{\alpha}} - i\frac{\partial}{\partial y'^{\alpha}}) - \frac{i}{2}\frac{\partial x'^{\alpha}}{\partial y^{\mu}}(\frac{\partial}{\partial x'^{\alpha}} - i\frac{\partial}{\partial y'^{\alpha}}) = \frac{1}{2}(\frac{\partial x'^{\alpha}}{\partial x^{\mu}} - i\frac{\partial x'^{\alpha}}{\partial y^{\mu}})\frac{\partial}{\partial z'^{\alpha}}, \tag{1.1.23}$$

where we used the Cauchy-Riemann equations from the first to the second line and defined $\frac{\partial}{\partial z'^{\alpha}} = \frac{\partial}{\partial x'^{\alpha}} - i\frac{\partial}{\partial y'^{\alpha}}$ as usual. This shows that if X is holomorphic in U, then it is holomorphic (with different components) in V as well. □

Definition 1.1.11. Let M be an almost complex manifold with almost complex structure J. If the Lie bracket of any holomorphic vector fields is again a holomorphic vector field, then J is said to be *integrable* or *involutive*.

Remark. Involutivity and integrability are not the same thing a priori. What we presented above is actually the definition of involutivity, while the condition for integrability is based on the existence of solutions to a certain set of differential equations. It is a fact that the two definitions are equivalent.[2] Hence, involutivity and integrability can be used interchangeably.

Definition 1.1.12. Let M be an almost complex manifold with almost complex structure J. The *Nijenhuis tensor field* is defined as:

$$\begin{aligned} N : \mathscr{T}_0^1(M) \times \mathscr{T}_0^1(M) &\to \mathscr{T}_0^1(M) \\ (X, Y) &\mapsto N(X,Y) = [X,Y] + J[JX,Y] + J[X,JY] - [JX,JY], \end{aligned} \tag{1.1.24}$$

where $[\cdot,\cdot]$ is the Lie bracket (see A.1.24 in the Appendix).

Theorem 1.1.6. Let M be an almost complex manifold with almost complex structure J. J is integrable/involutive if and only if $N(X,Y) = 0$ for any $X, Y \in \mathscr{T}_0^1(M)$.

[2] For the interested reader, please see [23].

Proof. Let $Z = (X + iY) \in \mathscr{T}_0^1(M)^{\mathbb{C}}$ and $W = (U + iV) \in \mathscr{T}_0^1(M)^{\mathbb{C}}$. We then extend the Nijenhuis tensor field as usual:

$$N(Z, W) = \big(N(X, U) - N(Y; V)\big) + i\big(N(X, V) + N(Y, U)\big). \tag{1.1.25}$$

We recall that $\mathscr{T}_0^1(M)^{\mathbb{C}} = \mathscr{T}_0^1(M)^+ \oplus \mathscr{T}_0^1(M)^-$ (see 1.1.14), so we can write Z and W accordingly as $Z = Z^+ + Z^-$ and $W = W^+ + W^-$. As usual, the superscripts $\pm$ denote elements of $\mathscr{T}_0^1(M)^{\pm}$.
Now suppose that J is integrable/involutive. This means that $[Z^+, W^+] \in \mathscr{T}_0^1(M)^+$, i.e. $J[Z^+, W^+] = +i[Z^+, W^+]$. Thus,

$$\begin{aligned} N(Z^+, W^+) =&[Z^+, W^+] + J[JZ^+, W^+] + J[Z^+, JW^+] - [JZ^+, JW^+] = \\ &[Z^+, W^+] + J[iZ^+, W^+] + J[Z^+, iW^+] - [iZ^+, iW^+] = 0. \end{aligned} \tag{1.1.26}$$

Analogously, $N(Z^-, W^-) = 0$. Also,

$$\begin{aligned} N(Z^+, W^-) =&[Z^+, W^-] + J[JZ^+, W^-] + J[Z^+, JW^-] - [JZ^+, JW^-] = \\ &[Z^+, W^-] + J[iZ^+, W^-] + J[Z^+, (-i)W^-] - [iZ^+, (-i)W^-] = 0, \end{aligned} \tag{1.1.27}$$

and analogously $N(Z^-, W^+) = 0$. Thus, the Nijenhuis tensor vanishes:

$$N(Z, W) = N(Z^+ + Z^-, W^+ + W^-) = N(Z^+, W^+) + N(Z^+, W^-) + N(Z^-, W^+) + N(Z^-, W^-) = 0. \tag{1.1.28}$$

Let us now show the opposite. We assume that $N(X, Y) = 0 \ \forall X, Y \in \mathscr{T}_0^1(M)$. It follows from 1.1.25 that $N(Z, W) = 0 \ \forall Z, W \in \mathscr{T}_0^1(M)^{\mathbb{C}}$. Suppose that $Z = Z^+, W = W^+$, with $Z^+, W^+ \in \mathscr{T}_0^1(M)^+$. Then,

$$\begin{aligned} 0 = N(Z^+, W^+) = [Z^+, W^+] + J[JZ^+, W^+] + J[Z^+, JW^+] - [JZ^+, JW^+] = \\ [Z^+, W^+] + J[iZ^+, W^+] + J[Z^+, iW^+] - [iZ^+, iW^+] = 2[Z^+, W^+] + 2iJ[Z^+, W^+] \\ \implies J[Z^+, W^+] = i[Z^+, W^+] \iff [Z^+, W^+] \in \mathscr{T}_0^1(M)^+. \end{aligned} \tag{1.1.29}$$

This completes the proof. $\square$

Theorem 1.1.7. Let M be an almost complex manifold with almost complex structure J. If J is integrable, then M is a complex manifold with the almost complex structure J.

Proof. There are various proofs for this theorem, known as Newlander-Nirenberg theorem. We refer to [20] for a quick sketch of the proof, which uses the concept of (r, s)-forms, that we will introduce shortly.[3] □

Remark. Before ending this section, we now turn to forms.

Definition 1.1.13. Let M be a differentiable manifold. A *complex q-form* γ at the point p is defined as $\gamma = \alpha + i\beta$, where α and β are two q-forms at the point p, i.e. $\alpha, \beta \in \Omega^q_p(M)$.

Remark. We denote the vector space of complex q-forms at p as $\Omega^q_p(M)^{\mathbb{C}}$.
We have that $\Omega^q_p(M)$ is a vector subspace of $\Omega^q_p(M)^{\mathbb{C}}$ defined by elements of the form $(\alpha+i0) \in \Omega^q_p(M)^{\mathbb{C}}$. The complex conjugate of $\gamma = \alpha + i\beta$ is, as usual, $\bar{\gamma} = \alpha - i\beta$. A complex q-form is real if and only if $\gamma = \bar{\gamma}$.

Definition 1.1.14. Let M be a differentiable manifold. A *complex q-form* γ is defined as $\gamma = \alpha + i\beta$, where α and β are two q-forms, $\alpha, \beta \in \Omega^q(M)$.

Remark. This is simply an extension of the previous definitions to forms defined over the whole manifold.

Definition 1.1.15. Let $\gamma = (\alpha + i\beta) \in \Omega^q_p(M)^{\mathbb{C}}$ and $\gamma' = (\alpha' + i\beta') \in \Omega^q_p(M)^{\mathbb{C}}$. Then, their *exterior product* is defined as:

$$\gamma \wedge \gamma' = (\alpha + i\beta) \wedge (\alpha' + i\beta') = (\alpha \wedge \alpha' - \beta \wedge \beta') + i(\alpha \wedge \beta' + \beta \wedge \alpha'). \tag{1.1.30}$$

Definition 1.1.16. Let $\gamma = (\alpha + i\beta) \in \Omega^q_p(M)^{\mathbb{C}}$. The *exterior derivative* d is defined as:

$$d\gamma = d\alpha + id\beta. \tag{1.1.31}$$

Definition 1.1.17. Let M be a complex manifold with $\dim_{\mathbb{C}} M = m$. Let $\alpha \in \Omega^q_p(M)^{\mathbb{C}}$. Let V_i be q vectors at p, with either $V_i \in T_pM^+$ or $V_i \in T_pM^-$ $(i = 1, ..., q)$.
α is called an *(r,s)-form* if $\alpha(V_1, ..., V_q) = 0$ unless r of the vectors are in T_pM^+ and s are in T_pM^-. Clearly, r, s are integers such that $r + s = q$. The set of (r,s)-forms at the point p is denoted by $\Omega^{r,s}_p(M)$. (r, s) specifies the *bidegree* of α.

Remark. A (r,s)-form is defined over M if there is a smooth assignment of (r,s)-forms to all points of M. The set of (r,s)-forms defined over M is, as usual $\Omega^{r,s}(M)$. If a form α belongs to $\Omega^{r,s}(M)$, we will sometimes make it explicit with the notation $\alpha^{r,s}$.

[3] In addition, [24] and [25] give a more detailed overview on the topic.

Example 1.1.2. Let (V, ψ) be a chart of a complex manifold M. Let $p \in V$ and $\psi(p) = z^\mu$. We choose $\{\frac{\partial}{\partial z^\mu}\}$ and $\{\frac{\partial}{\partial \bar{z}^\mu}\}$ as the basis for the tangent space. Accordingly, $\{dz^\mu\}$ and $\{d\bar{z}^\mu\}$ provide a basis for the cotangent space. We have that $\{\frac{\partial}{\partial z^\mu}\}$ span T_pM^+ and $\{\frac{\partial}{\partial \bar{z}^\mu}\}$ span T_pM^-. Hence, a (r,s)-form α in components reads:

$$\alpha = \frac{1}{r!s!}\alpha_{\mu_1 ... \mu_r \bar{\nu_1} ... \bar{\nu_s}} dz^{\mu_1} \wedge ... \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge ... \wedge d\bar{z}^{\nu_s}. \tag{1.1.32}$$

Here we use bars on indices to emphasize the contraction with anti-holomorphic basis vectors. Clearly, $\{dz^{\mu_1} \wedge ... \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge ... \wedge d\bar{z}^{\nu_s}\}$ forms a basis for $\Omega^{r,s}_p(M)$.

Remark. An (r,s)-form in the z^μ coordinates is also an (r,s)-form in any other overlapping coordinates. This can be quickly shown by using chain rule and holomorphic properties, in analogy with theorems 1.1.1 and 1.1.5.

Theorem 1.1.8. If $\alpha \in \Omega^{r,s}(M)$ and $\beta \in \Omega^{r',s'}(M)$, then $\bar{\alpha} \in \Omega^{s,r}(M)$ and $\alpha \wedge \beta \in \Omega^{r+r',s+s'}(M)$.

Proof. It immediately follows from the component expression. $\square$

Theorem 1.1.9. A complex p-form α has a unique decomposition given by:

$$\alpha = \sum_{r+s=p} \alpha^{r,s} \tag{1.1.33}$$

Proof. We prove this statement in components. We consider a general p-form α, which reads:

$$\alpha = \alpha_{n_1 ... n_p} dz^{n_1} ... dz^{n_p}, \tag{1.1.34}$$

where $n_i = (\mu_i, \bar{\mu}_i)$ and $dz^{n_i} = (dz^{\mu_i}, d\bar{z}^{\mu_i})$. Hence, we can split each index n_i, and the corresponding components, in the above expression as:

$$\begin{aligned} \alpha =& \alpha_{n_1 ... n_{i-1} n_i ... n_p} dz^{n_1} ... dz^{n_{i-1}} dz^{n_i} ... dz^{n_p} = \\ & \alpha_{n_1 ... n_{i-1} \mu_i ... n_p} dz^{n_1} ... dz^{n_{i-1}} dz^{\mu_i} ... dz^{n_p} + \alpha_{n_1 ... n_{i-1} \bar{\mu}_i ... n_p} dz^{n_1} ... dz^{n_{i-1}} d\bar{z}^{\mu_i} ... dz^{n_p}. \end{aligned} \tag{1.1.35}$$

Performing this splitting for all the indices, and denoting by r the number of indices without bars, each term in the sum has $0 \leq r \leq p$. Cearly the remaining $s = p - r$ indices have bars. Thus, we have a sum of (r,s)-forms, with $0 \leq r \leq p$ and $s = p - r$. $\square$

Theorem 1.1.10. The exterior derivative of an (r,s)-form is a sum of an $(r+1,s)$-form and an $(r,s+1)$-form.

Proof. Applying the exterior derivative to 1.1.32, and separating the new index as above, we obtain:

$$d\alpha = \frac{1}{r!s!}\partial_n \alpha_{\mu_1\dots\mu_r\bar{\nu_1}\dots\bar{\nu_s}} dz^n \wedge dz^{\mu_1} \wedge \dots \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge \dots \wedge d\bar{z}^{\nu_s} =$$
$$\frac{1}{r!s!}\left(\partial_\beta \alpha_{\mu_1\dots\mu_r\bar{\nu_1}\dots\bar{\nu_s}} dz^\beta + \partial_{\bar{\beta}} \alpha_{\mu_1\dots\mu_r\bar{\nu_1}\dots\bar{\nu_s}} d\bar{z}^\beta\right) \wedge dz^{\mu_1} \wedge \dots \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge \dots \wedge d\bar{z}^{\nu_s}. \quad (1.1.36)$$

The first term is an $(r+1,s)$-form, while the second one is an $(r,s+1)$-form. □

Definition 1.1.18. Let α be a complex (r,s)-form. Then, according to the above theorem, we can split the operator d as $d = \partial + \bar{\partial}$, where

$$\begin{gathered}\partial : \Omega^{r,s}(M) \to \Omega^{r+1,s}(M) \\ \alpha = \alpha_{\mu_1\dots\bar{\nu}_s} dz^{\mu_1} \wedge \dots \wedge d\bar{z}^{\nu_s} \mapsto \partial\alpha = \partial_\beta \alpha_{\mu_1\dots\bar{\nu}_s} dz^\beta \wedge dz^{\mu_1} \wedge \dots \wedge d\bar{z}^{\nu_s},\end{gathered} \quad (1.1.37)$$

and

$$\begin{gathered}\bar{\partial} : \Omega^{r,s}(M) \to \Omega^{r+1,s}(M) \\ \alpha = \alpha_{\mu_1\dots\bar{\nu}_s} dz^{\mu_1} \wedge \dots \wedge d\bar{z}^{\nu_s} \mapsto \partial\alpha = \bar{\partial}_{\bar{\beta}} \alpha_{\mu_1\dots\bar{\nu}_s} d\bar{z}^\beta \wedge dz^{\mu_1} \wedge \dots \wedge d\bar{z}^{\nu_s}.\end{gathered} \quad (1.1.38)$$

The operators ∂ and $\bar{\partial}$ are called *Dolbeault operators.*

Remark. Following 1.1.33, the action of a Dolbeault operator on any form α is specified by the action on its (r,s) components:

$$\partial\alpha = \sum_{r+s=p} \partial\alpha^{r,s} \quad \text{and} \quad \bar{\partial}\alpha = \sum_{r+s=p} \bar{\partial}\alpha^{r,s}. \quad (1.1.39)$$

Theorem 1.1.11. For any complex form α, the Dolbeault operators satisfy:

1. $\partial\partial = (\partial\bar{\partial} + \bar{\partial}\partial) = \bar{\partial}\bar{\partial} = 0.$
2. $\partial\bar{\alpha} = \overline{\bar{\partial}\alpha}$ and $\bar{\partial}\bar{\alpha} = \overline{\partial\alpha}.$

Proof. Let $\alpha \in \Omega^{r,s}(M)$.
1. follows directly from the nilpotency of the exterior derivative and the decomposition 1.1.33 of theorem 1.1.9. Explicitly:

$$0 = d^2\alpha = (\partial + \bar{\partial})(\partial + \bar{\partial})\alpha = \partial\partial\alpha + (\partial\bar{\partial} + \bar{\partial}\partial)\alpha + \bar{\partial}\bar{\partial}\alpha. \tag{1.1.40}$$

Since the three terms are of different bidegree from each other, they must vanish separately. To prove 2., we start by considering that $\bar{d} = \overline{\partial + \bar{\partial}} = \bar{\partial} + \partial = d$. Thus,

$$\partial\bar{\alpha} + \bar{\partial}\alpha = d\bar{\alpha} = \bar{d}\bar{\alpha} = \overline{d\alpha} = \overline{(\partial + \bar{\partial})\alpha} = \overline{\partial\alpha} + \overline{\bar{\partial}\alpha}. \tag{1.1.41}$$

The result is obtained by matching the terms on the two sides according to their bidegree. □

1.2 Hermitian Manifolds

In this section, we study a special case of (almost) complex manifolds: (almost) Hermitian manifolds. The key notion to remember before starting this section is that any (almost) complex manifold is a real differentiable manifold (see definition 1.1.10). As such, it may admit a Riemannian metric, which turns it into a Riemannian manifold. The conditions on such metric will play a crucial rule in the following definitions and theorems. We follow both [19] and [20] in this section.

Definition 1.2.1. Let M be a complex manifold and let g be a Riemannian metric of M as a differentiable manifold. We can extend (but *not* complexify) g from $T_pM \times T_pM \to \mathbb{R}$ to $T_pM^{\mathbb{C}} \times T_pM^{\mathbb{C}} \to \mathbb{R}$. For $Z = (X + iY) \in T_pM^{\mathbb{C}}$ and $W = (U + iV) \in T_pM^{\mathbb{C}}$, the *extended metric* g is defined as:

$$g_p(Z, W) = g_p(X, U) - g_p(Y, V) + i(g_p(X, V) + g_p(Y, U)), \tag{1.2.1}$$

for any $p \in M$.

Definition 1.2.2. Let M be an almost complex manifold with $\dim_{\mathbb{C}}(M) = m$ and with almost complex structure J. Let g be a Riemannian metric on M.

Denote by ${J_n}^k$ and g_{nk} the components of J and g, respectively, at some point p in a chart. Then, if for any $p \in M$, $X, Y \in T_pM$

$$g_p(J_pX, J_pY) = g_p(X,Y) \iff {J_m}^k {J_n}^l g_{kl} = g_{mn}, \tag{1.2.2}$$

the metric g is *hermitian wrt* J and M is an *almost Hermitian manifold.*

Theorem 1.2.1. Let M be an almost complex manifold with almost complex structure J and $X \in T_pM$. Then, X and J_pX are orthogonal wrt an Hermitian metric g.

Proof.

$$g_p(X, J_p) = g_p(J_pX, J_pJ_pX) = -g_p(J_pX, X) = -g_p(X, J_p). \tag{1.2.3}$$

□

Theorem 1.2.2. An almost complex manifold always admits a Hermitian metric.

Proof. Let g be any Riemannian metric on an almost complex manifold M. We define a new metric via

$$g'_p(X,Y) = \frac{1}{2}\big(g_p(X,Y) + g_p(J_pX, J_pY)\big). \tag{1.2.4}$$

Equivalently, in components,

$$g'_{mn} = \frac{1}{2}(g_{mn} + {J_m}^k {J_n}^l g_{kl}). \tag{1.2.5}$$

If g is positive definite, then g' is positive definite. Clearly, by construction:

$$g'_p(J_pX, J_pY) = g'_p(X,Y) \iff {J_m}^k {J_n}^l g'_{kl} = g'_{mn}. \tag{1.2.6}$$

Hence, we have constructed an Hermitian metric g' for the almost complex manifold. □

Remark. The above result emphasizes that hermiticity is a condition on the metric, not on the manifold.

Definition 1.2.3. If we use the complex basis (as we anticipated), then we can organise the *components of the metric* at a point p as follows:

$$g_{\mu\nu} = g_p(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}) \tag{1.2.7}$$
$$g_{\mu\bar{\nu}} = g_p(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^{\bar{\nu}}}), \tag{1.2.8}$$
$$g_{\bar{\mu}\nu} = g_p(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial z^\nu}), \tag{1.2.9}$$
$$g_{\bar{\mu}\bar{\nu}} = g_p(\frac{\partial}{\partial z^{\bar{\mu}}}, \frac{\partial}{\partial z^{\bar{\nu}}}). \tag{1.2.10}$$

Remark. As usual, $g_{\mu\nu} = g_{\nu\mu}$, $g_{\mu\bar{\nu}} = g_{\bar{\nu}\mu}$ and $g_{\bar{\mu}\bar{\nu}} = g_{\bar{\nu}\bar{\mu}}$, by symmetry. And it is immediate to see that we also have $\overline{g_{\mu\bar{\nu}}} = g_{\bar{\mu}\nu}$ and $\overline{g_{\mu\nu}} = g_{\bar{\mu}\bar{\nu}}$ (recall that the metric g is extended, but not complexified, so it is still a real operator).

Theorem 1.2.3. The only non-zero components of an Hermitian metric are of the form $g_{\mu\bar{\nu}}$, and the Hermitian metric in components reads:

$$g = g_{\mu\bar{\nu}} dz^\mu \otimes d\bar{z}^\nu + g_{\bar{\mu}\nu} d\bar{z}^\mu \otimes dz^\nu. \tag{1.2.11}$$

Proof.

$$g_{\mu\nu} = g_p(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}) = g_p(J_p\frac{\partial}{\partial z^\mu}, J_p\frac{\partial}{\partial z^\nu}) = -g_p(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}) = -g_{\mu\nu}. \tag{1.2.12}$$

Analogously,

$$g_{\bar{\mu}\bar{\nu}} = g_p(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial \bar{z}^\nu}) = g_p(J_p\frac{\partial}{\partial \bar{z}^\mu}, J_p\frac{\partial}{\partial \bar{z}^\nu}) = -g_p(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial \bar{z}^\nu}) = -g_{\bar{\mu}\bar{\nu}}. \tag{1.2.13}$$

The explicit forms in components immediately follows. $\square$

Remark. We now make a few comments on indices.
First, we summarise the notation introduced so far for a manifold M:

- We use lower case Roman letters for indices that take values from 1 to $\dim_{\mathbb{R}}(M)$.
- We used unbarred Greek letters for indices that run from 1 to $\dim_{\mathbb{C}}(M)$ and are naturally contracted with the holomorphic part of the basis (i.e. dz^μ or $\frac{\partial}{\partial z^\mu}$). We will sometimes refer to these as *holomorphic indices*.

- We used barred Greek letters for indices that run from 1 to $\dim_{\mathbb{C}}(M)$ and are naturally contracted with the holomorphic part of the basis (i.e. $d\bar{z}^\mu$ or $\frac{\partial}{\partial \bar{z}^\mu}$). We will sometimes refer to these as *anti-holomorphic indices*

In what follows, we will also use the following equivalent notations:

$$\partial_\mu = \frac{\partial}{\partial z^\mu} \ , \qquad \partial_{\bar{\mu}} = \frac{\partial}{\partial \bar{z}^\mu}. \tag{1.2.14}$$

Finally, it is an immediate consequence of the above result that when indices are raised/lowered, we need to add a "bar" on top of them (with the rule that two bars cancel).

Definition 1.2.4. Let M be an almost Hermitian manifold, with almost complex structure J and Hermitian metric g. If J is integrable (or equivalently M is a complex manifold, or also the Nijenhus tensor vanishes), then M is an *Hermitian manifold.*

Theorem 1.2.4. Let M be a Hermitian manifold. Then, there exist a unique metric connection, called the Hermitian connection, with the properties:

1. The covariant derivative of the complex structure vanishes.
2. The torsion ${\Gamma_{[mn]}}^r$ vanishes unless m and n are indices of the same type.
3. The covariant derivative of the metric vanishes.

Proof. We assume that the above properties hold, and derive a (unique) explicit expression for the connection. Since J is covariantly constant for all $p \in M$, so are $\mathcal{P}^\pm$. Specifically,

$$\nabla_k {(\mathcal{P}^+)_m}^n = 0. \tag{1.2.15}$$

The covariant derivative is defined according to A.1.45, in the Appendix. Taking $m = \mu$ and $n = \nu$ leads to

$$-{\Gamma_{k\mu}}^t {(\mathcal{P}^+)_t}^\nu + {\Gamma_{kt}}^\nu {(\mathcal{P}^+)_\mu}^t = -{\Gamma_{k\mu}}^\alpha {(\mathcal{P}^+)_\alpha}^\nu + {\Gamma_{k\alpha}}^\nu {(\mathcal{P}^+)_\mu}^\alpha = -{\Gamma_{k\mu}}^\nu + {\Gamma_{k\mu}}^\nu = 0, \tag{1.2.16}$$

where we used the explicit form of $\mathcal{P}^+$ in the complex basis, i.e. 1.1.22. Hence, the condition is automatically satisfied. However, choosing $m = \mu$ and $n = \bar{\nu}$ gives a non-trivial constraint. Going through the same calculation yields:

$$-{\Gamma_{k\mu}}^t {(\mathcal{P}^+)_t}^{\bar{\nu}} + {\Gamma_{kt}}^{\bar{\nu}} {(\mathcal{P}^+)_\mu}^t = +{\Gamma_{k\alpha}}^{\bar{\nu}} {(\mathcal{P}^+)_\mu}^\alpha = {\Gamma_{k\mu}}^{\bar{\nu}} = 0, \tag{1.2.17}$$

where the first term vanishes due to 1.1.22. Since $k = \alpha$ or $\bar{\alpha}$, we have that $\Gamma_{\alpha\mu}{}^{\bar{\nu}} = \Gamma_{\bar{\alpha}\mu}{}^{\bar{\nu}} = 0 = \Gamma_{\bar{\alpha}\bar{\mu}}{}^{\nu} = \Gamma_{\alpha\bar{\mu}}{}^{\nu}$, where the last two equalities have been obtained by complex conjugation. The vanishing of $\Gamma_{[mn]}{}^{r}$ with mixed components finally imposes $\Gamma_{\mu\bar{\alpha}}{}^{\bar{\nu}} = 0 = \Gamma_{\bar{\mu}\alpha}{}^{\nu}$. Hence, conditions 1. and 2. imply the vanishing of all mixed components, which we could have equally taken as a starting point (see following Remark).
We now provide a unique prescription for constructing the metric connection. Condition 3. in components reads

$$\nabla_m g_{rt} = 0 = \partial_m g_{rt} - \Gamma_{mr}^{\ n} g_{nt} - \Gamma_{mt}^{\ n} g_{rn}. \tag{1.2.18}$$

Specifically, we look at $m = \alpha$, $r = \mu$, $t = \bar{\nu}$, for which:

$$\partial_\alpha g_{\mu\bar{\nu}} - \Gamma_{\alpha\mu}{}^{\sigma} g_{\sigma\bar{\nu}} = 0. \tag{1.2.19}$$

We denote with $g^{\bar{\nu}\beta}$ the inverse of $g_{\sigma\bar{\nu}}$, so that $g_{\sigma\bar{\nu}} g^{\bar{\nu}\beta} = \delta^{\beta}_{\sigma}$. Using this, we find:

$$\Gamma_{\alpha\mu}{}^{\beta} = g^{\bar{\nu}\beta} \partial_\alpha g_{\mu\bar{\nu}}. \tag{1.2.20}$$

Either by choosing $m = \bar{\alpha}$, $r = \mu$, $t = \bar{\nu}$ and following the same steps, or simply by complex conjugation, we obtain:

$$\Gamma_{\bar{\alpha}\bar{\mu}}{}^{\bar{\beta}} = g^{\nu\bar{\beta}} \partial_{\bar{\alpha}} g_{\bar{\mu}\nu}. \tag{1.2.21}$$

□

Remark. We have shown that 1. and 2. lead to the vanishing of all mixed components of the connection. This condition has a natural interpretation. We are effectively imposing that a holomorphic vector at p remains holomorphic when parallely transported to another point q. Technically, we are assuming that given a point p with coordinates $\{z^\mu\}$ and a point q with coordinates $\{z^\mu + \delta z^\mu\}$, for $V = V^\mu(z) \frac{\partial}{\partial z^\mu}|_p \in T_p(M)^+$ and $\tilde{V} = \tilde{V}^\mu(z + \delta z) \frac{\partial}{\partial z^\mu}|_q \in T_q(M)^+$ we have:

$$\tilde{V}^\mu(z + \delta z) = V^\mu(z) - V^\alpha(z) \Gamma_{\alpha\beta}^{\ \mu} \delta z^\beta. \tag{1.2.22}$$

Theorem 1.2.5. Let M be a Hermitian manifold, with almost complex structure J and Hermitian connection Γ. Then, J is covariantly constant wrt the Hermitian connection.

Proof. We recall that the components of the covariant derivative of a (1,1)-tensor, such as J, read:

$$\nabla_n J_r{}^{t} = \partial_n J_r{}^{t} - \Gamma_{nr}^{\ s} J_s{}^{t} + \Gamma_{ns}^{\ t} J_r{}^{t}. \tag{1.2.23}$$

The first of the three contributions is zero, since J is constant. We now set $n = \alpha$, and recall that both J and Γ are non-zero only if all the indices are of the same kind. With this in mind, the three choices $r = \beta$ and $t = \bar{\sigma}$, $r = \bar{\beta}$ and $t = \sigma$, $r = \bar{\beta}$ and $t = \bar{\sigma}$ all lead to the contributions vanishing individually. The last option, $r = \beta$ and $t = \sigma$, vanish as well, this time using 1.1.20. The same argument as above goes through if we put a bar on each index. □

Theorem 1.2.6. The torsion of the Hermitian metric is pure in its indices.

Proof. This immediately follows from the definition of the components $T_{\mu\nu}{}^{\alpha} = 2\Gamma_{[\mu\nu]}^{\alpha}$, and the form of the connection found above. □

Theorem 1.2.7. The only non-zero components of the Riemann tensor with all the indices lowered are the ones that have mixed indices in both the first and second pair.

Proof. We start by recalling that the components of the Riemann tensor, according to our conventions, are given by:

$$R^{n}{}_{mrt} = \partial_r \Gamma_{tm}{}^{n} - \partial_t \Gamma_{rm}{}^{n} + \Gamma_{tm}{}^{s}\Gamma_{rs}{}^{n} - \Gamma_{rm}{}^{s}\Gamma_{ts}{}^{n}. \tag{1.2.24}$$

Note the symmetry $R^{n}{}_{mrt} = -R^{n}{}_{mtr}$.
If we set $n = \bar{\alpha}$ and $m = \beta$, then all four terms vanish since the connection components must be pure in their indices. Similarly, $n = \alpha$ and $m = \bar{\beta}$ lead to each contribution being zero. Thus, we deduce that n and m must both be indices of the same kind.
We now set $n = \alpha$ and $m = \beta$ (the case $n = \bar{\alpha}$ and $m = \bar{\beta}$ is reached by complex conjugation). If $r = \bar{\mu}$ and $t = \bar{\nu}$, all the connection components again vanish for the same reason. If we set $r = \mu$ and $t = \nu$, the four contributions are all non-zero. However, using the explicit form for the connection and the identity $g^{\bar{\alpha}\beta}\partial_\mu g_{\nu\bar{\alpha}} = -g_{\nu\bar{\alpha}}\partial_\mu g^{\bar{\alpha}\beta}$, it is quickly shown that they exactly cancel. Hence, the only non-trivial components, up to complex conjugation and symmetry on the last two indices, is given by:

$$R^{\alpha}{}_{\beta\bar{\mu}\nu} = \partial_{\bar{\mu}} \Gamma_{\nu\beta}^{\ \alpha} = \partial_{\bar{\mu}}(g^{\bar{\sigma}\alpha}\partial_\nu g_{\beta\bar{\sigma}}) = -R^{\alpha}{}_{\beta\nu\bar{\mu}}. \tag{1.2.25}$$

Via complex conjugation, we obtain:

$$R^{\bar{\alpha}}{}_{\bar{\beta}\mu\bar{\nu}} = \partial_{\mu} \Gamma_{\bar{\nu}\bar{\beta}}^{\ \bar{\alpha}} = \partial_{\mu}(g^{\sigma\bar{\alpha}}\partial_{\bar{\nu}} g_{\bar{\beta}\sigma}) = -R^{\bar{\alpha}}{}_{\bar{\beta}\bar{\nu}\mu}. \tag{1.2.26}$$

Lowering the upper index, we have that, as claimed, the non-zero components of the Riemann tensor with all indices lowered are:

$$R_{\bar{\alpha}\beta\bar{\mu}\nu}\,, \quad R_{\bar{\alpha}\beta\nu\bar{\mu}}\,, \quad R_{\alpha\bar{\beta}\mu\bar{\nu}}\,, \quad R_{\alpha\bar{\beta}\bar{\nu}\mu}. \tag{1.2.27}$$

□

Definition 1.2.5. We let $\mathfrak{R}_{\mu\bar{\nu}} = R^{\alpha}{}_{\alpha\mu\bar{\nu}}$. Using this, we define the *Ricci form* as:

$$\mathfrak{R} = i\mathfrak{R}_{\mu\bar{\nu}}\mathfrak{R}dz^{\mu}d\bar{z}^{\nu}. \tag{1.2.28}$$

Theorem 1.2.8. The Ricci form is a real, closed form.

Proof. We use the notation $g = \det(g_{mn})$, and also $\sqrt{g} = \bar{g} = \det(g_{\mu\bar{\nu}})$. Using the identity $\delta\bar{g} = \bar{g}g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}$ (in the last step), we obtain:

$$\mathfrak{R}_{\mu\bar{\nu}} = R^{\alpha}{}_{\alpha\mu\bar{\nu}} = -\partial_{\bar{\nu}}(g^{\bar{\sigma}\alpha}\partial_{\mu}g_{\alpha\bar{\sigma}}) = -\partial_{\bar{\nu}}\partial_{\mu}\log(\bar{g}). \tag{1.2.29}$$

Then,

$$\mathfrak{R} = i\mathfrak{R}_{\mu\bar{\nu}}dz^{\mu} \wedge d\bar{z}^{\nu} = -i\partial_{\bar{\nu}}\partial_{\mu}\log(\bar{g})dz^{\mu} \wedge d\bar{z}^{\nu} = -i\partial\bar{\partial}\log(\bar{g}). \tag{1.2.30}$$

This form satisfies $\bar{\mathfrak{R}} = +i\partial_{\nu}\partial_{\bar{\mu}}\log(\bar{g})d\bar{z}^{\mu} \wedge dz^{\nu} = -i\partial_{\bar{\mu}}\partial_{\nu}\log(\bar{g})dz^{\nu} \wedge d\bar{z}^{\mu} = \mathfrak{R}$. Hence, the Ricci form is real.
Using the identity $\partial\bar{\partial} = -\frac{1}{2}d(\partial - \bar{\partial})$, together with $d^2 = 0$, we also see that $\mathfrak{R}$ is closed. □

Remark. Following the last line of the previous proof, we can write $\mathfrak{R}$ as

$$\mathfrak{R} = d\Big(\frac{i}{2}(\partial - \bar{\partial})\log(\bar{g})\Big). \tag{1.2.31}$$

This might suggest that $\mathfrak{R}$ is globally exact. However, $\bar{g}$ is not a coordinate scalar, and therefore the above statement should be regarded as valid only within a single patch U_i of the manifold.

1.3 Kähler Manifolds

This section begins with the introduction of a 2-form, which can be constructed on any Hermitian manifold from the metric and the almost complex structure. We will explore how, by simply requiring the closure of such form, a geometry with peculiar properties emerges. Again, [19] and [20] are the main references for this section.

Definition 1.3.1. Let M be an Hermitian manifold, with Hermitian metric g. We can define a (0,2)-tensor Ω field as

$$\Omega_p(X, Y) = g_p(J_p X, Y) \tag{1.3.1}$$

for $X, Y \in T_pM$ and $p \in M$. We will refer to it as the *Kähler form*.

Remark. Sometimes, the above definition is recast as the commutativity of the following diagram $\forall p \in M$:

$$\begin{tikzcd} T_pM \arrow[r, "J"] \arrow[dr, "\Omega"'] & T_pM \arrow[d, "g"] \\ & T_p^*M. \end{tikzcd} \tag{1.3.2}$$

Theorem 1.3.1. Ω is antisymmetric in its entries, and hence it is a form.

Proof.

$$\Omega_p(X, Y) = g_p(J_pX, Y) = g_p(J_p^2X, J_pY) = -g_p(X, J_pY) = -g_p(J_pY, X) = -\Omega_p(Y, X). \tag{1.3.3}$$

□

Theorem 1.3.2. The only non-zero components of the Kähler form are $\Omega_{\mu\bar{\nu}} = ig_{\mu\bar{\nu}}$.

Proof.

$$\Omega_{\mu\nu} = \Omega_p(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}) = g_p(J\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}) = ig_p(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial z^\nu}) = ig_{\mu\nu}(p) = 0, \tag{1.3.4}$$

$$\Omega_{\bar{\mu}\bar{\nu}} = \Omega_p(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial \bar{z}^\nu}) = g_p(J\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial \bar{z}^\nu}) = -ig_p(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial \bar{z}^\nu}) = -ig_{\bar{\mu}\bar{\nu}}(p) = 0, \tag{1.3.5}$$

$$\Omega_{\mu\bar{\nu}} = \Omega_p(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial \bar{z}^\nu}) = g_p(J\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial \bar{z}^\nu}) = ig_p(\frac{\partial}{\partial z^\mu}, \frac{\partial}{\partial \bar{z}^\nu}) = ig_{\mu\bar{\nu}}(p), \tag{1.3.6}$$

$$\Omega_{\bar{\mu}\nu} = \Omega_p(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial z^\nu}) = g_p(J\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial z^\nu}) = -ig_p(\frac{\partial}{\partial \bar{z}^\mu}, \frac{\partial}{\partial z^\nu}) = -ig_{\bar{\mu}\nu}(p). \tag{1.3.7}$$

□

Remark. The explicit expression for the Kähler form in components is:

$$\Omega_p = \Omega_{mn} dz^m \otimes dz^n = i g_{\mu\bar{\nu}} dz^\mu \otimes d\bar{z}^\nu - i g_{\bar{\mu}\nu} d\bar{z}^\mu \otimes dz^\nu = i g_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu. \tag{1.3.8}$$

There is an intimate relation between the Kähler form and the almost complex structure J. We can quickly check using using 1.1.20 that:

$$\Omega_{mn} = J_m{}^k g_{kn} = J_{mn}, \tag{1.3.9}$$

where in the last line we employed the usual notation for the lowering of an index. Thus, we can also write the Kähler form as

$$\Omega = \Omega_{mn} dz^m \otimes dz^n = J_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu. \tag{1.3.10}$$

Definition 1.3.2. Let M be a Hermitian manifold, with Hermitian metric g and Kähler form Ω. If $d\Omega = 0$, then M is called a *Kähler manifold*, and g is a *Kähler metric*.

Theorem 1.3.3. The Kähler form defines a real, positive, nowhere vanishing $2m$-form via:

$$\Omega^m = \underbrace{\Omega \wedge \ldots \wedge \Omega}_{\text{m factors}}. \tag{1.3.11}$$

Proof. We first construct a suitable orthonormal basis for the Hermitian metric g, starting from the usual orthonormal basis $\{\hat{e}_n\}$ for the Riemannian metric. The procedure is the following. We consider $\hat{e}_1$ and $J\hat{e}_1$, and notice that by definition of orthonormal basis and Hermitian metric:

$$\begin{aligned} &g(\hat{e}_1, \hat{e}_1) = 1 \\ &g(J\hat{e}_1, J\hat{e}_1) = g(\hat{e}_1, \hat{e}_1) = 1 \\ &g(\hat{e}_1, J\hat{e}_1) = -g(J\hat{e}_1, \hat{e}_1) = 0 \end{aligned} \tag{1.3.12}$$

where we implicitly used $J^2 = -\mathbb{1}_{2m}$. Then, we take $\hat{e}_2$, that is orthogonal to $\hat{e}_1$ and $J\hat{e}_1$ and for which the same conditions as 1.3.12 clearly hold.[4] Proceeding this way, we obtain

[4] If $\hat{e}_2$ is proportional to $J\hat{e}_1$, we take $\hat{e}_3$.

the orthonormal basis for the Hermitian metric: $\{\hat{e}_1, J\hat{e}_1, ..., \hat{e}_m, J\hat{e}_m\}$.
It follows directly from the definition of Kähler form that:

$$\Omega(\hat{e}_i, J\hat{e}_j) = \delta_{ij} \qquad \text{and} \qquad \Omega(\hat{e}_i, \hat{e}_j) = 0 = \Omega(J\hat{e}_i, J\hat{e}_j). \tag{1.3.13}$$

Hence, we obtain

$$\begin{aligned}\underbrace{\Omega \wedge \ldots \wedge \Omega}_{\text{m factors}}(\hat{e}_1, J\hat{e}_1, \ldots, \hat{e}_m, J\hat{e}_m) &= \sum_P \Omega\left(\hat{e}_{P(1)}, J\hat{e}_{P(1)}\right) \ldots \Omega\left(\hat{e}_{P(m)}, J\hat{e}_{P(m)}\right) \\ &= m!\Omega\left(\hat{e}_1, J\hat{e}_1\right) \ldots \Omega\left(\hat{e}_m, J\hat{e}_m\right) = m!, \end{aligned} \tag{1.3.14}$$

where P stands for an element of the permutation group of m objects. This shows that $\Omega \wedge ... \wedge \Omega$ is a real, positive, nowhere vanishing $2m$-form. □

Remark. The above result shows that Kähler manifolds are automatically orientable (they admit a nowhere vanishing volume form). We could relate exactly the above form to the natural volume element for the underlying differentiable manifolds (A.1.3), but this would involve some unnecessary maths. We limit ourselves to stating that Ω^m is proportional to the natural volume element of M (see [26]) and we introduce a definition that will be used later.

Definition 1.3.3. The *holomorphic volume element* Φ is defined via:[5]

$$\frac{\Omega^m}{m!} = (-1)^{m(m-1)/2}(i)^m \Phi \wedge \bar{\Phi}. \tag{1.3.15}$$

Remark. Using the usual complex coordinates $\{z^\mu\}$, Ω^m reads:

$$\begin{aligned}\Omega \wedge ... \wedge \Omega = i^m g_{\mu_1\bar{\nu}_1}...g_{\mu_m\bar{\nu}_m} dz^{\mu_1} \wedge d\bar{z}^{\nu_1} \wedge ... \wedge dz^{\mu_m} \wedge d\bar{z}^{\nu_m} = \\ i^m \epsilon^{\mu_1...\mu_m} \epsilon^{\bar{\nu}_1...\bar{\nu}_m} g_{\mu_1\bar{\nu}_1}...g_{\mu_m\bar{\nu}_m} dz^1 \wedge d\bar{z}^1 \wedge ... \wedge dz^m \wedge d\bar{z}^m = \\ i^m m! \det(g_{\mu\bar{\nu}}) dz^1 \wedge d\bar{z}^1 \wedge ... \wedge dz^m \wedge d\bar{z}^m.\end{aligned} \tag{1.3.16}$$

It follows by counting the permutations of the bases that:

$$\Phi = \sqrt{\bar{g}} dz^1 \wedge ... \wedge dz^m. \tag{1.3.17}$$

[5] This is defined as in [13], up to irrelevant factors.

Theorem 1.3.4. On a Kähler manifold, the Kähler metric and the Kähler form are uniquely determined by a scalar, defined patch-wise.

Proof. We split the exterior derivative into Dolbeault operators, and then require the vanishing of the two contributions. For a given chart U_i with coordinates $\{z^\mu\}$, we have:

$$d\Omega = (\partial + \bar{\partial}) i g_{\mu\bar{\nu}} dz^\mu \wedge d\bar{z}^\nu = i\partial_\alpha g_{\mu\bar{\nu}} dz^\alpha \wedge dz^\mu \wedge d\bar{z}^\nu + i\partial_{\bar{\alpha}} g_{\mu\bar{\nu}} d\bar{z}^\alpha \wedge dz^\mu \wedge d\bar{z}^\nu = \frac{1}{2} i \Big[(\partial_\alpha g_{\mu\bar{\nu}} - \partial_\mu g_{\alpha\bar{\nu}}) dz^\alpha \wedge dz^\mu \wedge d\bar{z}^\nu + (\partial_{\bar{\alpha}} g_{\mu\bar{\nu}} - \partial_{\bar{\nu}} g_{\mu\bar{\alpha}}) d\bar{z}^\alpha \wedge dz^\mu \wedge d\bar{z}^\nu \Big] = 0. \tag{1.3.18}$$

The (1,2) and (2,1) parts must vanish separately, yielding

$$\partial_\alpha g_{\mu\bar{\nu}} = \partial_\mu g_{\alpha\bar{\nu}} \quad \text{and} \quad \partial_{\bar{\alpha}} g_{\mu\bar{\nu}} = \partial_{\bar{\nu}} g_{\mu\bar{\alpha}}. \tag{1.3.19}$$

From these equations we deduce that in every chart U_i, there exist a scalar ϕ_i ($\phi_i \in \mathcal{F}(U_i)$), such that

$$g_{\mu\bar{\nu}} = \partial_\mu \partial_{\bar{\nu}} \phi_i. \tag{1.3.20}$$

Equivalently,

$$\Omega = i\partial\bar{\partial}\phi_i \tag{1.3.21}$$

on U_i. □

Definition 1.3.4. The scalars ϕ_i, defined in every patch, are called *Kähler potential.*

Remark. We have that $\partial\bar{\partial} = -\frac{1}{2} d(\partial - \bar{\partial})$. This might seem to suggest that $\Omega = d\big(-\frac{i}{2}(\partial - \bar{\partial})\big)\phi_i$ is exact.

Theorem 1.3.5. Ω is not exact.

Proof. We prove the theorem by contradiction.
Let M be a Kähler manifold with $\dim_{\mathbb{C}} M = m$ and with a Kähler form Ω. Consider the 2m-form

$$\underbrace{\Omega \wedge ... \wedge \Omega}_{\text{m factors}}. \tag{1.3.22}$$

We already showed that this form is positive and nowhere vanishing. Now consider the integral

$$\int \Omega \wedge ... \wedge \Omega. \tag{1.3.23}$$

According to the above consideration, this is non-zero in general (specifically, it is proportional to the volume of the manifold). If Ω is exact, however, Stoke's theorem implies that this integral vanishes identically. This is a contradiction.
Hence Ω cannot be exact. □

Remark. The non-exactness of Ω lies in the fact that a globally defined Kähler potential ϕ does not exist, but only locally defined functions ϕ_i exist. Specifically, in some overlap of two charts $U_i \cap U_j$, ϕ_i and ϕ_j can differ by a sum of a holomorphic and a anti-holomorphic function. To see this, let U_i and U_j have coordinates z^μ and w^μ, respectively. Then, in the overlap:

$$\Big(\frac{\partial}{\partial z^\mu}\frac{\partial}{\partial \bar{z}^\nu}\phi_i\Big)dz^\mu \otimes d\bar{z}^\nu = \Big(\frac{\partial}{\partial w^\mu}\frac{\partial}{\partial \bar{w}^\nu}\phi_j\Big)dw^\mu \otimes d\bar{w}^\nu. \tag{1.3.24}$$

This implies:

$$\Big(\frac{\partial}{\partial z^\mu}\frac{\partial}{\partial \bar{z}^\nu}\phi_i\Big) = \Big(\frac{\partial}{\partial w^\alpha}\frac{\partial}{\partial \bar{w}^\beta}\phi_j\Big)\frac{\partial w^\alpha}{\partial z^\mu}\frac{\partial w^{\bar{\alpha}}}{\partial z^{\bar{\nu}}}, \tag{1.3.25}$$

which is satisfied for $\phi_j(w^\mu, \bar{w}^\mu) = \phi_i(z^\mu, \bar{z}^\mu) + \lambda(z^\mu) + \psi(\bar{z}^\mu)$, where $\lambda(z^\mu)$ $(\psi(\bar{z}^\mu))$ is a holomorphic (anti-holomorphic) function.

Theorem 1.3.6. The Kähler metric is torsion free.

Proof. Using 1.2.20 and 1.3.19, we can explicitly verify that

$$T_{\mu\nu}{}^{\alpha} = g^{\bar{\sigma}\alpha}(\partial_\mu g_{\nu\bar{\sigma}} - \partial_\nu g_{\mu\bar{\sigma}}) = 0 = T_{\bar{\mu}\bar{\nu}}{}^{\bar{\alpha}}. \tag{1.3.26}$$

All the other components are zero since the connection is pure in its indices. □

Remark. This makes the connection of the Kähler metric very similar to the Levi-Civita connection.

Theorem 1.3.7. The Riemann tensor for the Kähler metric is symmetric on the second and third indices of the same kind.

Proof. Using the expression for the Hermitian metric 1.2.20 and the Kähler metric condition 1.3.19, we obtain

$$R^{\alpha}{}_{\beta\mu\bar{\nu}} = -\partial_{\nu}(g^{\bar{\sigma}\alpha}\partial_{\mu}g_{\beta\bar{\sigma}}) = -\partial_{\bar{\nu}}(g^{\bar{\sigma}\alpha}\partial_{\beta}g_{\mu\bar{\sigma}}) = R^{\alpha}{}_{\mu\beta\bar{\nu}}. \tag{1.3.27}$$

By complex conjugation, we obtain:

$$R^{\bar{\alpha}}{}_{\bar{\beta}\bar{\mu}\nu} = R^{\bar{\alpha}}{}_{\bar{\mu}\bar{\beta}\nu}. \tag{1.3.28}$$

□

Definition 1.3.5. We define the *Ricci form* $\mathfrak{R}$ exactly as for the case of an Hermitian manifold:

$$\mathfrak{R} = i\mathfrak{R}_{\mu\bar{\nu}}dz^{\mu} \wedge d\bar{z}^{\nu} = -i\partial_{\bar{\nu}}\partial_{\mu}\log(\bar{g})dz^{\mu} \wedge d\bar{z}^{\nu} = -i\partial\bar{\partial}\log(\bar{g}). \tag{1.3.29}$$

Remark. Thanks to 1.3.27, we have that $\mathfrak{R}_{\mu\bar{\nu}} = R^{\alpha}_{\alpha\mu\bar{\nu}} = R^{\alpha}_{\mu\alpha\bar{\nu}} = R_{\mu\bar{\nu}}$ (see A.1.51).

1.4 Some More Concepts and Definitions

In this section we present some more concepts, ubiquitous to complex differential geometry, which will be used in the study of Kähler and Calabi-Yau manifolds. The relevant resource for this section is [19].

Definition 1.4.1. Let M be a complex manifold and $\alpha \in \Omega^{p,0}(M)$. If α satisfies $\bar{\partial}\alpha = 0$, then it is called a *holomorphic p-form* (or, more explicitly, a *holomorphic* $(p,0)$*-form*).

Theorem 1.4.1. The components of a holomorphic p-form are holomorphic functions.

Proof. Let α be a holomorphic p-form. Being a $(p,0)$-form, its coordinate expression is:

$$\frac{1}{p!}\alpha_{\mu_1\ldots\mu_p}dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_p}. \tag{1.4.1}$$

It obeys:

$$\bar{\partial}\alpha = 0 = \partial_{\bar{\beta}}\alpha_{\mu_1\ldots\mu_p}d\bar{z}^{\beta} \wedge dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_p}, \tag{1.4.2}$$

which implies (and is implied by)

$$\partial_{\bar{\beta}}\alpha_{\mu_1\ldots\mu_p} = 0. \tag{1.4.3}$$

□

Theorem 1.4.2. The Dolbeault operators are nilpotent, i.e. $\partial^2 = 0$ and $\bar{\partial}^2 = 0$.

Proof. It immediately follows from the symmetry of the indices in the component expression. □

Remark. The above result shows that, just like the exterior derivative operator, the Dolbeault operator define a cohomology problem.

Definition 1.4.2. Let M be a complex manifold, and let $\alpha \in \Omega^{r,p}(M)$. Then,

- If $\bar{\partial}\alpha = 0$, α is called $\bar{\partial}$-*closed*. The set of $\bar{\partial}$-closed (r,p)-forms is denoted by $Z^{r,p}_{\bar{\partial}}$.
- If $\alpha = \bar{\partial}\beta$ for some $\beta \in \Omega^{r,p-1}(M)$, α is called $\bar{\partial}$-*exact*. The set of $\bar{\partial}$-exact (r,p)-forms is denoted by $B^{r,p}_{\bar{\partial}}$.
- The (r,p)th $\bar{\partial}$-*cohomology group* is the complex vector space:

$$H^{r,p}_{\bar{\partial}}(M) = Z^{r,p}_{\bar{\partial}} / B^{r,p}_{\bar{\partial}}. \tag{1.4.4}$$

Remark. Phrasing it in more concrete terms, the elements of $H^{r,p}_{\bar{\partial}}(M)$ are equivalence classes of $\bar{\partial}$-closed (r,p)-forms that differ from each other by a $\bar{\partial}$-exact (r,p)-form. We can, as usual, choose a representative for the class, α, which provides a definition for the equivalence class:

$$[\alpha] = \{\beta \in \Omega^{r,p}(M) : \bar{\partial}\beta = 0, \quad \alpha - \beta = \bar{\partial}\gamma \quad \text{for} \quad \gamma \in \Omega^{r,p-1}(M)\}. \tag{1.4.5}$$

Definition 1.4.3. We define the *extended Hodge star* (or, simply *Hodge star*) $*$ on a complex manifold, by taking the definition for the real case (see A.1.1) and extending it to $\Omega^p(M)^{\mathbb{C}}$.

Remark. The extended Hodge star in components takes the form:

$$\begin{aligned} *dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_r} \wedge d\bar{z}^{\nu_1} \wedge \ldots \wedge d\bar{z}^{v_s} \sim \epsilon^{\mu_1 \cdots \mu_r}_{\bar{\mu}_{r+1} \cdots \bar{\mu}_m} \epsilon^{\bar{v}_1 \ldots \bar{v}_s}_{v_{s+1} \ldots v_m} \\ \times d\bar{z}^{\mu_{r+1}} \wedge \ldots \wedge d\bar{z}^{\mu_m} \wedge dz^{\nu_{s+1}} \wedge \ldots \wedge dz^{v_m}, \end{aligned} \tag{1.4.6}$$

where $\sim$ denotes proportionality, since we omitted the factors in front. We see from this that the Hodge star is a map of the form $* : \Omega^{r,s}(M) \to \Omega^{m-s,m-r}(M)$.

Theorem 1.4.3. The Hodge star on a Hermitian manifold defines an isomorphism between $H^{r,s}_{\partial}(M)$ and $H^{m-s,m-r}_{\partial}(M)$.

Proof. In the previous remark, we noted that $* : \Omega^{r,s}(M) \to \Omega^{m-s,m-r}(M)$. The isomorphism between $H^{r,s}_{\partial}(M)$ and $H^{m-s,m-r}_{\partial}(M)$ follows from the extended Hodge star exactly as Poicare duality in the case of a real manifold (see A.1.9) follows from the Hodge star.[6] □

Definition 1.4.4. We define the *inner product* between two (r,s)-forms $\alpha, \beta \in \Omega^{r,s}(M)$ as:

$$\begin{aligned}(\cdot,\cdot) : \Omega^{r,s}(M) &\to \mathbb{R} \\ \alpha, \beta \mapsto (\alpha,\beta) &= \int_M \alpha \wedge \bar{*}\beta,\end{aligned} \tag{1.4.7}$$

where $\bar{*}\beta = *\bar{\beta} \in \Omega^{m-r,m-s}(M)$.

Definition 1.4.5. We define the *adjoint operators* $\partial^\dagger$ and $\bar{\partial}^\dagger$ by:

$$(\alpha, \partial\beta) = (\partial^\dagger\alpha, \beta) \qquad \text{and} \qquad (\alpha, \bar{\partial}\beta) = (\bar{\partial}^\dagger\alpha, \beta). \tag{1.4.8}$$

Remark. This definition is analogue to the one for the adjoint exterior derivative in ordinary geometry (see A.1.6).
We also note that (almost) complex dimensional manifolds, which are even dimensional, $d^\dagger = - * d *$ (see A.1.6).

Theorem 1.4.4. The adjoint operators $\partial^\dagger$ and $\bar{\partial}^\dagger$ take the following form:

$$\partial^\dagger = - * \bar{\partial} * \qquad \text{and} \qquad \bar{\partial}^\dagger = - * \partial * . \tag{1.4.9}$$

Proof. Let $\alpha \in \Omega^{r-1,s}(M)$ and $\beta \in \Omega^{r,s}(M)$. Then, $\alpha \wedge \bar{*}\beta \in \Omega^{m-1,m}(M)$, and so $\bar{\partial}(\alpha \wedge \bar{*}\beta) = 0$. We also note that $\partial\bar{*}\beta \in \Omega^{m-r+1,m-s}(M) \subset \Omega^{2m-r-s+1}(M)$. Using this, together with $\bar{*}\bar{*} = **$ and A.1.2, we obtain:

$$\begin{gathered} d(\alpha \wedge \bar{*}\beta) = \partial\alpha \wedge \bar{*}\beta = \partial\alpha \wedge \bar{*}\beta + (-1)^{r-1+s}\alpha \wedge \partial \; \bar{*}\beta = \\ \partial\alpha \wedge \bar{*}\beta + (-1)^{r-1+s}\alpha \wedge (-1)^{r+s+1} \; \bar{*}\,\bar{*}\,\partial \; \bar{*}\beta = \partial\alpha \wedge \bar{*}\beta + \alpha \wedge \; \bar{*}\,\bar{*}\,\partial \; \bar{*}\beta. \end{gathered} \tag{1.4.10}$$

Then, we can integrate this equation over the manifold (which is assumed to be complex and compact, without boundary) and use Stokes theorem to find:

$$0 = (\partial\alpha, \beta) + (\alpha, \bar{*}\,\partial \; \bar{*}\beta)\,, \tag{1.4.11}$$

[6] See [27] for a detailed derivation.

where 1.4.7 has been used. Then, by recognising that $\bar{*}\,\partial\,\bar{*}\beta = *\,\bar{\partial}\,*\,\beta$, we arrive at the desired result:

$$(\partial\alpha, \beta) = (\alpha, -*\,\bar{\partial}\,*\,\beta) \implies \partial^\dagger = -*\,\bar{\partial}\,*\,. \tag{1.4.12}$$

Following exacly the same steps, but with $\alpha \in \Omega^{r,s-1}(M)$, leads to $\bar{\partial}^\dagger = -*\,\partial *$. □

Theorem 1.4.5. $\partial^\dagger$ and $\bar{\partial}^\dagger$ are nilpotent.

Proof.

$$\begin{aligned}(\partial^\dagger)^2 &= *\,\bar{\partial}\,**\,\bar{\partial}\,* = (-1)^\tau *\,\bar{\partial}^2 * = 0,\\ (\bar{\partial}^\dagger)^2 &= *\,\partial\,**\,\partial\,* = (-1)^\tau *\,\partial^2 * = 0,\end{aligned} \tag{1.4.13}$$

where the precise value of τ is irrelevant, and we used property 1. in theorem 1.1.11. □

Definition 1.4.6. The *Laplacian* for the Dolbeault operators are defined as:

$$\begin{aligned}\triangle_\partial &= (\partial + \partial^\dagger)^2 = \partial\partial^\dagger + \partial^\dagger\partial,\\ \triangle_{\bar{\partial}} &= (\bar{\partial} + \bar{\partial}^\dagger)^2 = \bar{\partial}\bar{\partial}^\dagger + \bar{\partial}^\dagger\bar{\partial}.\end{aligned} \tag{1.4.14}$$

Definition 1.4.7. Let α be a (r, s)-form.
If $\triangle_\partial\alpha = 0$, α is called *∂-harmonic.* The set of such forms is denoted by $\mathrm{Harm}^{r,s}_\partial(M)$.
If $\triangle_{\bar{\partial}}\alpha = 0$, α is called *$\bar{\partial}$-harmonic.* The set of such forms is denoted by $\mathrm{Harm}^{r,s}_{\bar{\partial}}(M)$.

Theorem 1.4.6. On a Kähler manifold, we have:

$$\frac{1}{2}\triangle = \triangle_\partial = \triangle_{\bar{\partial}}. \tag{1.4.15}$$

Proof. The proof involves a long calculation in components, which for reasons of space we omit here.[7] □

Definition 1.4.8. The dimensions of the (r, s)th-chomology groups are called the *Hodge numbers*:

$$h^{r,s}(M) = \dim_{\mathbb{C}}(H^{r,p}_{\bar{\partial}}(M)). \tag{1.4.16}$$

[7] The reader is referred to [28]

Definition 1.4.9. It is conventional to arrange these numbers in the *Hodge diamond*. For a manifold of complex dimension 3, it looks like:

$$\begin{array}{ccccccc} & & & h^{0,0} & & & \\ & & h^{1,0} & & h^{0,1} & & \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ & & h^{3,2} & & h^{2,3} & & \\ & & & h^{3,3} & & & \end{array}$$

Theorem 1.4.7. For a Kähler manifold M with $\dim_{\mathbb{C}}(M) = m$ the following relations between Hodge numbers hold:

1. $h^{r,s} = h^{m-r,m-s}$.
2. $b^{r,s} = b^{s,r}$.

Proof.

1. By theorem 1.4.3, we have an isomorphism between $H_{\bar{\partial}}^{r,s}$ and $H_{\bar{\partial}}^{m-r,m-s}$. It clearly follows that, by definition, $\dim_{\mathbb{C}}(H_{\bar{\partial}}^{r,s}) = h^{r,s} = h^{m-r,m-s} = \dim_{\mathbb{C}}(H_{\bar{\partial}}^{m-r,m-s})$.
2. Let $\alpha \in \Omega^{r,s}(M)$ be harmonic, i.e. $\triangle_{\partial}\alpha = \triangle_{\bar{\partial}}\alpha = 0$. Then, $0 = \overline{\triangle_{\partial}\alpha} = \triangle_{\bar{\partial}}\bar{\alpha}$, which means that $\bar{\alpha} \in \Omega^{s,r}$ is also harmonic. This shows a one-to-one correspondence between harmonic (r, s)-forms and harmonic (s, r)-forms. It follows by defintion that $h^{r,s} = h^{s,r}$.

□

Remark. Visually, the Hodge diamond of a Kähler manifold is symmetric about the vertical and horizontal axes.

Definition 1.4.10. Let M be a Riemannian manifold with affine connection ∇. Consider a point $p \in M$, and the set of closed loops at p: $s = \{c(t) : 0 \leq t \leq 1,\ c(0) = c(1) = p\}$. Choosing any $X \in T_pM$ and parallel transporting it along some curve $c(t) \in s$, we obtain a new vector $X_c \in T_pM$. The set of all such transformations, for all possible closed loops, is called *holonomy group* and denoted by $H(p)$.

Theorem 1.4.8. Kähler manifolds have $U(n)$ holonomy, i.e. $H(p) = U(n)$.

Proof. This proof is based on a result in group theory that we will not show.[8] It is the following:

$$U(n) = O(2n) \cap GL(n, \mathbb{C}) \cap Sp(2n, \mathbb{R}). \tag{1.4.17}$$

It is known as the "2-out-of-3" rule, since the intersection of any two factors on the rhs defines $U(n)$.
Let $X = X^\mu \partial/\partial z^\mu$ be a holomorphic vector at p, i.e. $X \in T_p(M)^+$. If we parallely transport it around a loop, we end up with another vector $X'^\mu = X^\nu h_\nu{}^\mu$, which must be holomorphic (see theorem 1.2.4 and the following remark). This implies that $h_\nu{}^\mu \in GL(m, \mathbb{C})$. It also true that the length of a vector is preserved upon parallel transport, (see A.1.47), which imposes $h_\nu{}^\mu \in O(2m)$. These two considerations lead to the fact that elements $h_\nu{}^\mu$ of the holonomy group $H(p)$ belong to $GL(m, \mathbb{C}) \cap O(2m) = U(m)$, according to the "2-out-of-3" rule. $\square$

Definition 1.4.11. We have proven (see theorem 1.2.8) that the Ricci form is closed. However, it is not exact, and thus it can be a representative of a non-trivial cohomology class. The cohomology class represented by the (scaled) Ricci form is called *1st Chern class*:

$$c_1 = [\frac{\mathfrak{R}}{2\pi}]. \tag{1.4.18}$$

Theorem 1.4.9. The first Chern class is invariant under a smooth change of metric $g \to g + \delta g$.

Proof. Using the identity $\delta log(\bar{g}) = g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}$, we obtain:

$$\delta\mathfrak{R} = -\delta(i\partial\bar{\partial} log(\bar{g})) = -i\partial\bar{\partial} g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}} = \frac{i}{2} d(\partial - \bar{\partial}) g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}. \tag{1.4.19}$$

$g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}$ is a scalar, making $(\partial - \bar{\partial}) g^{\mu\bar{\nu}}\delta g_{\mu\bar{\nu}}$ is a well-defined one-form over the whole manifold. Hence, $\delta\mathfrak{R}$ is exact and $(\mathfrak{R} + \delta\mathfrak{R})$ belong to the same cohomology class of $\mathfrak{R}$. Thus, $c_1(M)$ is unchanged under $g \to g + \delta g$.

$\square$

[8] The reader is referred to [29].

1.5 Calabi-Yau Manifolds

Calabi-Yau manifolds are ubiquitous to string theory. We present a number of equivalent definitions, and illustrate how they are linked with each other. In order to do so, we introduce some profound theorems in modern mathematics, which require long and complicated proofs. For this reason, we will sometimes omit them or only provide a sketch of the argument. In such cases, appropriate references are given. We also present some more special properties of Calabi-Yau manifolds. For this chapter, we followed mainly [20] and [30].

Definition 1.5.1. A *Calabi-Yau manifold* M of $\dim_{\mathbb{C}}(M) = m$ is a compact Kähler manifold with *one* of the following properties:

1. Its first Chern class vanishes.
2. It admits a Ricci flat metric.
3. It admits a nowhere vanishing holomorphic m-form.
4. It has $SU(m)$ holonomy.

Theorem 1.5.1. A compact Kähler manifold with a Ricci flat metric has $c_1 = 0$.

Proof. Let the Ricci flat metric be g. Then, $\mathfrak{R}(g) = 0$ by assumption, and thus by definition $c_1 = 0$ (i.e. all the elements of the cohomology class are reached by adding an exact form). As we have shown in theorem 1.4.9, this result holds for any metric which can be obtained by smooth variation. □

Theorem 1.5.2. A compact Kähler manifold with $c_1 = 0$ admits a unique Ricci flat metric.

Proof. This is one of the few results that we quote without providing a proof or a sketch of it (and there is a good reason for it).[9] This follows from Calabi conjecture. □

Remark. Theorems 1.5.1 and 1.5.2 prove $1 \iff 2$.

[9] A proof can be found in [5]

Theorem 1.5.3. Let M be a compact Kähler manifold with a Ricci flat metric g and $\dim_{\mathbb{C}}(M) = m$. Then, the holonomy group is contained in $SU(m)$.

Proof. The proof that we present here is not rigorous and complete, since it applies only to simply connected manifolds. However, the statement above is true also for multiply connected ones.[19]
We already showed that Kähler manifolds have $U(n)$ (theorem 1.4.8). To prove that this further reduces to $SU(m)$ in the presence of a Ricci flat metric, we need to be more specific. Let X be parallely transported along an infinitesimal parallelogram with sides given by ϵ and $\bar{\delta}$. The resulting vector, still sitting at p, is given by (see A.1.50):

$$X'^{\mu} = X^{\mu} + X^{\alpha} R^{\mu}{}_{\alpha\beta\bar{\gamma}} \epsilon^{\beta} \bar{\delta}^{\gamma}. \tag{1.5.1}$$

Hence, elements of the holonomy group take the form:

$$h_{\nu}{}^{\mu} = \delta_{\nu}{}^{\mu} + R^{\mu}{}_{\nu\beta\bar{\gamma}} \epsilon^{\beta} \bar{\delta}^{\gamma} \tag{1.5.2}$$

Close to unity (which is the case we are considering, since we are taking the parallelogram to be infinitesimal), $U(m)$ is decomposed as $SU(m) \times U(1)$. At the level of Lie algebra,

$$\mathfrak{u}(m) = \mathfrak{su}(m) \oplus \mathfrak{u}(1), \tag{1.5.3}$$

where $\mathfrak{su}(m)$ are traceless elements and $\mathfrak{u}(1)$ corresponds to the trace. For a Ricci flat metric, by definition, the trace vanishes. Hence, the Lie algebra is just $\mathfrak{su}(m)$ and the corresponding holonomy group (assuming that the manifold is simply connected) is $SU(m)$.

□

Remark. We have just shown that $2 \implies 4$.

Theorem 1.5.4. Let M be a compact Kähler manifold with $\dim_{\mathbb{C}}(M) = m$. If there exists a nowhere vanishing holomorphic m-form, then $c_1 = 0$.

Proof. Let $\Omega_{\mu_1 \dots \mu_m}$ be the components of the nowhere vanishing holomorphic m-form on M. Then, by antisymmetry, they must be proportional to the totally antisymmetric symbol, i.e. given a set of coordinates $\{z^{\mu}\}$:

$$\Omega_{\mu_1 \dots \mu_m}(z) = f(z) \epsilon_{\mu_1 \dots \mu_m}, \tag{1.5.4}$$

where $f(z)$ is clearly holomorphic and nowhere vanishing. Hence, we have:

$$\overline{\Omega}^{\mu_1...\mu_m} = \bar{f}\epsilon_{\bar{\alpha}_1...\bar{\alpha}_m} g^{\mu_1\bar{\alpha}_1}...g^{\mu_m\bar{\alpha}_m} = (\bar{g})^{-1}\bar{f}\epsilon^{\mu_1...\mu_m}, \tag{1.5.5}$$

where recall $\bar{g} = \det(g_{\bar{\mu}\nu}) = \sqrt{\det(g_{mn})}$. It follows that

$$||\Omega||^2 = \frac{1}{m!}\Omega_{\mu_1...\mu_m}\overline{\Omega}^{\mu_1...\mu_m} = |f|^2(\bar{g})^{-1}. \tag{1.5.6}$$

Plugging this into the expression for the Ricci form (1.2.30), we obtain:

$$\mathfrak{R} = i\partial\bar{\partial}\log(||\Omega||^2) = d\Big(-\frac{i}{2}(\partial - \bar{\partial})\log(||\Omega||^2)\Big). \tag{1.5.7}$$

But this time, by hypothesis $\log(||\Omega||^2)$ is a coordinate scalar globally defined, and thus $\mathfrak{R}$ is exact. This proves that $c_1 = 0$. □

Remark. We have just shown $3 \implies 1$.

Theorem 1.5.5. Let M be a compact Kähler manifold with $\dim_{\mathbb{C}}(M) = m$ and with $SU(m)$ holonomy. Then, it admits a nowhere vanishing holomorphic m-form.

Proof. Assuming $SU(m)$ holonomy, we can have a globally defined Hermitian metric of the form $g = dz^1\wedge d\bar{z}^1 + \ldots + dz^m \wedge d\bar{z}^m$ (which is preserved by the holonomy group). This implies that the associated Kähler form reads: $\Omega = i\,(dz^1 \wedge d\bar{z}^1 + \ldots + dz^m \wedge d\bar{z}^m)$. Constructing Ω^m, and extracting the associated holomorphic volume form, we obtain $\Phi = \mathrm{d}z^1 \wedge \ldots \wedge \mathrm{d}z^m$ (see definition 1.3.3 and remark after that). Φ is of type $(m, 0)$ globally defined and nowhere vanishing. □

Remark. We have just shown that $4 \implies 3$.
Thus, the chain of implications proved above reads $1 \iff 2 \implies 4 \implies 3 \implies 1$. This shows the equivalence of the four definitions. [10]

Theorem 1.5.6. A Calabi-Yau manifold with non-zero Euler number has $h^{1,0} = 0$.

Proof. This proof relies on the fact that on a manifold with Euler number χ, any vector field has at least $|\chi|$ zeroes (see [31], for instance). We have that $b^1 = 2h^{1,0}$, and hence $b^1 = 0$ if and only if $h^{1,0} = 0$. Since b^1 is a topological invariant, we can work with a Ricci flat metric

[10] Note that $1 \impliedby 2$ was not necessary in such a chain, but it was included for completeness since we did not provide any proof for $1 \implies 2$.

(that we have shown is admitted on a Calabi-Yau) without loss of generality. Let α be a harmonic 1-form, i.e. $\triangle\alpha = 0$. Using the explicit expression for the Laplacian operator, and after some algebra, we find:

$$-\nabla^\nu \nabla_\nu \alpha_\mu + R_\mu{}^\nu \alpha_\nu = 0. \tag{1.5.8}$$

Ricci flatness implies that the above equation reduces to $-\nabla^\nu \nabla_\nu \alpha_\mu = 0$. Contracting with α^μ and integrating, we obtain:

$$\int_M \sqrt{g}\alpha^\mu \nabla^\nu \nabla_\nu \alpha_\mu = -\int_M \sqrt{g}\nabla^\nu \alpha^\mu \nabla_\nu \alpha_\mu = 0. \tag{1.5.9}$$

Since the integrand is positive definite, $\nabla_\nu \alpha_\mu = 0$. But, since we stated that α_μ must have at least one zero and it is also covariantly constant, it vanishes everywhere. Thus, $\alpha = 0$, and $h^{1,0} = 0$. □

Theorem 1.5.7. The holomorphic m-form on a compact Kähler manifold M with $\dim_{\mathbb{C}}(M) = m$ is harmonic.

Proof. We need to prove that $\nabla\Omega = (dd^\dagger + d^\dagger d)\Omega = 0$.
First, we show that $d\Omega = (\partial + \bar{\partial})\Omega = 0$. Since Ω is an $(m,0)$-form, $\partial\Omega$ vanishes trivially (by antisymmetry). And given that it is holomorphic by assumption, $\bar{\partial}\Omega = 0$.
$d^\dagger\Omega = 0$ should be shown in components (see A.1.7):

$$\begin{aligned} d^\dagger\Omega &= -\frac{1}{(m-1)!}\nabla^\alpha \Omega_{\alpha\mu_1\ldots\mu_{m-1}} dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_{m-1}} \\ &= -\frac{1}{(m-1)!} g^{\alpha\bar{\beta}} \nabla_{\bar{\beta}} \Omega_{\alpha\mu_1\ldots\mu_{m-1}} dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_{m-1}}. \end{aligned} \tag{1.5.10}$$

Then, we recall that the connection is pure in its indices, which means that the above expression reduces to:

$$d^\dagger\Omega = -\frac{1}{(m-1)!} g^{\alpha\bar{\beta}} \partial_{\bar{\beta}} \Omega_{\alpha\mu_1\ldots\mu_{m-1}} dz^{\mu_1} \wedge \ldots \wedge dz^{\mu_{m-1}} = 0. \tag{1.5.11}$$

Thus, Ω is harmonic. □

Theorem 1.5.8. The holomorphic m-form on a compact Kähler manifold M with $\dim_{\mathbb{C}}(M) = m$ is unique (up to a constant).

Proof. This proof uses as standard result from complex analysis: the maximum modulus principle (see [20]). In its simplest version, it states that the modulus of a function of one complex variable holomorphic in some open set U cannot have a maximum or minimum at a point in the interior of U unless it is constant on the whole U. This naturally extends to the case of more variables, and implies that a holomorphic function (globally defined) on a compact manifold must be constant.
The theorem quickly follows given this result. Let Ω and Ω' both be holomorphic $(m, 0)$-forms, so that they are both proportional to the totally antisymmetric symbol. They can differ from each other only by a (non-vanishing) holomorphic function: $\Omega' = f(z)\Omega$. Such function, according to the above argument, must be a constant. This completes the proof. □

Theorem 1.5.9. Let M be Calabi-Yau manifold with $\dim_{\mathbb{C}}(M) = m$. Then, $h^{m,0} = 1$.

Proof. We showed that a Calabi-Yau manifold has a unique holomorphic harmonic (m,0)-form Ω. This means that $h^{m,0} \geq 1$, with equality if such form is the only harmonic (m,0)-form on M. We will show that this is the case.
Let Ω' be a different harmonic (m,0)-form. Its closeness implies

$$d\Omega' = \partial\Omega' + \bar{\partial}\Omega' = 0. \tag{1.5.12}$$

Since $\partial\Omega'$ is trivially zero, we obtain $\bar{\partial}\Omega' = 0$. The components of a (m,0)-form are simply a scalar, so let $f(z, \bar{z})$ be the ones of Ω'. Then, we obtain $\partial_{\bar{\mu}} f(z, \bar{z}) = 0$, which implies that f is holomorphic. Thus, there is a unique harmonic (m,0)-form on M (the holomorphic one), proving that $h^{m,0} = 1$. □

Theorem 1.5.10. Let M be Calabi-Yau manifold with $\dim_{\mathbb{C}}(M) = m$. Then, $h^{k,0} = h^{m-p,0}$.

Proof. To avoid a long and tedious calculation with indices, we prove the above statement for the case $m = 3$. Exactly the same steps can be followed for any m, leading to the general result claimed (see, for instance, [30]).
Let ω be a $(2, 0)$-form: $\omega = \omega_{\mu\nu} dz^\mu dz^\nu$. Then, we construct a $(0, 1)$-form v using the holomorphic volume element (specifically, its complex conjugate). Its components are defined as:

$$v_{\bar{\alpha}} = \frac{1}{2} \bar{\Omega}_{\bar{\alpha}\bar{\mu}\bar{\nu}} w^{\bar{\mu}\bar{\nu}}. \tag{1.5.13}$$

It follows from this definition that

$$\nabla^{\bar{\alpha}} v_{\bar{\alpha}} = g^{\bar{\alpha}\beta}\nabla_\beta\big(\frac{1}{2}\bar{\Omega}_{\bar{\alpha}\bar{\mu}\bar{\nu}} w^{\bar{\mu}\bar{\nu}}\big) = g^{\bar{\alpha}\beta}\frac{1}{2}\bar{\Omega}_{\bar{\alpha}\bar{\mu}\bar{\nu}}\nabla_\beta w^{\bar{\mu}\bar{\nu}} = \tag{1.5.14}$$

$$g^{\bar{\alpha}\beta}\frac{1}{2}\bar{\Omega}_{\bar{\alpha}}{}^{\mu\nu}\nabla_\beta w_{\mu\nu} = \frac{1}{2}\bar{\Omega}^{\beta\mu\nu}\nabla_\beta w_{\mu\nu} = \frac{1}{2}\bar{\Omega}^{\beta\mu\nu}\partial_\beta w_{\mu\nu}, \tag{1.5.15}$$

where we used, in order, that: Ω is holomorphic, so $\bar{\Omega}$ is anti-holomorphic; the connection is pure in its indices; the covariant derivative of the metric vanishes; the connection is symmetric in its lower indices. All these properties were derived in theorem 1.2.4 and 1.3.6. This equality is saying (in the language of components) that $\bar{\partial}^\dagger v = 0 \iff \partial\omega = 0$.
Now we invert 1.5.13, to find:

$$w^{\bar{\rho}\bar{\sigma}} = \frac{\Omega^{\bar{\alpha}\bar{\rho}\bar{\sigma}}}{\|\Omega\|^2} v_{\bar{\alpha}}. \tag{1.5.16}$$

Using this relation, and the same considerations listed above, we obtain:

$$\nabla_{\bar{\rho}} w^{\bar{\rho}\bar{\sigma}} = \frac{\Omega^{\bar{\kappa}\bar{\rho}\bar{\sigma}}}{\|\Omega\|^2}\nabla_{\bar{\rho}} v_{\bar{\kappa}}. \tag{1.5.17}$$

This equality implies that $\partial^\dagger\omega = 0 \iff \bar{\partial}v = 0$. Combining the two results, we have that

$$\partial\omega = 0 \quad \text{and} \quad \partial^\dagger\omega = 0 \iff \bar{\partial}v = 0 \quad \text{and} \quad \bar{\partial}^\dagger v = 0. \tag{1.5.18}$$

In words, ω is ∂-harmonic if and only if v is $\bar{\partial}$-harmonic. Having shown (see theorem 1.4.6) that on a Kähler manifold ∂-harmonic, $\bar{\partial}$-harmonic and d-harmonic coincide, the proof is completed: $h^{k,0} = h^{m-p,0}$ for $m = 3$. □

Remark. Diagrammatically, the Hodge diamond for a Calabi-Yau manifold M with $dim_{\mathbb{C}}(M) = 3$ looks like:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{1,2} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

Generalised Geometry 2

In this chapter, we introduce generalised geometry. Starting from the simple idea of extending the tangent bundle, we show how a completely new type of geometry emerges.
We start by a quick review of bundles from ordinary differential geometry in section 2.1. They are crucial for understanding generalised geometry, which, at the simplest level, can be described as an extension of the ordinary tangent bundle to include the cotangent bundle at each point.
In section 2.2, which is the most important in this chapter, we define this new framework and present its basic features. We introduce the natural canonical metric and its symmetries, the Dorfman bracket, the Courant bracket and its symmetries. We try to motivate the interpretation of the latter two objects by referring to ordinary geometry.
Section 2.3 presents the concepts of a Lie algebroid and a Leibniz algebroid, which is are crucial notions from a purely mathematical point of view, and the patching nature of the generalised tangent bundle.
Finally, we derive the form of the generalised metric in section 2.4, and construct some simple generalised structures.
This section is based mainly on [19] for the first section, and [13], [32], [33], [34] and [35] for the rest.

2.1 Bundles and G-Structures in Ordinary Geometry

The concepts treated in this section all revolve around bundles. These mathematical objects play a crucial role in the geometrical description of modern theoretical physics. In particular, we focus on tangent bundles and G-structures. The former is the starting point for constructing generalised geometry. The latter is of great use when dealing with global tensors and spinors. We present a significant number of definitions, but we try to avoid the presentation from being too abstract by providing relevant examples and giving a few "concrete" theorems the end of the section. The main reference is [19], and examples are inspired by [35].

Definition 2.1.1. Let M, E be topological manifolds, and π be a continuous surjective map $\pi : E \to M$. Then, the triple (E, π, M) is a *bundle*.

Remark. We will denote such triple as $E \xrightarrow{\pi} M$.

Definition 2.1.2. $E' \xrightarrow{\pi'} M'$ is called a *subbundle* of $E \xrightarrow{\pi} M$ if $E' \subseteq E$ and $M' \subseteq M$ are submanifolds and $\pi' = \pi|_{E'}$.

Definition 2.1.3. Let $E \xrightarrow{\pi} M$ be a bundle and let $p \in M$. Then, $F_p = \pi^{-1}(p)$ is called the *fibre* at p. Here π^{-1} denotes the pre-image of π.

Definition 2.1.4. Let $E \xrightarrow{\pi} M$ be a bundle and let F be a differentiable manifold. Let the following properties hold: [1]

- $\pi^{-1}(p) = F_p \cong F \quad \forall p \in M$.

- M is equipped with a set of open coverings $\{U_i\}$ with diffeomorphisms ϕ_i that obey:

$$\phi_i : U_i \times F \to \pi^{-1}(U_i) \tag{2.1.1}$$
$$\text{s.t. } \pi \circ \phi_i(p, f) = p \quad \forall f \in F, p \in U_i. \tag{2.1.2}$$

  ϕ_i^{-1} maps π^{-1} onto $U_i \times F$, and is called *local trivialisation*.

- Fixing a point $p \in U_i$, i.e. $\phi_{i,p}(f) = \phi_i(p, f)$, we obtain a diffeomorphism $\phi_{i,p}(f) : F \to F_p$. For any non-trivial intersection $U_i \cap U_j$, $t_{ij}(p) = \phi_{i,p}^{-1} \circ \phi_{j,p} : F \to F$ is an element of the structure group G. The maps $t_{ij} : U_i \cap U_j \to G$ are called transition functions. Finally, with these definitions, $\phi_i(p, t_{ij}(p)f) = \phi_j(p, f)$.

Then, the collection (E, π, M, F, G) is a differentiable *fibre bundle*.

[1] To be pedantic, we would first need to discuss coordinate bundles, and then present a fiber bundle as an equivalence class of coordinate bundles. We are not pedantic.[19]

Remark. The second property is sometimes written as the commutativity of the diagram

$$\begin{tikzcd} U_i \times F \arrow[r, "\phi_i"] \arrow[dr, "proj_1"'] & \pi^{-1}(U_i) \arrow[d, "\pi"] \\ & U_i \end{tikzcd} \quad , \tag{2.1.3}$$

where $proj_1 : U_i \times F \to U_i$ is the natural projection.
Fiber bundles are usually denoted by $F \to E \xrightarrow{\pi} M$. Since we will only be concerned with fiber bundles, we will use the short notation $E \xrightarrow{\pi} M$ instead.

Definition 2.1.5. Let $E \xrightarrow{\pi} M$ be a fibre bundle. A smooth map $s : M \to E$ such that $\pi \circ s = \mathbb{1}$ is called a (cross) *section*.

Definition 2.1.6. The set of sections on M is denoted by $\Gamma(M)$.

Theorem 2.1.1. Let M be a differentiable manifold, and let TM be the disjoint union of the tangent spaces at all points of M,

$$TM = \cup_{p \in M} T_p M. \tag{2.1.4}$$

Define the map

$$\pi : TM \to M \tag{2.1.5}$$

$$T_p M \ni V \mapsto p \quad \forall p \in M. \tag{2.1.6}$$

Then, $TM \xrightarrow{\pi} M$ is a fibre bundle, called the *tangent bundle*.

Proof. We begin by identifying the fibre: $F_p = \pi^{-1}(p) = T_p(M)\ \forall p \in M$. Let dim(M)=m, then $F_p \cong \mathbb{R}^m = F$.
Next, we consider an open covering $\{U_i\}$ on M. Let an arbitrary U_i have coordinates $\{x^\mu\}$, and constrcut the local trivialisation map as follows.

$$\phi_i : U_i \times F \to \pi^{-1}(U_i) \tag{2.1.7}$$

$$(p, V^\mu) \mapsto V = V^\mu \frac{\partial}{\partial x^\mu}|_p \quad , \tag{2.1.8}$$

where $V^\mu \in \mathbb{R}^m$ and $p \in U_i$. By construction, $\pi \circ \phi_i(p, V^\mu) = p$, as required.
Consider a non-trivial overlap of U_i with a second chart, U_j, with coordinates y^α. Then, the transition function given by $t_{ij}(p) = \phi_{i,p}^{-1} \circ \phi_{j,p}$ takes the form

$$t_{ij}(p) : F \to F \tag{2.1.9}$$

$$\tilde{V}^\mu \mapsto t_{ij}(p)(\tilde{V}^\mu) = \phi_{i,p}^{-1} \circ \phi_j(p, \tilde{V}^\mu) = \phi_{i,p}^{-1}(V = \tilde{V}^\mu \frac{\partial}{\partial y^\mu}|_p = V^\mu \frac{\partial}{\partial x^\mu}|_p) = V^\mu. \tag{2.1.10}$$

It is known that the components of a vector in two different coordinate frames are related by $V^\mu = \tilde{V}^\alpha \frac{\partial x^\mu}{\partial y^\alpha}$, where $\frac{\partial x^\mu}{\partial y^\alpha} \in GL(m, \mathbb{R})$. Thus, $t_{ij}(p)$ are elements of a group, $GL(m, \mathbb{R})$, which is the structure group for this fiber bundle.
Finally, for completeness, we point out that sections of the this fibre bundle, i.e. maps of the form $M \to TM$, are vector fields. □

Remark. An analogous construction shows that $T^*M \xrightarrow{\pi} M$ is a fibre bundle, where

$$T^*M = \cup_{p \in M} T_p^* M. \tag{2.1.11}$$

Example 2.1.1. An interesting example (which, to be precise, involves a slight modification) is given by almost complex manifolds. Let us consider replacing T_pM with the complexified tangent space $T_pM^{\mathbb{C}}$ in the above construction. Then, with appropriate modifications (the basis becomes the complex one, for instance), all the arguments still apply. Thus, we have obtained what is called the *complexification* of the tangent bundle. We know from theorem 1.1.3 that the $T_pM^{\mathbb{C}}$ splits into two disjoint subspaces at every point p of the manifold, i.e. $T_pM^{\mathbb{C}} = T_pM^+ \oplus T_pM^-$. It is easy to check that $TM^+ = \cup_{p \in M} T_pM^+$ and $TM^- = \cup_{p \in M} T_pM^-$ define two subbundles of $TM = \cup_{p \in M} T_pM$, with projection define by the same π as for $TM \xrightarrow{\pi} M$.

Definition 2.1.7. Let $E^1 \xrightarrow{\pi_1} M^1$ and $E^2 \xrightarrow{\pi_2} M^2$ be fibre bundles. Let f' be a smooth map $f' : E^2 \to E^1$. If for any given fibre F_p^2 of E^2 we have that $f'(u) \in F_q^1 \ \forall u \in F_p^2$ for some fibre F_q^1 of E^1, f' is called a *bundle map*. Then, f' induces a map $M^2 \to M^1$ such that $f(p) = q$.

Remark. Note that the following diagram commutes:

$$\begin{tikzcd} E^2 \arrow[r, "f'"] \arrow[d, "\pi^2"'] & E^1 \arrow[d, "\pi^1"'] \\ M^2 \arrow[r, "f"] & M^1 \end{tikzcd} \tag{2.1.12}$$

Definition 2.1.8. A fiber bundle $E \xrightarrow{\pi_1} M$ whose fiber bundle is a vector space is called a *vector bundle*.

Definition 2.1.9. Let $E \xrightarrow{\pi} M$ be a vector bundle with fibre $\mathbb{R}^n$. For any given chart U_i, we can choose n linearly independent sections over it: $\{e_1(p), ..., e_n(p)\}$. This set is called a *frame* over U_i.

Example 2.1.2. Consider the tangent bundle $TM \xrightarrow{\pi} M$. Let U_i be any chart with coordinates x^μ. Then, the set $\{\frac{\partial}{\partial x^\mu}\}$ is a frame over U_i.

Definition 2.1.10. Let $E \xrightarrow{\pi_1} M$ be a fiber bundle. If its fibre F is the same as the structure group G, it is called a *principal bundle*.

Theorem 2.1.2. Consider the tangent bundle $TM \xrightarrow{\pi} M$, with $\dim_{\mathbb{R}}(M) = m$. We define:

$$LM = \cup_{p\in M} L_p(M), \tag{2.1.13}$$

where $L_p(M)$ is the set of frames at p. Then, $LM \xrightarrow{\pi} M$ is a principal bundle (with π to be defined).

Proof. Let U_i be a chart with coordinates $\{x^\mu\}$ and corresponding frame $\{\frac{\partial}{\partial x^\mu}\}$ over U_i. Any frame at p, A=$\{a_1, ..., a_m\} = \{a_\nu\}$, can be written as

$$a_\nu = \gamma^\mu_\nu \frac{\partial}{\partial x^\mu}|_p, \tag{2.1.14}$$

with $\gamma^\mu_\nu \in GL(m, \mathbb{R})$. Thus, any frame at a given point is labelled by an element of $GL(m, \mathbb{R})$. We define the projection map as follows:

$$\begin{aligned} \pi : LM &\to M \\ A &\mapsto p \quad \text{for} \quad A \in L_p(M). \end{aligned} \tag{2.1.15}$$

Thus, the fiber at p is the set of frames at p, which, according to the result above, can be identified with $GL(m, \mathbb{R})$. Hence, the natural definition for the trivialisation map is:

$$\begin{aligned} \phi_i : U_i \times GL(m, \mathbb{R}) &\to \pi^{-1}(U_i) \\ (p, \gamma^\mu_\nu) &\mapsto \{a_\nu\}, \end{aligned} \tag{2.1.16}$$

where the elements of the frame $\{a_\nu\}$ are given by 2.1.14. They clearly satisfy the properties in definition 2.1.4.
Consider a non-trivial overlap of U_i with a second chart, U_j, with coordinates y^α. Then, the transition function given by $t_{ij}(p) = \phi_{i,p}^{-1} \circ \phi_{j,p}$ takes the form

$$\begin{aligned} t_{ij}(p) &: F \to F \\ &\tilde{\gamma}^\mu_\nu \mapsto t_{ij}(p)(\tilde{\gamma}^\mu_\nu) = \phi_{i,p}^{-1} \circ \phi_j(p, \tilde{\gamma}^\mu_\nu) = \phi_{i,p}^{-1}(V = \tilde{\gamma}^\mu_\nu \frac{\partial}{\partial y^\mu}|_p = \gamma^\mu_\nu \frac{\partial}{\partial x^\mu}|_p) = \gamma^\mu_\nu. \end{aligned} \tag{2.1.17}$$

Again, we have that the components in the two frames are related by $\gamma^\mu_\nu = \tilde{\gamma}^\alpha_\nu \frac{\partial x^\mu}{\partial y^\alpha}$, with $\frac{\partial x^\mu}{\partial y^\alpha} \in GL(m, \mathbb{R})$. Thus, the fiber coincides with the structure group. □

Definition 2.1.11. $LM \xrightarrow{\pi} M$ is called the *tangent frame bundle.*

Definition 2.1.12. Let M be a differentiable manifold with $\dim_{\mathbb{R}}(M) = n$ and let $G \subset GL(n, \mathbb{R})$. M has a *G-structure* if it is possible to reduce the tangent frame bundle such that it has structure group G.

Remark. A very common way of describing G-structures consists of imposing the existence of G-invariant tensors (or spinors) globally defined on the manifold. To see the equivalence of such descriptions[2], consider a tensor which takes the same form in all patches (this is what globally defined means). This implies that we are not free to use arbitrary frames, but only those which allow the tensor to be written in a specific form. It follows that we are not allowed to use arbitrary transition functions either, but only those that leave the tensor invariant. These define the group G of the G-structure.
The G-invariant tensors can be found using representation theory. For a real manifold, we decompose the representation of $GL(n, \mathbb{R})$ in which the tensor transforms into irredicible representations of G. If a singlet is present in the decomposition, then the existence of such tensor, globally defined on the manifold, imposes a G-structure.

Theorem 2.1.3. An almost complex manifold M with has with $\dim_{\mathbb{C}}(M) = m$ has a $GL(m, \mathbb{C})$-structure.

[2]Strictly speaking, it is not an equivalence. A globally defined non-trivial tensor implies the existence of a structure. But the inverse is not true, although the counter-examples are very rare (orientation is one of them).

Proof. Let M be an almost complex manifold with $\dim_{\mathbb{C}}(M) = m$ and with almost complex structure J. As we have shown in theorem 1.1.5, holomorphic vectors are such in any chart. In terms of $LM \xrightarrow{\pi} M$, this means that the transition functions are not elements of $GL(2m, \mathbb{R})$, but they belong to $GL(m, \mathbb{C})$ instead (note that this is consistent with the requirement of m being even, i.e. theorem 1.1.2). In other words, (anti-)holomorphic vectors in each chart are mapped to (anti-)holomorphic vectors in any overlapping chart. Thus, the structure group is reduced to $GL(m, \mathbb{C})$. This statement clearly holds at the level of basis vectors as well, hence the structure group of the associated tangent frame bundle is $GL(m, \mathbb{C})$, and M has a $GL(m, \mathbb{C})$-structure. □

Theorem 2.1.4. A Riemannian manifold M with $\dim_{\mathbb{R}}(M) = n$ and metric g has a natural $O(n)$-structure.

Proof. Let g be the Riemannian metric: a globally defined, postive-definite, symmetric tensor. We can define orthonormal frames in each patch of the manifold, such that $g_{\mu\nu} = \delta_{\mu\nu}$ everywhere on the manifold. Let us show how this corresponds to a $O(n)$-structure. Consider a point p in the overlap between two patches $U_i \cap U_j$. Let $\{e_\mu\}$ be the basis for U_i at p, and $\{\hat{e}_\mu\}$ be the basis for U_j at p. In general, the two are related by a $GL(n, \mathbb{R})$ transformation: $\hat{e}_\mu = A_\mu{}^\nu e_\nu$ with $A \in GL(n, \mathbb{R})$. But the existence of the globablly defined metric above imposes:

$$g(e_\mu, e_\nu) = \delta_{\mu\nu} = g(\hat{e}_\mu, \hat{e}_\nu). \tag{2.1.18}$$

Using $\hat{e}_\mu = A_\mu{}^\nu e_\nu$, we get:

$$\delta_{\mu\nu} = g(\hat{e}_\mu, \hat{e}_\nu) = A_\mu{}^\alpha A_\nu{}^\beta g(e_\alpha, e_\beta) = A_\mu{}^\alpha A_\nu{}^\alpha. \tag{2.1.19}$$

Hence, the transition functions of the tangent frame bundle are elements of the set $O(n) = \{A \in GL(n, \mathbb{R}) : AA^T = \mathbb{1}_n\}$. □

Remark. If the form of the globally defined metric were the Minkowski metric with signature (p,q), then clearly the resulting manifold would have been an $O(p, q)$-structure.

Theorem 2.1.5. A Hermitian manifold with $\dim_{\mathbb{C}}(M) = m$ and metric g has a natural $U(m)$-structure.

Proof. This proof is very similar to the one for theorem 1.4.8
Let M be a Hermitian manifold with $\dim_{\mathbb{C}}(M) = m$ and almost complex structure J. By

definition, M is also a Riemannian manifold with $\dim_{\mathbb{R}}(M) = 2m$. We have shown that this reduces the structure group to $O(2m)$. We have also shown that the existence of an almost complex structure J on the manifold gives a $GL(m, \mathbb{C})$-structure. According to the "two-out-of-three" theorem, and since by definition M admits a Riemannian metric and an almost complex structure, M has a $U(m)$-structure. □

Example 2.1.3. Let M be an orientable Riemannian manifold with $\dim_{\mathbb{R}}(M) = 6$. We showed above that it has naturally an $SO(6, \mathbb{R})$ structure. The double cover of $SO(6, \mathbb{R})$ is $Spin(6)$, which is isomorphic to $SU(4)$.[3] Suppose we want to find the condition for an $SU(3)$ structure in terms of globally defined spinors. As discussed above, we decompose the representation under which a general spinor transforms, i.e. $SU(4)$, into irreps of its subgroup $SU(3)$. The result is $\mathbf{4} \to \mathbf{1} + \mathbf{3}$. This is easy to see if we consider the embedding of $SU(3)$ into the fundamental rep of $SU(4)$ (see [36]):

$$SU(3) \subset SU(4) : \quad \left(\begin{array}{c|c} SU(3) & 0 \\ \hline 0 & 1 \end{array} \right) \implies \text{singlet} : \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.1.20}$$

Thus, we see explicitly that the existence of a globally defined spinor which is a singlet of $SU(3) \subset SU(4)$ is equivalent to having an $SU(3)$ structure on the manifold.
We will return back to this example in section 1 of chapter 5.

2.2 The Basic Objects in Generalised Geometry

In this section, we develop a new mathematical framework. The starting point is the idea of extending the tangent bundle to a generalised tangent bundle, which consists of the tangent space plus the cotangent space at each point of the manifold. We study the basic structures that emerge directly from this assumption: a natural canonical metric and two brackets. Since the number of results presented is quite large, we provide full calculations only for the most important ones. For the others, we give a sketch of the derivation. The theorems regarding the properties of the brackets are relevant for the next section, where they appear in a more formal context.

[3] It is not always possible to lift a $SO(n)$ bundle to a $SPIN(n)$ bundle. However, in this dissertation, we will not deal with this subtlety. See [19] for more details.

The resources for this section are [13], [33] and [34].

Remark. For the rest of the chapter, we assume to be dealing with a manifold M with $\dim_{\mathbb{C}} = n/2$ (and therefore $\dim_{\mathbb{R}} = n$).

Definition 2.2.1. We define the *generalised tangent bundle* as the sum of the tangent bundle and cotangent bundle:

$$E = \cup_{p \in M} T_p M \oplus T_p^* M. \tag{2.2.1}$$

Remark. We anticipate that this is a very convenient definition, but not a rigorous one. We will provide a flavour of how E should be really thought of in the next section.
The most rigorous and concise definition involves short exact sequences, which we do no deal with in this dissertation.[13]

Definition 2.2.2. The sections of the generalised tangent bundle are called *generalised vector fields* (or just *generalised vectors*, for short), and they are a (formal) sum of a vector field and a one-form, which we can also write as a column vector:

$$\mathbb{X} = X + \xi = \begin{pmatrix} X \\ \xi \end{pmatrix} \in \Gamma(E), \tag{2.2.2}$$

where $X \in \Gamma(TM)$ and $\xi \in \Gamma(T^*M)$.

Remark. We will use both of these equivalent notations in what follows.

Definition 2.2.3. Let $\mathbb{X} = \begin{pmatrix} X \\ \xi \end{pmatrix}, \mathbb{Y} = \begin{pmatrix} Y \\ \eta \end{pmatrix} \in \Gamma(E)$. Then, there exist a *natural canonical metric* on the generalised tangent bundle, which acts on its sections as:

$$\mathcal{I}(\mathbb{X}, \mathbb{Y}) = \frac{1}{2}(\xi(Y) + \eta(X)). \tag{2.2.3}$$

In matrix notation, 2.2.3 can be written as:

$$\mathcal{I}(\mathbb{X}, \mathbb{Y}) = \begin{pmatrix} X \\ \xi \end{pmatrix} \frac{1}{2} \begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix} \begin{pmatrix} Y \\ \eta \end{pmatrix} = \mathbb{X}^M \mathcal{I}_{MN} \mathbb{Y}^N, \tag{2.2.4}$$

where we introduced "generalised" indices $M, N = 1, ..., 2n$.

Remark. It clearly follows from 2.2.4 that the natural canonical metric has components given by the matrix:

$$\mathcal{I}_{MN} = \frac{1}{2}\begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix}_{MN}. \tag{2.2.5}$$

Then, we denote the components of the inverse matrix by:

$$\mathcal{I}^{MN} = 2\begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix}^{MN}. \tag{2.2.6}$$

By definition, the matrix multiplication of the two yields $\mathbb{1}_{2n}$.

Definition 2.2.4. The group $O(n,n)$ is defined as:

$$O(n,n) = \{A \in GL(2n,\mathbb{R}) : A^T \begin{pmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix} A = \begin{pmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}\}. \tag{2.2.7}$$

Remark. It is clear that, by definition, $O(n,n)$ is the group of symmetries of the natural canonical metric.
We will now study its features, not in the general setting of linear algebra, but in the context of generalised geometry. Concretely, we think of $O(n,n)$ elements as acting on generalised vectors, not general $2n$-dimensional vectors.[4]

Theorem 2.2.1. Generators of $O(n,n)$ at a given point take the form:

$$\begin{pmatrix} a & \beta \\ B & -a^T \end{pmatrix}, \tag{2.2.8}$$

with

- a being an endomorphism on the tangent bundle, i.e. a: $TM \to TM$.
- B being a skew symmetric map from the tangent bundle to the cotangent bundle (a 2-form), i.e. $B : TM \to T^*M$.

[4] There is a subtle point that is worth emphasizing here. $TM \oplus T^*M$ is associated to a $GL(n)$ principal bundle (according to theorem 2.1.2). However, since we introduced the metric $\mathcal{I}$, we can conveniently think of the natural structure group as being $O(n,n)$ (analogously to theorem 2.1.4) See [13].

- β being a skew symmetric map from the cotangent bundle to the tangent bundle, i.e. $\beta : T^*M \to TM$.

Proof. We write $g \in O(n,n)$ close to the identity as

$$g = \mathbb{1} + \epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} + O(\epsilon^2), \tag{2.2.9}$$

with $\epsilon \ll 1$. Then, we impose

$$g^T \begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix} g = \begin{pmatrix} & \mathbb{1} \\ \mathbb{1} & \end{pmatrix}, \tag{2.2.10}$$

which to order ϵ yields:

$$b = -b^T\,, \quad c = -c^T\,, \quad d = -a^T. \tag{2.2.11}$$

We can make the redefinitions $b = \beta$, $c = B$ to obtain exactly 2.2.8. $\square$

Definition 2.2.5. We define three subgroups of $O(n,n)$ as follows.

1. *B-transformations.*
   These elements are obtained by exponentiating the generators with $a = 0$ and $\beta = 0$. They are of the form:

   $$\exp\left[\begin{pmatrix} 0 & 0 \\ B & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}. \tag{2.2.12}$$

   To ease the notation, these transformations are usually written as e^B. This acts as a shear in the T^*M direction:

   $$\mathbb{X} = X + \xi \mapsto \mathbb{X}' = X + (\xi - i_X B). \tag{2.2.13}$$

   We will refer to this subgroup as G_B.

2. *β-transformations.*
   These elements are obtained by exponentiating the generators with $a = 0$ and $B = 0$. They are of the form:

   $$\exp\left[\begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}\right] = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}. \tag{2.2.14}$$

To ease the notation, these transformations are usually written as e^{β}. This acts as a shear in the TM direction:

$$\mathbb{X} = X + \xi \mapsto \mathbb{X}' = (X + \beta \cdot \xi) + \xi. \tag{2.2.15}$$

3. *$GL(n)$-transformations.* These elements are obtained by exponentiating the generators with $B = 0$ and $\beta = 0$. They are of the form

$$\exp\left[\begin{pmatrix} a & 0 \\ 0 & -a^T \end{pmatrix}\right] = \begin{pmatrix} e^a & 0 \\ 0 & (e^{a^T})^{-1} \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix}. \tag{2.2.16}$$

   We will write these transformations symbolically as e^a (this is one of the few instances of us using a notation that is not commonly employed). This subgroup is an embedding of $GL(n)$ inside $O(d,d)$. We will refer to this subgroup as $GL(n,\mathbb{R})$.

Remark. The above result has a very interesting interpretation. The natural structure group of the generalised tangent bundle can be thought of an "extension" of the structure group of the tangent bundle ($GL(n,\mathbb{R})$) by two other transformations (B and β).

Remark. B-transformations are more fundamental than β-transformations, both mathematically and physically. From the abstract point of view, $\beta \in \mathscr{T}_0^{\,2}(M)$ and thus it cannot be a connection. Conversely, $B \in \mathscr{T}_2^{\,0}(M)$, and therefore it can be a connection. Physically, the B field is intimately related to the 2-form B that appears in the NSNS sector of Type II supergravity, as we will show in chapter 5.

Definition 2.2.6. The map

$$\pi_{TM} : E \to TM \tag{2.2.17}$$

$$X + \xi \mapsto X \tag{2.2.18}$$

is called the *anchor map.*

Remark. We will show the role of this map inside a bigger, more formal picture in the next section. For now, we only point out that is a very natural projection map in the setting that we are describing.

Definition 2.2.7. Let $\mathbb{X} = (X + \xi)$, $\mathbb{Y} = (Y + \eta) \in \Gamma(E)$. Then, the *Dorfman derivative* of $\mathbb{Y}$ wrt $\mathbb{X}$ is defined as

$$\begin{aligned} L : \Gamma(E) \times \Gamma(E) &\to \Gamma(E) \\ (\mathbb{X}, \mathbb{Y}) &\mapsto L_{\mathbb{X}}\mathbb{Y} = \begin{pmatrix} \mathcal{L}_X Y \\ \mathcal{L}_X \eta - i_Y d\xi \end{pmatrix}. \end{aligned} \tag{2.2.19}$$

Remark. The operation just defined is sometimes called Dorfman bracket, and the notation $[\cdot, \cdot]$ is used instead of $L_{(\cdot)}(\cdot)$. It is just a matter of convention, and our choice is consistent with the interpretation (not yet motivated) of the Dorfman derivative as a generalisation of the Lie derivative.

Theorem 2.2.2. Let $\mathbb{X}, \mathbb{Y}, \mathbb{Z} \in \Gamma(E)$, $f \in \mathcal{F}(M)$ and a, b be constants. Then, the Dorfman derivative satisfies the following properties.

1. Bilinearity:

$$L_{\mathbb{X}}(a\mathbb{Y} + b\mathbb{Z}) = aL_{\mathbb{X}}\mathbb{Y} + bL_{\mathbb{X}}\mathbb{Z} \quad \text{and} \quad L_{a\mathbb{X}+b\mathbb{Y}}\mathbb{Z} = aL_{\mathbb{X}}\mathbb{Z} + bL_{\mathbb{Y}}\mathbb{Z}. \tag{2.2.20}$$

2. "Adjoint Leibniz" rule:

$$L_{\mathbb{X}}(L_{\mathbb{Y}}\mathbb{Z}) = L_{L_{\mathbb{X}}\mathbb{Y}}\mathbb{Z} + L_{\mathbb{Y}}(L_{\mathbb{X}}\mathbb{Z}). \tag{2.2.21}$$

3. Failure of skew symmetry:

$$L_{\mathbb{X}}\mathbb{Y} + L_{\mathbb{Y}}\mathbb{X} = 2d\big(\mathcal{I}(\mathbb{X}, \mathbb{Y})\big). \tag{2.2.22}$$

a. "Projected" Leibniz rule with a function:

$$L_{\mathbb{X}}(f\mathbb{Y}) = fL_{\mathbb{X}}\mathbb{Y} + \pi_{TM}(\mathbb{X})[f]\mathbb{Y}. \tag{2.2.23}$$

b. "Projected" reduction to directional derivative

$$L_{\mathbb{X}}(f) = \pi_{TM}(\mathbb{X})[f]. \tag{2.2.24}$$

c. "Projected" reduction to Lie derivative

$$\pi_{TM}(L_{\mathbb{X}}\mathbb{Y}) = \mathcal{L}_X Y. \tag{2.2.25}$$

Proof. These facts follow from direct computations starting from 2.2.19.
Property 1. follows quickly from the linearity property of the interior product and of the Lie derivative.
Property 2. is the less straightforward to prove. The vector part is obtained by using the "adjoint Liebniz" rule A.1.14. The form part is obtained by means of A.1.14, together with Cartan's "magic" formula A.1.31 and A.1.33 (remembering that $d^2 = 0$).
The proof of property 3. is a one-liner if one uses Cartan's "magic" formula A.1.31.
Property a. follows from A.1.18.
Property b. is a special case of property a., where we let $\mathbb{Y}$ be unity and omit it.
Property c. is true by the definitions of anchor map and Dorfman derivative. □

Remark. We can compare 1., 2. and 3. with A.1.13, A.1.14 and A.1.15, respectively. Skew symmetry is violated, which might suggest that the Dorfman derivative is not a suitable generalisation for the Lie derivative. However, as explained in the appendix, the fact that the Lie derivative coincides with the Lie bracket is not an a priori requirement, but rather a coincidence. Thus, the failure of the Dorfman bracket to be a Lie bracket is not a fundamental issue. To see why the Dorfman bracket generalises the Lie derivative, we review the properties denoted by letters. Those are the ones that involve the anchor map, and thus they are identities of a new kind with respect to the ones of ordinary geometry. We see that a.,b. are generalisations of A.1.18, A.1.17. Moreover, c. includes the Lie derivative as a "special case" of the Dorfman derivative. From these considerations, we are led to think that the Dorfman derivative can be though as a generalisation of the Lie derivative.

Definition 2.2.8. We provide an embedding of the partial derivative operator into the generalised tangent bundle with the operator ∂_M, defined as follows:

$$\partial_M = \begin{cases} \partial_\mu & \text{for } M = \mu + d \\ 0 & \text{for } M = \mu \end{cases} \quad \Longleftrightarrow \quad \partial_M = \begin{pmatrix} \partial_\mu \\ 0. \end{pmatrix}. \tag{2.2.26}$$

Theorem 2.2.3. The Dorfman derivative can be alternatively written as:

$$(L_{\mathbb{X}}\mathbb{Y})^M = \mathbb{X}^N \partial_N \mathbb{Y}^M - \mathbb{Y}^N \partial_N \mathbb{X}^M + \mathbb{Y}_N \partial^M \mathbb{X}^N, \tag{2.2.27}$$

where the indices in the last term are raised and lowered using the natural canonical metric (see 2.2.5 and 2.2.6).

Proof. We analyse the three terms separately, then it will be explicit that the result follows from their sum. Let $\mathbb{X} = X + \xi$ and $\mathbb{Y} = Y + \eta$. Then,

$$\begin{aligned}\mathbb{X}^N \partial_N \mathbb{Y}^M &= X^\alpha \partial_\alpha \begin{pmatrix} Y^\mu \\ \eta_\mu \end{pmatrix} = \begin{pmatrix} X^\alpha \partial_\alpha Y^\mu \\ X^\alpha \partial_\alpha \eta_\mu \end{pmatrix}, \\ -\mathbb{Y}^N \partial_N \mathbb{X}^M &= -Y^\alpha \partial_\alpha \begin{pmatrix} X^\mu \\ \xi_\mu \end{pmatrix} = \begin{pmatrix} -Y^\alpha \partial_\alpha X^\mu \\ -Y^\alpha \partial_\alpha \xi_\mu \end{pmatrix}.\end{aligned} \tag{2.2.28}$$

Hence,

$$\mathbb{X}^N \partial_N \mathbb{Y}^M - \mathbb{Y}^N \partial_N \mathbb{X}^M = \begin{pmatrix} (\mathcal{L}_X Y)^\mu \\ X^\alpha \partial_\alpha \eta_\mu - Y^\alpha \partial_\alpha \xi_\mu \end{pmatrix}. \tag{2.2.29}$$

The first/vector component already matches the Dorfman derivative definition (2.2.19). Finally, recalling that $\mathcal{I}$ and $\mathcal{I}^{-1}$ exchange the first n rows with the last n, and have numerical factors in front that multiply to one, we have:

$$\mathbb{Y}_N \partial^M \mathbb{X}^N = \mathbb{Y}^L \mathcal{I}_{LN} \partial_K \mathcal{I}^{KM} \mathbb{X}^N = \begin{pmatrix} \eta \\ Y \end{pmatrix}_N \begin{pmatrix} 0 \\ \partial_\mu \end{pmatrix} \begin{pmatrix} X \\ \xi \end{pmatrix}^N = \begin{pmatrix} 0 \\ \eta_\alpha \partial_\mu X^\alpha + Y^\alpha \partial_\mu \xi_\alpha \end{pmatrix}. \tag{2.2.30}$$

Summing the above with 2.2.29, we see that the bottom/form component reads:

$$X^\alpha \partial_\alpha \eta_\mu + \eta_\alpha \partial_\mu X^\alpha - Y^\alpha \partial_\alpha \xi_\mu + Y^\alpha \partial_\mu \xi_\alpha = (\mathcal{L}_X \eta)_\mu - (i_Y d\xi)_\mu. \tag{2.2.31}$$

This part also matches 2.2.19. □

Remark. This way of writing the Dorfman derivative might seem less elegant and more cluttered, but it is convenient for calculations and provides an easy generalisation for its action on generalised tenesors (see next definition).
Moreover, it shows the intimate connection between generalised geometry and double field theory (see chapter 6, section 4).

Definition 2.2.9. The action of the Dorfman derivative wrt a generalised vector $\mathbb{X}$ on a generalised tensor (field) $\mathbb{T}$ of type (p,q) is given by:

$$\begin{aligned}(L_{\mathbb{X}} \mathbb{T})^{M_1 \dots M_p}_{N_1 \dots N_q} = {} & \mathbb{X}^N \partial_N \mathbb{T}^{M_1 \dots M_p}_{N_1 \dots N_q} + \left(\partial^{M_1} \mathbb{X}^N - \partial^N \mathbb{X}^{M_1}\right) \mathbb{T}^{\;\; M_2 \dots M_p}_{N N_1 \dots N_q} + \dots + \left(\partial^{M_p} \mathbb{X}^N - \partial^N \mathbb{X}^{M_p}\right) \mathbb{T}^{M_1 \dots M_{p-1}}_{N_1 \, \dots \, N_q \;\; N} \\ & - \left(\partial_M \mathbb{X}_{N_1} - \partial_{N_1} \mathbb{X}_M\right) \mathbb{T}^{M M_1 \dots M_p}_{\;\; N_2 \dots N_q} - \dots - \left(\partial_M \mathbb{X}_{N_q} - \partial_{N_q} \mathbb{X}_M\right) \mathbb{T}^{M_1 \dots M_p M}_{N_1 \dots N_{q-1}}.\end{aligned} \tag{2.2.32}$$

Remark. It is immediate to check that such definition reduces to 2.2.27 for the case of a $(1,0)$ generalised tensor.

Theorem 2.2.4. The Dorfman derivative of the natural canonical metric vanishes.

Proof. Let $\mathbb{X} \in \Gamma(E)$. Then,

$$(L_{\mathbb{X}}\mathcal{I})_{MN} = \mathbb{X}^K \partial_K \mathcal{I}_{MN} - (\partial_K \mathbb{X}_M - \partial_M \mathbb{X}_K)\mathcal{I}^K{}_N - (\partial_K \mathbb{X}_N - \partial_N \mathbb{X}_K)\mathcal{I}_M{}^K. \tag{2.2.33}$$

The first term vanishes since $\mathcal{I}_{MN}$ is a constant. By definition of $\mathcal{I}^{MN}$ and $\mathcal{I}_{MN}$ (2.2.5, 2.2.6), $\mathcal{I}_M{}^K$ and $\mathcal{I}^K{}_N$ are just δ^K_M. Thus, the last four terms cancel in pairs. □

Definition 2.2.10. Let $\mathbb{X} = (X + \xi)$, $\mathbb{Y} = (Y + \eta) \in \Gamma(E)$. Then, the *Courant bracket* $[\![\mathbb{X}, \mathbb{Y}]\!]$ is defined as the anti-symmetrisation of the Dorfman derivative of $\mathbb{Y}$ wrt $\mathbb{X}$:

$$[\![\mathbb{X}, \mathbb{Y}]\!] = \frac{1}{2}(L_{\mathbb{X}}\mathbb{Y} - L_{\mathbb{Y}}\mathbb{X}). \tag{2.2.34}$$

Theorem 2.2.5. The explicit expression for the Courant bracket is:

$$[\![\mathbb{X}, \mathbb{Y}]\!] = \begin{pmatrix} [X, Y] \\ \mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi) \end{pmatrix}. \tag{2.2.35}$$

Proof. The vector part follows immediately by recalling that $\mathcal{L}_X Y = [X, Y]$ and $[X, Y] = -[Y, X]$ (see Appendix, A.1.28 and A.1.26). The form part is obtained by using Cartan's "magic" formula A.1.31 to rewrite the terms $i_Y d\xi$ and $i_X d\eta$. □

Theorem 2.2.6. Let $\mathbb{X}, \mathbb{Y}, \mathbb{Z} \in \Gamma(E)$. Then, the Courant bracket satisfies the following properties:

1. Bilinearity:

$$[\![\mathbb{X} + \mathbb{Y}, \mathbb{Z}]\!] = [\![\mathbb{X}, \mathbb{Z}]\!] + [\![\mathbb{Y}, \mathbb{Z}]\!] \quad \text{and} \quad [\![\mathbb{X}, \mathbb{Y} + \mathbb{Z}]\!] = [\![\mathbb{X}, \mathbb{Y}]\!] + [\![\mathbb{X}, \mathbb{Z}]\!]. \tag{2.2.36}$$

2. Skew symmetry:

$$[\![\mathbb{X}, \mathbb{Y}]\!] = -[\![\mathbb{Y}, \mathbb{X}]\!]. \tag{2.2.37}$$

3. Failure of the Jacobi identity:

$$[\![[\![\mathbb{X}, \mathbb{Y}]\!], \mathbb{Z}]\!] + [\![[\![\mathbb{Z}, \mathbb{X}]\!], \mathbb{Y}]\!] + [\![[\![\mathbb{Y}, \mathbb{Z}]\!], \mathbb{X}]\!] = \frac{1}{3}d\big(\mathcal{I}([\![\mathbb{X}, \mathbb{Y}]\!], \mathbb{Z}) + \mathcal{I}([\![\mathbb{Z}, \mathbb{X}]\!], \mathbb{Y}) + \mathcal{I}([\![\mathbb{Y}, \mathbb{Z}]\!], \mathbb{X})\big). \tag{2.2.38}$$

a. "Projected" reduction to Lie bracket:

$$\pi_{TM}([\![\mathbb{X},\mathbb{Y}]\!]) = [\pi_{TM}(\mathbb{X}), \pi_{TM}(\mathbb{Y})]. \tag{2.2.39}$$

b. Failure of "projected" Leibniz rule with a function:

$$[\![\mathbb{X}, f\mathbb{Y}]\!] = f[\![\mathbb{X},\mathbb{Y}]\!] + \pi_{TM}(\mathbb{X})[f]\mathbb{Y} - \mathcal{I}(\mathbb{X},\mathbb{Y})df. \tag{2.2.40}$$

Proof. Property 1. was already proven for the Dorfman derivative, hence it holds for the Courant bracket.
Property 2. is true by construction.
Property 3. involves a long calculation, which we do not present here, but can be found in [13].
Property a. follows immediately from the definitions of anchor map and Courant bracket.
Property b. can be obtained via direct calculation. The identities that need to be used are A.1.18, A.1.35 and the Leibniz rule for the exterior derivative. □

Remark. Property 2. is the crucial one, since it states that the Courant bracket is skew-symmetric by construction (just as any bracket over a Lie algebra). We see from a. that the Courant bracket reduces to the Lie bracket upon projection by the anchor map. However, it fails to obey the "projected" version of the Liebniz rule with a function. This, together with property 3., suggests that the Courant bracket should not interpreted as a derivation (a generalisation of a differential operator), but as the generalisation of the Lie bracket in the context of generalised geometry.
An additional property that confirms this interpretation is the following, which should be compared with A.1.30.

Theorem 2.2.7. The Dorfman derivative and Courant bracket satisfy the following property, for $\mathbb{X},\mathbb{Y},\mathbb{Z} \in \Gamma(E)$:

$$L_{\mathbb{X}}(L_{\mathbb{Y}}\mathbb{Z}) - L_{\mathbb{Y}}(L_{\mathbb{X}}\mathbb{Z}) = L_{[\![\mathbb{X},\mathbb{Y}]\!]}\mathbb{Z}. \tag{2.2.41}$$

Proof. The calculation is very similar to the one that leads to 2.2.21. If we assume 2.2.21, it just consists of showing that $L_{[\![\mathbb{X},\mathbb{Y}]\!]}\mathbb{Z} = L_{L_{\mathbb{X}}\mathbb{Y}}\mathbb{Z}$. The vector part is trivially true. For the form part, using $\mathcal{L}_Y\xi = i_Y d\xi + di_Y\xi$ and $d^2 = 0$ yields the result. □

Remark. We tried to motivate as much as possible the interpretations of the Dorfman derivative and Courant bracket by comparing them to the operations in ordinary geometry.

The results provided strongly suggest that our interpretation is correct, but clearly they do not *prove* it. To show formally that $L_{(\cdot)}(\cdot)$ and $[\![\cdot,\cdot]\!]$ in generalised geometry play the same roles as $\mathcal{L}_{(\cdot)}(\cdot)$ and $[\cdot,\cdot]$ in ordinary geometry, we need to introduce the notion of *derived brackets*. This is far too abstract for the aim of this dissertation, but the interested reader can consult [37] and [38].

Theorem 2.2.8. The Courant bracket is invariant under the $GL(n)$ transformations, i.e. given $\mathbb{X}, \mathbb{Y} \in \Gamma(E)$:

$$[\![e^a\mathbb{X}, e^a\mathbb{Y}]\!] = e^a[\![\mathbb{X},\mathbb{Y}]\!]. \tag{2.2.42}$$

Proof. We will save ourselves a lot of indices and matrices by showing this statement in a coordinate independent way. Let f be a diffeomorphism of the manifold. Then, its embedding in the generalised tangent bundle is given by

$$f_{GG} = \begin{pmatrix} f_* & 0 \\ 0 & (f^*)^{-1} \end{pmatrix}, \tag{2.2.43}$$

which acts on generalised vectors. At the level of components, this is nothing but 2.2.16. It quickly follows from A.1.36, A.1.37 and A.1.38 that

$$[\![f_{GG}\mathbb{X}, f_{GG}\mathbb{Y}]\!] = f_{GG}[\![\mathbb{X},\mathbb{Y}]\!], \tag{2.2.44}$$

which is equivalent to 2.2.42 once a choice of coordinates is made. □

Theorem 2.2.9. The B-transformation is a symmetry of the Courant bracket if and only if B is closed, i.e.

$$[\![e^B\mathbb{X}, e^B\mathbb{Y}]\!] = e^B[\![\mathbb{X},\mathbb{Y}]\!], \tag{2.2.45}$$

for B s.t. $dB = 0$ and $\mathbb{X}, \mathbb{Y} \in \Gamma(E)$.

Proof. Let $\mathbb{X} = (X+\xi)$, $\mathbb{Y} = (Y+\eta) \in \Gamma(E)$. Then, we have that $e^B\mathbb{X} = X + (\xi - i_X B))$ and $e^B\mathbb{Y} = Y + (\eta - i_Y B))$. The vector parts of the two generalised vectors are unchanged under the B-transformation. This implies that the first component of the Courant bracket between the transformed vectors is the same as the first component of the Courant bracket between the original ones. All the transformation takes place in the form part. Explicitly, the Courant bracket of the transformed generalised vectors reads:

$$\begin{aligned}[] [\![e^B\mathbb{X}, e^B\mathbb{Y}]\!] = [\![e^B\mathbb{X}, e^B\mathbb{Y}]\!] - \mathcal{L}_X i_Y B + \mathcal{L}_Y i_X B + \frac{1}{2}d(i_X i_Y B) - \frac{1}{2}d(i_Y i_X B) = \\ [\![e^B\mathbb{X}, e^B\mathbb{Y}]\!] - \mathcal{L}_X i_Y B + i_Y d(i_X B) = [\![e^B\mathbb{X}, e^B\mathbb{Y}]\!] - i_{[X,Y]}B - i_Y\mathcal{L}_X B + i_Y d(i_X B) = \\ [\![e^B\mathbb{X}, e^B\mathbb{Y}]\!] - i_{[X,Y]}B - i_Y i_X dB \end{aligned} \tag{2.2.46}$$

where in the first step we just expanded out, in the second one we used the antisymmetry of B to sum the last two terms and then used Cartan's "magic" formula A.1.31, the third one was obtained through A.1.33 and finally A.1.31 led to the final expression. If $dB = 0$, then

$$[\![e^B\mathbb{X}, e^B\mathbb{Y}]\!] = [\![e^B\mathbb{X}, e^B\mathbb{Y}]\!] - i_{[X,Y]}B = e^B[\![\mathbb{X}, \mathbb{Y}]\!]. \tag{2.2.47}$$

□

Theorem 2.2.10. The β-transformation is not a symmetry of the Courant bracket.

Proof. Let $\mathbb{X} = (X + \xi)$, $\mathbb{Y} = (Y + \eta) \in \Gamma(E)$. Then, we have that $e^\beta\mathbb{X} = (X + \beta\cdot\xi) + \xi$ and $e^\beta\mathbb{Y} = (Y + \beta\cdot\eta) + \eta$. Taking the Courant bracket of the transformed generalised vectors reads:

$$[\![e^\beta\mathbb{X}, e^\beta\mathbb{Y}]\!] = [\![\mathbb{X}, \mathbb{Y}]\!] + \begin{pmatrix} [\beta\cdot\xi, Y] + [X, \beta\cdot\eta] + [\beta\cdot\xi, \beta\cdot\eta] \\ \mathcal{L}_{\beta\cdot\xi}\eta - \mathcal{L}_{\beta\cdot\eta}\xi - \frac{1}{2}d(i_{\beta\cdot\xi}\eta - i_{\beta\cdot\eta}\xi) \end{pmatrix}. \tag{2.2.48}$$

Comparing this with

$$e^\beta[\![\mathbb{X}, \mathbb{Y}]\!] = [\![\mathbb{X}, \mathbb{Y}]\!] + \beta\cdot\left[\mathcal{L}_X\eta - \mathcal{L}_Y\xi - \frac{1}{2}d(i_X\eta - i_Y\xi)\right], \tag{2.2.49}$$

we see that they do not match in general. One easy way to explicitly confirm that is to look at the second component of 2.2.48, use Cartan's "magic" formula A.1.31 and obtain the sum of two terms that do not cancel in general. □

Remark. Thus, we have shown that two subgroups of $O(n, n)$ out of three are symmetries of the Courant bracket. Hence, we have shown the following theorem.

Theorem 2.2.11. The symmetries of the Courant bracket are given by:

$$G_{gen\,diff} = GL(n, \mathbb{R}) \rtimes G_{B,\,closed}, \tag{2.2.50}$$

where $G_{B,\,closed}$ is the subgroup 1. in definition 2.2.5, with $dB = 0$.

Remark. The above notation (semidirect product) means that any of the elements of $G_{gen\,diff}$ can be written as:

$$e^B \begin{pmatrix} A & \\ & (A^T)^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \\ B & 1 \end{pmatrix} \begin{pmatrix} A & \\ & (A^T)^{-1} \end{pmatrix} = \begin{pmatrix} A & \\ BA & (A^T)^{-1} \end{pmatrix}. \tag{2.2.51}$$

2.3 Some Formal and Subtle Aspects

In this section, we try to give an overview of what generalised geometry formally is. We introduce the concepts of Courant algebroid and Leibniz algebroid, and study the local nature of the generalised tangent bundle E. The former two are introduced for completeness, but they will not play a role in the rest of the dissertation. The latter is an analysis which will reveal important connections with the physics in chapter 5, section 2. To improve the readability and allow for a large number of comments, we will insert all the discussion about the generalised bundle into a single, long, remark. For the first part of the section, we follow [13] and [39]. For the second one, we refer the reader to [15], [40], [33] and [34].

Definition 2.3.1. A *Courant algebroid* is a collection of objects:

- A vector bundle, E.
- A nondegenerate symmetric bilinear form, $\mathcal{I}$.
- A smooth bundle map $\pi_{TM} : E \to T$.
- A skew-symmetric bracket, $[\![\cdot, \cdot]\!]$.

This induces a natural differential operator $d : \mathcal{F}(M) \to \Gamma(E)$ defined by $\mathcal{I}(df, \mathbb{X}) = \frac{1}{2}\pi_{TM}(\mathbb{X})f$ for any $f \in \mathcal{F}(M), \mathbb{X} \in \Gamma(E)$. The above objects must also satisfy:

1. $\pi_{TM}([\![\mathbb{X}, \mathbb{Y}]\!]) = [\pi_{TM}(\mathbb{X}), \pi_{TM}(\mathbb{Y})]$
2. $[\![[\![\mathbb{X}, \mathbb{Y}]\!], \mathbb{Z}]\!] + [\![[\![\mathbb{Z}, \mathbb{X}]\!], \mathbb{Y}]\!] + [\![[\![\mathbb{Y}, \mathbb{Z}]\!], \mathbb{X}]\!] = \frac{1}{3}d\big(\mathcal{I}([\![\mathbb{X}, \mathbb{Y}]\!], \mathbb{Z}) + \mathcal{I}([\![\mathbb{Z}, \mathbb{X}]\!], \mathbb{Y}) + \mathcal{I}([\![\mathbb{Y}, \mathbb{Z}]\!], \mathbb{X})\big)$
3. $[\![\mathbb{X}, f\mathbb{Y}]\!] = f[\![\mathbb{X}, \mathbb{Y}]\!] + \pi_{TM}(\mathbb{X})[f]\mathbb{Y} - \mathcal{I}(\mathbb{X}, \mathbb{Y})df.$
4. $\mathcal{I}(df, dg) = 0 \quad \forall f, g \in \mathcal{F}(M)$

5. $\pi_{TM}(\mathbb{X})(\mathcal{I}(\mathcal{Y},\mathcal{Z}) = \mathcal{I}([\![\mathbb{X},\mathbb{Y}]\!] + d\mathcal{I}(\mathbb{X},\mathbb{Y}),\mathbb{Z}) + \mathcal{I}([\![\mathbb{X},\mathbb{Z}]\!] + d\mathcal{I}(\mathbb{X},\mathbb{Z}),\mathbb{Y}) \qquad \forall \mathbb{X},\mathbb{Y},\mathbb{Z} \in \Gamma(E)$.

Theorem 2.3.1. The collection $(E, \mathcal{I}, \pi_{TM}.[\![\cdot,\cdot]\!])$, where the symbols correspond to the objects defined in the previous section, form a Courant algebroid (with the induced differential operator being the usual exterior derivative d).

Proof. Properties 1., 2. and 3. have already been proven as properties of the Courant bracket. Property 4. clearly follows from the form of the natural canonical metric. For property 5., we refer to [13]. □

Definition 2.3.2. A *Leibniz algebroid* is a collection of objects:

- A vector bundle, E.
- A bundle ma $\pi_{TM} : R \to T$.
- A bilinear map $L : \Gamma(E) \otimes \Gamma(E) \to \Gamma(E)$.

They must satisfy:

1. $L_{\mathbb{X}}(L_{\mathbb{Y}}\mathbb{Z}) = L_{L_{\mathbb{X}}\mathbb{Y}}\mathbb{Z} + L_{\mathbb{Y}}(L_{\mathbb{X}}\mathbb{Z})$.
2. $L_{\mathbb{X}}(f\mathbb{Y}) = fL_{\mathbb{X}}\mathbb{Y} + \pi_{TM}(\mathbb{X})[f]\mathbb{Y}$.

Theorem 2.3.2. The collection $(E, \pi_{TM}, L_{(\cdot)}(\cdot))$, where the symbols correspond to the objects defined in the previous section, form a Leibniz algebroid

Proof. We showed bilinearity, 1. and 2. in the previous section (see theorem 2.2.2). □

Remark. So far we have always be been working with $E = TM \oplus T^*M$. This is a useful simplification, which we will show to be valid, but it is not rigorous. We now try to explain what E really is, its isomorphism with $TM \oplus T^*M$ and the role of the B-transformation.
Consider a manifold M with open covering given by $\{U_i\}$. Let $\mathbb{X}$ be a global section of $TM \oplus T^*M$, i.e. a sum of a vector field X and a one form ξ defined smoothly over the whole manifold. For a point p inside some patch U_i, we can write $\mathbb{X}$ as:

$$\mathbb{X}(p) = X_i(p) + \xi_i(p), \tag{2.3.1}$$

where the roman subscript specifies the patch chosen. For a point $p \in U_i \cap U_j$, we have that:

$$X_i(p) + \xi_i(p) = X_j(p) + \xi_j(p). \tag{2.3.2}$$

This holds for all patches and all intersections, and it can be used as a way to define $TM \oplus T^*M$ from a local point of view.
Analogously, we now try to define E from a local point of view. We start with $\mathbb{X}$ defined only in some patch U_i, as before:

$$\mathbb{X}(p) = X_i(p) + \xi_i(p) \qquad \forall p \in U_i. \tag{2.3.3}$$

But for the patching, we make a small modification. We have shown that, in addition to diffeomorphisms, the Courant bracket is invariant under B-transformations with B closed. Therefore, we try to include them into our patching rules. And since we are working at the level of patches, we assume that the globally closed B-form is (locally) exact. With this in mind, we define E by requiring that, in the intersection between U_i and another patch U_j, the following holds:

$$X_i(p) + \xi_i(p) = X_j(p) + \xi_j(p) - i_{X_j(p)} d\Lambda_{ij}(p). \tag{2.3.4}$$

As mentioned above, $\{\Lambda_{ij}\}$ is a set of one forms defined in the intersections, which locally satisfy $B = d\Lambda_{ij}$. We also see (by simply moving $-i_{X_j(p)} d\Lambda_{ij}(p)$ to the lhs) that $\Lambda_{ji} = -\Lambda_{ij}$. Everything so far is coordinate-free. But if we were to work in components, the above would read:

$$X_i^\mu = (A_{ij})^\mu{}_\nu (X_j)^\nu \qquad \text{(j is not summed over)}, \tag{2.3.5}$$
$$\xi_{i\mu} = (A_{ij})_\mu{}^\nu (\xi_j)_\nu + (X_j)^\nu \partial_{[\nu} (\Lambda_{ij})_{\mu]} \qquad \text{(j is not summed over)}, \tag{2.3.6}$$

where A is the usual Jacobian matrix for diffeomorphisms. We emphasize that the patching rules for the vector part define an ordinary section of TM, while the form part does not correspond to a section of T^*M. We will shortly see how to deal with this. But before, one needs to consider a second condition, in addition to 2.3.4, regarding in triple intersections. In an intersection between three patches $U_i \cap U_j \cap U_k$, the one-form part of $\mathbb{X}$ reads (according to 2.3.4):

$$\xi_i(p) = \xi_j(p) - i_{X_j(p)} d\Lambda_{ij}(p) = \xi_k(p) - i_{X_k(p)} d\Lambda_{ik}(p) = \tag{2.3.7}$$
$$\big(\xi_j(p) - i_{X_j(p)} d\Lambda_{kj}(p)\big) - i_{X_k(p)} d\Lambda_{ik}(p), \tag{2.3.8}$$

where in the last step we wrote $\xi_k(p)$ in terms $\xi_j(p)$. Since the vector part is unchanged from patch to patch, the above yields:

$$i_{v_{(i)}}\left(d\Lambda_{ij} + d\Lambda_{jk} + d\Lambda_{ki}\right) = 0, \tag{2.3.9}$$

which implies

$$\Lambda_{ij} + \Lambda_{jk} + \Lambda_{ki} = d\Lambda_{ijk}, \tag{2.3.10}$$

for some function Λ_{ijk}. It is common to define $g_{ijk} = \exp(i\Lambda_{ijk})$, so that the above condition becomes[5]

$$\Lambda_{ij} + \Lambda_{jk} + \Lambda_{ki} = -ig_{ijk}^{-1} dg_{ijk} \quad \text{(no sums)}. \tag{2.3.11}$$

This completes the construction of E from the local prospective. It is definitely different from $TM \oplus T^*M$, because of how the form part is patched. However, it is possible to construct an isomorphism between the two. To do so, we introduce a set of two-forms defined patchwise: $\{B_i\}$, where B_i is defined on U_i. We define them so that they obey:

$$B_i - B_j = d\Lambda_{ij} \tag{2.3.12}$$

in all overlaps. This clearly respects the antisymmetry of Λ_{ij}. The collection $\{B_i\}$ defines a global closed three form H, where H restricted to U_i is given by dB_i. Specifically, $H_i = H_j$ in the overlap $U_i \cap U_j$ thanks to 2.3.12. With these new ingredients, 2.3.4 now reads:

$$X_i(p) + \xi_i(p) = X_j(p) + \xi_j(p) - i_{X_j(p)} d\Lambda_{ij}(p) = X_j(p) + \xi_j(p) - i_{X_i(p)} B_i + i_{X_j(p)} B_j, \tag{2.3.13}$$

where we used $X_i(p) = X_j(p)$. It is now clear that we can define a new section $\mathbb{X}'$, given by $X_i(p) + \xi_i(p) + i_{X_i(p)} B_i$ on U_i, which will take the same form in all other patches:

$$X_i(p) + \xi_i(p) + i_{X_i(p)} B_i = X_j(p) + \xi_j(p) + i_{X_j(p)} B_j, \tag{2.3.14}$$

for $p \in U_i \cap U_j$. Thus, we obtained a section of $TM \oplus T^*M$ defined on a single patch, which matches with other patches on the intersections, just as 2.3.2. We have therefore "constructed" $TM \oplus T^*M$.

[5] This defines a connective structure on a gerbe. For more details, in the context that we are studying, see [41].

Summarising, we see that there exist an isomorphism between E and $TM \oplus T^*M$, which, however, relies on the choice of $\{B_i\}$. Hence, it is not canonical. We note that $\mathbb{X}'$ is related by $\mathbb{X}$ by a B-transformation, i.e. $\mathbb{X}' = \exp(-B)\mathbb{X}$. Sometimes E is referred as the "twisted" version of $TM \oplus T^*M$, and the procedure we have just performed is known as "untwisting" E.
We labelled this derivation as a remark because although being really instructive, it is not very formal. To be completely rigorous, one should consider the isomorphism not at the level of E and $TM \oplus T^*M$, but at the level of algebroids (introduced earlier in this section). One needs to introduce the "twisted" version of the operations described in the previous section. Given a three form H, and $\mathbb{X} = (X + \xi), \mathbb{Y} = (Y + \eta)$, the "twisted" Dorfman derivative is given by

$$L^H_{\mathbb{X}}\mathbb{Y} = \mathcal{L}_X Y + \mathcal{L}_X \eta - i_Y d\xi - i_Y i_X H, \tag{2.3.15}$$

and the "twisted" Courant bracket is given by:

$$[\![\mathbb{X}, \mathbb{Y}]\!]_H = [\![\mathbb{X}, \mathbb{Y}]\!] + i_X i_Y H. \tag{2.3.16}$$

Then, the isomorphism defined by H is the following: the "twisted" generalised vector bundle E together with the "untwisted" operations is isomorphic to the "untwisted" $TM \oplus T^*M$ together with the "twisted" operations (that we just defined).[42]
The above discussion on the local nature of E will be relevant for section 5.2, where we will show how the NSNS sector of type II supergravity resembles the above construction. The parenthesis on isomorphisms will appear again in section 1 of chapter 6, where we will work with twisted structures.

2.4 Generalised Metric

In this section, we find the form for a generalised metric living on E. The construction will be motivated by the analogy with ordinary geometry. In such framework, the structure group is naturally $GL(n, \mathbb{R})$, which is reduced to its maximal compact subgroup $O(n)$ by introducing a metric. In generalised geometry, the natural starting point (given the natural canonical metric) is an $O(n, n)$ structure group. Its maximal compact subgroup is $O(n - p, p) \times O(p, n - p)$. We therefore want to construct an object which reduces the structure group from $O(n, n)$ to $O(n - p, p) \times O(p, n - p)$, and then call it the generalised metric.

The relevant references are [13], [34] and [35].

Theorem 2.4.1. The maximal compact subgroup of $O(n,n)$ is $O(n-p,p) \times O(p,n-p)$.

Proof. This can be found in [13], and we do not prove it here. □

Theorem 2.4.2. Reducing the structure group $O(n,n)$ to its maximal compact subgroup is equivalent to splitting the generalised tangent bundle into two n-dimensional subbundles C_+, C_- with metric of opposite signature.

Proof. We anticipate that this proof is quick and sketchy.
We define the two subbundles starting from their frames. We define the two sets of "ordinary" frames $\{\hat{E}^+_a\}$ and $\{\hat{E}^-_{\bar{a}}\}$ $(a, \bar{a} = 1, ..., n)$ such that

$$\mathcal{I}(\hat{E}^+_a, \hat{E}^+_b) = \eta_{ab}, \qquad \mathcal{I}(\hat{E}^-_{\bar{a}}, \hat{E}^-_{\bar{b}}) = -\eta_{\bar{a}\bar{b}}, \qquad \mathcal{I}(\hat{E}^+_a, \hat{E}^-_{\bar{b}}) = 0, \tag{2.4.1}$$

where $\eta_{ab}, \eta_{\bar{a}\bar{b}}$ are two (identical) flat metrics with signature $(p, n-p)$. From this, we build a set of frames for the generalised tangent bundle as

$$\hat{E}_A = \begin{cases} \hat{E}^+_a & \text{for } A = a \\ \hat{E}^-_{\bar{a}} & \text{for } A = \bar{a} + n \end{cases} . \tag{2.4.2}$$

This construction makes the $O(n-p,p) \times O(p,n-p)$ evident: the first factor acts on $\hat{E}^+_a$ and the second on $\hat{E}^-_{\bar{a}}$, i.e. $O(n-p,p) \times O(p,n-p)$ simply rotates each set of frames. Letting $\{\hat{E}^+_a\}$ specify the subbundle C_+ and $\{\hat{E}^-_{\bar{a}}\}$ the subbundle C_-, we have that $E = C_+ \oplus C_-$. □

Remark. To continue the analogy with ordinary geometry, this proof should be compared with theorem 2.1.4.
Generalised frames will be developed more in detail in chapter 6.

Definition 2.4.1. We define the *metric-projection operators* Π_+ and Π_- as

$$\Pi_\pm = \frac{1}{2}(\delta_M{}^N \pm P_M{}^N) \qquad \Longleftrightarrow \Pi_\pm = \frac{1}{2}(\mathbb{1}_{2n} \pm P) \tag{2.4.3}$$

Theorem 2.4.3. $\Pi_\pm$ are well-defined projection operators if they satisfy:

$$P^2 = \mathbb{1}_{2n}. \tag{2.4.4}$$

Proof. Assuming that $P^2 = \mathbb{1}_{2n}$, we have:

$$\Pi_+ + \Pi_- = \mathbb{1}_{2n}, \qquad \Pi_\pm^2 = \frac{1}{4}(\mathbb{1}_{2n} \pm 2P + P^2) = \frac{1}{2}(\mathbb{1}_{2n} \pm P) = \Pi_\pm \tag{2.4.5}$$

$$\Pi_+\Pi_- = \frac{1}{4}(\mathbb{1}_{2n} - P^2) = 0 = \Pi_-\Pi_+. \tag{2.4.6}$$

Thus, $\Pi_\pm$ are projection operators. □

Remark. We have just constructed two projectors, which take generalised vectors into two disjoint subbundles of the generalised tangent bundle. We have seen that reducing the structure group of the generalised tangent bundle to its maximal compact subgroup splits it into two subbundles. The motivation for proceeding in the study of the projectors is the hope that they project generalised vectors exactly into those subbubdles.

Definition 2.4.2. We define the *generalised metric* as

$$\mathcal{G}_{MN} = P_M{}^K \mathcal{I}_{KN} \iff \mathcal{G} = P\mathcal{I}, \tag{2.4.7}$$

satisfying $\mathcal{G}_{MN} = \mathcal{G}_{NM}$.

Theorem 2.4.4. The generalised metric has the form:

$$\begin{pmatrix} g - Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix}, \tag{2.4.8}$$

where B is antisymmetric and g is symmetric.

Proof. Since this is a crucial result, we provide a constructive proof. We let the generalised metric take the generic form

$$\mathcal{G} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tag{2.4.9}$$

with all of its components be invertible. The condition 2.4.4 in terms of the generalised metric can be written as

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ac + ba & ad + b^2 \\ c^2 + da & cd + db \end{pmatrix} = \frac{1}{4} \cdot \begin{pmatrix} 0 & \mathbb{1}_n \\ \mathbb{1}_n & 0 \end{pmatrix}. \tag{2.4.10}$$

The symmetry condition implies $d = d^T$, $a = a^T$, $c = b^T$. We define $d = g^{-1}$, where g is symmetric. We also define $B = d^{-1}c$. The bottom-right equation of 2.4.10 reads:

$dc^T = (cd)^T = -cd$. Using this and the fact that d is symmetric, we can quickly see that B is antisymmetric:

$$dB^T d = (dBd)^T = (cd)^T = -cd = -dBd. \tag{2.4.11}$$

The bottom-left equation of 2.4.10 in terms of B yields:

$$dBdB + da = \frac{1}{4} \cdot \mathbb{1}_n \implies a = \frac{1}{4} d^{-1} - BdB = \frac{1}{4} g - Bg^{-1}B. \tag{2.4.12}$$

Also,

$$b = c^T = (dB)^T = B^T d = -Bd = -Bg^{-1} \implies c = g^{-1} b. \tag{2.4.13}$$

Hence, by simply scaling $g \to 2g$, we obtain

$$\mathcal{G} = \frac{1}{2} \begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1} \end{pmatrix}. \tag{2.4.14}$$

$\square$

Remark. Now, we introduce generalised structures on the manifold inspired by the constructions in ordinary complex geometry that we presented in the previous chapter. We will restrict ourselves to the Riemannian metrics for the rest of this section.

Definition 2.4.3. A *generalised almost complex structure* is a map $\mathcal{J} : E \to E$, which satisfies:

$$\begin{aligned} \pi(\mathcal{J}\mathbb{X}) &= \pi(\mathbb{X}) \\ \mathcal{J}^2 &= -\mathbb{1}_{2n} \\ \mathcal{I}(\mathcal{J}\mathbb{X}, \mathcal{J}\mathbb{Y}) &= \mathcal{I}(\mathbb{X}, \mathbb{Y}). \end{aligned} \tag{2.4.15}$$

The underlying manifold is called an *generalised almost complex manifold.*

Remark. The first condition means that the map preserves the bundle structure. The second one is exactly the requirement we imposed for an almost complex structure (cf. definition 1.1.10). The third one is the hermiticity condition of the natural canonical metric wrt the generalised almost complex structure (cf. definition 1.2.2).
Just as in the case of ordinary geometry (see theorem 2.1.1), a generalised almost complex structure defines two subbundles: $E^{\pm} \subset (E)^{\mathbb{C}}$, with fibers being the $(\pm i)$ eigenspaces defined by J at each point. The reduction of the associated structure group to $GL(n, \mathbb{C})$ also follows by the same argument as in the ordinary case.

Theorem 2.4.5. The structure group of a generalised almost complex manifold is $U(n/2, n/2)$.

Proof. The proof follows directly from a result in group theory, which we are not going to prove:

$$U(n/2, n/2) = O(n, n) \cap GL(n, \mathbb{C}), \tag{2.4.16}$$

provided that n is even. This can be found in [13], for instance.
$O(n, n)$ is the structure group that preserves the natural canonical metric, and $GL(n, \mathbb{C})$ is the structure group associated with the generalised complex structure (defined by its eigenspaces). $\square$

Example 2.4.1. Let M be an almost complex manifold with almost complex structure J. Then,

$$(\mathcal{J}_J)_{MN} = \begin{pmatrix} -J & 0 \\ 0 & J^T \end{pmatrix}_{MN} \tag{2.4.17}$$

define the components of a generalised complex structure. This is true since:

$$(J^2)_m{}^n = J_t{}^n J_m{}^t = -\delta_m{}^n, \tag{2.4.18}$$

$$((J^T)^2)_m{}^n = J_m{}^t J_t{}^n = -\delta_m{}^n. \tag{2.4.19}$$

Example 2.4.2. Let M be a manifold admitting a non-degenerate 2-form Ω. Then, it is clear that

$$(\mathcal{J}_\Omega)_{MN} = \begin{pmatrix} 0 & \Omega^{-1} \\ -\Omega & 0 \end{pmatrix}_{MN} \tag{2.4.20}$$

define the components of a generalised complex structure.

Definition 2.4.4. An generalised almost complex structure is called *integrable* (or simply, we remove the "almost") if the Courant bracket of two holomorphic generalised vectors is an holomorphic generalised vector.

Remark. Note how this definition is analogous to definition 1.1.11, but with the Courant bracket playing the role of the Lie bracket in ordinary geometry (as we anticipated).

Definition 2.4.5. Let M be a manifold that admits a two commuting generalised complex structures, $\mathcal{J}_1$ and $\mathcal{J}_2$. Then, if $\mathcal{G}' = -\mathcal{J}_1\mathcal{J}_2$ is a positive-definite metric on E, the pair $(\mathcal{J}_1, \mathcal{J}_2)$ is called a *generalised Kähler structure* and M is a *generalised Kähler manifold.*

Example 2.4.3. This example justifies the above definition. Let M be a Kähler manifold with almost complex structure J, Kähler form Ω and Kähler metric g. Let $\mathcal{J}_J$ and $\mathcal{J}_\Omega$ be the two generalised complex structures constructed as in the previous two examples. Then, we have that

$$\mathcal{G}' = -\mathcal{J}_J\mathcal{J}_\Omega = -\mathcal{J}_\Omega\mathcal{J}_J = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix}. \tag{2.4.21}$$

This is a positive-definite metric on E, showing that a Kähler manifold is a special case of a generalised Kähler manifold. Note that 2.4.21 follows from:

$$-\mathcal{J}_J\mathcal{J}_\Omega = \begin{pmatrix} 0 & J\Omega^{-1} \\ J^T\Omega & 0 \end{pmatrix} \quad , \quad -\mathcal{J}_\Omega\mathcal{J}_J = \begin{pmatrix} 0 & -\Omega^{-1}J^T \\ -\Omega J & 0 \end{pmatrix}, \tag{2.4.22}$$

together with the following calculations in components:

$$\begin{aligned} (J^T\Omega)_{mn} &= J_m{}^t\Omega_{nt} = g_{nm}, \\ -(\Omega J)_{mn} &= -\Omega_{tm}J_n{}^t = \Omega_{mt}J_n{}^t = g_{mn} = g_{nm}, \\ (J\Omega^{-1})^{mn} &= J_t{}^m\Omega^{nt} = g^{mn}, \\ -(\Omega^{-1}J^T)^{mn} &= -\Omega^{tm}J_t{}^n = \Omega^{mt}J_t{}^n = g^{nm} = g^{mn}. \end{aligned} \tag{2.4.23}$$

Theorem 2.4.6. By applying a B-transformation to the generalised Kähler metric, we obtain $\frac{1}{2}\mathcal{I}^{-1}\mathcal{G}$.

Proof. Using 2.2.12, we have that

$$e^B\mathcal{G}'e^{-B} = \begin{pmatrix} -g^{-1}B & g^{-1} \\ g - Bg^{-1}B & Bg^{-1} \end{pmatrix} = \frac{1}{2}\mathcal{I}^{-1}\mathcal{G}. \tag{2.4.24}$$

□

Theorem 2.4.7. A generalised Kähler manifold has a $U(n) \times U(n)$- structure.

Proof. The result is quickly obtained by considering that a generalised Kähler manifold has a $U(n/2, n/2)$-structure associated with the generalised complex structures and a $O(n) \times O(n)$ strucure coming from the generalised metric. The intersection is given by ([13]):

$$U(n/2, n/2) \cap O(n) \times O(n) = U(n/2) \times U(n/2). \tag{2.4.25}$$

□

Part II

The Physics

Strings and Superstrings

3

This is a very brief review of string theory and superstring theory, which is aimed at introducing the conventions used and presenting the known results that will be used in this dissertation. We assume that the reader is already somewhat familiar with bosonic string theory and very familiar with Quantum Field Theory (QFT).

In section 1, we introduce the classical bosonic string. We present the Polyakov action, describe its properties and derive the dynamics of classical closed bosonic strings.

Section 2 shows how to quantise the bosonic string via canonical quantisation, in analogy with QFT. This section is only illustrative, and should be regarded as a preparation for the following one.

In section 3, we quantise bosonic string theory via light-cone quantisation. The massless spectrum of closed strings is derived and the critical number of dimensions is determined.

Section 4 presents classical superstring theory (RNS formalism) and the basic features of its dynamics.

In section 5, we describe the quantisation of closed superstrings. The massless spectrum of type IIA/B theories is derived.

String theory is such a vast subject that we feel the need to list the many fundamental concepts that we do not treat in this quick review. What you will *not* find in the following pages is: details on conformal field theories, operator product expansions, path integrals, partition functions, open strings, branes, interactions, vertex operators, scattering amplitudes, heterotic strings, type I supestring theory. These topics were (hopefully always) deliberately avoided to keep this work self-contained, reasonable in length and tailored to the applications discussed in the rest of the dissertation.

For extensive and complete discussions about the topics that we are about to present we refer to [21], [43], [44], [45], [46], [47] and [48] which inspired this review.

3.1 Classical Bosonic String Theory

We now study string theory at the classical level. By analogy with the particle case, we introduce the action for strings moving in space time (Nambu-Goto). We then find an equivalent action, the Polyakov action, which is more convenient for quantisation. We study its equations of motion, symmetries and associated conserved quantities. Finally, we provide boundary conditions for closed strings, and give a general solution to the equations of motion.
For this section, we followed all of the sources mentioned above.

We take the Minkowski metric to be $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. A particle sweeps out a line in Minkowski space, called worldline. A string is an extended one dimensional object. Thus, it sweeps a surface, called worldsheet. This can be parametrised by two parameters. The first one is the usual time-like coordinate, τ, which is the only one we would need if we were to consider a particle. Since strings are extended objects, we need to choose a second space-like coordinate σ. Let the string have length l, then $\sigma \in [0, l]$ simply tells us where we are along the string. We anticipate that we will be interested in closed strings, i.e. those that do not have separated ends (one can go around them).
A (point-)particle's trajectory in Minkowski space is specified by $X^\mu(\tau)$, where $\mu = 0, 1, 2, 3$. We can think of a string as a collection of point(-particle)s glued together, so it follows that its motion will be described by $X^\mu(\tau, \sigma)$, where σ labels which point of the string we are looking at.
The action for a particle moving in Minkowski space is given by the integral of its line element:

$$S_{particle}[X(\tau)] = -m \int ds = \int d\tau \sqrt{X^\mu(\tau) X^\nu(\tau) \eta_{\mu\nu}} \tag{3.1.1}$$

When trying to generalise to a string, we follow the above considerations and replace

1. The single integral with two integrals
2. The line element with the area element

In doing so, we obtain the Nambu-Goto action, which reads:

$$S_{NG}[X(\tau, \sigma)] = -\frac{1}{2\pi\alpha'} \int d\sigma d\tau \sqrt{-\det(h_{ab})}, \tag{3.1.2}$$

where we used the compact notation

$$h_{ab} = \partial_a X^\mu(\tau,\sigma)\partial_b X^\nu(\tau,\sigma)\eta_{\mu\nu}, \quad a,b = 0,1 \quad \text{and} \quad \partial_a = (\partial_\tau, \partial_\sigma).$$

We will also often use the shorthands $X^2 = X^\mu X_\mu = X^\mu X^\nu \eta_{\mu\nu}$ and $\sigma^a = (\tau, \sigma)$. From now on, we will keep the number of dimensions of space-time general. We will denote it by D, so that $\mu, \nu = 0, ..., D-1$.

The square root in the Nambu-Goto action is not suitable for path integral calculations. Thus, we introduce the equivalent Polyakov action, which is given by:

$$S_P[X,\gamma] = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \sqrt{-\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu, \tag{3.1.3}$$

where $\gamma = \det(\gamma_{ab})$. The equivalence is quickly shown considering the equations of motion obtained by varying γ_{ab}. They yield:

$$T_{ab} = h_{ab} - \frac{1}{2}\gamma_{ab}\gamma^{cd}h_{cd} = 0 \implies h_{ab}(-h)^{-\frac{1}{2}} = \gamma_{ab}(-\gamma)^{-\frac{1}{2}}, \tag{3.1.4}$$

where we defined the stress energy tensor T_{ab} and $h = \det(h_{ab})$. Substituting the second identity into the action yields back the Nambu-Goto action. Varying X^μ we get the other set of equations of motion:

$$\partial_a(\sqrt{-\gamma}\gamma^{ab}\partial_b X^\mu) = \sqrt{-\gamma}\nabla^2 X^\mu = 0. \tag{3.1.5}$$

These are non-linear equations, very difficult to solve. In order to make progress, we consider the symmetries of the Polyakov action. They are:

1. **Poincare**. Under this symmetry, fields transform as:

$$X^\mu \to \Lambda^\mu{}_\nu X^\nu + a^\mu, \quad \text{with} \quad \Lambda^\mu{}_\nu \in SO(1, D-1), \quad \text{and} \quad \gamma^{ab} \to \gamma^{ab}. \tag{3.1.6}$$

2. **Reparametrisation**. Under this symmetry, fields transform as:

$$\sigma^a \to \tilde{\sigma}^a(\sigma^a), \quad \gamma_{ab} \to \frac{\partial\sigma^c}{\partial\tilde{\sigma}^a}\frac{\partial\sigma^d}{\partial\tilde{\sigma}^b}\gamma_{cd} \quad \text{and} \quad X^\mu \to X^\mu. \tag{3.1.7}$$

3. **Weyl**. Under this symmetry, fields transform as:

$$\gamma_{ab} \to \Omega^2(\sigma^a)\gamma_{ab} \quad \text{and} \quad X^\mu \to X^\mu. \tag{3.1.8}$$

Using our gauge freedom from 2. and 3., we can set $\gamma_{ab} = \eta_{ab}$.[1] Then, the action simplifies to

$$S = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \eta_{\alpha\beta} \partial^\alpha X^\mu \partial^\beta X_\mu = \frac{1}{4\pi\alpha'} \int d\sigma d\tau (\dot{X}^2 - X'^2), \tag{3.1.9}$$

with EoM for X^μ that become linear:

$$\ddot{X} - X'' = 0. \tag{3.1.10}$$

However, we still need to ensure that the vanishing of the stress energy tensor (3.1.4) is satisfied. This gives two constraints:

$$\begin{aligned} \dot{X} \cdot X' &= 0, \\ \dot{X}^2 + X'^2 &= 0. \end{aligned} \tag{3.1.11}$$

Now we make a quick analysis of the classical quantities associated with this action. The canonical momentum conjugate to X^μ is:

$$\Pi^\mu = \frac{\delta S}{\delta \dot{X}^\mu} = \frac{1}{2\pi\alpha'} \dot{X}^\mu. \tag{3.1.12}$$

As we saw, Poincare is a global symmetry of 3.1.9, thus it leads to conserved quantities. Infinitesimally, it takes the form $\delta X^\mu = \epsilon^\mu_\nu X^\nu + \epsilon^\mu$, with $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$. The Noether currents associated to this symmetry are:

$$P^\mu_\alpha = \frac{1}{2\pi\alpha'} \partial_\alpha X^\mu, \tag{3.1.13}$$

$$J^{\mu\nu}_\alpha = \frac{1}{2\pi\alpha'} (X^\mu \partial_\alpha X^\nu - X^\nu \partial_\alpha X^\mu), \tag{3.1.14}$$

where the first one corresponds to translations and the second one to Lorentz transformations. Note that $P^\mu_0 = \Pi^\mu$, as expected. The Hamiltonian is then

$$H = \int_0^l d\sigma (X^\mu \Pi_\mu - \mathcal{L}) = \frac{1}{4\pi\alpha'} \int d\sigma (\dot{X}^2 + X'^2). \tag{3.1.15}$$

Coming back to the dynamics, we have to solve 3.1.10, subject to the two conditions 3.1.11. In order to do so, we first provide the boundary conditions for the closed string:

$$\begin{aligned} X^\mu(\tau, l) &= X^\mu(\tau, 0), \\ \partial^\sigma X^\mu(\tau, l) &= \partial^\sigma X^\mu(\tau, 0), \\ \gamma_{ab}(\tau, l) &= \gamma_{ab}(\tau, 0). \end{aligned} \tag{3.1.16}$$

[1]We are (reasonably) assuming that there are no topological obstruction.

We point out a small detail just for completeness. When varying the action to obtain 3.1.10, we discard a boundary term after integration by parts. These boundary conditions ensure that such boundary term really vanishes.
The final step before solving the dynamics is to define "light-cone" parametrisation for the world-sheet, i.e.

$$\sigma^+ = \tau + \sigma \quad \text{and} \quad \sigma^- = \tau - \sigma. \tag{3.1.17}$$

This choice is common to [44], [21], [46] and [45], while in [43] it is τ that changes sign. Accordingly, we define $\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma)$, which lead to:

$$\text{EoM:} \qquad \partial_+\partial_- X^\mu = 0, \tag{3.1.18}$$

$$\text{Constraints:} \quad \begin{cases} \partial_+ X^\mu \partial_+ X_\mu = 0, \\ \partial_- X^\mu \partial_- X_\mu = 0. \end{cases} \tag{3.1.19}$$

With these definitions, the most general solution to the EoM consistent with the boundary conditions reads:

$$X^\mu(\tau,\sigma) = X^\mu_L(\sigma^+) + X^\mu_R(\sigma^-), \tag{3.1.20}$$

with

$$X^\mu_L(\sigma^+) = \frac{1}{2}x^\mu + \frac{\pi}{l}\alpha' p^\mu \sigma^+ + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n \exp(-i2\pi n\sigma^+/l), \tag{3.1.21}$$

$$X^\mu_R(\sigma^-) = \frac{1}{2}x^\mu + \frac{\pi}{l}\alpha' p^\mu \sigma^- + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n \exp(-i2\pi n\sigma^-/l). \tag{3.1.22}$$

The subscript L stands for left-moving, while R stands for right-moving. We can regard this as a simple Fourier expansion, justified by the periodic boundary conditions, where α^μ_n and $\tilde{\alpha}^\mu$ are the Fourier coefficients. The choice of the symbol p^μ for the coefficient of $\sigma^{+/-}$ is not a coincidence: it really is the total momentum of the string. Substituting this expansion into the expression for the momentum 3.1.13 (with $\alpha = 0$) and integrating along the string yields exactly p^μ. The same is true for x^μ, which is the classical centre-of-mass position.
This is the most general ansatz, as found in [45]. It agrees with [44] and [46], where $l = 2\pi$, and with [21], where $l = \pi$.
Having found the most general solution to the EoM, we now need to impose the constraints.

To simplify the calculations, we define $\alpha_0^\mu = \tilde{\alpha}_0^\mu = \sqrt{\frac{\alpha'}{2}}p^\mu$. The constraints then read:

$$\sum_n L_n \exp(-i2\pi n\sigma^-/l) = 0 \implies L_n = 0, \tag{3.1.23}$$

$$\sum_n \tilde{L}_n \exp(-i2\pi n\sigma^+/l) = 0 \implies \tilde{L}_n = 0. \tag{3.1.24}$$

where we defined

$$L_n = \frac{1}{2}\sum_m \alpha_{n-m}\cdot\alpha_m, \tag{3.1.25}$$

$$\tilde{L}_n = \frac{1}{2}\sum_m \tilde{\alpha}_{n-m}\cdot\tilde{\alpha}_m. \tag{3.1.26}$$

Since $p^2 = -M^2$ we have the following two equations (level matching conditions):

$$\frac{4}{\alpha'}\sum_{n>0}\alpha_n\cdot\alpha_{-n} = M^2 = \frac{4}{\alpha'}\sum_{n>0}\tilde{\alpha}_n\cdot\tilde{\alpha}_{-n} \tag{3.1.27}$$

3.2 Canonical Quantisation of the Bosonic String

In this section, we quantise bosonic string theory following the canonical procedure from QFT. We identify two sets of harmonic oscillators and construct the Fock space. Then, we show the existence of negative-norm states and describe how to impose the physical condition on the Fock space. We mainly follow [21], [44] and [47] for this section.

In analogy with the usual QFT quantisation, we impose the Equal Time Commutation Relations for the fields X^μ and $\Pi_\mu = (1/2\pi\alpha')\dot{X}^\mu$ (the conjugate momentum):

$$\begin{aligned}
&[X^\mu(\tau,\sigma), \Pi_\nu(\tau,\sigma')] = i\delta(\sigma-\sigma')\delta^\mu_\nu,\\
&[X^\mu(\tau,\sigma), X^\nu(\tau,\sigma')] = 0,\\
&[\Pi_\mu(\tau,\sigma), \Pi_\nu(\tau,\sigma')] = 0.
\end{aligned} \tag{3.2.1}$$

Using the mode expansions 3.1.21 and 3.1.22, these imply:

$$\begin{aligned}
&[x^\mu, p_\nu] = i\delta^\mu_\nu,\\
&[\alpha^\mu_n, \alpha^\nu_m] = n\eta_{\mu\nu}\delta_{m+n,0},\\
&[\tilde{\alpha}^\mu_n, \tilde{\alpha}^\nu_m] = n\eta_{\mu\nu}\delta_{m+n,0},\\
&\text{All others zero.}
\end{aligned} \tag{3.2.2}$$

Using these commutators, one can show that:

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{D}{12}m\left(m^2-1\right)\delta_{m+n,0}, \tag{3.2.3}$$

where D is the number of space-time dimensions.
Again, in analogy with QFT, we interpret α_n^μ and $\tilde{\alpha}_n^\mu$ (scaled by n) as two sets of creation/annihilation operators. We let those with $n > 0$ ($n < 0$) be annihilators (creators). The construction of the Fock space is then the usual one. We define the vacuum state to obey:

$$\alpha_n^\mu \left|0,p\right\rangle = \tilde{\alpha}_n^\mu \left|0,p\right\rangle = 0 \quad \text{for} \quad n > 0, \tag{3.2.4}$$

where p labels the momentum eigenvalue. It is evident that the vacuum of string theory is not the same object as the vacuum state of space-time from QFT, see [44] for a clarification on this point. The most general state is obtained by acting creation operators on $|0,p\rangle$:

$$\left(\alpha_{-1}^{\mu_1}\right)^{n_{\mu_1}}\left(\alpha_{-2}^{\mu_2}\right)^{n_{\mu_2}}\ldots\left(\tilde{\alpha}_{-1}^{\nu_1}\right)^{n_{\nu_1}}\left(\tilde{\alpha}_{-2}^{\nu_2}\right)^{n_{\nu_2}}\ldots|0,p\rangle. \tag{3.2.5}$$

The Fock space that we just constructed, unfortunately, has negative norm states. For instance, $\alpha_{-1}^0 \left|0,p\right\rangle$ is an evident example. To eliminate those, we introduce a condition for physical states (cf. Gupta-Bleuler quantisation). This is nothing but the quantum version of the constraints 3.1.23 and 3.1.24. We require

$$L_n \left|\text{phys}\right\rangle = 0 \quad \text{for} \quad n > 0, \tag{3.2.6}$$

where $|\text{phys}\rangle$ is any physical state of the theory. We restricted to $n > 0$ since, when performing quantisation, there is an ambiguity in the ordering of the operators appearing in L_0. Thus, we let $L_0 = \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n + \frac{1}{2}\alpha_0^2$ and $\tilde{L_0} = \sum_{n=1}^{\infty}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n + \frac{1}{2}\alpha_0^2$ (i.e. they are normal ordered), and then add an undetermined a in the constraints:

$$(L_0 - a)\left|\text{phys}\right\rangle = 0 \quad \text{and} \quad (\tilde{L}_0 - a)\left|\text{phys}\right\rangle = 0. \tag{3.2.7}$$

Defining $N = \sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n$ and $\tilde{N} = \sum_{n=1}^{\infty}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n$, we also have:

$$N - a = \frac{\alpha'}{4}M^2 = \tilde{N} - a, \tag{3.2.8}$$

and using 3.2.7 we obtain

$$(L_0 - \tilde{L}_0)\left|\text{phys}\right\rangle = 0 \implies N = \tilde{N}, \tag{3.2.9}$$

which is the level-matching condition.
We will fix the constant a, together with the number of dimensions D, at the end of the next section. One of the options for doing that is to look at *spurious* states (see [21]), but such approach would introduce more definitions which are not relevant for our next discussions.

3.3 Light-cone Quantisation

In this section, we present an alternative approach to quantisation, starting from the classical theory again. This approach is better than canonical quantisation in the sense that it makes the dynamical degrees of freedom evident. We start from the free theory again and develop a specific solution where the residual gauge symmetry of the theory has been fixed. Only at that point, we quantise and discuss both the ground state and the first excited state. Finally, we fix the number of space-time dimensions and the ordering constant a to ensure consistency of the theory. The relevant resources for this section are [21], [44] and [46].

We start by recalling that we used symmetries 2. and 3. of the Polyakov action to set $\gamma_{ab} = \eta_{ab}$, i.e. $ds^2 = -d\tau^2 + d\sigma^2$. However, there is still some remaining symmetry, which is evident in the "light-cone" parametrisation 3.1.17. If we make the transformations:

$$\sigma^+ \to \tilde{\sigma}^+(\sigma^+) \quad \text{and} \quad \sigma^- \to \tilde{\sigma}^-(\sigma^-), \tag{3.3.1}$$

then $ds^2 = -d\sigma^+ d\sigma^-$ is changed by an overall factor. By Weyl symmetry (3. in our list), this change is irrelevant. This symmetry is a "set of measure zero" of the full symmetry, this is why it is still present after gauge fixing (see [44]). To exploit the full power of this remaining symmetry, we introduce light-cone coordinates for space-time:

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1}). \tag{3.3.2}$$

We will refer to the other coordinates as the *transverse coordinates* X^i, $i = 1, ..., D-2$. The equation of motion 3.1.18 imposes, as before, $X^+ = X^+_L(\sigma^+) + X^+_R(\sigma^-)$. We now use the residual symmetry mentioned above to set:

$$X^+_L = \frac{1}{2}x^+ + \frac{\pi}{l}\alpha' p^+ \sigma^+ \quad , \quad X^+_R = \frac{1}{2}x^+ + \frac{\pi}{l}\alpha' p^+ \sigma^-. \tag{3.3.3}$$

For X^-, we again use the EoM to write $X^- = X_L^-(\sigma^+) + X_R^-(\sigma^-)$. And then, we look at the constraints 3.1.19. The first of those, using the light-cone coordinates, reads:

$$2\partial_+ X^- \partial_+ X^+ = \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i. \tag{3.3.4}$$

Using 3.3.3, this becomes

$$\partial_+ X_L^- = \frac{l}{2\pi\alpha' p^+} \sum_{i=1}^{D-2} \partial_+ X^i \partial_+ X^i. \tag{3.3.5}$$

Similarly, the second constraint in 3.1.19 becomes:

$$\partial_- X_R^- = \frac{l}{2\pi\alpha' p^+} \sum_{i=1}^{D-2} \partial_- X^i \partial_- X^i. \tag{3.3.6}$$

Then, we see that X^+ and X^- are not degrees of freedom of the theory. They are fully specified: the first one by the surviving reparametrisation symmetry, and the second by the constraints. The dynamical variables are the transverse coordinates X^i.
As the last thing before quantising the theory, we look at the level-matching conditions. We start by writing X^- as a general solution of the wave equation (cf. 3.1.21 and 3.1.22):

$$X_L^-\left(\sigma^+\right) = \frac{1}{2}x^- + \frac{\pi}{l}\alpha' p^- \sigma^+ + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}_n^- \exp(-i2\pi n\sigma^+/l), \tag{3.3.7}$$

$$X_R^-\left(\sigma^-\right) = \frac{1}{2}x^- + \frac{\pi}{l}\alpha' p^- \sigma^- + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n}\alpha_n^- \exp(-i2\pi n\sigma^-/l). \tag{3.3.8}$$

As we discussed, everything that appears in those solutions is determined in terms of X^i. Specifically, if we define $\alpha_0^- = \tilde{\alpha}_0^- = \sqrt{\frac{\alpha'}{2}}p^-$, the constraints 3.3.5 and 3.3.6 imply:

$$\frac{\alpha' p^-}{2} = \frac{1}{2p^+} \sum_{i=1}^{D-2} \left(\frac{1}{2}\alpha' p^i p^i + \sum_{n\neq 0} \tilde{\alpha}_n^i \tilde{\alpha}_{-n}^i \right), \tag{3.3.9}$$

$$\frac{\alpha' p^-}{2} = \frac{1}{2p^+} \sum_{i=1}^{D-2} \left(\frac{1}{2}\alpha' p^i p^i + \sum_{n\neq 0} \alpha_n^i \alpha_{-n}^i \right), \tag{3.3.10}$$

respectively. Then, $p^2 = -M^2$ gives the level-matching:

$$M^2 = 2p^+p^- - \sum_{i=1}^{D-2} p^i p^i = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \alpha^i_{-n}\alpha^i_n = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n, \tag{3.3.11}$$

Another confirmation that X^i are the only physical excitation of the string.
Having identified the real degrees of freedom, we proceed to quantise exactly as in the previous section, but restricting to the transverse coordinates and their associate quantities. In practice, equations from 3.2.6 to 3.2.9 still hold, but now with:

$$N = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \alpha^i_{-n}\alpha^i_n \quad \text{and} \quad \tilde{N} = \frac{4}{\alpha'} \sum_{i=1}^{D-2} \sum_{n>0} \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n. \tag{3.3.12}$$

The Fock space is constructed exactly in the usual way, and we will now study its lowest state. We note that the ground state $|0, p\rangle$ has a mass $M^2 = -\frac{4a}{\alpha'}$, according to 3.2.8. The first excited state is given by acting $\alpha^i_{-1}\tilde{\alpha}^i_{-1}$ (recall that creation operators must always come in pairs, according to the level-matching condition 3.2.9). We write it as:

$$\Omega^{ij}(p) = \alpha^i_{-1}\tilde{\alpha}^i_{-1} |0, p\rangle . \tag{3.3.13}$$

Since $i, j = 1, ..., D-2$, this state forms a representation (module, if one wants to be precise) for $SO(D-2)$. In words, it is a tensor of type $(0, 2)$, on which $SO(D-2)$ acts via its $(D-2)^2$ dimensional representation. As always, we can split Ω^{ij} into its irreps by considering the symmetries of the indices. We have the following decomposition:

$$\Omega^{ij} = \underbrace{\frac{1}{D-2}\Omega^{kk}\delta^{ij}}_{=\phi} + \underbrace{\Omega^{[ij]}}_{=B^{ij}} + \big(\underbrace{\Omega^{(ij)} - \frac{1}{D-2}\Omega^{kk}\delta^{ij}}_{=G^{ij}}\big). \tag{3.3.14}$$

More in detail:

- The trace part of Ω^{ij} defines a singlet scalar field, which we call *dilatino*.
- The antisymmetric part of Ω^{ij} defines an antisymmetric $(0, 2)$ tensor B_{ij}, which we call *B-field* (more correctly the Kalba-Ramond field).
- The traceless symmetric part of Ω^{ij} defines a traceless symmetric $(0, 2)$ tensor G_{ij} which we call *graviton*.

Finally, let us conclude this section by fixing the constants a and D. We have just determined that the first excited states corresponds to a representation of $SO(D-2)$. This is the little group for massless particles in D dimensions, and hence we require this state to have zero mass. According to 3.2.8, this yields $\frac{\alpha'}{4}M^2 = 1 - a = 0$. Therefore, for consistency, we impose $a = 1$. Now recall how a was defined: it was introduced to account the ordering ambiguity of L_0 after quantisation. Choosing to bring L_0 to normal order yields:

$$\frac{1}{2}\sum_{i=1}^{D-2}\sum_{n=-\infty}^{+\infty}\alpha^i_{-n}\alpha^i_n = \frac{1}{2}\sum_{i=1}^{D-2}\sum_{n=-\infty}^{+\infty} :\alpha^i_{-n}\alpha^i_n: + \underbrace{\frac{1}{2}(D-2)\sum_{n=1}^{\infty} n}_{-a}, \tag{3.3.15}$$

where we split the sum into $n > 0$ and $n < 0$ pieces, used 3.2.2 on the $n < 0$ one and then combined them into a single one again. The term $\sum_{n=1}^{\infty} n$ is divergent, so it must regularised. The quickest way of doing this is to use zeta function regularisation, which gives $\sum_{n=1}^{\infty} n = -\frac{1}{12}$. Hence, $-a = -\frac{D-2}{24}$, which using $a = 1$ yields $D = 26$. This is called the *critical dimension* for bosonic string theory.

3.4 Classical Superstring Theory

In this section, we introduce classical type II superstring theory. To prevent this dissertation from becoming a book, we do not give an introduction on supersymmetry, even though it is required to derive carefully the action for superstrings. We instead take it as our starting point, with some justification on why it is the correct one. Then, we discuss the boundary conditions and general solutions of the theory.
For this section, we mainly refer to [21], [46] and [45].

Our starting point is the action:

$$S = -\frac{1}{2\pi}\int d^2\sigma\big(\partial_\alpha X^\mu \partial^\alpha X_\mu + \bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu - B^\mu B_\mu\big), \tag{3.4.1}$$

where

$$\psi^\mu = \begin{pmatrix}\psi^\mu_- \\ \psi^\mu_+\end{pmatrix}, \quad \bar{\psi} = \psi^\dagger i \rho^0. \tag{3.4.2}$$

and ρ^0, ρ^1 are the Dirac matrices in two dimensions. They satisfy $\{\rho^a, \rho^b\} = 2\eta^{ab}$. They can be explicitly chosen to be:

$$\rho^0 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{3.4.3}$$

The action 3.4.1 defines the so-called RNS formalism, and it is obtained very naturally from the Polyakov action. Since our aim is to include fermions, we simply add the standard Dirac action for a free fermion ψ^μ: $\bar{\psi}^\mu \rho^\alpha \partial_\alpha \psi_\mu$. Then, the extra term is added to make the supersymmetry manifest with the superspace formalism.[2] We now define the usual coordinates $\sigma^+ = \tau + \sigma$ and $\sigma^- = \tau - \sigma$ and adopt the notation $\partial_+ = \frac{\partial}{\partial\sigma^+}$ and $\partial_- = \frac{\partial}{\partial\sigma^-}$. Then, the equations of motion are given by

$$\begin{aligned} \partial_+\partial_- X^\mu &= 0, \\ \partial_+\psi^\mu_- &= 0, \\ \partial_-\psi^\mu_+ &= 0. \end{aligned} \tag{3.4.4}$$

By construction, the first one is exactly the same as for the case of the bosonic string. Before proposing a general solution for ψ, we need to establish the boundary conditions. In varying the action and integrating by parts, we obtain the following boundary terms in the fermionic sector:

$$\int d\tau \big[\psi_+ \cdot \delta\psi_+ - \psi_- \cdot \delta\psi_-\big]_0^l, \tag{3.4.5}$$

which must vanish. This happens if, at each end of the string, we have that $\psi_+ = \pm\psi_-$. The overall sign between ψ_+ and ψ_- leads to a simple redefinition, but what encodes the real physical difference is the relative sign between the two ends. This leads to two possibilities. Given the conventional choice $\psi^\mu_+(\tau, 0) = \psi^\mu_-(\tau, 0)$, we can have:

1. Ramond (R) boundary conditions, which read: $\psi^\mu_+(\tau, l) = \psi^\mu_-(\tau, l)$.
   The solution to the EoM consistent with these boundary conditions is (cf. 3.1.21 and 3.1.22):

$$\psi^\mu_-(\tau, \sigma) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} d^\mu_n \mathrm{exp}\Big(- in\frac{\pi}{l}(\tau - \sigma)\Big), \tag{3.4.6}$$

$$\psi^\mu_+(\tau, \sigma) = \frac{1}{\sqrt{2}} \sum_{n\in\mathbb{Z}} d^\mu_n \mathrm{exp}\Big(- in\frac{\pi}{l}(\tau + \sigma)\Big). \tag{3.4.7}$$

[2] We do not provide the supersymmetry transformations here, since they are not relevant for our discussion. They can be found in [21].

2. Naveu-Schwarz (NS) boundary conditions, which read: $\psi_+^\mu(\tau, l) = -\psi_-^\mu(\tau, l)$. The solution to the EoM consistent with these boundary conditions is:

$$\psi_-^\mu(\tau, \sigma) = \frac{1}{\sqrt{2}} \sum_{u \in \mathbb{Z} + \frac{1}{2}} b_u^\mu \exp\Big(- iu\frac{\pi}{l}(\tau - \sigma)\Big), \tag{3.4.8}$$

$$\psi_+^\mu(\tau, \sigma) = \frac{1}{\sqrt{2}} \sum_{u \in \mathbb{Z} + \frac{1}{2}} b_u^\mu \exp\Big(- iu\frac{\pi}{l}(\tau + \sigma)\Big). \tag{3.4.9}$$

We have not assumed that the strings are closed so far, but we are about to do it.
This time we are dealing with fermions, not bosons, and thus we have more freedom on our boundary conditions. Since fermionic quantities are not observables, but only their bosonic products are, we can choose periodic or anti-periodic boundary conditions for closed strings. If we let 0 and l label the same point, we find that ψ_+ and ψ_- are not related anymore. In other words, given

$$\psi_+(\tau, 0) = \pm\psi_+(\tau, l) \quad \text{and} \quad \psi_-(\tau, 0) = \pm\psi_-(\tau, l), \tag{3.4.10}$$

we can choose the signs independently, and still ensure that the boundary term vanishes. Thus, we have two options for the right-moving solution:

$$\psi_-(\tau, \sigma) = \begin{cases} \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} d_n^\mu \exp\Big(- in\frac{2\pi}{l}(\tau - \sigma)\Big) & (R) \quad \text{or} \\ \frac{1}{\sqrt{2}} \sum_{u \in \mathbb{Z} + \frac{1}{2}} b_u^\mu \exp\Big(- iu\frac{\pi}{l}(\tau - \sigma)\Big) & (NS), \end{cases} \tag{3.4.11}$$

and, independently, two options for the left-moving one

$$\psi_+(\tau, \sigma) = \begin{cases} \frac{1}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \tilde{d}_n^\mu \exp\Big(- in\frac{2\pi}{l}(\tau + \sigma)\Big) & (R) \quad \text{or} \\ \frac{1}{\sqrt{2}} \sum_{u \in \mathbb{Z} + \frac{1}{2}} \tilde{b}_u^\mu \exp\Big(- iu\frac{\pi}{l}(\tau + \sigma)\Big) & (NS). \end{cases} \tag{3.4.12}$$

It is not surprising that, according to the four possible choices of boundary conditions, we divide the theory into four sectors: (R, R), (R, NS), (NS, R), (NS, NS).

3.5 Quantisation of the Superstring

We now move onto quantising this theory. To do it, we follow the canonical quantisation procedure. We initially focus only on one sector at a time, either R or NS. States in the

closed string theory are obtained from tensor products of the above. The massless spectrum, which defines the low energy limit of the theory, is derived.
This is the most important section of the chapter, hence it deserves many references. They are: [21], [46], [48], [45], [49], [50] and [51].

For the bosonic part, we have the same result as before:

$$[\alpha_n^\mu, \alpha_m^\nu] = n\eta_{\mu\nu}\delta_{m+n,0},$$
$$[\tilde{\alpha}_n^\mu, \tilde{\alpha}_m^\nu] = n\eta_{\mu\nu}\delta_{m+n,0}. \tag{3.5.1}$$

For the fermionic sector, as usual, we require the equal time *anti*-commutation relation

$$\{\psi_a^\mu(\tau,\sigma), \psi_b^\nu(\tau,\sigma')\} = l\delta_{a+b,0}\delta(\sigma-\sigma'), \tag{3.5.2}$$

which implies:

$$\{b_u^\mu, b_v^\nu\} = \eta^{\mu\nu}\delta_{u+v,0} \quad (NS), \tag{3.5.3}$$
$$\{d_m^\mu, d_n^\nu\} = \eta^{\mu\nu}\delta_{m+n,0} \quad (R). \tag{3.5.4}$$

The ground state of the R sector is given by:

$$\alpha_m^\mu|0_R>= 0 = d_m^\mu|0_R> \quad \text{for } \ m>0, \tag{3.5.5}$$

while the ground state for the NS sector obeys

$$\alpha_m^\mu|0_R>= 0 = b_u^\mu|0_R> \quad \text{for } \ m,u>0. \tag{3.5.6}$$

Also, we have that for the R sector:

$$L_n = \frac{1}{2} : \sum_{m\in\mathbb{Z}} \alpha_{n+m}\cdot\alpha_{-m} : + \frac{1}{2}\sum_{m\in\mathbb{Z}}(m+\frac{n}{2}) : d_{n+m}\cdot d_{-m} : \tag{3.5.7}$$

and

$$F_n = \sum_{m\in\mathbb{Z}} \alpha_m\cdot d_{n-m}. \tag{3.5.8}$$

For the NS sector:

$$L_n = \frac{1}{2} : \sum_{m\in\mathbb{Z}} \alpha_{n+m}\cdot\alpha_{-m} : + \frac{1}{2}\sum_{u\in\mathbb{Z}+\frac{1}{2}}(u+\frac{n}{2}) : b_{n+u}\cdot b_{-u} : \tag{3.5.9}$$

and

$$G_u = \sum_{n\in\mathbb{Z}} : \alpha_n \cdot b_{-n+u} : . \tag{3.5.10}$$

We point out that the only normal order ambiguity arises in L_0, while normal ordering can be removed from all other quantities. To flag this problem, we add appropriate constants in the definitions above, as in the bosonic case. We include a_R in the rhs of 3.5.7 (for $n=0$) and a_{NS} in the rhs of 3.5.9 (for $n=0$). Then we have the following two algebras. For the R sector:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{D}{8} m^3 \delta_{m+n,0}, \\
[L_m, F_n] &= \left(\frac{m}{2} - n\right) F_{m+n}, \\
\{F_m, F_n\} &= 2L_{m+n} + \frac{D}{2} m^2 \delta_{m+n,0}.
\end{aligned} \tag{3.5.11}$$

For the NS sector:

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{D}{8} m \left(m^2 - 1\right) \delta_{m+n,0}, \\
[L_m, G_r] &= \left(\frac{m}{2} - r\right) G_{m+r}, \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{D}{2} \left(r^2 - \frac{1}{4}\right) \delta_{r+s,0}.
\end{aligned} \tag{3.5.12}$$

We are now (almost) ready to describe the spectrum for each of the two sectors. We first need a small parenthesis on light-cone coordinates.
Exactly as in the bosonic case, the degrees of freedom are less than one might be tempted to think, and they correspond to the transverse modes. For the bosonic sector, the same argument presented in the previous section applies, leaving only X^i as the dynamical fields (where $i = 1, ..., D-2$). The same is true for the fermionic sector, where, if one introduces light-cone coordinates, ψ^+ can be fixed by a residual symmetry and ψ^- is determined in terms of the transverse fields ψ^i.
Now it is time for building the quantum theory.
Let us start with the NS sector. As we just argued, the ground state satisfies:

$$\begin{aligned}
\alpha_n^i \left|0;p\right\rangle &= 0 \quad \text{for } n > 0, \\
b_u^i \left|0;p\right\rangle &= 0 \quad \text{for } r > 0.
\end{aligned} \tag{3.5.13}$$

The physical state conditions reads:

$$\begin{aligned}
G_u \left|\text{phys}\right\rangle &= 0 \quad \text{for } \; u > 0, \\
L_n \left|\text{phys}\right\rangle &= 0 \quad \text{for } \; n > 0, \\
(L_0 - a_{NS}) \left|\text{phys}\right\rangle &= 0,
\end{aligned} \tag{3.5.14}$$

where a_{NS} is the constant introduced before to keep track of the order ambiguity upon quantisation. From the last equation we obtain the mass formula:

$$\alpha' M^2 = \sum_{n=1}^{\infty} \alpha^i_{-n} \alpha^i_n + \sum_{u=\frac{1}{2}}^{\infty} u b^i_{-u} b^i_u - a_{NS}. \tag{3.5.15}$$

As usual, the first excited states are obtained by acting the raising operators on the vacuum. The smallest excitation above the ground state is given by

$$b^i_{-\frac{1}{2}} \left|0; p\right\rangle . \tag{3.5.16}$$

This state naturally belongs to the vector representation of $SO(D-2)$, since i labels the transverse coordinates. This is consistent with a massless vector in D dimensions. Hence, we see that $\alpha' M^2 = \frac{1}{2} - a_{NS}$ imposes $a_{NS} = \frac{1}{2}$. We immediately infer from this equation and 3.5.13 that the ground state is tachyonic, since $\alpha' M^2 = -\frac{1}{2}$.
All the other states obtained by the action of raising operators will be massive, so we have exhausted the massless states for this sector.
Before moving to the spectrum of the R sector, let us determine D. We will do it analogously to the bosonic case. According to 3.5.9, a_{NS} has two contributions. One comes from the α's and it is the same as the bosonic case, i.e. $\frac{D-2}{24}$. The second one, from the b's, can be obtained analogously using regularisation, and it turns out to be $\frac{D-2}{48}$ (see [45]). Hence, from the condition on a_{NS} that we just derived, we have $-\frac{1}{2} = -(D-2)(\frac{1}{24} + \frac{1}{48}) = -\frac{D-2}{16}$ which gives $D = 10$. This is the *critical dimension* for superstring theory.
Moving now to the R sector, we take the vacuum to satisfy:

$$\begin{aligned}
F_n \left|0; p\right\rangle &= 0 \quad \text{for } \; n \geq 0, \\
L_n \left|0; p\right\rangle &= 0 \quad \text{for } \; n > 0, \\
(L_0 - a_R) \left|0; p\right\rangle &= 0.
\end{aligned} \tag{3.5.17}$$

Note that the last equation is the same condition as the one in 3.5.13, but L_0 here refers to the operator in the R sector and so we have a different constant for the order ambiguity.

This time, it can be determined immediately. It follows from 3.5.11 that $L_0 = F_0^2$. Hence, the first equation above implies $a_R = 0$.
Defining the ground state is not as trivial as in the NS case. The reason has to do with the first relation in 3.5.4:

$$\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}. \tag{3.5.18}$$

If we let $d_0^\mu = \frac{1}{\sqrt{2}}\Gamma^\mu$, then we recover the Clifford algebra $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$ (see appendix A2). Thus, these operators form a representation of the Clifford algebra. This implies that states, which are acted upon, must be spinors. The ground state is no exception. We will now build explicitly a suitable space (of ground states), starting from the definition of an appropriate ground state. Firstly, we recall that the number of independent operators is $D-2$. We can combine them in the following way:

$$\begin{aligned} D^i &= \frac{1}{\sqrt{2}}\big(d_0^{2i} + i d_0^{2i-1}\big), \\ \bar{D}^i &= \frac{1}{\sqrt{2}}\big(d_0^{2i} - i d_0^{2i-1}\big) \quad \text{for } \; i = 1, 2, ..., \frac{D-2}{2}. \end{aligned} \tag{3.5.19}$$

They satisfy:

$$\{D^i, \bar{D}^j\} = \delta^{ij}, \tag{3.5.20}$$

$$\{D^i, D^j\} = 0 = \{\bar{D}^i, \bar{D}^j\}. \tag{3.5.21}$$

Hence, we have obtained a set of $D-2$ fermionic creation and annihilation operators.[48] We then follow the usual procedure from quantum field theory. We define the ground state (of the ground state) $|\zeta\rangle$ such that $D^i|\zeta\rangle = 0$. Then, we construct the space of (ground) states by acting the operators $\bar{D}^i$ on it in all possible ways, with each $\bar{D}^i$ appearing at most once. This gives $2^{(D-2)/2}$ states. For $D = 10$, we can label the ground states as $|\pm, \pm, \pm, \pm >$, where $+$ $(-)$ in the i^{th} entry means that $\bar{D}^i$ is (not) acting on $|\zeta\rangle$. In this notation, $|\zeta\rangle = |-, -, -, - >$. The space that we constructed is actually made of two smaller spaces that do not "talk to each other", i.e. the representation is reducible (see A2). Specifically, states with an even number of $+$'s do not mix with states with an odd number of $+$'s under a Lorentz transformation, i.e. chirality is preserved. Thus, we have two sets of ground states, each being 8-dimensional if $D = 10$.
So far we carefully explored the nature of the ground state, which is the only one we will need for this discussion. The other states of the spectrum can be obtained by applying the

raising operators on the ground state, as usual.
There are two undesirable features in the spectrum obtained so far. The first one is the presence of the Tachyonic mode. The second one is the absence of space-time supersymmetry. We can solve both problems in one go by eliminating some states in the spectrum, with a mechanism called GSO projection.
We define the GSO projection operators as follows:

$$P_{NS} = (-1)^F = (-1)^{\sum_{u=1/2}^{\infty} b^i_{-u} b^i_u + 1} \qquad \text{for the NS sector,} \tag{3.5.22}$$

$$P_R = (-1)^F = \Gamma_{11}(-1)^{\sum_{n=1}^{\infty} d^i_{-n} d^i_n} \qquad \text{for the R sector,} \tag{3.5.23}$$

where $\Gamma_{11} = \Gamma_0 ... \Gamma_9$. The action of $(-1)^F$ on the NS sector is simple: it counts the number of b-excitations (plus one). What we require is that states must satisfy $(-1)^F = 1$. In other words, they must have an odd number of b-excitations. We immediately see that the tachyonic state is removed from this spectrum this way.
The GSO operator on the R sector has a simple interpretation as well: imposing $(-1)^F = \pm 1$ requires the spinor to be chiral. Which specific sign we choose is irrelevant at this point, and it is just a matter of convention.
Hence, in summary, the effect of GSO projection is to remove the tachyon mode in the NS and pick a definite chirality ground state in the R sector. We are left with a bosonic vacuum with 8 DoF in the NS sector and a fermionic vacuum with 8 DoF in the R sector. Although we will not repeat this analysis, the same happens for the massive states: fermionic and bosonic DoF match for every state. This is a necessary condition for supersymmetry, which is a strong suggestion that supersymmetry holds. A full proof requires a different formalism, which can be found in [46], [21].
Up to now, we have focused on open strings, which only have the two sectors R and NS. As we saw, for closed strings we have two independent choices of boundary conditions, since there are two independent set of modes. Thus, a general state in the spectrum of closed superstring theory is constructed from the tensor product of two states: one left-moving and one right-moving. This implies the need for a classification, which we will now explain.
The GSO projection selects a ground state for the R sector with a definite chirality. The choice of which specific chirality is irrelevant for open strings, since both lead to the same spectrum. Now however, we have to choose the chirality twice: once for the left-movers and once for the right-movers. If they are the same, the theory is said to be of type IIB (this is equivalent to requiring $(-1)^F = 1$ for both R sectors). If they are different, it is of type IIA (this corresponds to choosing $(-1)^F = 1$ for the right-moving R sector and $(-1)^F = -1$ for the left moving one,or vice-versa). With this clarification made, we can now analyse the

spectra of these two theories.

1. **Type IIB**
   We choose a ground state of definite chirality, say $|\psi_+\rangle$, for the R sector of both right-movers and left-movers. The massless spectrum is then:

$$\begin{aligned} &\text{NS-NS}: \quad \tilde{b}^i_{-\frac{1}{2}}\,|0\rangle \otimes\ b^j_{-\frac{1}{2}}\,|0\rangle \\ &\text{NS-R}: \quad \tilde{b}^i_{-\frac{1}{2}}\,|0\rangle \otimes\ |\psi_+\rangle \\ &\text{R-NS}: \quad |\psi_+\rangle \otimes\ b^i_{-\frac{1}{2}}\,|0\rangle \\ &\text{R-R}: \quad |\psi_+\rangle \otimes\ |\psi_+\rangle\,. \end{aligned} \tag{3.5.24}$$

   We now carefully study its field content.

   - NS-NS: The state consists of a vector times a vector, i.e.

$$\tilde{b}^i_{-\frac{1}{2}} b^i_{-\frac{1}{2}}\,|0,0\rangle \equiv A^{ij}. \tag{3.5.25}$$

     As a representation of $SO(8)$, it decomposes as usual according to

$$\underbrace{A^{ij}}_{\mathbf{64}_V} = \underbrace{\frac{1}{8}A^{kk}\delta^{ij}}_{\mathbf{1}} + \underbrace{A^{[ij]}}_{\mathbf{28}_V} + \underbrace{\left(A^{(ij)} - \frac{1}{8}A^{kk}\delta^{ij}\right)}_{\mathbf{35}_V}. \tag{3.5.26}$$

     The subscript V emphasizes that we are dealing with a vector representation. Since it is common to denote the representation with its dimension, we will identify the two things and always use the bold notation with subscripts (if necessary). Thus, the spectrum is given by:

     – A scalar field, $\frac{1}{8}A^{kk}\delta^{ij} \equiv \phi$, which is a singlet of $SO(8)$, the dilaton.
     – An antisymmetric tensor field $A^{[ij]} \equiv B_{ij}$, with dimension $(7\times 8)/2 = \mathbf{28}_V$, the B-field.
     – A traceless symmetric tensor field $A^{(ij)} - \frac{1}{8}A^{kk}\delta^{ij} \equiv g_{ij}$, with dimensions $(8\times 9)/2 - 1 = \mathbf{35}_V$, the graviton.

     These are all old friends. And now come the new ones.

- NS-R: The state consists of a vector times a chiral spinor, i.e.

$$\tilde{b}^i_{-\frac{1}{2}} |0, \psi_+\rangle \equiv \psi^i_+, \tag{3.5.27}$$

where the spinor index is suppressed. Before proceeding to the decomposition, we need to establish the notation associated with different chiralities. We leave the representations with chirality "+" without any subscripts or superscripts. For the ones that correspond to chirality "-", we reserve a prime. Using this notation, 3.5.27 can be decomposed into the following two pieces:

  - A chiral spin 1/2 spinor $\Gamma^i \psi^i_+ \equiv \lambda$, which has dimension $\mathbf{8}'$ and it is called the *dilatino*.[3]
  - A spin 3/2 field Ψ_i that satisfies $\Gamma^i \Psi_i = 0$, has dimensions $8 \times 8 - 8 = \mathbf{56}'$ and is called the *gravitino*.

- R-NS: The state consists of a chiral spinor times a vector, i.e.

$$b^i_{-\frac{1}{2}} |\psi_+, 0\rangle \equiv \psi^i_+. \tag{3.5.28}$$

Exactly as before, we have:

  - A chiral spin 1/2 spinor $\Gamma^i \psi^i_+ \equiv \lambda'$ with dimension $\mathbf{8}'$ (another dilatino).
  - A spin 3/2 field Ψ'_i satisfying $\Gamma^i \Psi'_i = 0$ with dimensions $\mathbf{56}'$ (another gravitino).

- R-R: The state consists of a chiral spinor times a chiral spinor of the same chirality, i.e.

$$|\psi_+, \psi_+\rangle\,. \tag{3.5.29}$$

To determine the decomposition of this state, we write first it as $|\psi_+\rangle \langle\psi_+|$. Then, we study its components by sandwitching strings of gamma matrices as $\langle\psi_+| \Gamma^{i_1} ... \Gamma^{i_n} |\psi_+\rangle$. Since inner products between states of opposite chirality vanish, gamma matrices can appear only in an even number (recall that acting with a gamma matrix changes the chirality). This, we are left with three possibilities:

  - A scalar field $\langle\psi_+|\psi_+\rangle \equiv C_0$, which is a singlet.
  - An antisymmetric (0,2)-tensor field $\langle\psi_+| \Gamma^i \Gamma^j |\psi_+\rangle \equiv C_{ij}$ with dimension $(7 \times 8)/2 = \mathbf{28}$.

[3] Note that the chirality has changed due to the action of the gamma matrix.

– A self-dual antisymmetric (0,4)-tensor field $\langle\psi_+|\,\Gamma^i\Gamma^j\Gamma^k\Gamma^l\,|\psi_+\rangle \equiv C_{ijkl}$ with dimension $\frac{1}{2}(8\times 7\times 6\times 5)/24 = \mathbf{35}$.

Hence, the low energy limit (only massless states are excited) of type IIB superstring theory contains the following fields:

$$\phi, B_{ij}, g_{ij}, \lambda, \Psi_i, \lambda', \Psi'_i, C_0, C_{ij}, C_{ijkl}. \tag{3.5.30}$$

2. **Type IIA**
We choose ground states of opposite parity for left-movers and right-movers, say $|+\rangle$ and $|+\rangle$. The massless spectrum is then:

$$\begin{aligned}
&\text{NS-NS}: \quad \tilde{b}^i_{-\frac{1}{2}}\,|0\rangle \otimes\; b^j_{-\frac{1}{2}}\,|0\rangle\\
&\text{NS-R}: \quad \tilde{b}^i_{-\frac{1}{2}}\,|0\rangle \otimes\; |+\rangle\\
&\text{R-NS}: \quad |+\rangle \otimes\; b^i_{-\frac{1}{2}}\,|0\rangle\\
&\text{R-R}: \quad |+\rangle \otimes\; |+\rangle\,.
\end{aligned} \tag{3.5.31}$$

The field content is the following.

- NS-NS: The state consists of a vector times a vector, i.e.

$$\tilde{b}^i_{-\frac{1}{2}} b^i_{-\frac{1}{2}}\,|0,0\rangle \equiv A^{ij}. \tag{3.5.32}$$

  Exactly as in type IIB, the decomposition yields:

  – A scalar field, the dilaton ϕ.

  – An anstisymmetric (0,2)-tensor field, the B-field B_{ij}.

  – A traceless symmetric (0,2)-tensor field, the graviton g_{ij}.

- NS-R: The state consists of a chiral spinor times a vector, i.e.

$$\tilde{b}^i_{-\frac{1}{2}}\,|\psi_+, 0\rangle \equiv \psi^i_+. \tag{3.5.33}$$

  Exactly as before, we have:

  – A chiral spin 1/2 spinor $\Gamma^i\psi^i_+$ with dimension $\mathbf{8}'$, the dilatino λ.

  – A spin 3/2 field Ψ_i satisfying $\Gamma^i\Psi_i = 0$ with dimensions $\mathbf{56}'$, the gravitino Ψ_i.

- R-NS: The state consists of a chiral spinor times a vector, i.e.

$$b^i_{-\frac{1}{2}} \left|\psi_-, 0\right\rangle \equiv \psi^i_+. \tag{3.5.34}$$

  Analogously, the decomposition reads:

  - A chiral spin 1/2 spinor $\Gamma^i \psi^i_-$ with dimension $\mathbf{8}'$ (another dilatino λ').
  - A spin 3/2 field Ψ'_i satisfying $\Gamma^i \Psi'_i = 0$ with dimensions $\mathbf{56}'$ (another gravitino Ψ'_i).

  Note that the chiralities of these states are opposite to the NS-R ones. This was not the case for type IIB.

- R-R: The state consists of a chiral spinor times a chiral spinor of the opposite chirality, i.e.

$$\left|\psi_-, \psi_+\right\rangle . \tag{3.5.35}$$

  As before, we write the state first as $\left|\psi_+\right\rangle \left\langle\psi_-\right|$. Its components are again found by sandwitching strings of gamma matrices. Since inner products between states of opposite chirality vanish, gamma matrices can appear only in an odd number this time. Thus, we are left with two possibilities:

  - A vector field $\left\langle\psi_-\right| \Gamma^i \left|\psi_+\right\rangle \equiv A_i$ with dimension $\mathbf{8}_V$. We include such subscript to stress that we are dealing with a vector representation.
  - An antisymmetric (0,3)-tensor field $\left\langle\psi_-\right| \Gamma^i \Gamma^j \Gamma^k \left|\psi_+\right\rangle \equiv A_{ijk}$ with dimension $(8 \times 7 \times 6)/3! = \mathbf{56}_V$.

Thus, the low energy limit (only massless states are excited) of type IIA superstring theory contains the following fields:

$$\phi, B_{ij}, g_{ij}, \lambda, \Psi_i, \lambda', \Psi'_i, A_i, A_{ijk}. \tag{3.5.36}$$

3.6 Sigma Model

In this section, we quickly introduce the sigma model for bosonic string theory and its low energy effective action. This section is based on [44].

We have only dealt so far with strings moving in a flat space-time. A natural question to ask would be how to generalise this to general curved backgrounds. A natural answer is: just replace η with a general metric G. As a result of such replacement, we obtain the simplest *sigma-model* action:

$$S = \frac{1}{4\pi\alpha'}\int d^2\sigma\sqrt{\gamma}\gamma^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X). \tag{3.6.1}$$

We can interpret $G_{\mu\nu}$ as the graviton obtained in the massless spectrum (more precisely, as a "condensate" of gravitons, see [44]), which leads to the second natural question: is it possible to couple the other massless fields to the strings as well? The answer is yes, and the result is the most generic sigma model for bosonic string theory:

$$S = -\frac{1}{4\pi}\int d\tau d\sigma\big[(\sqrt{\gamma}\gamma^{ab}G_{\mu\nu} + \epsilon^{ab}B_{\mu\nu})\partial_a X^\mu \partial_b X^\nu + \sqrt{\gamma}\phi R^{(2)}\big], \tag{3.6.2}$$

where $R^{(2)}$ is the Ricci scalar associated with the worldsheet and the convention is $\epsilon^{01} = -1$. Before proceeding, we study the B-field part to emphasize a crucial property. Let us consider the gauge transformation of B:

$$B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu\lambda_\nu - \partial_\nu\lambda_\mu. \tag{3.6.3}$$

Then, using the product rule on ∂_a, the change in the Lagrangian is given by:

$$\begin{aligned}\delta\mathcal{L} = \epsilon^{ab}(\partial_\mu\lambda_\nu - \partial_\nu\lambda_\mu)\partial_a X^\mu \partial_b X^\nu = \partial_a(\epsilon^{ab}(\partial_\mu\lambda_\nu - \partial_\nu\lambda_\mu)X^\mu\partial_b X^\nu) - \epsilon^{ab}\partial_a(\partial_\mu\lambda_\nu - \partial_\nu\lambda_\mu)X^\mu\partial_b X^\nu \\ -\epsilon^{ab}(\partial_\mu\lambda_\nu - \partial_\nu\lambda_\mu)X^\mu\partial_a\partial_b X^\nu,\end{aligned} \tag{3.6.4}$$

where the last term vanishes due to symmetry. Using now $\partial_a = \partial_a X^\alpha \partial_\alpha$ and denoting by $\partial_a(...)$ total derivative terms, we have:

$$\delta\mathcal{L} = \partial_a(...) - \epsilon^{ab}\partial_a X^\alpha \partial_b X^\nu \partial_\alpha\partial_\mu\lambda_\nu X^\mu + \epsilon^{ab}\partial_a X^\alpha \partial_b X^\nu \partial_\alpha\partial_\nu\lambda_\mu X^\mu. \tag{3.6.5}$$

The last term vanishes due to symmetry, while we can use the product rule on ∂_b in the second one to obtain:

$$\delta\mathcal{L} = \partial_a(...) + \epsilon^{ab}(\partial_b\partial_a X^\alpha)X^\nu\partial_\alpha\partial_\mu\lambda_\nu X^\mu + \epsilon^{ab}\partial_a X^\alpha X^\nu(\partial_b\partial_\alpha\partial_\mu\lambda_\nu)X^\mu + \epsilon^{ab}\partial_a X^\alpha X^\nu \partial_\alpha\partial_\mu\lambda_\nu(\partial_b X^\mu), \tag{3.6.6}$$

where the new total derivative obtained from the product rule is again contained within $\partial_a(...)$. The terms that are not total derivatives all vanish due to symmetry (for the second, this become explicit with the use of $\partial_b = (\partial_b X^\beta)\partial_\beta)$. Thus, the transformation 3.6.3 is a symmetry of the action, the B field is a proper gauge field and the dynamics only depends on its gauge-independent field strength $H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$. We emphasize that there is also a "gauge symmetry of the gauge symmetry":

$$\lambda_\mu \to \lambda_\mu + \partial_\mu \phi, \tag{3.6.7}$$

where ϕ is a scalar field. It is easy to check from 3.6.3 that adding $\partial_\mu\phi$ to the gauge parameter λ_μ gives the same gauge transformation (since partial derivatives commute). This is why we can (somewhat ironically) identify 3.6.7 as a gauge symmetry of the gauge symmetry.
We now go into one of the few discussions that we do not accompany with detailed calculations: anomalies upon quantisation. We will just present the result, which will be useful for the first section of the next chapter. In order for the theory 3.6.2 to retain Weyl invariance at the quantum level, we require the vanishing of the following three beta functions:

$$\begin{aligned}
\beta_{\mu\nu}(G) &= \alpha' \mathcal{R}_{\mu\nu} + 2\alpha' \nabla_\mu \nabla_\nu \Phi - \frac{\alpha'}{4} H_{\mu\lambda\kappa} H_\nu^{\lambda\kappa}, \\
\beta_{\mu\nu}(B) &= -\frac{\alpha'}{2} \nabla^\lambda H_{\lambda\mu\nu} + \alpha' \nabla^\lambda \Phi H_{\lambda\mu\nu}, \\
\beta(\Phi) &= -\frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla_\mu \Phi \nabla^\mu \Phi - \frac{\alpha'}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda}.
\end{aligned} \tag{3.6.8}$$

These equations are not easy to derive, and it can be shown that they exactly the equations of motion coming from the following action:

$$S_{eff} = \frac{1}{k_0^2} \int d^{26}X (-G)^{1/2} e^{-2\phi} \Big(R - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + 4\partial_\mu \phi \partial^\mu \phi \Big). \tag{3.6.9}$$

This is called the (low energy) effective action for bosonic string theory.

Compactification

4

We have shown that the number of space-time dimensions is 26 for bosonic string theory and 10 for superstring theories. In our universe we only observe 4. This chapter focuses on compactification, which is a mathematical tool to obtain a world that is D-dimensional, with $D > 4$, but looks 4-dimensional at low energies. Conceptually, the procedure is very simple. We split the target space into M_4 and an internal space X_{D-4}, that we want to get rid of at low energies. The subscripts indicate the dimensions of the two manifolds. The total space is thus a D-dimensional product of the form $M_{tot} = M_4 \times X_{D-4}$. A crucial property that X_{D-4} must have to be "invisible" at low energies is compactness. Given a compact manifold X_{D-4} with a typical size L, at energies $E << 1/L$ the physics essentially takes place in M_4 without reference to the internal space.[1][52] We anticipate that this is the only paragraph where the word "manifold" appears in this chapter. We keep the discussion as detached from differential geometry as possible.

To illustrate the above mechanism, we start in section 1 with the simple case of quantum field theory. First, we take the Klein-Gordon field as a toy model, and then move to the case of the low energy effective field theory of the bosonic sigma model.

In section 2, we study the compactification of bosonic string theory. The case of a single internal dimension is initially reviewed, before studying an arbitrary toroidal compactification. T-duality is carefully studied in both contexts.

Section 3 presents an alternative derivation of a particular type of T-duality (factorised duality), via Buscher's appraoch.

In section 4, we present compactification of type II superstring theory, both for its low energy limit and at the level of the full theory.

Finally, section 5 quickly introduces 11 dimensional supergravity, and type II supergravity (the low energy limit of type II superstring theory).

[1] If X_{D-4} was not compact we could not have a notion of size.

In this chapter, we follow [21], [43], [45], [46] for compactifications in general and [53], [54] for the corresponding T-duality.

4.1 S^1 Compactification in Field Theory

In this section, we present the simplest type of compactification, S^1, for the simplest kind of theory, field theory. We will start from the case of a free scalar field theory and then move to the effective field theory of bosonic string theory.
The sources relevant for this section are [45], [44] and [43].

A toy model: S^1 Compactification in Field Theory

Let us consider a massless Klein-Gordon scalar field ϕ living in a 5-dimensional space-time M_{tot}. Its action is:

$$S_{5d} = \int_{M_{tot}} d^5x \partial_M \phi \partial^M \phi, \tag{4.1.1}$$

with $M = 0, 1, 2, 3, 4$ and $\phi = \phi(x_0, ..., x_4)$. Compactifying this theory works as follows. We assume that space-time takes the form $M_{tot} = M_4 \times S^1$, where S^1 is a circle of radius R.[2] Then, we have coordinates x^μ associated to M_4 ($\mu = 0, 1, 2, 3$) and one coordinate $x^5 \equiv y$ associated to S^1. Clearly, y is periodic: $y = y + 2\pi R$. Hence, we can expand the field into its Fourier modes as:

$$\phi(x_1, ..., y) = \sum_{k\in\mathbb{Z}} e^{2\pi iky/R} \phi_k(x_0, x_1, x_2, x_3). \tag{4.1.2}$$

The wave equation $\partial_M \partial^M \phi(x_1, ..., y) = 0$ then becomes:

$$\partial_\mu \partial^\mu \phi_k(x_0, x_1, x_2, x_3) = (2\pi k/R)^2 \phi_k(x_0, x_1, x_2, x_3) \tag{4.1.3}$$

for all $k \in \mathbb{Z}$. In other words, each fourier component ϕ_k of the 5-dimensional field ϕ behaves as a 4-dimensional field with mass given by $(m_k)^2 = (2\pi k/R)^2$. Thus, any field living in 5-dimensions is nothing but a tower of more and more massive fields in 4-dimensions: the Kaluza-Klein (KK) tower. If we can only probe energies much smaller than $1/R$, than the

[2]Note that S^1 is compact.

only observable field of the KK tower is the one labelled by $k = 0$. Hence, the dynamics at low energies is described by the effective action:

$$S_{eff} = \int_{M_4} d^4x \partial_\mu \phi_0 \partial^\mu \phi_0, \tag{4.1.4}$$

where the dependence on x^4 has dropped out. In summary, we have started with a 5-dimensional massless scalar field, and compactification has led to a 4-dimensional massless scalar field at low energies. This process goes under the name of Kaluza-Klein (KK) reduction.

S^1 Compactification for Bosonic String Effective Field Theory

The title of this subsection is self-explicative. We have just seen the main features of S^1 compactification for the simplest field theory. We will now see them in a more concrete and useful scenario: the effective field theory for the light modes of the bosonic string.

We start by considering the last result of the previous chapter (cf. 3.6.9):

$$S_{eff} = \frac{1}{k_0^2} \int d^{26}X (-G)^{1/2} e^{-2\phi} \Big(R - \frac{1}{12} H_{NMP} H^{NMP} + 4 \partial_M \phi \partial^M \phi \Big), \tag{4.1.5}$$

where $M, N = 0, ..., 25$. This is the low energy effective action for bosonic string theory, to first order in α'. In order to perform compactification, i.e. extract a 25-dimensional theory from 4.1.5, we now need to focus on the metric and B-field to see how they behave in the lower-dimensional world. We follow the same procedure as for the scalar field in the previous subsection, but with an extra step. We first separate the metric/B-field in components according to their behaviour under the 25-dimensional Lorentz group. This is the extra step, that we did not have before since we were dealing with a scalar. Then, we perform KK reduction for each of the objects separately. This is summarised in the schemes below:

$$
G_{MN}(x_0,...,x_{25}) \xrightarrow[\text{Separate components}]{} \begin{cases} \underbrace{G_{\mu\nu}(x_0,...,x_{25})}_{\text{Metric for SO(25)}} \xrightarrow[\text{KK Reduction}]{} G^{(0)}_{\mu\nu}(x_0,...,x_{24}) \\ \underbrace{G_{\mu\,26}(x_0,...,x_{25})}_{\text{Vector for SO(25)}} \xrightarrow[\text{KK Reduction}]{} G^{(0)}_{\mu\,26}(x_0,...,x_{24}) \\ \underbrace{G_{25\,25}(x_0,...,x_{25})}_{\text{Scalar for SO(25)}} \xrightarrow[\text{KK Reduction}]{} G^{(0)}_{25\,25}(x_0,...,x_{24}), \end{cases} \tag{4.1.6}
$$

$$
B_{MN}(x_0,...,x_{25}) \xrightarrow[\text{Separate components}]{} \begin{cases} \underbrace{B_{\mu\nu}(x_0,...,x_{25})}_{\text{2-form for SO(25)}} \xrightarrow[\text{KK Reduction}]{} B^{(0)}_{\mu\nu}(x_0,...,x_{24}), \\ \underbrace{B_{\mu\,26}(x_0,...,x_{25})}_{\text{Vector for SO(25)}} \xrightarrow[\text{KK Reduction}]{} B^{(0)}_{\mu\,25}(x_0,...,x_{24}). \end{cases} \tag{4.1.7}
$$

The superscript (0) emphasizes that KK reduction has been performed and we are keeping only the massless modes of the KK tower. Hence, by decomposing the metric under the 25-dimesional Lorentz group we obtain a graviton, a U(1) gauge boson and a scalar. All living in 25 dimensions. By decomposing the B-field we get a 2-form and a U(1) gauge boson, again living in 25 dimensions. To explicitly see the emergence of gauge fields, we parametrise the metric as follows:

$$
ds^2 = G^{(0)}_{MN}dx^M dx^N = g_{\mu\nu}dx^\mu dx^\nu + g_{25\,25}\big(dx^{25} + A_\mu dx^\mu\big)^2. \tag{4.1.8}
$$

The relation between this parametrisation and the decomposition in 4.1.6 is given by:

$$
\begin{aligned}
G^{(0)}_{25\,25} &= g_{25\,25} = e^{2\sigma}, \\
G^{(0)}_{\mu\,25} &= e^{2\sigma}A_\mu, \\
G^{(0)}_{\mu\,\nu} &= g_{\mu\nu} + e^{2\sigma}A_\mu A_\nu,
\end{aligned} \tag{4.1.9}
$$

where we *defined* $g_{25\,25} = e^{2\sigma}$. Looking at 4.1.8, it is evident that under a diffeomorphism of the compact coordinate

$$
(x^{25})' = x^{25} + \lambda(x^\mu) \tag{4.1.10}
$$

the vector field A_μ transforms as

$$A'_\mu = A_\mu - \partial_\mu \lambda, \tag{4.1.11}$$

to ensure invariance of the metric. Hence, we have found that the U(1) gauge transformation follows from the reparametrisation of the compact coordinate. This is an instance of the Kaluza-Klein mechanism, i.e. the emergence of gauge invariance from higher-dimensional diffeomorhpism invariance. From very similar considerations follows the appearance of a U(1) gauge field in the decomposition of the B-field, that we will call $\hat{A}_\mu$.
Once the KK reduction has been performed on both G and B, we can finally reduce 4.1.5 to a 25 dimensional theory.
The Ricci scalar for the metric of the form 4.1.8 is given by:

$$R = R_{25} - 2e^{-\sigma}\nabla^2 e^{\sigma} - \frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu}, \tag{4.1.12}$$

where R_{25} is constructed from $g_{\mu\nu}$ and $F_{\mu\nu}$ is the field strength of A_μ. This, together with the definitions $G_{25} = \det(g_{\mu\nu})$ and $\phi_{25} = \phi - \sigma/2$, leads to the expression:

$$\begin{aligned} S_{eff} = \frac{\pi R}{k_0^2}\int dX^{25}(-G_{25})^{1/2}e^{-2\phi_{25}}\Big(&R - \partial_\mu\sigma\partial^\mu\sigma + 4\partial_\mu\phi_0\partial^\mu\phi_0 \\ &-\frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu v} - \frac{1}{12}\hat{H}_{\mu\nu\lambda}\hat{H}^{\mu\nu\lambda} + \frac{1}{4}e^{-2\sigma}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu}\Big), \end{aligned} \tag{4.1.13}$$

where $\hat{H}_{\mu\nu\lambda} = \partial_{[\mu}B_{\nu\lambda]} - A_{[\mu}\hat{F}_{\nu\lambda]}$ and $\hat{F}$ is the field strength of $\hat{A}_\mu$.[43] We did not provide the full calculation since this result will not appear anywhere else in this dissertation. This derivation should just serve as an example of KK reduction on a general quantum field theory, since it illustrates its main features. In particular, we emphasize that the effective field theory shows coupling with a $U(1)$ gauge theory. This was the original idea of Kaluza and Klein: obtaining Maxwell from gravity in higher dimensions.[8]

4.2 Toroidal Compactification in Bosonic String Theory

In the previous two sections we have shown how compactification affects field theories. We will now study the effects of compactification in string theory instead. We will start from S^1 compactification, as before, and then move to the more general case of toroidal compactification.
We follow mainly [21],[46] and [45] throughout the section. [44] and [43] are specific for the first part of the section [53], while [54] and [55] are relevant for the second one.

S^1 Compactification

String theory is a fundamentally different beast from field theory, and compactifying it leads to new features. Restricting to the bosonic case, we consider strings propagating in $M_{25} \times S^1$, with X_{25} being the compact coordinate. The boundary conditions read:

$$X^\mu(\sigma + l, \tau) = X^\mu(\sigma, \tau) \quad \text{for } \mu = 1, ..., 24 \tag{4.2.1}$$

$$X^{25}(\sigma + l, \tau) = X^{25}(\sigma, \tau) + 2\pi R w. \tag{4.2.2}$$

It is 4.2.2 that plays a crucial role in changing the theory. It says that the string can wrap around the circular coordinate, and w counts the number of times this happens. The special choice $w = 0$ brings the theory back to the uncompactified one. We stress that this feature is new and specific to string theory: we can only have winding if we are dealing with extended objects. To make the following expressions less cumbersome, we make the conventional choice $l = 2\pi$. Then, it is easy to check that the mode expansion satisfying 4.2.2 is given by:

$$X^{25}(\sigma, \tau) = x^{25} + \alpha' p^{25} \tau + wR\sigma + \text{oscillator modes}. \tag{4.2.3}$$

Now, as in the field theory case, p^{25} must be of the form k/R for $k \in \mathbb{Z}$. To see why, recall that the wave function contains the factor $\exp(ip^{25}x^{25})$. This, together with the fact that x^{25} and $x^{25} + 2\pi R$ are the same point, implies $p^{25} = k/R$. Mathematically,

$$e^{ip^{25}x^{25}} = e^{ip^{25}(x^{25}+2\pi R)} = e^{ip^{25}x^{25}} e^{ip^{25}2\pi R} \implies e^{ip^{25}2\pi R} = 1 \implies p^{25} = \frac{k}{R} \quad \text{with } k \in \mathbb{Z}. \tag{4.2.4}$$

We can, as usual, separate left-moving and right moving modes so that $X^{25}(\sigma, \tau) = X^{25}_L(\sigma^+) + X^{25}_R(\sigma^-)$. We achieve this by defining

$$X^{25}_L(\sigma + \tau) = \frac{x^{25}}{2} - \frac{\tilde{x}^{25}}{2} + \frac{1}{2}\alpha' p^{25}_L \sigma^+ + \text{oscillator modes}, \tag{4.2.5}$$

$$X^{25}_R(\sigma - \tau) = \frac{x^{25}}{2} + \frac{\tilde{x}^{25}}{2} + \frac{1}{2}\alpha' p^{25}_R \sigma^- + \text{oscillator modes}, \tag{4.2.6}$$

with

$$p^{25}_L = \frac{k}{R} + \frac{wR}{\alpha'}, \tag{4.2.7}$$

$$p^{25}_R = \frac{k}{R} - \frac{wR}{\alpha'}. \tag{4.2.8}$$

Let us now focus on the spectrum of this theory, from the point of view of a 25-dimensional observer. The 25-dimensional mass squared is given by:

$$M^2 = -\sum_{\mu=0}^{24} p^\mu p_\mu. \tag{4.2.9}$$

Once we quantise this theory, the physical-state condition gives us:

$$(p_L^{25})^2 + \frac{4}{\alpha'}(\tilde{N} - 1) = M^2 = (p_R^{25})^2 + \frac{4}{\alpha'}(N - 1). \tag{4.2.10}$$

The difference of these equations reads:

$$N - \tilde{N} = wk, \tag{4.2.11}$$

which is a totally new level-matching condition. The sum of the two equations yields:

$$M^2 = \left(\frac{k}{R}\right)^2 + \left(\frac{wR}{\alpha'}\right)^2 + \frac{2}{\alpha'}(N + \tilde{N} - 2). \tag{4.2.12}$$

T-duality - A first appearance

In this subsection we introduce the simplest instance of T-duality, a symmetry that arises from compactification of string theory.

The last two equations that we derived (4.2.11 and 4.2.11) provide a fundamental description of the theory we developed. T-duality is a transformation that acts on both, leaving them unchanged:

$$R \leftrightarrow \frac{\alpha'}{R} \quad \text{and} \quad k \leftrightarrow w. \tag{4.2.13}$$

This result deserves some comments on its physical implications. First of all, let us express T-duality in words: a 25-dimensional observer is not able to distinguish a universe where strings have radius R from one where they have radius $\frac{\alpha'}{R}$. This statement is not only true at the level of the free theory, but it holds for interactions as well (even though we will not prove it here). Thus, string theory in general exhibits T-duality.

This symmetry implies that we have a concept of minimum distance in string theory. Suppose that we start with a large R and start decreasing its value. Once we reach $\sqrt{\alpha'}$, the theory will look like one where R is increasing, with momentum numbers and winding numbers

exchanged.[3]
Another interesting feature appears when we consider the decompactification limit, i.e. $R \to \infty$. In this regime, the theory has an infinite number of states becoming massless (those with zero winding number). T-duality implies that the same must happen for $R \to 0$, which is indeed the case, since now for $k = 0$ all the possible winding modes have vanishing mass. This happens since once the compact dimensions are very small, winding requires almost no energy. Thus the decompactification limit coincides with the limit of "infinite compactification".
We finish this subsection by presenting an equivalent way of expressing T-duality. It is evident from 4.2.7 and 4.2.8 that the transformation 4.2.13 can be equivalently written as:

$$p_L^{25} \to p_L^{25} \quad \text{and} \quad p_R^{25} \to -p_R^{25}. \tag{4.2.14}$$

More generally, we can flip the entire right-moving part of the compact coordinate, without affecting the physical quantities. Then, T-duality reads:

$$X_L^{25} \to X_L^{25} \quad \text{and} \quad X_R^{25} \to -X_R^{25}. \tag{4.2.15}$$

The resulting solution for the compact coordinate is given by:

$$\tilde{X}^{25}(\sigma,\tau) = X_L^{25}(\sigma^+) - X_R^{25}(\sigma^-) = \tilde{x}^{25} + \omega R\tau + \alpha' \frac{k}{R}\sigma + \text{oscillator modes}. \tag{4.2.16}$$

This is clearly related to 4.2.3 by the interchanges in 4.2.13. This formulation of T-duality will show its advantages when addressing the same symmetry in superstring theory.

T^n Compactification

In this subsection, we make yet another step to a more realistic model. Compactifying only one dimension out of 26 is not satisfying if we aim at describing a 4-dimensional world. This is why we now explore how to compactify multiple dimensions. We do it in the easiest way, i.e. making the internal space a product of circles. Explicitly, $M_{tot} = M_4 \times T^{D-4}$, where $D = 26$ for the current discussion and T^{D-4} is the $(D-4=22)$-torus. The metric for this space reads:

$$ds^2 = G_{MN}dX^M dX^N = G_{\mu\nu}dX^\mu dX^\nu + G_{IJ}dY^I dY^J, \tag{4.2.17}$$

[3] This is just a hand-wavy argument, which needs further discussion once branes are taken into account.[45]

where X^μ are the coordinates on the 4-dimensional space (with $\mu, \nu = 0, 1, 2, 3$) and Y^I are the coordinates on the internal torus (with $I, J = 4, ..., D-1$). The geometry of the torus T^{D-4} is specified by G_{IJ}. It takes the form $G_{IJ} = R_I^2 \delta_{IJ}$ for a rectangular torus (all the circles are perpendicular to each other) with circles of radius R_I.
Our aim is compactifying the general sigma model given by:

$$S = -\frac{1}{4\pi} \int d\tau d\sigma (\sqrt{\gamma}\gamma^{\alpha\beta} G_{MN} + \epsilon^{\alpha\beta} B_{MN}) \partial_\alpha X^M \partial_\beta X^N, \tag{4.2.18}$$

where we have absorbed the constant α' in the definition of the background fields for convenience. We will focus on the compact coordinate part of such action:

$$S = -\frac{1}{4\pi} \int d\tau d\sigma (\sqrt{\gamma}\gamma^{\alpha\beta} G_{IJ} + \epsilon^{\alpha\beta} B_{IJ}) \partial_\alpha Y^I \partial_\beta Y^J, \tag{4.2.19}$$

where $\epsilon^{01} = -1$. We can simplify it choosing the usual gauge so that $\gamma_{\alpha\beta} = \eta_{\alpha\beta}$, obtaining

$$S = -\frac{1}{4\pi} \int d\tau d\sigma (\eta^{\alpha\beta} G_{IJ} + \epsilon^{\alpha\beta} B_{IJ}) \partial_\alpha Y^I \partial_\beta Y^J. \tag{4.2.20}$$

In analogy with the S^1 compactification case, the boundary conditions read:

$$X^\mu(\sigma + l, \tau) = X^\mu(\sigma, \tau) \quad \text{ for } i = 1, ..., 24 \tag{4.2.21}$$

$$Y^I(\sigma + l, \tau) = X^I(\sigma, \tau) + 2\pi R w^I, \tag{4.2.22}$$

with $w^I \in \mathbb{Z}$ being the winding numbers for each compact dimension. To ease the notation, we make the conventional choice $l = 2\pi$ and $R = 1$. Then, the mode expansions take the same form as 4.2.3:

$$Y^I(\tau, \sigma) = Y^I_L(\sigma^+) + Y^I_R(\sigma^-), \tag{4.2.23}$$

$$Y^I_L(\sigma^+) = \frac{1}{2} y^I + \frac{1}{2} p^I_L \sigma^+ + \frac{i}{\sqrt{2}} \sum_{n \neq 0} \frac{1}{n} \tilde{\alpha}^I_n \exp(-in\sigma^+), \tag{4.2.24}$$

$$Y^I_R(\sigma^-) = \frac{1}{2} y^I + \frac{1}{2} p^I_R \sigma^- + \frac{i}{\sqrt{2}} \sum_{n \neq 0} \frac{1}{n} \alpha^I_n \exp(-in\sigma^-). \tag{4.2.25}$$

We now determine expressions for p^I_L and p^I_R. Since $Y^I(\tau, \sigma) = \frac{1}{2}(p^I_L - p^I_R)\sigma + ...$, the boundary condition 4.2.22 implies that

$$p^I_L - p^I_R = 2w^I. \tag{4.2.26}$$

The canonical momentum density is given by:

$$p_I = \frac{\delta S}{\delta \dot{Y}^I} = \frac{1}{2\pi}(G_{IJ}\dot{Y}^J + B_{IJ}Y'^J). \tag{4.2.27}$$

Since Y^I are periodic, the total momentum vector k_I will have integer entries. Using the mode expansion yields

$$k_I = \int_0^{2\pi} p_I d\sigma = \frac{1}{2}\Big(G_{IJ}(p_L^J + p_R^J) + B_{IJ}(p_L^J - p_R^J)\Big). \tag{4.2.28}$$

Using this result, together with 4.2.26 we obtain

$$p_L^I = w^I + G^{IJ}(k_J - B_{JK}w^K), \tag{4.2.29}$$
$$p_R^I = -w^I + G^{IJ}(k_J - B_{JK}w^K). \tag{4.2.30}$$

We can now use the physical-state condition:

$$(L_0 - 1\,|\text{phys}\rangle) = 0 = (\tilde{L}_0 - 1)\,|\text{phys}\rangle\,, \tag{4.2.31}$$

to obtain

$$\frac{1}{2}p_L^I p_L^J G_{IJ} + \tilde{N} - 1 = \frac{1}{8}M^2 = \frac{1}{2}p_R^I p_R^J G_{IJ} + N - 1. \tag{4.2.32}$$

Taking the difference of the two equations yields:

$$N - \tilde{N} = \frac{1}{2}G_{IJ}(p_L^I p_L^J - p_R^I p_R^J) = w^I k_I. \tag{4.2.33}$$

This can be re-written in matrix notation as:

$$N - \tilde{N} = \frac{1}{2}\begin{pmatrix} w & k \end{pmatrix}\begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}\begin{pmatrix} w \\ k \end{pmatrix}. \tag{4.2.34}$$

On the other hand, the sum of the two equations gives

$$M^2 = 2G_{IJ}(p_L^I p_L^J + p_R^I p_R^J) + 4(N + \tilde{N} - 2). \tag{4.2.35}$$

To make the symmetries of the spectrum manifest, we let $M_0^2 = 2G_{IJ}(p_L^I p_L^J + p_R^I p_R^J)$, and use 4.2.29, 4.2.30 to write it as:

$$\frac{1}{2}M_0^2 = (w\ \ k)\mathcal{G}\begin{pmatrix} w \\ k \end{pmatrix}, \tag{4.2.36}$$

where we defined

$$\mathcal{G} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}. \tag{4.2.37}$$

T-duality - A Quick Derivation

Here we introduce T-duality with a very brief derivation.
A necessary condition for a symmetry of the theory is to leave 4.2.34 and 4.2.36 invariant. Inspired by the S^1 symmetry, which exchanges momentum and winding number, it is natural to propose a general transformation of the form:

$$\begin{pmatrix} w \\ k \end{pmatrix} \to \begin{pmatrix} w \\ k \end{pmatrix}' = (A^{-1})^T \begin{pmatrix} w \\ k \end{pmatrix} \quad \text{for some } A \in GL(2n), \tag{4.2.38}$$

where $n = D - 4$. We include the transpose and inverse in the transformation just for convenience. It is just a conventional choice, but it has no physical meaning: if $A \in GL(2n)$, then clearly $(A^{-1})^T \in GL(2n)$ too.
We now derive the conditions for A to be a symmetry.
The level-matching condition 4.2.34 after the transformation becomes[4]

$$N - \tilde{N} = \frac{1}{2} \begin{pmatrix} w & k \end{pmatrix} A^{-1} \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} (A^{-1})^T \begin{pmatrix} w \\ k \end{pmatrix}. \tag{4.2.39}$$

Thus, the symmetry condition is $A^{-1} \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} (A^{-1})^T = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$. This condition is exactly the definition of the elements A^{-1} of the group $O(n,n)$[5], meaning that $(A^{-1}) \in O(n,n)$. By virtue of $O(n,n)$, we also have that $(A^{-1})^T \in O(n,n)$ and $A \in O(n,n)$.
The second condition for this transformation to be an automorphism of the theory is that $(A^{-1})^T$ must have integer elements. In other words, winding numbers and momenta in the compact directions must remain quantised after the transformation 4.2.38. This requirement implies that $(A^{-1})^T$ belongs to the discrete subgroup of $O(d,d)$: $O(d,d,\mathbb{Z})$. Thanks to the properties of $O(n,n)$, which we will explore in the next subsection, we can again infer from the above that $A^{-1} \in O(n,n,\mathbb{Z})$ and $A \in O(n,n,\mathbb{Z})$.
Up to now we have ensured that level-matching and quantised variables are preserve. We now inspect the mass condition. We let the transformation act on $\mathcal{R}$ simply as

$$\mathcal{G} \to \mathcal{G}' = A\mathcal{G}A^T, \tag{4.2.40}$$

[4]Note: we have assumed that the lhs is not affected by this transformation, which is intuitively true. The number operators are primitive objects in string theory: they count the number of excitations, but do not care about the specific data of our theory. Thus, they must be independent of the choice of backgrounds, winding numbers or momentum numbers. A more careful mathematical justification can be found in [53].

[5]Strictly speaking, it is not *the* definition. Putting the transpose on the other matrix defines exactly the same group, as we shall discuss in the next subsection.

which is the reason behind the transpose and inverse before. Then, with this choice, 4.2.36 is also invariant. Explicitly,

$$\frac{1}{2}M_0^2 \to (w \;\; k)A^{-1}A\mathcal{G}A^T(A^{-1})^T \begin{pmatrix} w \\ k \end{pmatrix} = \frac{1}{2}M_0^2. \tag{4.2.41}$$

Hence, any $A \in O(n,n,\mathbb{Z})$ is a symmetry transformation of the theory, i.e. the T-duality group for the toroidally compactified bosonic string is $O(n,n,\mathbb{Z})$.

T-duality - A more general(ised?) Derivation

In this subsection we provide a more careful derivation of the same result. We do this with a mathematical formalism that is definitely more complicated, but alludes for the first time to the framework of generalised geometry.
As the first thing, we define the *background metric* E as the sum of G and B:

$$E_{MN} = G_{MN} + B_{MN}. \tag{4.2.42}$$

As before, we focus only on the compactified sector of the action. Thus from now on we will refer to the background metric as to its internal components, i.e. we will only be concerned with $E_{IJ} = G_{IJ} + B_{IJ}$. It follows that the symmetric part of E yields the toroidal metric and the antisymmetric part of E gives the (internal) B-field. In other words, E contains all the information about the background of the theory, and hence the name. All the quantities that we introduced previously depend on E, for instance $\mathcal{G} = \mathcal{G}(E)$, $M_0^2 = M_0^2(E)$ and $\alpha_n^I = \alpha_n^I(E)$. Hence, it is more elegant to formulate T-duality purely in terms of a transformation of E. We know from the previous section that the symmetry group is a discrete subgroup of $O(n,n)$. For completeness, we provide its formal definition,

$$O(n,n) = \{A \in GL(2n,\mathbb{R}) : A^T \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} A = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}\} \tag{4.2.43}$$

and present some of its key features.
Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(n,n)$, then the following conditions must hold:

$$\begin{aligned} &a^Tc + c^Ta = 0, \\ &b^Td + d^Tb = 0, \\ &a^Tb + c^Td = \mathbb{1}_n. \end{aligned} \tag{4.2.44}$$

It is quick to show from the definition that $A^T \in O(n,n)$. Starting from the condition in 4.2.43, this is done by first inverting it, then multiplying by A on the left and by A^T on the right. Hence, this implies a similar set of conditions:

$$\begin{aligned} &ab^T + ba^T = 0, \\ &cd^T + dc^T = 0, \\ &d^T a + b^T c = \mathbb{1}_n. \end{aligned} \tag{4.2.45}$$

It follows from the first two equations of 4.2.44 and the last one of 4.2.45 that

$$A^{-1} = \begin{pmatrix} d^T & b^T \\ c^T & a^T \end{pmatrix}. \tag{4.2.46}$$

The fact that $A^{-1} \in O(n,n)$ follows again from the defining condition. Taking its inverse shows that $(A^{-1})^T$ belongs to $O(n,n)$, and then the above result (i.e. that $A \in O(n,n) \implies A^T \in O(n,n)$) shows $A^{-1} \in O(n,n)$. Note that this one is just a check, rather than a new result.

The properties derived above clarify a few things that we glossed over before. Firstly, as we mentioned in the footnote, it shows why we can immediately infer from 4.2.39 that the two undetermined matrices appearing in it both belong to $O(n,n)$, as well as their inverses. Secondly, it shows that the properties derived for $O(n,n)$ also hold for $O(n,n,\mathbb{Z})$, which is not just a subset of $O(n,n)$, but a subgroup of it. Specifically, it is obvious that matrices with integer entries give other matrices with integer entries when multiplied. The fact that their inverses still have integer entries seem a bold statement, but it is nevertheless true for matrices in $O(n,n)$, according to 4.2.46. Hence, $O(n,n,\mathbb{Z})$ forms a group by its own, as claimed.

Let us now return to our recently defined backgound metric. Since by $O(n,n)$ is a group of $2n \times 2n$ matrices, the immediate question is how to act it on the $n \times n$ matrix E. This can be achieved via a fractional linear transformation. Given an element $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(n,n)$, we let it act on E as:

$$E \to E' = A(E) = (aE + b)(cE + d)^{-1}. \tag{4.2.47}$$

This choice is such that if $E' = A(E)$ for any $A \in O(n,n)$, then:

$$\mathcal{G}(E') = A\mathcal{G}(E)A^T = \mathcal{G}(E)', \tag{4.2.48}$$

where the prime on the RHS refers the transformation found in the previous section, 4.2.40. To show this, let us take 4.2.47 as our starting point. For a given $A \in O(n,n)$, it is a map of the form $GL(n) \to GL(n)$. We then define a vielbein e for the toroidal metric (see appendix):

$$G_{IJ} = \sum_{\alpha} e_I^{\alpha} e_J^{\alpha} \quad \text{or} \quad G = ee^T, \tag{4.2.49}$$

and use it to define the $2n \times 2n$ matrix

$$g_E = \begin{pmatrix} e & B(e^T)^{-1} \\ 0 & (e^T)^{-1} \end{pmatrix}. \tag{4.2.50}$$

The subscript E emphasizes that such matrix is built entirely from the data contained in E. It is easy to check that $g_E \in O(n,n)$. Thus, it labels a $GL(n) \to GL(n)$ map described 4.2.47, which has the following property:

$$g_E(1_n) = E. \tag{4.2.51}$$

Moreover, g_E is itself the vielbein of $\mathcal{G}(E)$, since it satisfies

$$g_E(g_E)^T = \mathcal{G}(E). \tag{4.2.52}$$

Now we make good use of the relations derived so far. Firstly, have that

$$\mathcal{G}(E)' = A\mathcal{G}(E)A^T = Ag_E g_E^T A^T, \tag{4.2.53}$$

where we used 4.2.40 for the first step and 4.2.52 for the second step. By 4.2.52 we also have that

$$\mathcal{G}(E') = g_{E'}(g_{E'})^T. \tag{4.2.54}$$

Requiring that these two transformations are equal (which is the aim of this discussion) implies

$$g_{E'} = Ag_E. \tag{4.2.55}$$

Then, by 4.2.51:

$$E' = g_{E'}(1_n) = Ag_E(1_n) = A(E) = (aE+b)(cE+d)^{-1}, \tag{4.2.56}$$

as claimed.
In summary, we encoded the transformation of $\mathcal{G}$ (4.2.40) in a transformation of a more fundamental object: the background metric. Hence, the symmetry group $O(n,n,\mathbb{Z})$ encodes all the transformations that leave bosonic string theory compactified on a torus invariant. It is worth studying this discrete group in more detail, and we will do so via the discussion of its generators. $O(d,d,\mathbb{Z})$ is generated by three type of elements:

1. **Integer theta-parameter shift**
   These elements take the form:
   $$A_\Theta = \begin{pmatrix} 1_n & \Theta \\ 0 & 1_n \end{pmatrix} \quad \text{with} \quad \Theta \in GL(D-4,\mathbb{Z}) \quad \text{and} \quad \Theta_{IJ} = -\Theta_{JI}. \tag{4.2.57}$$
   It is easy to check they belong to $O(n,n,\mathbb{Z})$ (it just follows from matrix multiplication and antisymmetry of Θ). Let us examine how this transformation acts on the fields. Just by performing matrix multiplication, we find:
   $$\begin{aligned} A_\Theta \mathcal{G} A_\Theta^T &= \begin{pmatrix} G - BG^{-1}B + BG^{-1}\Theta^T - \Theta G^{-1}B + \Theta G^{-1}\Theta^T & BG^{-1} + \Theta G^{-1} \\ -G^{-1}B + G^{-1}\Theta^T & G^{-1} \end{pmatrix} = \\ &\begin{pmatrix} G - (B+\Theta)G^{-1}(B+\Theta) & (B+\Theta)G^{-1} \\ -G^{-1}(B+\Theta) & G^{-1} \end{pmatrix} = \mathcal{G}(G, B+\Theta), \end{aligned} \tag{4.2.58}$$
   where in the second step we used the antisymmetry of Θ. Hence, its action is that of shifting B by an antisymmetric matrix with integer entries. At the same time, A_Θ also acts on the momentum and winding numbers. We have that $(A_\Theta^{-1})^T$ is the same, with Θ moved to the other off-diagonal block, which implies the transformations:
   $$w^I \to w^I \quad \text{and} \quad k_I \to k_I + \Theta_{IJ} w^J. \tag{4.2.59}$$
   By looking at equations 4.2.29 and 4.2.30, we see that they are left invariant by the above transformations.

2. **Basis change**
   These elements can be written as:
   $$A_M = \begin{pmatrix} M & 0 \\ 0 & (M^T)^{-1} \end{pmatrix} \text{with} \quad M_{IJ} \in GL(D-4,\mathbb{Z}). \tag{4.2.60}$$
   This symmetry corresponds to a change of the compactification lattice, which we did not introduce, and is not as easily visualised as the previous ones.

3. **Factorised Duality**
   These are elements of the form:
$$\begin{pmatrix} \mathbb{1}_n - e_i & e_i \\ e_i & \mathbb{1}_n - e_i \end{pmatrix}, \tag{4.2.61}$$
   where e_i is a matrix of zeroes except for the component ii, which is one. This transformation is the generalisation of 4.2.13 to the case of toroidal compactification. It is immediate to check that its effect on momentum and winding numbers corresponds to the exchange $w^i \leftrightarrow k_i$. Moreover, it takes $R_i \to \frac{1}{R_i}$, where R_i is the radius of the ith compact direction. Its effect on the background in components is given by:
$$g_{00} \to \frac{1}{g_{00}}, \quad g_{0M} \to \frac{b_{0M}}{g_{00}}, \quad g_{MN} \to g_{MN} - \frac{g_{0M}g_{0N} - b_{0M}b_{0N}}{g_{00}}, \tag{4.2.62}$$
$$b_{0M} \to \frac{g_{0M}}{g_{00}}, \quad b_{MN} \to b_{MN} - \frac{g_{0M}b_{0N} - b_{0M}g_{0N}}{g_{00}}. \tag{4.2.63}$$
   These are called *Buscher rules*, and we derive them in next section (with $i = 0$) following the original construction.

4.3 Buscher's Approach

In this section, we study an approach to T-duality that was originally formulated by Buscher. The discussion is based on his original work [56], along with [46] and [57].

Let us consider the non-linear sigma-model in D-dimensions:
$$S_\sigma = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \Big[\sqrt{-\gamma}\gamma^{ab}G_{MN}\partial_a X^M \partial_b X_N - \epsilon^{ab}B_{MN}\partial_a X^M \partial_b X^N\Big]. \tag{4.3.1}$$

We assume that there is a $U(1)$ (abelian) isometry, represented by a translation in one of the space-like coordinates. We denote such coordinate by θ, so that $X^M = (\theta, X^\mu)$ with $\mu = 0, ..., n-1$; the metric and the B-field are independent of θ.
The same physics expressed in 4.3.1 can be equivalently encoded in a new action:
$$S'_\sigma = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \Big[\sqrt{-\gamma}\gamma^{ab}\big(G_{00}V_aV_b + 2G_{0\mu}V_a\partial_b X^\mu + G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\big) - \tag{4.3.2}$$
$$-\epsilon^{ab}\big(2B_{0\mu}V_a\partial_b X^\mu + B_{\mu\nu}\partial_a X^\mu\partial_b X^\nu\big) - 2\epsilon^{ab}\tilde{\theta}\partial_a V_b\Big]. \tag{4.3.3}$$

This can be easily seen with a hint: $\tilde{\theta}$ acts as a Lagrange multiplier. Specifically, varying wrt $\tilde{\theta}$ one obtains $V_a = \partial_a \theta$,[6] and we recover S_σ.
Once the equivalence between 4.3.1 and 4.3.3 has been shown, we can use 4.3.3 to produce yet another action: the dual to S_σ. To do this, we start by varying wrt V_a. The resulting EoM (obtained by integrating by parts and discarding boundary terms) is only algebraic, and reads:

$$V_a = -\frac{1}{G_{00}}\big(G_{0\mu}\partial_a X^\mu - \frac{\epsilon_a{}^b}{\sqrt{-\gamma}}(B_{0\mu}\partial_b X^\mu + \partial_b\tilde{\theta})\big). \tag{4.3.4}$$

Substituting this back into 4.3.3 (which is equivalent to integrating out V_a in the path integral) yields

$$\begin{aligned} S^d_\sigma = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \Big[&\sqrt{-\gamma}\gamma^{ab}\big(\tilde{G}_{00}\partial_a\tilde{\theta}\partial_b\tilde{\theta} + 2\tilde{G}_{0\mu}\partial_a\tilde{\theta}\partial_b X^\mu + \tilde{G}_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\big) - \\ &-\epsilon^{ab}\big(2\tilde{B}_{0\mu}\partial_a\tilde{\theta}\partial_b X^\mu + \tilde{B}_{\mu\nu}\partial_a X^\mu\partial_b X^\nu\big)\Big], \end{aligned} \tag{4.3.5}$$

where

$$\begin{aligned} \tilde{G}_{00} = \frac{1}{G_{00}}, \qquad \tilde{G}_{0\mu} &= \frac{B_{0\mu}}{G_{00}}, \qquad \tilde{G}_{\mu\nu} = G_{\mu\nu} - \frac{G_{0\mu}G_{0\nu} - B_{0\mu}B_{0\nu}}{G_{00}} \\ \tilde{B}_{0\mu} &= \frac{G_{0\mu}}{G_{00}}, \qquad \tilde{B}_{\mu\nu} = B_{\mu\nu} - \frac{G_{0\mu}B_{0\nu} - G_{0\nu}B_{0\mu}}{G_{00}}. \end{aligned} \tag{4.3.6}$$

These are the Buscher rules.[7] We will come back to this topic for a deeper analysis in the next chapter (section 3), since we have not unlocked the power of differential geometry yet.

4.4 Torodial Compactification in Type II Superstring Theory

This section describes toroidal compactification for type II superstring theory. In this chapter, it represents the last step towards a more realistic description of the world. Let us

[6] More precisely, we get $\partial_{[a}V_{b]}$. If we let V_a be the components of a 1-form V, this reads $dV = 0$. For topologically trivial world-sheets (i.e. where Poincare lemma can be applied), this implies that $V = d\theta$ for some scalar θ, so that $V_a = \partial_a \theta$

[7] The inclusion of the dilaton in this discussion needs discussions about regularization and conformal invariance, and is not relevant for our purposes. For more detail, see [58].

quickly review the ones that we have already done. We have started with bosonic string theory, which has no fermions and predicts 26 space-time dimensions. Then, we have obtained a theory in 25 dimensions via compactification of one dimension, and subsequently we described how to compactify any number of dimensions. This brought the number of dimensions to potentially 4. Now, we introduce fermions on our theory (i.e. consider type II superstrings), and try to again reduce the number of dimensions. We find that, by compactifying one dimension, type IIA and type IIB theories are related by T-duality. A striking result indeed.
The relevant references for this section are [45], [46] and [21].

S^1 Compactification of the Fields - A Hint of T-duality

We have already discussed how compactifying a single dimension cannot produce a realistic model of the world. But, at the same time, we showed that novel and interesting features arise when we do so for the bosonic string theory. Motivated by this, we will now perform S^1 compactification for type II superstring theory. And we will be lucky enough to find more novel and more interesting features arising.
At the end of the previous chapter, we have derived the massless spectrum of Type II superstring theories. We will assume that one of the dimensions is compactified on a circle, and we will perform a KK reduction of the fields in the spectrum. This may seem inappropriate, since we are studying what happens to the low energy limit of superstring theory without having gone through an analysis of what happens to the full theory. But note that:

1. This is just a shortcut. Compactifying superstring theory and then looking at its massless spectrum would give the same answer.[59]

2. It is a very common procedure to look at the effective theory to study the symmetries of the full theory itself. We are going to repeat this procedure in the next chapter.

As we saw, the KK reduction consists of Fourier expanding the fields wrt the compact coordinate and then keeping only the massless mode of the KK tower that emerges. In practice, this means decomposing the SO(8) representations of the 10-dimensional theory into SO(7) representations of the 9-dimensional theory. The decomposition for the two open-string massless states is given in table 4.1. The decomposition of the vector representation clearly contains a singlet, which can be thought as a vector with only the last entry being nonzero. What is left is the fundamental of $SO(7)$. Such decomposition does not apply to

Sector	State	SO(8)	SO(7)
NS	$b^i_{-1/2}\vert 0\rangle$	$\mathbf{8}_V$	$\mathbf{7}+\mathbf{1}$
R	$\vert\psi_+\rangle$	$\mathbf{8}$	$\mathbf{8}$
R	$\vert\psi_-\rangle$	$\mathbf{8}$’	$\mathbf{8}$

Table 4.1:

the spinorial case, where the singlet does not exist. Note that, since in odd dimensions there is no chirality, the two representations of different chiralities in $SO(8)$ reduce to the same one for $SO(7)$.

For the closed string we need to take tensor products of the above states, and then decompose them. Alternatively, we can firstly decompose them and secondly take the tensor products. Both ways lead to the same answer, which is shown in the tables below (inspired by [45]). Horizontal arrows correspond to taking the tensor product, while vertical ones correspond to performing the decomposition. We start from type IIB:

$$
\begin{array}{ll}
NS-NS: & \left\{ \begin{array}{ccrccccc}
8_V, 8_V & \rightarrow & 8_V \times 8_V = & 35_V & + & 28_V & + & 1 \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow \\
 & & 7\times 7 = & 27 & + & 21 & + & 1 \\
7+1, 7+1 & \rightarrow & 7\times 1 + 1\times 7 = & 7 & + & 7 & & \\
 & & 1\times 1 = & 1 & & & &
\end{array} \right. \\
NS-R: & \left\{ \begin{array}{ccrccc}
8_V, 8 & \rightarrow & 8_V \times 8 = & 56' & + & 8' \\
\downarrow & & & \downarrow & & \downarrow \\
7+1, 8 & \rightarrow & 7\times 8 = & 48+8 & & \\
 & & 1\times 8 = & & & 8
\end{array} \right. \\
R-NS: & \left\{ \begin{array}{ccrccc}
8, 8_V & \rightarrow & 8 \times 8_V = & 56' & + & 8' \\
\downarrow & & & \downarrow & & \downarrow \\
8, 7+1 & \rightarrow & 8\times 7 = & 48+8 & & \\
 & & 8\times 1 = & & & 8
\end{array} \right. \\
R-R: & \left\{ \begin{array}{ccrccccc}
8, 8 & \rightarrow & 8\times 8 = & 35 & + & 28 & + & 1 \\
\downarrow & & & \downarrow & & \downarrow & & \downarrow \\
8, 8 & \rightarrow & 8\times 8 = & 35 & + & 21+7 & + & 1
\end{array} \right.
\end{array}
\tag{4.4.1}
$$

The field content of the field theory after KK reduction appears with a grey background. We now perform the same analysis for type IIA:

$$
\begin{array}{ll}
NS-NS: & \left\{\begin{array}{ccccccc}
8_V, 8_V & \rightarrow & 8_V \times 8_V = & 35_V & + & 28_V & + \; 1 \\
\downarrow & & & \downarrow & & \downarrow & \downarrow \\
 & & 7 \times 7 = & 27 & + & 21 & + \; 1 \\
7+1, 7+1 & \rightarrow & 7 \times 1 + 1 \times 7 = & 7 & + & 7 & \\
 & & 1 \times 1 = & 1 & & &
\end{array}\right. \\
NS-R: & \left\{\begin{array}{ccccc}
8_V, 8 & \rightarrow & 8_V \times 8 = & 56' & + \; 8' \\
\downarrow & & & \downarrow & \downarrow \\
7+1, 8 & \rightarrow & 7 \times 8 = & 48+8 & \\
 & & 1 \times 8 = & & 8
\end{array}\right. \\
R-NS: & \left\{\begin{array}{ccccc}
8', 8_V & \rightarrow & 8' \times 8_V = & 56 & + \; 8 \\
\downarrow & & & \downarrow & \downarrow \\
8, 7+1 & \rightarrow & 8 \times 7 = & 48+8 & \\
 & & 8 \times 1 = & & 8
\end{array}\right. \\
R-R: & \left\{\begin{array}{cccccc}
8, 8' & \rightarrow & 8 \times 8' = & 56_V & + & 8_V \\
\downarrow & & & \downarrow & & \downarrow \\
8, 8 & \rightarrow & 8 \times 8 = & 35+21 & + & 7+1
\end{array}\right.
\end{array}
\tag{4.4.2}
$$

Comparing the two, we see that the field content exactly matches. This tedious exercise of representation theory proves that, at least at the massless level, type IIA and type IIB superstring theories compactified on a circle are the same theory. This is an example of how to extract useful information about the full theory from its low energy limit. This result hints at the fact that superstrings of type IIA and IIB are indistinguishable once we take one coordinate to be periodic. We show that this is indeed the case in the next section.

S^1 Compactification of the Strings - The Power of T-duality

We now proceed to the compactification of type II superstrings, following closely the bosonic case. The bosonic sector of superstring theory is extremely similar to bosonic string theory. In fact, if we change the number of dimensions from 10 to 26, the two are identical. It follows that all the analysis performed in section 4.2 applies here, with the only difference that compactification is performed along the direction X^9 instead of X^{25}. Following this logic, T-duality for the bosonic sector of superstring theory takes the form:

$$X_L^9 \to X_L^9 \quad \text{and} \quad X_R^9 \to -X_R^9. \tag{4.4.3}$$

As before, this corresponds to interchanging momentum and winding numbers. Writing the transformation in this form is very convenient, since it immediately fixes the transformation of the fermionic partner. This because, according to world-sheet supersymmetry, X^9 and ψ^9 must transform in the same way. Thus, we infer the effect of T-duality on the fermionic fields:

$$\psi_L^9 \to \psi_L^9 \quad \text{and} \quad \psi_R^9 \to -\psi_R^9. \tag{4.4.4}$$

This transformation changes the chirality of ground state in the right-moving R sector. To see why, consider the effect that T-duality has on d_0^μ ($d_0^\mu \to -d_0^\mu$) and recall that we defined $d_0^\mu = \frac{1}{\sqrt{2}}\Gamma^\mu$. It follows that, under T-duality,

$$\Gamma_{10} = \Gamma_0 ... \Gamma_9 \to -\Gamma_0 ... \Gamma_9 = -\Gamma_{10}. \tag{4.4.5}$$

Thus, the chirality of the ground state in the right-moving R sector is flipped. This means that if we start with type IIB, with both R ground states having the same chirality, we end up with them having different chiralities, i.e. IIA. Conversely, starting with IIA we obtain IIB. Hence, T-duality shows that type IIA and type IIB superstring theories compactified on a circle are equivalent. This is a *striking* result.

S^n Compactification

The natural conclusion of this section is the study of how to compactify several dimensions in type II superstring theory. We would like to start with a theory analogous to 4.2.18, but with supersymmetry, and then compactify it on a torus, study its moduli space and so on. Unfortunately, it is not known how to quantise such theory when the backgrounds for the RR fields are nontrivial ([45]). All we could do is follows the above procedure for the NS-NS sector only, which is exactly what we did in section 4.2. Thus, to make any progress, we have to look at the low energy approximation of type II string theory: type II supergravity. This leads us to the next section, which aims at introducing it.

4.5 Type II Supergravity

In this section there are no strings. It is dedicated to their low energy effective actions, i.e. supergravity theories. We start by introducing eleven-dimensional supergravity, which is a

theory with very special features. Then, we illustrate how to obtain type IIA supergravity in 10 dimensions via dimensional reduction, and present type IIB supergravity. Finally, we give an overview of the democratic formalism for the two theories.
The relevant sources for this section are: [21], [48], [46] for the first two subsections and [35], [60] for the last one.

11-d Supergravity

11-dimensional supergravity is a special theory. Firstly, it is remarkably simple, as we shall see. Secondly (and most importantly), eleven is the maximum number of dimensions in which one can have a supergravity theory, i.e. a theory with particles of spin 2 or less.[8] We will first present the bosonic part action, then explain the notation and describe its field content. The bosonic sector of 11-d sugra is:

$$S = \frac{1}{2\kappa_{11}^2}\left[\int d^{11}x\sqrt{-G}\Big(R - \frac{1}{2}|F_4|^2\Big) - \frac{1}{6}\int A_3 \wedge F_4 \wedge F_4\right], \tag{4.5.1}$$

where:

- κ_{11} is the coupling constant.
- G^{MN} is the metric, G its determinant, and R is the Ricci scalar (see A.1.52).
- F_4 is the field strength of A_3, i.e. $F_{M_1M_2M_3M_4} = 4\partial_{[M_1}A_{M_2M_3M_4]}$ (or, in short, $F_4 = dA_3$). Subscripts indicate the rank of the tensor. This notation (and its generalisation) is ubiquitous to this section.
- $\int A_3 \wedge F_4 \wedge F_4 = \int d^{11}x \frac{1}{(12)^4}\epsilon^{M_1\cdots M_{11}}F_{M_1\cdots M_4}F_{M_5\cdots M_8}A_{M_9M_{10}M_{11}}$.
- $|F_n|^2 = \frac{1}{n!}G^{M_1N_1}...G^{M_nN_n}F_{M_1...M_n}F_{N_1...N_n}$.

All indices run from 0 to 10. We promised to avoid differential geometry in this chapter. However, its notation is so convenient that we could not resist using it. For a very detailed derivation, which has no reference at all to differential geometry, see [46].
It is useful (and necessary every time we deal with spinors), to introduce a vielbein for

[8] This result will not be proven here. The relevant reference is [61]. Also, the action that we provide below is written in a modern fashion. To link this to the normalisation adopted in the original papers, please see [46].

the metric (see relevant section in the Appendix, for a quick introduction in the context of differential geometry). This consists of defining a matrix E^A_M such that:

$$G_{MN} = \eta_{AB} E^A_M E^B_N. \tag{4.5.2}$$

As we mentioned, the bosonic field content of the theory is remarkably simple. We have a graviton G_{MN} and a three-form A_{MNP}. The first one is a symmetric traceless tensor of $SO(D-2)$, the little group for massless particles. It has $\frac{(D-1)(D-2)}{2} - 1 = 44$ DoF, for $D = 11$. The second one is a rank-3 antisymmetric tensor, which carries 84 DoF. Thus, as for any supersymmetric theory, there must be other 128 fermionic DoF in the spectrum. These are contained in the gravitino field Ψ_M. The full action includes the kinetic term for the gravitino, and two other terms. We will not present it here, since it is not relevant for the purpose of this work, but it can be found in [62]. What really concerns us are the supersymmetry transformations that leave such action invariant. In order to present them, we introduce the eleven gamma matrices in eleven dimensions, Γ_A. We let $\Gamma_M = E^A_M \Gamma_A$. We also introduce the following definitions:

$$\begin{aligned} \mathbf{F}^{(4)} &= \frac{1}{4!} F_{MNPQ} \Gamma^{MNPQ}, \\ \mathbf{F}^{(4)}_M &= \frac{1}{3!} F_{MNPQ} \Gamma^{NPQ}, \end{aligned} \tag{4.5.3}$$

where $\Gamma^{M_1 M_2 \ldots M_n} = \Gamma^{[M_1} \Gamma^{M_2} \ldots \Gamma^{M_n]}$. With this notation, the supersymmetry transformation then read:

$$\begin{aligned} \delta E^A_M &= \bar{\epsilon} \Gamma^A \Psi_M, \\ \delta A_{MNP} &= -3 \bar{\epsilon} \Gamma_{[MN} \Psi_{P]}, \\ \delta \Psi_M &= \nabla_M \epsilon + \frac{1}{12} \big(\Gamma_M \mathbf{F}^{(4)} - 3 \mathbf{F}^{(4)}_M \big) \epsilon. \end{aligned} \tag{4.5.4}$$

The covariant derivative ∇_M uses the spin connection ω, and takes the form:

$$\nabla_M \epsilon = \partial_M \epsilon + \frac{1}{4} \omega_{MAB} \Gamma^{AB} \epsilon, \tag{4.5.5}$$

where

$$\omega_{MAB} = \frac{1}{2} (+\Omega_{ABM} - \Omega_{MAB} - \Omega_{BMA}), \tag{4.5.6}$$

and

$$\Omega^A_{MN} = 2 \partial_{[N} E^A_{M]}. \quad [21] \tag{4.5.7}$$

Type IIA Supergravity

We now perform a KK reduction on this field theory, from 11 dimensions to 10 dimensions. The resulting theory is type IIA supergravity. The name is not just coincidental, as this theory is the low energy approximation of Type IIA closed superstring theory. We will show how its spectrum is the same as the massless spectrum derived in section 3.5.
We start by reducing the 11-dimensional gravitino Ψ_N to 10 dimensions. This can be obtained by using only group theory and representation theory, but we will try to give a short physical interpretation. As we discussed, Ψ_N belongs to the **128** of $Spin(9)$. Then, the first ten components give the two 10-dimensional gravitions, while Ψ_{10} gives two 10-dimensional dilationos. Each has only 8 DoF, due to the Dirac equation. Hence, we infer that the two gravitinos have 56 DoF each. What we have effectively done is decomposing the **128** of $Spin(9)$ into irreps of its subgroup $Spin(8)$.
The dimensional reduction of the bosonic fields is more familiar. We will use greek letters for the 10-dimensional theory: $\mu, \nu = 0, ..., 9$. Following exactly the same procedure as in section 4.1, we obtain

$$ds^2 = e^{-2\phi/3} g_{\mu\nu} dx^\mu dx^\nu + e^{4\phi/3}(dx^{10} + A_\mu dx^\mu)^2. \tag{4.5.8}$$

Note that this equation is identical to 4.1.8, with $\phi \to \sigma$ (σ is defined in 4.1.9) and an overall scaling by $e^{-2\phi/3}$. Thus, we have a 10-dimensional metric $g_{\mu\nu}$, a $U(1)$ gauge field A_μ and a scalar (dilaton) field ϕ. We denote the field strength of the gauge field, as usual, by $F_{\mu\nu}$. In terms of these objects, the veilbein reads:

$$E_M^A = \begin{pmatrix} e^{-\phi/3} e_\mu^a & 0 \\ e^{2\phi/3} A_\mu & e^{2\phi/3} \end{pmatrix}, \tag{4.5.9}$$

where e_μ^a is the vielbein for the 10-dimensional metric $g_{\mu\nu}$. It is easy to see that its inverse is given by:

$$E_B^M = \begin{pmatrix} e^{\phi/3} e_b^\mu & 0 \\ -e^{\phi/3} A_\nu e_b^\nu & e^{-2\phi/3} \end{pmatrix}. \tag{4.5.10}$$

$$\tag{4.5.11}$$

The 11-dimensional three form decomposes into a 2-form and a 3-form:

$$A_{MNP} \to \begin{cases} A_{\mu\nu\sigma} & \text{(3-form)}, \\ A_{\mu\nu 10} \equiv B_{\mu\nu} & \text{(2-form)}, \end{cases} \tag{4.5.12}$$

and we denote their field strength by $F_{\mu\nu\sigma\rho}$ and $H_{\mu\nu\sigma}$, respectively. Hence, the 10-dimensional fields that appear are exactly those that we obtained from the massless states of type IIA superstring theory in section 3.5 (cf. 3.5.36). This is how we see that type IIA supergravity is the low energy limit of type IIA superstring theory. We will shortly show that the same applies to type IIB.

What we derived above are all the ingredients needed to perform the KK reduction of 4.5.1. We will not go through the full calculation, but the reader is referred to chapter 10 of [46] and references therein. The result is (setting the coupling constant to one):

$$S_{IIA} = S_{NS} + S_R + S_{CS}, \tag{4.5.13}$$

with

$$\begin{aligned} S_{NS} &= \frac{1}{2}\int d^{10}x\sqrt{-g}e^{-2\phi}\Big(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}|H_3|^2\Big), \\ S_R &= -\frac{1}{4}\int d^{10}x\sqrt{-g}\Big(|F_2|^2 + |\tilde{F}_4|^2\Big), \\ S_{CS} &- \frac{1}{4}\int B_2 \wedge F_4 \wedge F_4. \end{aligned} \tag{4.5.14}$$

The $\tilde{F}_4$ that appears above is given by $\tilde{F}_{\mu\nu\sigma\rho} = 4(\partial_{[\mu}A_{\nu\sigma\rho]} + A_{[\mu}H_{\nu\sigma\rho]})$, or simply $\tilde{F}_4 = dA_3 + A_1 \wedge H_3$.

Finally, we can also obtain the supersymmetry transformations. They read ([21]):

$$\begin{aligned} \delta\lambda &= \Big(-\frac{1}{3}\Gamma^\mu\partial_\mu\phi\Gamma_{11} + \frac{1}{6}\mathbf{H}^{(3)} - \frac{1}{4}e^\phi\mathbf{F}^{(2)} + \frac{1}{12}e^\phi\tilde{\mathbf{F}}^{(4)}\Gamma_{11}\Big)\epsilon, \\ \delta\Psi_\mu &= \Big(\nabla_\mu - \frac{1}{4}\mathbf{H}^{(3)}_\mu\Gamma_{11} - \frac{1}{8}e^\phi F_{\nu\alpha}\Gamma^{\nu\rho}_\mu\Gamma_{11} + \frac{1}{8}e^\phi\mathbf{F}^{(4)}\Gamma_\mu\Big)\epsilon. \end{aligned} \tag{4.5.15}$$

Type IIB Supergravity

Unfortunately, type IIB supergravity cannot be obtained by dimensional reduction from 11-dimensional supergravity. And this not all. It does not have an action either (it has a pseudo-action, which we will define). This theory is specified by the equations of motion and the supersymmetry transformations. It should not come as a surprise, given the above premises, that obtaining type IIB supergravity is not straightforward. And, since a detailed introduction on supersymmetry is outside the scope of this dissertation, we will just quote the result for this time.

The pseudo-action for type II supergravity is of the form:

$$S_{IIA} = S_{NS} + S'_R + S'_{CS}, \tag{4.5.16}$$

where S_{NS} is the same as in 4.5.14, but S_R and S_3 are different. The new degrees of freedom for type IIB are: a scalar C_0, a 2-form C_2 and a 4-form C_4. Again, the fields so far match those that we obtained from the massless states of type IIB superstring theory 3.5.30. The fermionic part of the pseudo-action is given by:

$$S'_R = -\frac{1}{4\kappa^2}\int d^{10}X\sqrt{-g}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right), \tag{4.5.17}$$

$$S'_{CS} = \int C_4 \wedge H_3 \wedge F_3, \tag{4.5.18}$$

with the following definitions:

$$\begin{array}{ccc} F_1 = dC_0, & F_3 = dC_2, & F_5 = dC_4, \\ \tilde{F}_3 = F_3 - C_0 \wedge H_3, & \tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H_3 + \frac{1}{2}B_2 \wedge F_3. & \end{array} \tag{4.5.19}$$

The action is only "pseudo" because the constrain $\tilde{F}_5 = \star\tilde{F}_5$ must be imposed at the level of equations of motion, to obtain the correct dynamics. As for the case of type IIA, there is an equal number of fermionic DoF associated to the bosonic ones. Once again, they are associated with a dilatino λ and a gravitino Ψ_μ, which completes the field content of the low energy limit of type IIB superstrings (3.5.30). To conclude this subsection, we provide the supersymmetry variations of those fields:

$$\delta\lambda = \frac{1}{2}\left(\partial_\mu\Phi - ie^{\Phi}\partial_\mu C_0\right)\Gamma^\mu\varepsilon + \frac{1}{4}\left(ie^{\Phi}\widetilde{\mathbf{F}}^{(3)} - \mathbf{H}^{(3)}\right)\varepsilon^\star, \tag{4.5.20}$$

$$\delta\Psi_\mu = \left(\nabla_\mu + \frac{i}{8}e^{\Phi}\mathbf{F}^{(1)}\Gamma_\mu + \frac{i}{16}e^{\Phi}\widetilde{\mathbf{F}}^{(5)}\Gamma_\mu\right)\varepsilon - \frac{1}{8}\left(2\mathbf{H}^{(3)}_\mu + ie^{\Phi}\widetilde{\mathbf{F}}^{(3)}\Gamma_\mu\right)\varepsilon^\star. \tag{4.5.21}$$

Democratic Formalism

We have presented the IIA and IIB supergravity theories. We now introduce a different formalism to express the two theories, called the "democratic formalism". While the fields in NS sector of the action are unchanged, the RR sector of type IIA (IIB) in the democratic formalism consists of a formal sum of forms (polyform) of even (odd) degrees:[9]

$$F = (F_0+)F_2 + F_4 + F_6 + F_8(+F_{10}) \qquad \text{for type IIA,} \tag{4.5.22}$$

$$F = F_1 + F_3 + F_5 + F_7 + F_9 \qquad \text{for type IIB.} \tag{4.5.23}$$

This formulation immediately raises a problem: we are doubling the degrees of freedom. In order to avoid that, we impose a self duality condition, so that only the fields F_n with $n \leq 5$ are really dynamical. Such constraint reads:

$$F = \lambda(*F), \tag{4.5.24}$$

with

$$\lambda(F_n) = (-1)^{\frac{n+1}{2}} F_n. \tag{4.5.25}$$

The way that these conditions enter in the formalism is described as follows. We first use all the available degrees of freedom (including the redundant ones) to write a pseudo-action:

$$S_{RR} = -\frac{1}{8} \int \left[F \wedge *F\right]_{10} \ , \tag{4.5.26}$$

where $[\ ]_{10}$ instructs to take the form of degree 10. Again, this is a pseudo-action, because it contains a number of unphysical degrees of freedom, and it is just a tool for obtaining the EoM. Such DoF are eliminated only after the on-shell conditions have been found, by imposing 4.5.24. This procedure, using 4.5.22 (4.5.23) as the starting point, yields the same theory as IIA (IIB).

[9] F_0 and F_{10} appear in parenthesis because they must be set to zero to obtain massless supergravity. See the Appendix of [60] for a detailed discussion.

Part III

The Mathematics and the Physics

Applications of Ordinary Geometry

5

Since we begun the discussion on string theory, we did not explicitly use notions from differential geometry. Let alone the concepts from the first two chapters, which are even more advanced. However, differential geometry is the mathematical framework underlying string theory, and now we use it in order to present some results that would be inaccessible without it.

In section 1, we perform the compactification of type II supergravity without fluxes. We show how this imposes a Calabi-Yau geometry on the internal manifold, and give an overview on how to obtain the 4-dimensional theory.

Section 2 explores the geometrical local nature of the fields appearing in the NSNS sector of type II supergravity, and their symmetries. This analysis illustrates how two links with generalised geometry emerge.

In section 3, we study the conditions for Buscher's duality more rigorously, and show how the notion of generalised vector naturally appears. Then, we review a more elegant approach to Buscher's duality.

There are no global references for this chapter, since it gives an overview of different applications.

5.1 Calabi-Yau Compactification in Type II Superstring Theory

We did not use differential geometry in sections 4.2, 4.3 and 4.4 because essentially we compactified "ad muzzum", i.e. in a very naive way. We did not really worry about topological issues, and we did not carefully study the nature of the internal space. This was justified since we only worked with the simplest type of manifolds, S^1 and its products. But such choice does not brings us very far.

Compactification on a torus leaves all supersymmetries unbroken. The supersymmetric standard model predicts $\mathcal{N} = 1$ supersymmetry, therefore our aim is to find a compactification that eliminates some supersymmetry. To do so we abandon circles, we take a step back and try to figure out which manifolds can serve as internal spaces.
As we have discussed in section 4.4, a very useful tool for probing a theory is the study of its low energy limit. The low energy limit of type II superstring theory is type II supergravity, as we have shown. Then, type II supergravity will be the starting point for this section discussion.
[21], [22], [48], [45] and [63] are the relevant references for the first subsection. The second subsection is based on [52], [64] and [63].

Fluxless Compactification and Calabi-Yau manifolds

In this subsection, we examine how the requirement of an internal Calabi-Yau space emerges. We study the properties discussed in section 1.5 and their relations in a more concrete context. We recall that the starting point of compactification was the requirement of a product manifold structure. Assuming that we are looking at superstring theory, and hence at a 10-dimensional space, we have $M_{tot} = M_4 \otimes X_6$. In doing so, we would like to preserve some supersymmetry. Although it is not observed at the energies currently probed, there are strong arguments for believing in supersymmetry at scales above the TeV.[65] We now discuss how the requirement of some supersymmetry being preserved constraints the choice of the internal manifold X_6.
We focus on a classical background, where the fermionic fields are taken to be zero.[1] As we saw in the last section of the previous chapter, the supersymmetry variation of any bosonic field involves a fermionic field. Thus, we do not need to worry about them. What we need to address is the variation of the fermionic fields, which we shall impose to vanish. We saw a full expression for the variation of the gravitino in type IIA (IIB) in equation 4.5.15 (4.5.21). In the absence of fluxes (i.e. in accordance with our assumption), it simply becomes:

$$\delta\Psi_M = \nabla_M \epsilon. \tag{5.1.1}$$

This is a profound statement, which carries two constraints on the internal space, one differential and one topological. To appreciate it, we first split the spinor over the total space

[1] This assumption is motivated, for instance, in [35]. The existence of non-zero of spinorial fields would break Poincare invariance in the 4D Minkowski external space.

into a product:

$$\epsilon = \xi \otimes \eta. \tag{5.1.2}$$

We also split the indices accordingly, with $\mu = 0, ..., 3$ and $m = 4, ..., 9$. Then, 5.1.2 implies

$$\delta\Psi_m = \nabla_m \eta = 0. \tag{5.1.3}$$

In words, this constraint implies that

- There mus exist a spinor on X_6 (topological condition).
- It must be covariantly conserved (differential condition).

We will examine both of them at once, but it is important to recognise that they are statements of different nature.
A Majorana-Weyl spinor in 10 dimensions has 16 real components (see appendix A2, and recall that a Majorana condition is a reality condition), giving a 16-dimensional irreducible representation of $Spin(9,1)$. The subgroup defined by 5.1.2 is $SL(2,\mathbb{C}) \times SU(4)$, i.e. one spinor in 4 dimensions cross another independent spinor in 6 dimensions. Under this subgroup, the 16-dimensional irrep of $Spin(9,1)$ decomposes as follows:

$$\mathbf{16} = (\mathbf{2}, \mathbf{4}) \oplus (\bar{\mathbf{2}}, \bar{\mathbf{4}}). \tag{5.1.4}$$

Let η_+ be a covariantly constant spinor on X_6. We will find an explicit form for it shortly. $\nabla_m \eta_+ = 0$ implies that $\eta_+^\dagger \eta_+$ is constant over X_6, and thus it can be normalised so that $\eta_+^\dagger \eta_+ = 1$. We can then defined a tensor as

$$J_m{}^n = -i\eta_+^\dagger \gamma_m{}^n \eta. \tag{5.1.5}$$

A calculation involving the Fierz identity (one of the few we will neither present nor outline, see [21]) shows that:

$$J_m{}^n J_n{}^p = -\delta_m{}^p. \tag{5.1.6}$$

Hence, J is an almost complex structure. We also have that the Nijenhuis tensor vanishes:

$$N^p{}_{mn} = J_m{}^q \partial_{[q} J^p_{n]} - J_n{}^q \partial_{[q} J^p_{m]} = 0. \tag{5.1.7}$$

Hence, we showed the following result.

Theorem 5.1.1. X_6 is a complex manifold, with complex structure J.

We can also choose an Hermitian metric (as we have shown in theorem 1.2.2, the existence of any Riemannian metric implies the existence of an Hermitian metric), making X_6 an Hermitian manifold. Finally, we construct a two form from J by lowering its upper index:

$$\Omega = J_m{}^k g_{kn} dx^m \otimes dx^n = J_{mn} dx^m \wedge dx^n = i g_{\mu\bar{\nu}} dx^\mu \wedge d\bar{x}^\nu. \tag{5.1.8}$$

The exterior derivative of this form gives:

$$d\Omega = \partial\Omega + \bar{\partial}\Omega = i\partial_\alpha g_{\mu\bar{\nu}} dx^\alpha \wedge dx^\mu \wedge d\bar{x}^\nu + i\partial_{\bar{\alpha}} g_{\mu\bar{\nu}} d\bar{x}^\alpha \wedge dx^\mu \wedge d\bar{x}^\nu. \tag{5.1.9}$$

Assuming a torsion-free connection, the total anti-symmetry on the lower indices implies that partial derivatives in this expression can be exchanged with covariant ones. But, given that metric is covariantly constant (see theorem 1.2.4), the expression vanishes, showing that Ω is closed. Hence, we showed the following result.

Theorem 5.1.2. X_6 is a Kähler manifold, with Ω being the Kähler form.

We now show that this Kähler manifold is Calabi-Yau. As we have proven, it is enough to verify one of the properties in definition 1.5.1, but here we study all of them for completeness.

4 (Holonomy). Consider a general spinor η on X_6, which has definite chirality and thus it transforms in the fundamental **4** of $SU(4)$. We can always choose a basis such that

$$\eta_+ = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \eta_0. \end{pmatrix} \tag{5.1.10}$$

Generally, this spinor will undergo a $SU(4)$ transformation when parallely transported along a closed loop. However, if we require η_+ to be covariantly constant, it must remain unchanged after such parallel transport. We see that $SU(3) \subset SU(4)$ indeed leaves η_+ invariant, since it acts only on the first three components (see example 2.1.3). Hence, the holonomy group of X_6 must be $SU(3)$.

2 (Ricci flatness). The condition 5.1.3 implies the integrability condition:

$$[\nabla_m, \nabla_n]\eta = 0 \tag{5.1.11}$$

Using

$$\nabla_m \eta = \partial_m \eta + \frac{1}{4}\omega_{mpq}\gamma^{pq}\eta, \tag{5.1.12}$$

where ω_{mpq} are the components of the spin connection, and the symmetry properties of the connection components, we have:

$$[\nabla_m, \nabla_n]\,\eta = \left[\partial_m + \frac{1}{4}\omega_{mpq}\gamma^{pq}, \partial_n + \frac{1}{4}\omega_{nrs}\gamma^{rs}\right]\eta. \tag{5.1.13}$$

Expanding out and using $[\gamma_{rs}, \gamma^{pq}] = -8\delta^{[p}_{[r}\gamma^{q]}_{s]}$ leads to:

$$[\nabla_m, \nabla_n]\,\eta = \frac{1}{4}\left(\partial_m\omega_{nrs} - \partial_n\omega_{mrs} + \omega_{mrp}\omega_n{}^p{}_s - \omega_{nrp}\omega_m{}^p{}_s\right)\gamma^{rs}\eta = \frac{1}{4}R_{mnrs}\gamma^{rs}\eta. \tag{5.1.14}$$

From this result, via the Bianchi identity and some properties of the gamma matrices, one obtains [2]

$$R_{mn} = 0. \tag{5.1.15}$$

Thus, X_6 is Ricci flat.

3 (Holomorphic form). We define a nowhere vanishing (3,0)-form as:

$$\Omega = \frac{1}{6}\Omega_{mnp}dz^m \wedge dz^n \wedge dz^p, \qquad \text{with} \qquad \Omega_{mnp} = \eta_-^T\gamma_{mnp}\eta_-. \tag{5.1.16}$$

This construction is enough to show that X_6 admits a nowhere vanishing holomorphic $(3,0)$-form.

1 (Vanishing c_1). The components of the above holomorphic can be written as

$$\Omega_{mnp} = f(z)\epsilon_{mnp}, \tag{5.1.17}$$

where $f(z)$ is nowhere vanishing and holomorphic. The norm reads

$$\|\Omega\|^2 = \frac{1}{3!}\Omega_{mnp}\bar{\Omega}^{mnp}, \tag{5.1.18}$$

[2] The details of this derivation are provided in exercises 9.6 and 9.7 (page 383) of [21].

and using 5.1.17 it gives:

$$\|\Omega\|^2 = |f|^2(\bar{g})^{-1}. \tag{5.1.19}$$

$\bar{g} = \sqrt{g}$, as in section 1.2. The Ricci form is then

$$\mathcal{R} = i\partial\bar{\partial}\log\sqrt{g} = -i\partial\bar{\partial}\log\|\Omega\|^2. \tag{5.1.20}$$

Since $\|\Omega\|^2$ is globally defined, so is $\log\|\Omega\|^2$. Thus, recalling that $\partial\bar{\partial} = -\frac{1}{2}d(\partial - \bar{\partial})$, we see that $\mathcal{R}$ is exact. This proves that $c_1 = 0$ on X_6.

These observations (redundantly) prove that the following theorem.

Theorem 5.1.3. X_6 is a Calabi-Yau manifold.

The Spectrum of the Compactified Theory

We have established that the suitable internal space for compactifying type II string theory is a Calabi-Yau manifold. We now study what fields a four dimensional observer sees when the internal space is Calabi-Yau. We will (present how to) show that type II theories (without fluxes) compactified on a Calabi-Yau manifold lead to $\mathcal{N} = 2$ supergravity in four dimensions. This subsection is only illustrative, and should not be seen as a rigorous presentation of the topic, since we did not discuss deformations.
The Hodge diamond for a Calabi-Yau manifold of complex dimension 3 will prove to be of vital importance in the following discussion. This was the last result of Chapter 1. We report it here:

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{1,2} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array}$$

We recall that it just depends on two topological numbers. Given the dimensions of the cohomology groups, we can choose a basis for them, since they are vector spaces. It follows directly from the Hodge theorem that for any cohomology class we can choose one harmonic form as its representative. Hence, the space of harmonic r-forms is isomorphic to the r-th cohomology vector space. Thus, we can choose a set of harmonic forms as basis for any cohomology space. This is the choice that we will make, and is summarised in the following definition.

Definition 5.1.1. We define the bases of the cohomology groups/harmonic forms as follows: With the following normalisation choices:

Cohomology Group	Basis
$H^{0,0}$	1
$H^{1,1}$	ω_a, with $a = 1, ..., h^{1,1}$
$H^{1,2}$	α_L, with $L = 1, ..., h^{1,2}$
$H^{2,2}$	$\bar{\omega}^a$, with $a = 1, ..., h^{1,1}$
$H^{1,1}$	1

$$\int \omega_a \wedge \tilde{\omega}^b = \delta_a^b \tag{5.1.21}$$

$$\int \alpha_K \wedge \beta^L = \delta_K^L. \tag{5.1.22}$$

We are now ready to expand the 10-dimensional fields using the above basis. Since supersymmetry is preserved, the fermionic part of the spectrum can be restored by supersymmetric transformations from the bosonic part sector. For this reason, we will focus only on the latter. We begin with the NSNS sector of type IIA. The expansion for the dilaton is trivial, since the only scalar harmonic form is "1". There are no harmonic one forms on a Calabi-Yau manifold. Hence, the B field cannot have a single internal index (in which case it would be a one form on the internal space): it needs to have zero internal indices or both. In the first case, it is a scalar just like ϕ, and it has a trivial expansion. In the second case, it is expanded in the basis ω_a. We now make a very pleasant coherence check. B is a gauge field, i.e. B and $B + d\lambda$ give the same physics, so they can be identified. This exactly the definition of cohomology class, to which B indeed belongs, and thus can be expanded as above. We can summarise what we have found so far:

$$\phi(x,y) = \phi(x), \tag{5.1.23}$$
$$B_2(x,y) = B_2(x) + b^a(x)\omega_a(y). \tag{5.1.24}$$

For the RR sector, following the same procedure, we find:[3]

$$\begin{aligned}
&\text{Type IIA:} \quad \begin{cases} C_1(x,y) &= C_1^0(x) \\ C_3(x,y) &= C_1^a(x)\omega_a(y) + \xi^L(x)\alpha_L(y) + \tilde{\xi}_L(x)\beta^L(y) \end{cases} \\
&\text{Type IIA:} \quad \begin{cases} C_0(x,y) &= C_0(x) \\ C_2(x,y) &= C_2(x) + c^a(x)\omega_a(y) \\ C_4(x,y) &= V_1^K(x)\alpha_K(y) + \rho_a(x)\tilde{\omega}^a(y). \end{cases}
\end{aligned} \tag{5.1.25}$$

In this way, we have obtained a set of fields on the external space (i.e. the coefficients in the expansion). And it is not any set. If we include the metric (which we have deliberately avoided), we find that the field content of the four-dimensional effective field theory is the one of $\mathcal{N}=2$ ungauged supergravity in 4 dimensions.
We stated at the very beginning that supersymmetry does not play a crucial role in this dissertation. Since we intend to maintain the promise, we just give the above result as a fact, without giving extra details on supergravity in four dimensions. However, we emphasize that the above result is remarkable, since it shows that compactifying the low energy limit of type II superstring theory on a Calabi-Yau manifold leads to a quasi-realistic description of the world. There are a few reasons behind the "quasi". One of them is that from a phenomenological point of view, $\mathcal{N}=1$ supersymmetry in four dimensions is much more desirable for explaining the world that we observe. That is a subject for the next chapter (section 6.1), where even more geometry will get involved.

Mirror Symmetry

This microscopic subsection is aimed at simply giving a taste of a very vast subject.
We have proved that the Hodge diamond of Calabi-Yau manifolds is not only symmetric wrt its horizontal and vertical axes, but it is symmetric wrt the diagonal as well. Physically, this means that the fields (which correspond to the cohomology classes in which they are

[3] We have deliberately skipped the metric in this analysis, because it involves deformations, which we did not introduced. For more details, see [64].

expanded) can be exchanged accordingly and still give rise to the same theory. This a very simple istance of a crucial duality in string theory: mirror symmetry. In general, we talk about mirror symmetry whenever a pair of Calabi-Yau manifolds lead to the same string theory. For more information on the subject, the reader is referred to [66] and [67].

5.2 Symmetries of the NSNS Sector for type II - A hint of Generalised Geometry

This section should lie halfway between this chapter and the next one. We examine the nature of the NSNS fields in the low energy limit of type II superstring theories from a geometrical prospective. We also determine their symmetries. We only use concepts and tools from ordinary geometry (this is why this section belongs to this chapter), but we make evident connections with the generalised geometry framework (this is why it could belong to the next one as well).
In this section, we mainly follow [15], [34] and [68].

The NSNS sector of type IIA supergravity is given by (see 4.5.14):

$$S_{NS} = \frac{1}{2}\int d^{10}x\sqrt{-g}e^{-2\phi}\Big(R + 4\partial_\mu\phi\partial^\mu\phi - \frac{1}{12}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}\Big), \tag{5.2.1}$$

where we have dropped the subscript 3 on H for convenience, since this is the only field strength that we will deal with in this section. As usual, this action is invariant under diffeomorphisms, which can be parametrised by a vector X, and are given by:

$$g_{\mu\nu} \to g_{\mu\nu} + \mathcal{L}_X g_{\mu\nu}, \tag{5.2.2}$$

$$b_{\mu\nu} \to b_{\mu\nu} + \mathcal{L}_X b_{\mu\nu}, \tag{5.2.3}$$

$$\phi \to \phi + \mathcal{L}_X \phi. \tag{5.2.4}$$

We could check explicitly that these transformations are indeed symmetries of the action. But such check should not be performed if one trusts differential geometry: all terms in the action are coordinate scalars on the manifold, so the coordinate transformations do not affect them. In addition to diffeomorphisms, there is a gauge symmetry due to the fact that B enters the action only through its field strength H. We now study the geometrical implications of this fact. H is required to be closed,

$$dH = 0 \iff \partial_{[\mu}H_{\alpha\beta\gamma]}, \tag{5.2.5}$$

which is the usual Bianchi identity. But since H is not globally exact, B can differ between patches, i.e. it is only defined locally. Let us be more precise. Given an open covering $\{U_i\}$ of the manifold, and B_i be defined in the patch U_i. Then, we allow the various B_i's to differ from patch to patch up to an exact factor:

$$B_i - B_j = d\Lambda_{ij} \qquad \forall i, j. \tag{5.2.6}$$

Thus Λ_{ij} provides the patching for B, and this construction holds since H is insensitive to such exact term. Moreover, we are also allowed to shift the B-field by an exact term within the same patch, provided that this shift agrees with the neighbouring patches in the intersections. These transformations read:

$$B_i' = B_i + d\lambda_i \ , \qquad \text{with} \quad d\lambda_i = d\lambda_j \quad \text{for} \quad U_i \cap U_j \neq \emptyset. \tag{5.2.7}$$

This is the local bosonic gauge symmetry. The condition on the intersection implies that the set of λ_i's define a global closed two-form. Thus, if we combine this symmetry with diffeomorphism invariance, noting that they do not commute, we find the group of symmetries of the NSNS sector to be $G_{gen\,diff} = GL(n, \mathbb{R}) \rtimes G_{B,\,closed}$. Explicitly, they read

$$\begin{aligned} g &\to g + \mathcal{L}_X g, \\ B_i &\to B_i + \mathcal{L}_X B_i + d\lambda_i, \\ \phi &\to \phi + \mathcal{L}_X \phi. \end{aligned} \tag{5.2.8}$$

The patching condition 5.2.6 implies:

$$\mathcal{L}_X B_i + d\lambda_i = \mathcal{L}_X (B_j + d\Lambda_{ij}) + d\lambda_i = \mathcal{L}_X B_j + d i_X d\Lambda_{ij} + d\lambda_i = \mathcal{L}_X B_j + d(i_X d\Lambda_{ij} + \lambda_i), \tag{5.2.9}$$

where we used A.1.31. We read from the rhs that:

$$d\lambda_j = d(i_X d\Lambda_{ij} + \lambda_i). \tag{5.2.10}$$

This yields (recalling the gauge transformations of the gauge transformation, i.e. that we have the freedom to make the shift $\lambda_i' = \lambda_i + d\phi_i$):

$$\lambda_j = \lambda_i + i_X d\Lambda_{ij}. \tag{5.2.11}$$

This discussion should be compared with section 2.3, to appreciate the intimate relation between generalised geometry and the bosonic sector of type II supergravity.

We can explore this link further. Let us return to 5.2.8, and denote the action of the combined symmetry transformation (diffeomorphism plus gauge transformation) by $\delta_{(X,\lambda)}$. Let us focus on the metric and B-field only:

$$\delta_{(X,\lambda)}G = \mathcal{L}_X G \ , \qquad \delta_{(X,\lambda)}B = \mathcal{L}_X B + d\lambda. \tag{5.2.12}$$

We now study how successive symmetries combine. Specifically, the commutator of two transformations acting on the metric reads:

$$[\delta_{(X,\lambda)}, \delta_{(Y,\sigma)}]G = (\delta_{(X,\lambda)}\delta_{(Y,\sigma)} - \delta_{(Y,\sigma)}\delta_{(X,\lambda)})G = (\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X)G = \mathcal{L}_{[X,Y]}G, \tag{5.2.13}$$

where we used A.1.30. This is nothing new. If we act the commutator on B, we obtain:

$$\begin{aligned}[\delta_{(X,\lambda)}, \delta_{(Y,\sigma)}]B = (\delta_{(X,\lambda)}\delta_{(Y,\sigma)} - \delta_{(Y,\sigma)}\delta_{(X,\lambda)})B = \delta_{(X,\lambda)}(\mathcal{L}_Y B + d\sigma) - \delta_{(Y,\sigma)}(\mathcal{L}_X B + d\lambda) = \\ \mathcal{L}_X\mathcal{L}_Y B + \mathcal{L}_X d\sigma + d\lambda + \mathcal{L}_Y\mathcal{L}_X B + \mathcal{L}_Y d\lambda - d\sigma = (\mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X)B + d\mathcal{L}_X\sigma - d\mathcal{L}_Y\lambda + d(\lambda - \sigma),\end{aligned} \tag{5.2.14}$$

where in the last step we used Cartan's "magic" formula A.1.31 twice and $d^2 = 0$. Ignoring the exact term and using A.1.30, the above result may be written as:

$$[\delta_{(X,\lambda)}, \delta_{(Y,\sigma)}]B = \delta_{([X,Y],\mathcal{L}_X\sigma - \mathcal{L}_Y\lambda)}B. \tag{5.2.15}$$

We recall that one can always shift the one-form determining the gauge transformation by an exact form ($\lambda \to \lambda + d\phi$ leaves 5.2.12 invariant). We use this freedom on the rhs of the above equation to make a shift by $-\frac{1}{2}d(i_X\sigma - i_Y\lambda)$, so that it becomes:

$$\begin{aligned}[\delta_{(X,\lambda)}, \delta_{(Y,\sigma)}]B = \delta_{[\![(X,\lambda),(Y,\sigma)]\!]}B, \qquad \text{with} \\ [\![(X,\lambda),(Y,\sigma)]\!] = ([X,Y], \mathcal{L}_X\sigma - \mathcal{L}_Y\lambda) - \frac{1}{2}d(i_X\sigma - i_Y\lambda).\end{aligned} \tag{5.2.16}$$

Thus, we have obtained the Courant bracket (see 2.2.35) from purely physical arguments on the symmetries of the NSNS sector.

5.3 Rocek-Verlinde Approach to Buscher's Duality - Another Hint of Generalised Geometry

Here we present a different route to derive the Buscher rules. We will be more rigorous in the statement of the problem, as well as providing an equivalent, but more elegant, solution.

This section will provide a second instance of the appearance of geometry in string theory. It is based on the following resources: [57], [58] and [68].

Consider again the sigma model 4.3.1:

$$S_\sigma = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau\Big[\big(\sqrt{-\gamma}\gamma^{ab}G_{MN} - \epsilon^{ab}B_{MN}\big)\partial_a X^M \partial_b X^N\Big]. \tag{5.3.1}$$

What does it mean for it to admit an isometry? Let us choose the gauge choice $\gamma_{ab} = \eta_{ab}$ to simplify the calculations, and consider the transformation of coordinates:

$$\delta X^N = \epsilon K^N. \tag{5.3.2}$$

The corresponding transformation of the metric part of the action is:

$$\delta S_\sigma = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau\Big[\partial_K G_{MN}\delta X^K \partial_a X^M \partial^a X^N + 2g_{MN}\partial_a X^M \partial^a \delta X^N\Big].$$

We can integrate by parts the second term (discarding the boundary term) and use $\delta x^M = \epsilon K^M$ to obtain

$$\delta S_\sigma = -\frac{\epsilon}{4\pi\alpha'}\int d\sigma d\tau\Big[\partial_J G_{MN}K^J \partial_a X^M \partial^a X^N - 2g_{MN}\partial_a\partial^a(X^M)K^N - 2\partial^a g_{MN}\partial_a X^M K^N\Big].$$

Integrating by parts again the second term and using $g_{MN} = \frac{1}{2}(g_{MN} + g_{NM})$, we find:

$$\delta S_\sigma = -\frac{\epsilon}{4\pi\alpha'}\int d\sigma d\tau\Big[\big(\partial_J G_{MN}K^J + \partial_M g_{JN}K^J + \partial_N g_{JM}K^J\big)\partial_a X^M \partial^a X^N\Big].$$

We recognise that what sits within the round bracket are the components of the Lie derivative of G wrt K, which means that requiring $\delta S_\sigma = 0$ implies $\mathcal{L}_K G = 0$.
For the B-field part, we follow analogous steps, but with a major difference since the beginning. As shown in section 3.6, the B-field part of the action enjoys a gauge symmetry. Expressing it in terms of generalise geometry, we can shift the B field by the exterior derivative of an *arbitrary* one-form ζ, without changing the equations of motion. Thus, we need to go through the same calculation, but with an extra $\epsilon^{ab}2\partial_{[M}\zeta_{N]}\partial_a X^M \partial_b X^N$ inside the square brackets. This yields $\mathcal{L}_K B = d\zeta$.
Alternatively, we can give a more "geometrical" derivation. Referring to section 3.6 again, one can simply recognise that it is not B which should be left invariant by the isometry, but its gauge invariant field strength H: $\mathcal{L}_K H = 0$. Since we are free to change B via local

gauge transformations, $B' = B + d\zeta'$, we can adjust ζ' to achieve $\mathcal{L}_K B' = 0$. We do this as follows:

$$0 = \mathcal{L}_K B' = \mathcal{L}_K B + \mathcal{L}_K d\zeta' = \mathcal{L}_K B + d i_K d\zeta',$$

where we used Cartan's "magic" formula A.1.31. We now define:

$$\zeta = -i_K d\zeta' + df, \tag{5.3.3}$$

where f is the gauge transformation of the gauge transformation (see section 3.6), and so we have the freedom to add it. Then, we have $\mathcal{L}_K B - d\zeta = 0$, which matches the previous result.

Summarising, the coordinate transformation parametrised by K together with the gauge transformation of B parametrised by ζ generate an isometry if

$$\mathcal{L}_k G = 0 = \mathcal{L}_k B - d\zeta. \tag{5.3.4}$$

We see that to define T-duality we need the pair (K, ξ), which can be naturally organised into a generalised vector:

$$\mathbb{K} = \begin{pmatrix} K \\ \zeta \end{pmatrix}. \tag{5.3.5}$$

The conditions just derived and the above discussion will play an important role in phrasing Buscher's duality in the framework of generalised geometry (see section 6.3).

Now we assume that there is such an isometry and we gauge it, i.e. achieve invariance under 5.3.2 with $\epsilon = \epsilon(X^\mu)$. In order to do it, we introduce the gauge field A_a with transformation law given by

$$\delta A_a = -\partial_a \epsilon \tag{5.3.6}$$

and a covariant derivative D_a that obeys:

$$D_a X^\mu = \partial_a X^\mu + A_a k^\mu \to \partial_a (X^\mu + \epsilon k^\mu) + (A_a - \partial_a \epsilon) k^\mu = \partial_a X^\mu + A_a k^\mu = D_a X^\mu. \tag{5.3.7}$$

Then, the sigma model equivalent to S_σ (4.3.1) with gauged isometry reads:

$$S_\sigma^G = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \Big[\big(\sqrt{-\gamma}\gamma^{ab} G_{MN} - \epsilon^{ab} B_{MN} \big) D_a X^M D_b X^N - 2\epsilon^{ab} \tilde{\theta} \partial_a A_b \Big]. \tag{5.3.8}$$

Analogously to section 4.3, the equivalence to the original action is due to the last term, with Lagrange multiplier $\tilde{\theta}$. It is enforcing $dA = F = 0$, where F is the field strength of A. Again, as long as Poincare lemma applies, this gives $A = d\chi$, i.e. the gauge field is pure gauge. Thus, we can make the choice $A = 0$, which turns the covariant derivatives into partial ones, recovering the original action.
To get the dual action, we first make a suitable choice of coordinates so that the isometry is represented by the shift of just one of them: $\theta \to \theta + \epsilon$. Note that this was implicitly assumed in our discussion in section 4.3. 5.3.6 remains the gauge field, and the covariant derivative only appears together with the θ coordinate. Then, S_σ^G reads:

$$S_\sigma^G = -\frac{1}{4\pi\alpha'}\int d\sigma d\tau \Big[\sqrt{-\gamma}\gamma^{ab}\big(G_{00}(\partial_a\theta + A_a)(\partial_b\theta + A_b) + 2G_{0\mu}(\partial_a\theta + A_a)\partial_b X^\mu + G_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\big) - \\ -\epsilon^{ab}\big(2B_{0\mu}(\partial_a\theta + A_a)\partial_b X^\mu + B_{\mu\nu}\partial_a X^\mu \partial_b X^\nu\big) - 2\epsilon^{ab}\tilde{\theta}\partial_a A_b\Big]. \tag{5.3.9}$$

Varying wrt A, we obtain (cf. 4.3.4)

$$A_a = -\frac{1}{G_{00}}\Big(G_{0\mu}\partial_a X^\mu - \frac{\epsilon_a{}^b}{\sqrt{-\gamma}}(B_{0\mu}\partial_b X^\mu + \partial_b\tilde{\theta})\Big). \tag{5.3.10}$$

Using the isometry to set $\theta = 0$ and substituting 5.3.10 back into S_σ^G yields the dual action 4.3.5, related to the original one via the Buscher rules 4.3.6.

Applications of Generalised Geometry

6

This is the main chapter of this dissertation. It merges the most advanced results from part I and part II, showing how the framework of generalised geometry can be naturally applied to superstring theory. We present applications in various contexts. For reasons of time (and space), we chose to discuss them with different level of detail and rigour.

In section 6.1, we introduce spinors and use them to classify generalised complex manifolds. Then, we show the link with flux compactification of type II supergravity. The discussion is illustrative, but not exhaustive and complete.

Section 2 focuses on the geometrisation of type II supergravity. We carefully construct generalised objects in analogy with ordinary geometry, and re-write the NSNS sector of type II supergravity in terms of the generalised Ricci scalar. This section describes many of the steps in detail.

In section 3, we review how factorised duality (Buscher's duality) can be written in the context of generalised geometry. It is a short, but self-contained, discussion.

Finally, section 4 quickly introduces double field theory and its action. It consists of a brief overview on the subject

Again, there are no global references for this chapter. References can be found at the beginning of each section.

6.1 Generalised Calabi-Yau in Type II Compactification

In this section, we develop the language of (generalised) structures in terms of spinors, and present a classification of manifolds based on this. Then, we show where the internal manifold needed for flux compactification of the low energy limit of type II superstring theory (supergravity) fits into such classification. This discussion is concludes the journey through compactifications in string theory, which took place in sections 4.2, 4.4, 4.5, 5.1. In those

sections, we worked towards a more and more realistic model of the universe. We eliminated the extra dimensions via compactification and added fermions. This led to the appearance of Calabi-Yau manifolds as internal spaces. However, in the absence of fluxes, the number of supersymmetries preserved is $\mathcal{N} = 2$.[52] To reach the "realistic" $\mathcal{N} = 1$, we must study flux compactification, which is the subject of this section. We warn the reader that we do not provide a detailed and exhaustive overview of the subject. We limit ourselves to presenting the main relevant results and how they come about. This choice is due to time (and space) reasons.
We mainly follow [35], [52] [13], [32] and [69].

We start by introducing the concept of "twisting", which we mentioned in section 2.3.

Definition 6.1.1. Let $\mathbb{X} = (X + \xi), \mathbb{Y} = (Y + \eta) \in \Gamma(E)$ and $H \in \Omega^3(M)$ s.t. $dH = 0$. Then, the H-twisted Courant bracket is given by:

$$[\![\mathbb{X}, \mathbb{Y}]\!]_H = [\![\mathbb{X}, \mathbb{Y}]\!] + i_X i_Y H. \tag{6.1.1}$$

This bracket has the advantage of being symmetric under B-transformations of general nature (not only those with B closed), in the sense:

$$[\![e^B \mathbb{X}, e^B \mathbb{Y}]\!]_{H-dB} = e^B [\![\mathbb{X}, \mathbb{Y}]\!]_H. \tag{6.1.2}$$

The above result follows directly from 2.2.35 and 2.2.46. The choice of the letter H to denote the three form is not random. At the end of this section, we will be able to identify it with the three form H of the NSNS sector of type II supergravity.
We quickly recall that, as discussed in section 2.3, using the twisted operations implies that we are working with the untwisted generalised tangent bundle.
Analogously to the twisting of the Courant bracket, we introduce the twisted version of the exterior derivative.

Definition 6.1.2. The H-twisted exterior derivative is defined as:

$$d_H = d + H \wedge . \tag{6.1.3}$$

Thanks to $dH = 0$ and $d^2 = 0$, it is evident from this definition that $d_H^2 = 0$.
The H-twisted world is totally analogous to the original one. We just need to add a prefix. See for instance the following definition.

Definition 6.1.3. A generalised almost complex structure $\mathcal{J}$ is called H-integrable (and thus it is a complex structure)[1] if the H-twisted Courant bracket of two holomorphic generalised vectors is an holomorphic generalised vectors; i.e.

$$[\![\mathbb{X}, \mathbb{Y}]\!] \in E^+ \tag{6.1.4}$$

for $\mathbb{X}, \mathbb{Y} \in E^+$.

Note that $E^{\pm}$ were introduced at the end of section 2.4.
Let us now present a new type of object: polyforms. A polyform is a sum of forms of different dimensions. We denote the set of polyforms on a manifold M by $\Omega^\bullet(M)$. We can make generalised vectors act on polyforms, by defining the following map:

$$\mathbb{X} \cdot \phi = i_X \phi + \xi \wedge \phi. \tag{6.1.5}$$

for $\mathbb{X} = X + \xi \in \Gamma(E)$ and $\phi \in \Omega^\bullet(M)$. Equipped with these notions, we can prove something very interesting.

Theorem 6.1.1. $\Omega^\bullet(M)$ is a module over the Clifford algebra of $\Gamma(E)$.

Proof. Let $\mathbb{X} = (X + \xi), \in \Gamma(E)$ and $\phi \in \Omega^\bullet(M)$. Then,

$$\mathbb{X}^2 \cdot \phi = i_X \left(i_X \phi + \xi \wedge \phi\right) + \xi \wedge \left(i_X \phi + \xi \wedge \phi\right) = \left(i_X \xi\right) \phi = \mathcal{I}(\mathbb{X}, \mathbb{X})\phi. \tag{6.1.6}$$

□

The one above is the rigorous mathematical definition of the Clifford algebra, which is why the theorem is proved this way in [13], [32]. Now, we give a more physical phrasing of it. What we are saying is that sections of the generalised tangent bundle behave as gamma matrices when acting on $\Omega^\bullet(M)$ via the operation defined above. To see this, let $\mathbb{X} = (X + \xi) \in \Gamma(E)$, $\mathbb{Y} = (Y + \eta) \in \Gamma(E)$ and $\phi \in \Omega^\bullet(M)$. Then,

$$\begin{aligned} \{\mathbb{X}, \mathbb{Y}\} \cdot \phi = (\mathbb{X} \cdot \mathbb{Y} + \mathbb{Y} \cdot \mathbb{X}) \cdot \phi = i_X \left(i_Y \phi + \eta \wedge \phi\right) + \xi \wedge \left(i_Y \phi + \eta \wedge \phi\right) + \\ i_Y \left(i_X \phi + \xi \wedge \phi\right) + \eta \wedge \left(i_X \phi + \xi \wedge \phi\right) = \left(i_X \eta \wedge \phi + i_Y \xi \wedge \phi\right) = 2\mathcal{I}(\mathbb{X}, \mathbb{Y})\phi, \end{aligned} \tag{6.1.7}$$

[1] To be pedantic, we should say "H-complex structure".

where we used the antisymmetry properties of ϕ and the Wedge product as well as A.1.34. We obtained exactly the usual relation that we expect from gamma matrices (cf. A.2.1). The usual Majorana and Weyl conditions that we impose to find the irreps can be translated in the context of polyforms. The first one amounts to simply restricting to real forms. The second one restricts polyforms to be either of only odd degrees (negative chirality) or only even degrees (positive chirality). However, despite the appearance, polyforms by themselves are not in a 1-1 correspondence with spinors. Such isomorphism is achieved if we also specify a volume form, and define the Mukai pairing, which we will not construct.[2]
So far, we have seen how to construct spinors in the language of generalised geometry. We now give the last few definitions before starting our classification of manifolds using spinors.

Definition 6.1.4. A subbundle L of E is isotropic if

$$\mathcal{I}(\mathbb{X}, \mathbb{Y}) = 0 \quad \forall \mathbb{X}, \mathbb{Y} \in L. \tag{6.1.8}$$

Definition 6.1.5. The null space of a spinor ϕ is the subbundle L_ϕ that consists of all the annihilators of ϕ:

$$L_\phi = \{\mathbb{X} \in E : \mathbb{X} \cdot \phi = 0\}. \tag{6.1.9}$$

It follows from this definition that any null space is isotropic, since:

$$2\mathcal{I}(\mathbb{X}, \mathbb{Y})\phi = (\mathbb{X} \cdot \mathbb{Y} + \mathbb{Y} \cdot \mathbb{X}) \cdot \phi = 0 \implies \mathcal{I}(\mathbb{X}, \mathbb{Y}) = 0, \tag{6.1.10}$$

for any $\mathbb{X}, \mathbb{Y} \in L_\phi$ (and ϕ non-trivial).

Definition 6.1.6. A spinor ϕ is called pure if its null space L_ϕ is maximally isotropic, i.e. it has rank n.

This concept has an analogue for the usual spinors of $Spin(n)$, with n even. In such case, pure spinors are those that are annihilated by $n/2$ gamma matrices. A pure spinor is nothing but the Fock vacuum.[70] We recall that any Weyl spinor for $n = 6$ is pure.[35]
Finally, a pure spinor induces a decomposition in the complexified space of polyforms.

Definition 6.1.7. Let ϕ be a complex pure spinor and L_ϕ be the associated null space. Then, the space of polyforms can be written as:

$$\Omega^\bullet(M)^{\mathbb{C}} = \bigoplus_{-d/2 \le k \le d/2} U_k, \tag{6.1.11}$$

[2] The reader is referrred to [35] for more information.

with

$$U_k = \Lambda^{d/2-k}\bar{L}_\phi \cdot \phi. \tag{6.1.12}$$

The notation $\Lambda^{d/2-k}\bar{L}$ stands for the antisymmetric product of $d/2-k$ elements of $\bar{L}$. Such decomposition is called a *filtration*.

We can now revisit the manifold definitions based on generalised geometry (presented in section 2.4) and make some new ones, by simply referring to the properties of spinors. From now on, we assume that everything is complexified ($\Omega^\bullet, E,$ etc.).
We start with the case of a generalised almost complex structure/manifold. Given a generalised almost complex structure $\mathcal{J}$, we can associate to it a complex pure spinor $\phi_\mathcal{J}$, such that:

$$L_{\phi_\mathcal{J}} = E^+. \tag{6.1.13}$$

In words, the null space of $\phi_\mathcal{J}$ is the subbundle of E defined by the $+i$ eigenspace of $\mathcal{J}$. Note that $\phi_\mathcal{J}$ is then determined only up to an overall factor. This makes $\phi_\mathcal{J}$ a *pure spinor line bundle.*
Thus, we can summarise this result with a schematic theorem.

Theorem 6.1.2. Pure spinor line bundle $\iff$ Generalised almost complex structure.

As well as rewriting the definition, we can also review the definition of integrability.

Theorem 6.1.3. An almost complex structure is H-integrable if and only if its associated pure spinor is non-degenerate and satisfies:

$$d_H\phi = \mathbb{X} \cdot \phi, \tag{6.1.14}$$

for some generalised vector $\mathbb{X}$.

Proof.

$$[\mathbb{X}, \mathbb{Y}]_H \cdot \phi = [\{\mathbb{X}, d_H\}, \mathbb{Y}] \cdot \phi - d(\mathcal{I}(\mathbb{X}, \mathbb{Y})) \wedge \phi. \tag{6.1.15}$$

Restricting to the null space, i.e. $\mathbb{X}, \mathbb{Y} \in \Gamma(L_\phi)$, the second term vanishes, and we have:

$$[\mathbb{X}, \mathbb{Y}]_H \cdot \phi = \mathbb{X} \cdot \mathbb{Y} \cdot d_H\phi. \tag{6.1.16}$$

Integrability is equivalent to the vanishing of the above expression, which yields $d_H\phi = \mathbb{Z} \cdot \phi$ for some $\mathbb{Z}$. $\square$

Theorem 6.1.4. An almost Kähler structure is equivalent to the existence of two pure spinor line bundles ψ_1, ψ_2 such that:

$$\psi_2 \in \Gamma(U_0), \tag{6.1.17}$$

where U_i is the filtration defined by ψ_1.

The last condition is the statement that $\mathcal{J}_1$ and $\mathcal{J}_2$ should commute.
We have shown in section 2.4 that an almost Kähler manifold has an $U(n/2) \times U(n/2)$. structure. We can reduce this to $SU(n/2) \times SU(n/2)$ by removing the ambiguity in the spinor's definition (which makes them line bundles). In other words, by imposing a definite normalisation.

Definition 6.1.8. There is an $SU(n/2) \times SU(n/2)$ structure if there exist two pure spinors ψ_1, ψ_2, such that:

$$< \psi_1, \psi_1 >=< \psi_1, \psi_1 >\neq 0 \tag{6.1.18}$$

and

$$\psi_2 \in \Gamma(U_0), \tag{6.1.19}$$

where U_i is the filtration defined by ψ_1.

And now it is the turn for some new interesting definitions.

Definition 6.1.9. A complex spinor ϕ is called generalised Calabi-Yau a la Hitchin if it is pure and it satisfies:

$$d_H\phi = 0. \tag{6.1.20}$$

A manifold that admits such spinor is called *generalised Calabi-Yau a la Hitchin.*

This should not be taken as a generalisation in the usual sense: a Calabi-Yau manifold is not a special case of generalised Calabi-Yau manifolds.

Definition 6.1.10. Let a manifold admit a $SU(n/2) \times SU(n/2)$ defined by the pure spinors ψ_1 and ψ_2 (see definition 6.1.8). Then, if it satisfies:

$$d_H \psi_1 = 0 = d_H \psi_2, \tag{6.1.21}$$

it is called *generalised Calabi-Yau a la Gualtieri.*

This is a "proper" generalisation of ordinary Calabi-Yau manifolds, since they appear as the simplest case of generalised Calabi-Yau manifolds a la Gualtieri.

Now that we have seen the classification of manifolds based on their spinors' properties (as promised), we can determine which one is suitable for compactifying type II superstring theory.
The supersymmetry variations can be rearranged as (see [17]):[3]

$$\mathrm{d}_H \left(e^{4A-\phi} \, \mathrm{Re}\, \Psi_1 \right) = e^{4A} \tilde{F}, \tag{6.1.22}$$

$$\mathrm{d}_H \left(e^{3A-\phi} \Psi_2 \right) = 0, \tag{6.1.23}$$

$$\mathrm{d}_H \left(e^{2A-\phi} \, \mathrm{Im}\, \Psi_1 \right) = 0. \tag{6.1.24}$$

Thus, we immediately see from 6.1.23 that the geometry required for flux compactification of type II supergravity can be understood in terms of generalised manifolds. Specifically, we showed the following result.

Theorem 6.1.5. The internal manifold required for flux compactification of the low energy limit of type II superstring theory is a Generalised Calabi-Yau manifold a la Hitchin, where the pure closed spinor is $e^{3A-\phi}\Psi_2$.
The RR fluxes spoil the integrability of the second spinor, preventing the manifold from being generalised Calabi-Yau a la Gualtieri.

6.2 Geometrising Type II Supergravity

Generalised geometry is an extremely natural language to formulate type II supergravity theories. In this section, we show that using generalised geometry, the NSNS sector of type II sugra action can be geometrised, i.e. written in a general relativity fashion.
This section is based on [15], [16], [33] and [34].

[3] This is a very complicated calculation, which can be found in [71].

O(d,d) Structures

In analogy to ordinary geometry, we can introduce a generalised vielbein for generalised geometry. We denote it by $\hat{E}^A_M$, and we use the notation:

$$\hat{E}^A_M \hat{E}^M_B = \delta^A_B\,, \quad \hat{E}^A_M \hat{E}^N_A = \delta^N_M. \tag{6.2.1}$$

Our requirements for them are summarised in the following definition.

Definition 6.2.1. We define the *generalised frames* E^M_A via the conditions:

$$\hat{E}^M_A \mathcal{I}_{MN} \hat{E}^N_B = \begin{pmatrix} \eta & 0 \\ 0 & -\eta \end{pmatrix}_{AB}, \tag{6.2.2}$$

$$\hat{E}^M_A G_{MN} \hat{E}^N_B = \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix}_{AB}, \tag{6.2.3}$$

where η is the flat metric. Clearly, for the Riemannian case, η reduces to the identity matrix.

They are met by the following expression:

$$\hat{E} = \begin{pmatrix} \hat{e}_+ & \hat{e}_- \\ -B\hat{e}_+ + e_+ & B\hat{e}_- + e_- \end{pmatrix}. \tag{6.2.4}$$

We can split the "flat indices" as:

$$\begin{cases} A = a \ \text{ for } A = 1, ..., n \ , \\ A = \bar{a} + n \ \text{ for } A = n+1, ..., 2n \ , \end{cases} \tag{6.2.5}$$

which defines a splitting in the generalised vielbein:

$$\hat{E}_A = \begin{cases} \hat{E}^+_a & \text{for } A = a \\ \hat{E}^-_{\bar{a}} & \text{for } A = \bar{a} + n. \end{cases} \tag{6.2.6}$$

Explicitly, making the redefinition $\hat{e}_- \to -\hat{e}_-$, the two generalised frames read:

$$\hat{E}^+_a = \hat{e}^+_a + e^+_a + i_{\hat{e}^+_a} B \iff \hat{E}^+_a = \begin{pmatrix} \hat{e}^+_a \\ e^+_a - B\hat{e}^+_a \end{pmatrix}, \tag{6.2.7}$$

$$\hat{E}^-_{\bar{a}} = \hat{e}^-_{\bar{a}} - e^-_{\bar{a}} + i_{\hat{e}^-_{\bar{a}}} B \iff \hat{E}^-_{\bar{a}} = \begin{pmatrix} , \hat{e}^-_{\bar{a}} \\ -e^-_{\bar{a}} + -B\hat{e}^-_{\bar{a}} \end{pmatrix}. \tag{6.2.8}$$

With these definitions, we can write 6.2.2 and 6.2.3 as:

$$\mathcal{I}(\hat{E}^+_a, \hat{E}^+_b) = \eta_{ab}, \qquad \mathcal{I}(\hat{E}^-_{\bar{a}}, \hat{E}^-_{\bar{b}}) = -\eta_{\bar{a}\bar{b}}, \qquad \mathcal{I}(\hat{E}^+_a, \hat{E}^-_{\bar{b}}) = 0, \tag{6.2.9}$$

$$\mathcal{G}(\hat{E}^+_a, \hat{E}^+_b) = \eta_{ab}, \qquad \mathcal{G}(\hat{E}^-_{\bar{a}}, \hat{E}^-_{\bar{b}}) = \eta_{\bar{a}\bar{b}}, \qquad \mathcal{G}(\hat{E}^+_a, \hat{E}^-_{\bar{b}}) = 0. \tag{6.2.10}$$

The introduction of such frames makes the division of E into two subbundles $C^{\pm}$ that we introduced in section 2.4 even more manifest. We have that $\hat{E}^+_a$ serves as frame for one of them, while $\hat{E}^-_{\bar{a}}$ for the other one. Taken individually, they are governed by ordinary geometry.
With these tools at hand, it is natural to generalise the concepts of connection, torsion and Riemann tensor. We define the generalised connection by simply changing $\mathscr{T}^1_0(M) = TM$ into $\Gamma(E)$ in the definition of the ordinary connection A.1.39.

Definition 6.2.2. The *generalised connection* is map:

$$D_{\cdot}(\cdot) : \Gamma(E) \times \Gamma(E) \to \Gamma(E). \tag{6.2.11}$$

We also require it to satisfy the same properties as the ordinary connection, which are given in A.1.40. Given a coordinate system $\{\hat{E_N}\}$, the generalised connection components are also defined analogously to the ordinary ones:

$$D_M \hat{E}_N = \Gamma_{MN}{}^P \hat{E}_P \tag{6.2.12}$$

We can also define the generalised connection in the non-coordinate basis $\hat{E}_A$ analogously. We specify its components by:

$$D_M \hat{E}_A = \Omega_M{}^B{}_A \hat{E}_B. \tag{6.2.13}$$

$\Omega_M{}^B{}_A \hat{E}_B$ are called the generalised spin connection components, and we have that:

$$D_M \mathbb{X}^A = \partial_M \mathbb{X}^A + \Omega_M{}^B{}_A \mathbb{X}^B. \tag{6.2.14}$$

Thus, for a general (generalised) tensor with different type of indices we have:

$$D_P \mathbb{T}^{MA}{}_{NB} = \partial_P \mathbb{T}^{MA}{}_{NB} + \Gamma_P{}^M{}_Q \mathbb{T}^{QA}{}_{NB} + \Omega_P{}^A{}_C \mathbb{T}^{MC}{}_{NB} - \Gamma_P{}^Q{}_N \mathbb{T}^{MA}{}_{QB} - \Omega_P{}^C{}_B \mathbb{T}^{MA}{}_{NC}. \tag{6.2.15}$$

We would like to anticipate that, even though the definition of the connection and torsion components with "curved" indices (M, N, *etc*) are more natural and more fundamental, we shall be working almost exclusively with the spin connection and the "flat" indices A, B, *etc.* Unlike in ordinary geometry, generalised geometry has an intrinsic $O(n, n)$ structure. We therefore impose the connection to be compatible with such structure with the following constraint:

$$D_M \mathcal{I} = 0 \tag{6.2.16}$$

We have a second compatibility condition, which this time comes from the $O(n, n-p) \times O(p-n, p)$ structure. This is given by:

$$D_M \mathcal{G} = 0. \tag{6.2.17}$$

Finally, inspired by the Levi-Civita connection, there is a third condition that we can choose for the connection. This is clearly the vanishing of its torsion, which we are about to define.

Definition 6.2.3. The *generalised torsion* is given by:

$$T(\mathbb{X}, \mathbb{Y}) = L^D_{\mathbb{X}} \mathbb{Y} - L_{\mathbb{X}} \mathbb{Y}, \tag{6.2.18}$$

for $\mathbb{X}, \mathbb{Y} \in \Gamma(E)$. In components, it reads:

$$T_{MN}{}^P \mathbb{X}^M \mathbb{Y}^N = (L^D_{\mathbb{X}} \mathbb{Y})^P - (L_{\mathbb{X}} \mathbb{Y})^P. \tag{6.2.19}$$

Letting $\mathbb{X} = \hat{E}_A$ and $\mathbb{Y} = \hat{E}_B$, we obtain:

$$T(\hat{E}_A, \hat{E}_B) = L^D_{\hat{E}_A} \hat{E}_B - L_{\hat{E}_A} \hat{E}_B. \tag{6.2.20}$$

We now show how the three conditions above constrain the form of the connection. Firstly, we have that the compatibility with the $O(n, n)$ structure gives (according to 6.2.15):

$$D_M \mathcal{I}_{AB} = -\Omega_M{}^C{}_A \mathcal{I}_{CB} - \Omega_M{}^C{}_B \mathcal{I}_{AC} = 0. \tag{6.2.21}$$

Using the components in 6.2.9, we have:

$$\begin{pmatrix} \Omega_M{}^c{}_a & \Omega_M{}^{\bar{c}}{}_a \\ \Omega_M{}^c{}_{\bar{a}} & \Omega_M{}^{\bar{c}}{}_{\bar{a}} \end{pmatrix} \begin{pmatrix} \eta_{cb} & 0 \\ 0 & -\eta_{\bar{c}\bar{b}} \end{pmatrix} + \begin{pmatrix} \eta_{ac} & 0 \\ 0 & -\eta_{\bar{a}\bar{c}} \end{pmatrix} \begin{pmatrix} \Omega_M{}^c{}_b & \Omega_M{}^{\bar{c}}{}_b \\ \Omega_M{}^c{}_{\bar{b}} & \Omega_M{}^{\bar{c}}{}_{\bar{b}} \end{pmatrix} =$$

$$\begin{pmatrix} \Omega_{Mba} & -\Omega_{M\bar{b}a} \\ \Omega_{Mb\bar{a}} & -\Omega_{M\bar{b}\bar{a}} \end{pmatrix} + \begin{pmatrix} \Omega_{Mab} & \Omega_{Ma\bar{b}} \\ -\Omega_{M\bar{a}b} & -\Omega_{M\bar{a}\bar{b}} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{6.2.22}$$

Very similarly (the only difference from above is that $-\eta_{\bar{a}\bar{b}}$ is replaced by $\eta_{\bar{a}\bar{b}}$, see 6.2.10), the metric compatibility gives:

$$\begin{gathered}D_M\mathcal{I}_{AB} = -\Omega_M{}^C{}_A\mathcal{G}_{CB} - \Omega_M{}^C{}_B\mathcal{G}_{AC} = \\ -\begin{pmatrix}\Omega_M{}^c{}_a & \Omega_M{}^{\bar{c}}{}_a \\ \Omega_M{}^c{}_{\bar{a}} & \Omega_M{}^{\bar{c}}{}_{\bar{a}}\end{pmatrix}\begin{pmatrix}\eta_{cb} & 0 \\ 0 & \eta_{\bar{c}\bar{b}}\end{pmatrix} - \begin{pmatrix}\eta_{ac} & 0 \\ 0 & \eta_{\bar{a}\bar{c}}\end{pmatrix}\begin{pmatrix}\Omega_M{}^c{}_b & \Omega_M{}^{\bar{c}}{}_b \\ \Omega_M{}^c{}_{\bar{b}} & \Omega_M{}^{\bar{c}}{}_{\bar{b}}\end{pmatrix} = \\ -\begin{pmatrix}\Omega_{Mba} & \Omega_{M\bar{b}a} \\ \Omega_{Mb\bar{a}} & \Omega_{M\bar{b}\bar{a}}\end{pmatrix} - \begin{pmatrix}\Omega_{Mab} & \Omega_{Ma\bar{b}} \\ \Omega_{M\bar{a}b} & \Omega_{M\bar{a}\bar{b}}\end{pmatrix} = \begin{pmatrix}0 & 0 \\ 0 & 0\end{pmatrix}.\end{gathered} \tag{6.2.23}$$

From the off-diagonal blocks of 6.2.22 and 6.2.23 we infer that the components of the generalised connection with the last two indices lowered and mixed vanish. From the diagonal ones, we see that the components of the generalised connection with the last two indices lowered and pure are anti-symmetric those indices. Summarising:

$$\Omega_{M\bar{a}b} = 0 = \Omega_{Ma\bar{b}} \ , \quad \Omega_{Mab} = -\Omega_{Mba} \ , \quad \Omega_{M\bar{a}\bar{b}} = -\Omega_{M\bar{b}\bar{a}} \ . \tag{6.2.24}$$

Unfortunately, the hard part is yet to come. Up to this point, the generalised spin connection components are far from being uniquely determined, and we rely on the torsionless condition for accomplishing such task. We express it as the vanishing of 6.2.20 for all A, B. This yields:

$$L^D_{\hat{E}_A}\hat{E}_B = L_{\hat{E}_A}\hat{E}_B \qquad \forall A, B. \tag{6.2.25}$$

We will study one side at a time. Starting with the lhs, we have that:

$$\begin{aligned}\left(L^D_{\hat{E}_A}\hat{E}_B\right)^M &= \hat{E}^N_A D_N \hat{E}^M_B + \left(D^M\hat{E}^N_A - D^N\hat{E}^M_A\right)E_{BN} = \hat{E}^N_A\Omega_N{}^C{}_B\hat{E}^M_C + \\ &\left(\Omega^{MC}{}_A\hat{E}^N_C - \Omega^{NC}{}_A\hat{E}^M_C\right)E_{NB} = \Omega_A{}^C{}_B\hat{E}^M_C + \Omega^{MC}{}_A\eta_{CB} - \Omega_B{}^C{}_A\hat{E}^M_C.\end{aligned} \tag{6.2.26}$$

The rhs reads:

$$\left(L_{\hat{E}_A}\hat{E}_B\right)^M = \hat{E}^N_A\partial_N\hat{E}^M_B + \left(\partial^M\hat{E}^N_A - \partial^N\hat{E}^M_A\right)\hat{E}_{BN}. \tag{6.2.27}$$

We can now contract both sides with E_{MD} to extract useful information about the connection components. We will do it by splitting the flat indices into barred and unbarred ones, according to 6.2.5. Clearly, this involves going through a number of analogous lengthy calculations. We will show the steps in glory detail for the case of all the flat indices being unbarred, and simply quote the result of the other cases.[4] Letting $A = a, B = b, C = c$ we have:

$$\left(L^D_{\hat{E}_a}\hat{E}_b\right)^M E_{Md} = \eta\left(L^D_{\hat{E}_a}\hat{E}_b, \hat{E}_d\right) = \Omega^C_{a\ b}\hat{E}^M_C E_{Md} + \Omega^{MC}{}_a\eta_{Cb}E_{Md} - \Omega^C_{b\ a}\hat{E}^M_C E_{Md} =$$
$$\Omega_a{}^c{}_b\eta_{cd} + \Omega_d{}^c{}_a\eta_{cb} - \Omega_b{}^c{}_a\eta_{cd} = \Omega_{adb} + \Omega_{dba} - \Omega_{bda} = \Omega_{adb} + \Omega_{dba} + \Omega_{bad} = 3\Omega_{[adb]}. \tag{6.2.28}$$

This is the contraction of the lhs of 6.2.25 (with the above choice of indices). Contracting the rhs yields:

$$\begin{aligned}\left(L_{\hat{E}_A}\hat{E}_B\right)^M E_{MD} &= \hat{E}^N_A\left(\partial_N\hat{E}^M_B\right)E_{MD} + \hat{E}^M_D\left(\partial_M\hat{E}^N_A\right)E_{BN} - \hat{E}^N_B\left(\partial_N\hat{E}^M_A\right)E_{MD}\\ &= \hat{E}^N_A\left(\partial_N\hat{E}^M_B\right)E_{MD} + \hat{E}^M_D\left(\partial_M\hat{E}^N_A\right)E_{BN} + \hat{E}^N_B\left(\partial_N\hat{E}^M_D\right)E_{MA}\\ &= 3\hat{E}^N_{[A}\left(\partial_{|N|}\hat{E}^M_B\right)E_{D]M}.\end{aligned} \tag{6.2.29}$$

We now set $A = a, B = b, D = d$ again. Using 6.2.8, we find:

$$E^N_a\partial_N\hat{E}^M_b = \begin{pmatrix}\hat{e}^{+\mu}_a\left(\partial_\mu\hat{e}^{+\nu}_b\right)\\ \hat{e}^{+\mu}_a\partial_\mu\left(e^+_{b\nu} + \hat{e}^{+\rho}_b B_{\rho\nu}\right)\end{pmatrix}. \tag{6.2.30}$$

Contracting this with E_{dM} gives:

$$\hat{E}^N_a\left(\partial_N\hat{E}^M_b\right)E_{dM} = \frac{1}{2}\left(2\hat{e}^{+\mu}_a\left(\partial_\mu\hat{e}^{+\nu}_b\right)e^+_{d\nu} + \hat{e}^{+\mu}_a\left(\partial_\mu g_{\rho\nu}\right)e^{+\rho}_b\hat{e}^{+\nu}_d + \left(\partial_\mu B_{\rho\nu}\right)\hat{e}^{+\mu}_a\hat{e}^{+\rho}_b\hat{e}^{+\nu}_d\right). \tag{6.2.31}$$

We now impose antisymmetry on a, b, d, as for 6.2.29. We immediately see that the third term gives $\frac{1}{6}H_{abc}$, while the second vanishes. The first term involves a few lines of algebra, after which we find that it equals ω^+_{abc}. The superscript just emphasizes that this spin connection was obtained using the $\{\hat{e}^+_a\}$, which is evident from the indices.
Thus, we finally obtain the identity:

$$\Omega_{[abc]} = \omega^+_{[abc]} - \frac{1}{6}H_{abc}. \tag{6.2.32}$$

[4]The calculations that we skip, together with a more detailed presentation of this derivation, can be found in [34]. This subsection is based mainly on such source.

If we had chosen $A = \bar{a}, B = \bar{b}, C = \bar{c}$, exactly the same steps (with bars over the indices and $-$'s instead of $+$'s) would have led to:

$$\Omega_{[\bar{a}\bar{b}\bar{c}]} = \omega^{-}_{[\bar{a}\bar{b}\bar{c}]} + \frac{1}{6} H_{\bar{a}\bar{b}\bar{c}}. \tag{6.2.33}$$

The other non-vanishing components of the generalised torsion are of the form $\Omega_{a\bar{b}\bar{c}}$ and $\Omega_{\bar{a}bc}$. They are also constrained by 6.2.25. If we choose $A = \bar{a}, B = b, d = \bar{d}$ and go through the same kind of tedious calculations as before, we obtain:

$$\Omega_{a\bar{b}\bar{c}} = \omega^{-}_{a\bar{b}\bar{c}} + \frac{1}{2} H_{a\bar{b}\bar{c}}. \tag{6.2.34}$$

Similarly, choosing $A = \bar{a}, B = b, D = d$, yields:

$$\Omega_{\bar{a}bc} = \omega^{+}_{\bar{a}bc} - \frac{1}{2} H_{\bar{a}bc}. \tag{6.2.35}$$

From the above results, we find that the conditions imposed highly constrain the connection. However, they do not specify *uniquely* the components with all three indices of the same type. We see from 6.2.32 and 6.2.33 that they are only defined up to arbitrary tensors as follows:

$$\begin{aligned} \Omega_{abc} &= \omega^{+}_{abc} - \frac{1}{6} H_{abc} + A^{+}_{abc}, \\ \Omega_{\bar{a}\bar{b}\bar{c}} &= \omega^{-}_{\bar{a}\bar{b}\bar{c}} + \frac{1}{6} H_{\bar{a}\bar{b}\bar{c}} + A^{-}_{\bar{a}\bar{b}\bar{c}}, \end{aligned} \tag{6.2.36}$$

where $A^{\pm}$ satisfy $A^{+}_{[abc]} = 0 = A^{-}_{[\bar{a}\bar{b}\bar{c}]}$, $A^{+}_{abc} = -A^{+}_{acb}$ and $A^{-}_{\bar{a}\bar{b}\bar{c}} = -A^{-}_{\bar{a}\bar{c}\bar{b}}$. We summarise these results into a theorem.

Theorem 6.2.1. The components of a torsionless connection, compatible with the natural canonical metric and with the generalised metric are given by:

$$\begin{aligned} \Omega_{a\bar{b}\bar{c}} &= \omega^{-}_{a\bar{b}\bar{c}} + \frac{1}{2} H_{a\bar{b}\bar{c}}, \\ \Omega_{\bar{a}bc} &= \omega^{+}_{\bar{a}bc} - \frac{1}{2} H_{\bar{a}bc}, \\ \Omega_{abc} &= \omega^{+}_{abc} - \frac{1}{6} H_{abc} + A^{+}_{abc} \\ \Omega_{\bar{a}\bar{b}\bar{c}} &= \omega^{-}_{\bar{a}\bar{b}\bar{c}} + \frac{1}{6} H_{\bar{a}\bar{b}\bar{c}} + A^{-}_{\bar{a}\bar{b}\bar{c}}, \end{aligned} \tag{6.2.37}$$

where $A^{\pm}$ are undetermined tensors satisfying $A^{+}_{[abc]} = 0 = A^{-}_{[\bar{a}\bar{b}\bar{c}]}$, $A^{+}_{abc} = -A^{+}_{acb}$ and $A^{-}_{\bar{a}\bar{b}\bar{c}} = -A^{-}_{\bar{a}\bar{c}\bar{b}}$.

$O(d,d) \times \mathbb{R}$ and Type II Gravity

Where is the dilaton? In this subsection, we (give a sketch of how to) integrate the remaining scalar field of the NSNS sector inside the framework developed above.
To include the dilaton into our formalism, we need to consider a slight generalisation $\tilde{E}$ of the generalised tangent bundle E. This is obtained by weighting by $\det(T^*M)$.

Definition 6.2.4. The *weighted generalised tangent bundle* $\tilde{E}$ is given by:

$$\tilde{E} = \det(T^*M) \otimes E. \tag{6.2.38}$$

This leads to a natural principal bundle with fibre $O(n,n) \times \mathbb{R}^+$. We can obtain it by restricting to the *conformal frames*:

$$\mathcal{I}(\hat{E}_A, \hat{E}_B) = \phi^2 \begin{pmatrix} \eta & 0 \\ 0 & -\eta \end{pmatrix}_{AB}, \tag{6.2.39}$$

where $\phi \in \Gamma(\det(T^*M))$. In words, $\{\hat{E}_A\}$ is orthonormal up to a frame dependent conformal factor. This means that we allow change of basis that preserve the natural canonical metric up to an overall positive factor, and hence the $O(n,n) \times \mathbb{R}^+$ fibre.
Tensors of $\tilde{E}$ are representations of $O(n,n) \times \mathbb{R}^+$, i.e. representations of $O(n,n)$ with given weight under $\mathbb{R}^+$.
In addition to 6.2.39, we also impose [5]

$$\mathcal{G}(\hat{E}_A, \hat{E}_B) = \phi^2 \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix}_{AB}. \tag{6.2.40}$$

The solution found in the previous subsection (6.2.4) is clearly still valid, once scaled by ϕ. This defines an $(O(n,n-p) \times O(n-p,n)) \times \mathbb{R}^+$ structure.
Let us now present how to construct an analogous connection to the one developed in the

[5] This is slightly different from [15].

previous subsection, in the new formalism. We define again (see 6.2.13) the spin connection components as:

$$D_M \hat{E}_A = \Omega_M{}^B{}_A \hat{E}_B, \tag{6.2.41}$$

where now $\hat{E}_A$ are conformal orthonormal frames. Regarding the constraints, we require again compatibility with the natural canonical metric and generalised metric (6.2.16 and 6.2.17), but now we also add the compatibility with the dilaton. Thus, we impose:

$$D\mathcal{I} = D\mathcal{G} = D\phi = 0. \tag{6.2.42}$$

The calculations 6.2.21 and 6.2.23 remain valid, thanks to the condition $D\phi = 0$ that eliminates the extra term that would arise from the ϕ dependence otherwise. Thus, the results that followed from those calculations still apply:

$$\Omega_{M\bar{a}b} = 0 = \Omega_{Ma\bar{b}} \ , \quad \Omega_{Mab} = -\Omega_{Mba} \ , \quad \Omega_{M\bar{a}\bar{b}} = -\Omega_{M\bar{b}\bar{a}} \ . \tag{6.2.43}$$

The torsionless condition, however, is changed. And, as before, it is the hard bit.
The Lie derivative on weighted objects takes a slightly different form, with an extra term. For a vector field $X \in \Gamma(TM)$ with components X^μ, a vector field Y with weight w and one-form α with weight w, the components of the *Lie derivative* read:

$$\begin{aligned}(\mathcal{L}_X Y)^\mu &= V^\beta \partial_\beta Y^\mu - Y^\beta \partial_\beta X^\mu + w Y^\mu \partial_\beta X^\beta,\\ (\mathcal{L}_X \alpha)_\mu &= V^\beta \partial_\beta \alpha_\mu + \alpha_\beta \partial_\mu X^\beta + w \alpha_\mu \partial_\beta X^\beta.\end{aligned} \tag{6.2.44}$$

Accordingly, we make the following definition.

Definition 6.2.5. The Dorfman derivative of a generalised vector $\mathbb{Y}$ of weight w wrt a generalised vector $\mathbb{X}$ reads:

$$(L_{\mathbb{X}} \mathbb{Y})^M = \mathbb{X}^N \partial_N \mathbb{Y}^M - \mathbb{Y}^N \partial_N \mathbb{X}^M + \mathbb{Y}_N \partial^M \mathbb{X}^N + w Y^M \partial_N \mathbb{X}^N, \tag{6.2.45}$$

With this modification, are almost ready impose the torsionless condition. We need to consider one last caveat. The Dorfman derivative 6.2.45 is still taken wrt to a generalised vector field, i.e. a section of E, not $\tilde{E}$. However, we are working with *conformal* orthonormal frames $\{\hat{E}_A\}$, which are sections of $\tilde{E}$. We can deal with this by using $\{\phi^{-1}\hat{E}_A\} \in \Gamma(E)$ for the first entry of the Dorfman derivative. With this in mind, the torsionless condition reads:

$$L^D_{\phi^{-1}\hat{E}_A} \hat{E}_B = L_{\phi^{-1}\hat{E}_A} \hat{E}_B, \qquad \forall A, B. \tag{6.2.46}$$

This is the corresponds to 6.2.25 in the setting without the dilaton. Again, we can proceed to extract useful information by contracting with $\phi^{-1}E_D$ (note the extra factor of ϕ^{-1} compared to before) and performing similar manipulations. The result is:

$$3\Omega_{[ADB]} = 3\left(\Phi^{-1}\hat{E}^N_{[A}\right)\left(\partial_{|N|}\left(\Phi^{-1}\hat{E}^M_B\right)\right)\left(\Phi^{-1}E_{D]M}\right), \tag{6.2.47}$$

$$\Omega_D{}^D{}_A\eta_{BC} = \Phi^{-1}\left(\partial_N\hat{E}^N_A\right)\eta_{BC}. \tag{6.2.48}$$

6.2.47 is exacly 6.2.29 (and the corresponding lhs) with $\hat{E}_A \to \phi^{-1}\hat{E}_A$. 6.2.48 is a new condition that comes from the inclusion of the dilaton. By splitting the indices and using the explicit form of the vielbein, one obtains:

$$\Omega_d{}^d{}_a = \omega_d{}^d{}_a - 2\partial_a\varphi, \tag{6.2.49}$$

$$\Omega_{\bar{d}}{}^{\bar{d}}{}_{\bar{a}} = \omega_{\bar{d}}{}^{\bar{d}}{}_{\bar{a}} - 2\partial_{\bar{a}}\varphi. \tag{6.2.50}$$

Including these new conditions, in addition to the previous ones, we can write the general form of the generalised connection in this new setting.

Theorem 6.2.2. A torsionless connection in $\tilde{E}$, compatible with the natural canonical metric, with the generalised metric and with the dilaton, is given by:

$$D_a w^b_+ = \nabla_a w^b_+ - \frac{1}{6}H_a{}^b{}_c w^c_+ - \frac{2}{9}\left(\delta^b_a\partial_c\phi - \eta_{ac}\partial^b\phi\right)w^c_+ + A^{+b}_a{}_c w^c_+, \tag{6.2.51}$$

$$D_{\bar{a}} w^b_+ = \nabla_{\bar{a}} w^b_+ - \frac{1}{2}H_{\bar{a}}{}^b{}_c w^c_+, \tag{6.2.52}$$

$$D_a w^{\bar{b}}_- = \nabla_a w^{\bar{b}}_- + \frac{1}{2}H_a{}^{\bar{b}}{}_{\bar{c}} w^{\bar{c}}_-, \tag{6.2.53}$$

$$D_{\bar{a}} w^{\bar{b}}_- = \nabla_{\bar{a}} w^{\bar{b}}_- + \frac{1}{6}H_{\bar{a}}{}^{\bar{b}}{}_{\bar{c}} w^{\bar{c}}_- - \frac{2}{9}\left(\delta^{\bar{b}}_{\bar{a}}\partial_{\bar{c}}\phi - \eta_{\bar{a}\bar{c}}\partial^{\bar{b}}\phi\right)w^{\bar{c}}_- + A^{-\bar{b}}_{\bar{a}}{}_{\bar{c}} w^{\bar{c}}_-, \tag{6.2.54}$$

where $A^{\pm}$ are undetermined tensors. They must satisfy $A^+_{[abc]} = 0 = A^-_{[\bar{a}\bar{b}\bar{c}]}$, $A^+_{abc} = -A^+_{acb}$, $A^-_{\bar{a}\bar{b}\bar{c}} = -A^-_{\bar{a}\bar{c}\bar{b}}$ and $A^{+a}_a{}_b = 0 = A^{-\bar{a}}_{\bar{a}}{}_{\bar{b}}$.

It is clear from the presence of $A^{\pm}$ that the connection is not uniquely specified. The presence of such ambiguity constitutes a problem for any physical application. Therefore, we now proceed to construct unique operators from the above expressions.

6.2.52 and 6.2.53 are already fully determined, so they do not need any manipulation. 6.2.51 and 6.2.54 have ambiguities, but they can be removed by contracting the two free indices. This leads to:

$$D_a w_+^a = \nabla_a w_+^a - 2\left(\partial_a \phi\right) w_+^a, \tag{6.2.55}$$
$$D_{\bar{a}} w_-^{\bar{a}} = \nabla_{\bar{a}} w_-^{\bar{a}} - 2\left(\partial_{\bar{a}} \phi\right) w_-^{\bar{a}}. \tag{6.2.56}$$

At this point, we introduce spinors. We lift the $O(n-p,p) \times O(p,n-p)$ (which corresponds to the splitting $E = C_+ \oplus C_-$) to $Spin(n-p,p) \times Spin(p,n-p)$, which allows us to introduce $Spin(n-p,p)$ spinors. Thus, we have two spinor bundles $S(C_\pm)$ associated with the subbundles $C_\pm$. We denote by γ^a and $\gamma^{\bar{a}}$ the gamma matrices associated with them,[6] and with $\epsilon^\pm$ the corresponding spinors, i.e. $\epsilon^\pm \in \Gamma(S(C_\pm))$. Then, the generalised connection reads:

$$\begin{aligned} D_M \epsilon^+ &= \partial_M \epsilon^+ + \frac{1}{4} \Omega_M{}^{ab} \gamma_{ab} \epsilon^+, \\ D_M \epsilon^- &= \partial_M \epsilon^- + \frac{1}{4} \Omega_M{}^{\bar{a}\bar{b}} \gamma_{\bar{a}\bar{b}} \epsilon^-. \end{aligned} \tag{6.2.57}$$

Analogously to the previous discussion, we build the following unique operators:

$$D_{\bar{a}} \epsilon^+ = \left(\nabla_{\bar{a}} - \frac{1}{8} H_{\bar{a}bc} \gamma^{bc}\right) \epsilon^+, \tag{6.2.58}$$
$$D_a \epsilon^- = \left(\nabla_a + \frac{1}{8} H_{a\bar{b}\bar{c}} \bar{\gamma}^{\bar{b}\bar{c}}\right) \epsilon^-, \tag{6.2.59}$$
$$\gamma^a D_a \epsilon^+ = \left(\gamma^a \nabla_a - \frac{1}{24} H_{abc} \gamma^{abc} - \gamma^a \partial_a \phi\right) \epsilon^+, \tag{6.2.60}$$
$$\gamma^{\bar{a}} D_{\bar{a}} \epsilon^- = \left(\gamma^{\bar{a}} \nabla_{\bar{a}} + \frac{1}{24} H_{\bar{a}\bar{b}\bar{c}} \gamma^{\bar{a}\bar{b}\bar{c}} - \gamma^{\bar{a}} \partial_{\bar{a}} \phi\right) \epsilon^-. \tag{6.2.61}$$

The first two are unique because of 6.2.52 and 6.2.53. The last two turn out to be free of ambiguities when the identity $\gamma^a \gamma^{bc} = \gamma^{abc} + \eta^{ab} \gamma^c - \eta^{ac} \gamma^b$ is used. The expressions just derived can be used, together with 6.2.51-6.2.54, to define a unique generalised Ricci tensor. We can do it in two different ways when acting on generalised vectors, which turn out to be equivalent.

[6] We do not use Γ for the gamma matrices, as we do in the appendix, because this is a section about connection components and we do not want to raise unnecessary ambiguities

Definition 6.2.6. The components of the *generalised Ricci tensor* $R_{a\bar{b}}$ are equivalently defined as:

$$\begin{aligned}\mathcal{R}_{a\bar{b}} w^a_+ &= [D_a, D_{\bar{b}}]\, w^a_+, \qquad \text{or} \\ \mathcal{R}_{\bar{a}b} w^{\bar{a}}_- &= [D_{\bar{a}}, D_b]\, w^{\bar{a}}_-. \end{aligned} \tag{6.2.62}$$

Alternatively, we can also define it using spinors, in which case we have other two expressions, both equivalent to the above.

Definition 6.2.7. The components of the *generalised Ricci tensor* $\mathcal{R}_{a\bar{b}}$ can be equivalently defined also as:

$$\begin{aligned}\frac{1}{2}\mathcal{R}_{a\bar{b}}\gamma^a\epsilon^+ &= [\gamma^a D_a, D_{\bar{b}}]\,\epsilon^+, \qquad \text{or} \\ \frac{1}{2}\mathcal{R}_{\bar{a}b}\gamma^{\bar{a}}\epsilon^- &= [\gamma^{\bar{a}} D_{\bar{a}}, D_b]\,\epsilon^-. \end{aligned} \tag{6.2.63}$$

Finally, we define a unique scalar.

Definition 6.2.8. The *generalised the Ricci scalar* is defined as:

$$\begin{aligned} -\frac{1}{4}S\epsilon^+ &= \left(\gamma^a D_a \gamma^b D_b - D^{\bar{a}} D_{\bar{a}}\right)\epsilon^+, \qquad \text{or} \\ -\frac{1}{4}S\epsilon^- &= \left(\gamma^{\bar{a}} D_{\bar{a}} \gamma^{\bar{b}} D_{\bar{b}} - D^a D_a\right)\epsilon^-. \end{aligned} \tag{6.2.64}$$

These expressions can be evaluated explicitly, using the choice $\hat{e}^+_a = \hat{e}^-_{\bar{a}}$ (aligned frames):

$$\begin{aligned}\mathcal{R}_{ab} &= R_{ab} - \frac{1}{4}H_{acd}H_b{}^{cd} + 2\nabla_a\nabla_b\phi + \frac{1}{2}e^{2\phi}\nabla^c\left(e^{-2\phi}H_{cab}\right), \\ S &= R + 4\nabla^2\phi - 4(\partial\phi)^2 - \frac{1}{2}|H|^2. \end{aligned} \tag{6.2.65}$$

Hence, comparing with S_{NS} in 4.5.14, we see that the NSNS sector of type II supergravity is given by the generalised Ricci scalar (up to integration by parts). This shows that we have successfully "geometrised" the theory. We rename variables as follows: $\phi = \Phi$ and

$$\Phi = e^{-2\phi}\sqrt{g}, \tag{6.2.66}$$

where the last ϕ is a totally new variable. This allows an elegant presentation of the result.

Theorem 6.2.3. The action for the NSNS sector of type II supergravity is given by

$$S_{NS} = \frac{1}{2} \int d^{10}x \, \Phi S. \tag{6.2.67}$$

6.3 Generalised Geometry and T-duality

This (short) section presents how to rephrase T-duality in the framework of generalised geometry.
It is based on [72].

The setting is the same as in section 5.3, where we derived Buscher rules. We have an action of the form 5.3.1, where the following conditions apply:

$$\mathcal{L}_K G = 0, \tag{6.3.1}$$

$$\mathcal{L}_K B - d\zeta = 0, \tag{6.3.2}$$

for some vector K and one-form ζ. We can incorporate both in a single generalised vector $\mathbb{K} = K + \zeta$.
Motivated by the definition of isometries in ordinary geometry, let us evaluate the Lie derivative of the generalised metric G wrt $\mathbb{K}$. Using 2.2.32, we obtain:

$$L_{\mathbb{K}}\mathcal{G} = \begin{pmatrix} \mathcal{L}_K g - (\mathcal{L}_K B - \mathrm{d}\zeta)\, g^{-1} B & \\ -B\,(\mathcal{L}_K g^{-1})\, B - B g^{-1}\,(\mathcal{L}_K B - \mathrm{d}\zeta) & (\mathcal{L}_K B - \mathrm{d}\zeta)\, g^{-1} + B\,(\mathcal{L}_K g^{-1}) \\ -g^{-1}\,(\mathcal{L}_K B - \mathrm{d}\zeta) - (\mathcal{L}_K g^{-1})\, B & \mathcal{L}_K g^{-1} \end{pmatrix}. \tag{6.3.3}$$

Setting this to zero is equivalent to the two conditions 6.3.1 and 6.3.2. Specifically, the bottom-right component is equivalent to 6.3.1. With this result, the bottom-left and top-right components are both equivalent to 6.3.2. The top-left component is then automatically satisfied. In other words, the necessary conditon for T-duality is that $\mathbb{K}$ must be a generalised Killing vector, i.e.

$$\mathcal{L}_{\mathbb{K}}\mathcal{G} = 0. \tag{6.3.4}$$

To complete the analysis, we can also define Buscher's duality in this generalised picture. Firstly, we note that $\mathbb{K}$ can be normalised so that

$$\mathcal{I}(\mathbb{K},\mathbb{K}) = 1 \iff \mathbb{K}^T\mathcal{I}\mathbb{K} = 1, \tag{6.3.5}$$

where the one on the right is the component expression, with matrix multiplication implicit. This can be achieved by choosing coordinates to that $\mathbb{K} = \frac{\partial}{\partial t}$. From section 5.3, we have that $\zeta = -i_K d\zeta' + df$ (5.3.3), so that setting $f = t$ we have

$$\mathbb{K} = \partial/\partial t + \left(\mathrm{d}t - i_{\partial/\partial t}\,\mathrm{d}\zeta'\right), \tag{6.3.6}$$

from which we clearly see that $\mathcal{I}(\mathbb{K},\mathbb{K}) = 1$.
Then we construct the $2n \times 2n$ matrix

$$T_{\mathbb{K}} = \mathbb{1}_{2n} - 2\mathbb{K}\mathbb{K}^T\mathcal{I}. \tag{6.3.7}$$

We see that it is an $O(d,d)$ element by explicit computation (using 6.3.5):

$$\begin{aligned} T_{\mathbb{K}}^T\mathcal{I}T_{\mathbb{K}} = \mathcal{I} - (2\mathbb{K}\mathbb{K}^T\mathcal{I})^T\mathcal{I} - \mathcal{I}(2\mathbb{K}\mathbb{K}^T\mathcal{I}) + (2\mathbb{K}\mathbb{K}^T\mathcal{I})^T\eta(2\mathbb{K}\mathbb{K}^T\mathcal{I}) = \\ \mathcal{I} - 2\mathcal{I}\mathbb{K}\mathbb{K}^T\mathcal{I} - 2\mathcal{I}\mathbb{K}\mathbb{K}^T\mathcal{I} + 4\mathcal{I}\mathbb{K}(\mathbb{K}^T\mathcal{I}\mathbb{K})\mathbb{K}^T\mathcal{I} = \mathcal{I} \\ \implies T_{\mathbb{K}}^T\mathcal{I}T_{\mathbb{K}} = \mathcal{I} \iff \mathcal{I}(T_{\mathbb{K}}\mathbb{X}, T_{\mathbb{K}}\mathbb{X}) = \mathcal{I}(\mathbb{X},\mathbb{X}) \qquad \forall \mathbb{X} \in \Gamma(E). \end{aligned} \tag{6.3.8}$$

We can choose a local basis for TM and T^*M such that $\frac{\partial}{\partial t}$ and dt are the first elements of the corresponding basis. Then, setting $\zeta' = 0$, the T-duality matrix is explicitly given by:

$$T_{\mathbb{K}} = \begin{pmatrix} \mathbb{1} - m & m \\ m & \mathbb{1} - m \end{pmatrix}, \quad \text{with} \quad m = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix}. \tag{6.3.9}$$

This matrix acts on the generalised metric $\mathcal{G}$ as:

$$\mathcal{G}' = T_{\mathbb{K}}^T\mathcal{G}T_{\mathbb{K}} \iff \mathcal{G}(\mathbb{X},\mathbb{X}) = \mathcal{G}'(T_{\mathbb{K}}\mathbb{X}, T_{\mathbb{K}}\mathbb{X}) \qquad \forall \mathbb{X} \in \Gamma(E). \tag{6.3.10}$$

It is conventional to define T-duality in the gauge where $\mathcal{L}_K B' = 0$ (see 5.3.3 and the discussion that leads to it). According to the definition of B-transform 2.2.13, if we let $\mathbb{K}_0 = \frac{\partial}{\partial t} + dt$, then we have:

$$T_{\mathbb{K}} = e^{d\xi'}T_{\mathbb{K}_0}e^{-d\xi'}. \tag{6.3.11}$$

This shows that of $T_{\mathbb{K}}$ acts as a combination of a gauge transformation on $\mathcal{G}$ and the usual factorised duality.
Summarising, we have proved the following result.

Theorem 6.3.1. Consider pair of symmetries: a diffeomorphism and a gauge symmetry, parametrised by K and ζ, respectively. The conditions for such pair to be an isometry is equivalent to $L_{\mathbb{K}}\mathcal{G} = 0$, where $\mathbb{K} = K + \zeta$.
The corresponding Buscher's transformation, is encoded in generalised geometry by the element $T_{\mathbb{K}} = \mathbb{1}_{2n} - 2\mathbb{K}\mathbb{K}^T\mathcal{I}$, which combines a gauge transformation and a factorised duality transformation.

6.4 Double Field Theory

In this section we give a brief overview of double field theory. The idea behind this subject is to not only double the tangent space (as we did for generalise geometry), but to double the underlying manifold as well. Physically, double field theory is strongly linked with T-duality, which originally motivated its study.
We follow mainly [68], [73] and [74].

When performing toroidal compactification of bosonic string theory, we found that momenta along the compactified dimensions had to be quantised. We denoted them by k_I, and let us now change the notation to P_I. We found that the theory contained a second set of integer numbers: the winding numbers w^I. For this section, we will refer to them as $\tilde{P}^I$. Finally, for the momentum along the uncompactified dimensions K_μ, we had no constraints and no associated winding numbers. Thus, a field would take the form $\phi(K_\mu, P_I, \tilde{P}^I)$.[7] The amazing thing happens when taking the Fourier transform of such field. This leads to $\phi(X^\mu, Y^I, \tilde{Y}^I)$, where X^μ is the conjugate variable to K_μ, Y^I is conjugate to P_i and $\tilde{Y}^I$ to $\tilde{P}^I$. But while X^μ and Y^I have a physical interpretation (they are the coordinates for the target space that we started with), the same in not true for $\tilde{Y}^I$. These are "fictitious" coordinates associated to the winding modes. Since T-duality mixes momentum numbers and winding numbers, we have a mixing of Y^I and $\tilde{Y}^I$ as well. A sense for aesthetic beauty motivates us to introduce

[7] To discuss this motivation properly we would need string field theory, but we will not go into it.

also a dual set of coordinates for the uncompactified ones: $\tilde{X}_\mu$. This is the starting point for double field theory.
Making a new choice for the indices, we begin our discussion with a doubled set of $2n$ coordinates $(X^\mu, \tilde{X}_\mu)$ that parametrises an extended space-time. The above construction implies that we can naturally make the following definition.

Definition 6.4.1. The *double derivative operator* reads:

$$\partial_M = \begin{pmatrix} \partial/\partial X^\mu \\ \partial/\partial \tilde{X}_\mu \end{pmatrix} = \begin{pmatrix} \partial_\mu \\ \tilde{\partial}^\mu \end{pmatrix}. \tag{6.4.1}$$

The objects on which this operator acts must have $2n$ components, just like generalised vectors. We have already seen in section 5.3 that a generalised vector, $\mathbb{K}$ arises when we try to organise the symmetries of T-duality into a single object, so this is a nice consistency check. We would now like to construct a generalise Lie derivative. Starting with the two ingredients above, and inspired by the generalised Lie derivative in differential geometry (see 2.2.27 in particular), we make the following definitions.

Definition 6.4.2. The *D-derivative* is defined as as:[8]

$$(L_{\mathbb{X}}\mathbb{Y})^M = \mathbb{X}^N \partial_N \mathbb{Y}^M - \mathbb{Y}^N \partial_N \mathbb{X}^M + \mathbb{Y}_N \partial^M \mathbb{X}^N, \tag{6.4.2}$$

where $\mathbb{X}, \mathbb{Y}$ are any two generalise vectors and ∂_M is the double derivative operator (6.4.1).

In the above, it is implicit that indices are contracted using the natural canonical metric, which we recall, is given by:

$$I_{MN} = \frac{1}{2}\begin{pmatrix} \mathbb{1} & \\ & \mathbb{1} \end{pmatrix}. \tag{6.4.3}$$

A point of concern might be that we are using the same notation for the D-derivative, L, as for the Dorfman derivative in generalised geometry. However, this should not create any confusion, since we will only be dealing with double field theory in this section. Moreover, we will see that actually there is not much need for a new notation.
It is straightforward to check that the Lie derivative is not antisymmetric, which, as in generalised geometry, motivates the introduction of new antisymmetric operation.

[8] A small spoiler: the "D" in D-derivative is not just a random letter.

Definition 6.4.3. The *C-bracket*[9] is defined in components as:

$$[\![\mathbb{X}, \mathbb{Y}]\!]^M = \frac{1}{2}\big(L_{\mathbb{X}}\mathbb{Y} - L_{\mathbb{Y}}\mathbb{X}\big)^M, \tag{6.4.4}$$

where $\mathbb{X}, \mathbb{Y}$ are any two generalised vectors.

We can constrain the two brackets by considering an analogy with ordinary differential geometry. Since the ordinary Lie derivative forms a closed algebra under the Lie bracket (see A.1.30), we require that the D-derivative and the C-bracket show the same behaviour:

$$[L_{\mathbb{X}}, L_{\mathbb{Y}}] = L_{\mathbb{X}}L_{\mathbb{Y}} - L_{\mathbb{Y}}L_{\mathbb{X}} = L_{[\![L_{\mathbb{X}}, L_{\mathbb{Y}}]\!]}. \tag{6.4.5}$$

Computing both sides acting on an arbitrary generalised vector $\mathbb{Z}$ (which is a tedious but rewarding calculation, that we omit for a matter of space), we obtain:

$$([L_{\mathbb{X}}, L_{\mathbb{Y}}]\mathbb{Z})^M =$$
$$L_{[\![L_{\mathbb{X}}, L_{\mathbb{Y}}]\!]} + \big(- \mathbb{Z}_N\partial^K\mathbb{Y}^N\partial_K\mathbb{X}^M + \mathbb{Z}_N\partial^K\mathbb{X}^N\partial_K\mathbb{Y}^M - \frac{1}{2}\mathbb{Y}_N\partial^K\mathbb{X}^N\partial_K\mathbb{Z}^M + \frac{1}{2}\mathbb{X}_N\partial^K\mathbb{Y}^N\partial_K\mathbb{Z}^M\big). \tag{6.4.6}$$

For the term in brackets to vanish, the standard choice is to impose the so-called *strong constraint*:

$$\partial^K \Pi\partial_K \Sigma = \mathcal{I}^{MN}\partial_M \Pi\partial_N \Sigma 0, \tag{6.4.7}$$

where Π, Σ can be any field or gauge parameter. This constraint is integrated with a second one, called the *weak constraint*:

$$\partial_K\partial^K\Pi = \mathcal{I}^{MN}\partial_M\partial_N\Pi = 0, \tag{6.4.8}$$

where Π can be any field or gauge parameter. The weak constraint follows directly from 4.2.34. We just note that the bosonic fields that we are interested in (B, g and ϕ), we have that $N = \tilde{N}$, and that we can use the "Schrodinger" representation, where the momenta become derivative operators.[10]

Together, the two constraint yield the *section condition*:

$$\mathcal{I}^{MN}\partial_M \otimes \partial_N = 0. \tag{6.4.9}$$

[9] A small spoiler: the "C" in C-bracket is not a random letter either.

[10] For a more detailed and rigorous derivation, see [73].

Note that all these constraints are $O(n,n)$-invariant, since they only involve the natural canonical metric, which is (by definition) $O(n,n)$-invariant. Using the explicit expression for $\mathcal{I}^{MN}$, it reads $\partial_\mu \otimes \tilde{\partial}_\mu + \tilde{\partial}_\mu \otimes \partial_\mu = 0$. A solution (which can be shown to be the unique one, up to $O(n,n)$ transformations) is clearly given by $\tilde{\partial}_\mu(\Pi) = 0$, where again Π can be anything. This discussion leads to a very simple result.

Theorem 6.4.1. If we impose the section condition condition, then the D-derivative (6.4.2) reduces to the Dorfman derivative (2.2.27), and consequently the C-bracket to the Courant bracket.

The above theorem seem to suggest that everything discussed so far is useless: nothing depends on the extra coordinates that we introduced, and therefore we are left "only" with generalised geometry. However, we should see the glass half full.
We have found a constraint that can always be solved, i.e. we can always find a frame in which the fields are independent of half of the coordinates. Moreover, we note that this constraint breaks $O(n,n)$ invariance. The key realisation is that there is no need to solve it explicitly, i.e. providing the frame mentioned above. Instead, we can retain $O(n,n)$ invariance and simply know that we can solve it whenever we like.[75] This way, we proceed to develop double field theory, keeping in mind that only half of the coordinates have physical meaning. The generalised metric $\mathcal{G}$ is defined and parametrised exactly as we discussed when dealing with generalised geometry. Equipped with these tools, the aim is now to construct an action for double field theory. In order to do that, we first need a notion of measure. Thus, we introduce a scalar field d s.t. $\exp(-2d)$ has weight 1. Under generalised diffeomorphisms, we have:

$$e^{-2d} \to e^{-2d} + \partial_M(\mathbb{K}^M e^{-2d}). \tag{6.4.10}$$

Therefore, it is natural make the following identification:

$$e^{-2d} = e^{-2\phi}\sqrt{|g|}, \tag{6.4.11}$$

with ϕ being a scalar field.
There are two possible approaches to building an action for double field theory. One is to develop the same formalism as in section 6.2: connections, curvatures and Riemann tensors in the new framework. The other one is to include all terms quadratic in derivatives (with all indices contracted) involving $\mathcal{G}$ and d, and then constrain the result by demanding generalised diffeomorphisms to be symmetries of the action. Following this simple procedure (and making use of the section condition), surprisingly leads to a unique combination.

Theorem 6.4.2. The (unique) action S_{DFT} with only terms involving $\mathcal{G}$ and d quadratic in derivatives and generalised diffeomorphism invariance is given by:

$$S_{DFT} = \int d^n x d^n \tilde{x} e^{-2d} \mathcal{R}_{DFT}(\mathcal{H}, d), \tag{6.4.12}$$

with

$$\begin{aligned} \mathcal{R}_{\text{DFT}} = \frac{1}{8}\mathcal{G}^{MN}\partial_M \mathcal{G}^{PQ}\partial_N \mathcal{G}_{PQ} - \frac{1}{2}\mathcal{G}^{MN}\partial_M \mathcal{G}^{PQ}\partial_P \mathcal{G}_{QN} + \\ 4\partial_M \mathcal{G}^{MN}\partial_N d - 4\mathcal{G}^{MN}\partial_M d \partial_N d - \partial_M \partial_N \mathcal{G}^{MN} + 4\mathcal{G}^{MN}\partial_M \partial_N d. \end{aligned} \tag{6.4.13}$$

And now comes the main result of this section.

Theorem 6.4.3. Using the explicit form for the generalised metric, 6.4.11 and setting $\tilde{\partial}_\mu = 0$ yields:

$$S_{DFT} \xrightarrow{\tilde{\partial}=0} \int \mathrm{d}^n x \sqrt{-g} e^{-2\Phi} \left(R(g) - \frac{1}{12} H_{\mu\nu\rho} H^{\mu\nu\rho} + 4\nabla_\mu \nabla^\mu \Phi - 4\nabla_\mu \Phi \nabla^\mu \Phi \right), \tag{6.4.14}$$

which is (again) the NSNS sector of type II supergravity.[74]

Conclusion

In this dissertation, we reviewed some applications of generalised geometry to type II superstring theory, specifically within its low energy limit and in the context of T-duality symmetry. In order to do so, we first introduced all the relevant material, belonging to both mathematics and physics, assuming familiarity with quantum field theory and (real) differential geometry.

We started by providing an introduction to complex manifolds. After presenting the basic tools in complex geometry, we introduced the complex structures and studied (almost) complex manifolds. Then, we discussed Hermitian manifolds and Kähler manifolds, focusing on some of their properties. To conclude chapter 1, we defined Calabi-Yau manifolds and derived their main features.

Still on the mathematical side, we reviewed bundles and G-structures, which play a crucial role in the construction of generalised geometry. We then defined the latter, with its most basic objects: the natural canonical metric, the Dorfman derivative and the Courant bracket. We also discussed the local nature of the generalised tangent bundle, mentioned its rigorous definition, and derived the general metric. This ended the first part of the dissertation.

The second part was purely devoted to the physics, with no mention of differential geometry at all. In chapter 3 we introduced bosonic string theory and type II superstring theory. For both, we started with the classical action, derived the dynamics and then proceeded to quantising. We obtained the critical number of space-time dimensions and the low energy limit of the spectrum for both theories. As the last thing in the chapter, we wrote down the bosonic sigma model and its low energy effective theory.

Chapter 4 focused on compactifications. The Kaluza-Klein reduction mechanism was presented in Klein-Gordon quantum field theory and for the low energy effective theory of the bosonic sigma model. Then, we performed S^1 and T^n compactification on the bosonic string theory. In both cases, we derived and discussed the resulting T-duality symmetry,

introducing Buscher's approach. S^1 compactification of type II superstring theory was also performed, both at the level of low energy spectrum and for the full theory. To conclude the chapter, we quickly reviewed how type II supergravity is obtained from 11-dimensional supergravity.
The third part of the dissertation was aimed at merging the tools developed from the first two. In chapter 5, we performed the fluxless compactification of type II supergravity, and briefly presented the spectrum after Calabi-Yau compactification. Then, we studied the geometric nature of the NSNS sector in type II supergravity, comparing it with generalised geometry. The same comparison was also made starting from the bosonic sigma model, before reviewing a different approach to T-duality.
Chapter 6 is the most important one, in terms of results presented, since we described some applications of generalised geometry to superstring theory. We showed how to recast the bosonic sector of type II supergravity in the language of generalised geometry. We developed an elegant formulation of Buscher's duality in terms of generalised objects. We studied the generalised geometric structures that result from flux compactification. Finally, we gave a brief outline of double field theory.
Of the four applications just mentioned, the former two were described in detail, while the latter two were quickly summarised. For this reason, a more systematic development of double field theory and a rigorous treatment of flux compactification are two natural extensions of this dissertation. In addition to them, the study of mirror symmetry, which was briefly outlined in section 5.1, is another candidate for further research. Finally, future investigations could also include exceptional geometry, both from a mathematical prospective and in the context of 11-dimensional supergravity. [76] [77]

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Appendix A

A.1 Conventions and facts from Ordinary Differential Geometry

This appendix summarises some important results from differential geometry. They are taken from [19] and [78].

Hodge Dual and Related Constructions

Let α be a p-form on a Riemannian manifold M with $dim_{\mathbb{R}}(M) = m$ and with Riemannian metric g. In a patch with coordinates $\{x^\mu\}$, it reads: $\alpha_p = \frac{1}{p!} A_{\mu_1 \dots \mu_p} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_p}$. The Hodge dual is defined as the following map:

$$\begin{aligned} * : \; & \Omega^p(M) \to \Omega^{m-p}(M) \\ & \alpha \mapsto *\alpha = \frac{\sqrt{|g|}}{p!(m-p)!} \alpha_{\mu_1 \dots \mu_p} \epsilon^{\mu_1 \dots \mu_p}{}_{\mu_{p+1} \dots \mu_m} d\mu^{\mu_{p+1}} \wedge \dots \wedge dx^{\mu_m}. \end{aligned} \tag{A.1.1}$$

Here, $\epsilon_{\mu_1 \dots \mu_m}$ is the totally antisymmetric *symbol*, i.e. it takes values of ± 1, and its indices are raised/lowered using the metric. Note that we have

$$* * \alpha = (-1)^{p(m-p)+t} \alpha, \tag{A.1.2}$$

where $t = 0$ for a Riemannian manifold, while $t = 1$ for a Lorentzian one.
With these conventions, the natural volume element is given by:

$$*1 = \frac{\sqrt{|g|}}{m!} \epsilon_{\mu_1 \dots \mu_m} dx^{\mu_1} \wedge \dots \wedge dx^{\mu_m} = \sqrt{|g|} dx^1 \wedge \dots \wedge dx^m. \tag{A.1.3}$$

We can use the Hodge star to define a symmetric inner product:

$$(\cdot,\cdot): \quad \Omega^r(M) \times \Omega^r(M) \to \mathbb{R} \tag{A.1.4}$$
$$\alpha, \beta \mapsto (\alpha, \beta) = \int_M \alpha \wedge *\beta. \tag{A.1.5}$$

We also use the Hodge star to define the adjoint exterior derivative (sometimes called the co-differential) $d^\dagger$, as:[1]

$$\Omega^p(M) \to \Omega^{p-1}(M)$$
$$\alpha \mapsto d^\dagger = (-1)^{mp+m+1} * d * \alpha. \tag{A.1.6}$$

Note that for m even, the pre-factor is simply (-1). In components, we have:

$$d^\dagger \alpha = -\frac{1}{(p-1)!} \nabla^\beta \alpha_{\beta\mu_2 \ldots \mu_p} dx^{\mu_2} \wedge \ldots \wedge dx^{\mu_p}. \tag{A.1.7}$$

The adjoint exterior derivative $d^\dagger$ is the adjoint of d wrt the inner product defined above, i.e.

$$(d\beta, \alpha) = (\beta, d^\dagger \alpha). \tag{A.1.8}$$

Another interesting result that follows from the Hodge operator is *Poincare Duality*:

$$\text{A p-form } \alpha \text{ is harmonic if and only if } *\alpha \text{ is harmonic.} \tag{A.1.9}$$

A quick proof can be found in [30].

Hodge Decomposition and Harmonic Forms

Any r-form $\omega \in \Omega^r(M)$ can be written uniquely as:

$$\omega = d\alpha + d^\dagger \beta + \gamma, \tag{A.1.10}$$

where $\alpha \in \Omega^{r-1}(M)$, $\beta \in \Omega^{r+1}(M)$ and $\gamma \in$ Harm $^r(M)$. An important consequence of this theorem is that every cohomology class has a unique harmonic representative on a compact Riemannian manifold.

[1] This definition applies to Riemannian manifolds. For the Lorentzian case, we include an extra factor of (-1).

Lie Bracket and Lie Derivative

The Lie derivative is a generalisation of the ordinary derivative to the case where the underlying manifold is not flat. Let X be a vector field, which generates a flow σ_X, and Y another vector field. Also, let $p \in M$. Then, The Lie derivative of Y wrt X is given by

$$\mathcal{L}_X[Y]|_p = lim_{\epsilon\to 0}\left(\frac{\sigma_X(-\epsilon)_* Y|_{p'} - Y|_p}{\epsilon}\right), \tag{A.1.11}$$

with $p' = \sigma_X(\epsilon)p$. Let U_i be some chart with coordinates $\{x^\mu\}$. Then, given two vector fields $X = X^\mu(x)\frac{\partial}{\partial x^\mu}$ and $Y = Y^\mu(x)\frac{\partial}{\partial x^\mu}$, the $\mathcal{L}_X Y$ is given by:

$$\mathcal{L}_X Y = \Big(X^\alpha(x)\frac{\partial}{\partial x^\alpha}Y^\mu(x) - Y^\alpha(x)\frac{\partial}{\partial x^\alpha}X^\mu(x)\Big)\frac{\partial}{\partial x^\mu}. \tag{A.1.12}$$

It is easy to check from the component form that it is bilinear, i.e. for X,Y,$Z \in \mathcal{T}^1_{0,p}$ and a, b constants, it satisfies:

$$\mathcal{L}_{aX+bY}Z = a\mathcal{L}_X Z + b\mathcal{L}_Y Z \qquad \text{and} \qquad \mathcal{L}_X(aY + bZ) = a\mathcal{L}_X Y + b\mathcal{L}_X Z. \tag{A.1.13}$$

The Lie derivative also satisfies the "adjoint Leibniz" rule

$$\mathcal{L}_X(\mathcal{L}_Y Z) = \mathcal{L}_{\mathcal{L}_X Y}Z + \mathcal{L}_Y(\mathcal{L}_X Z), \tag{A.1.14}$$

and it is antisymmetric (or skew symmetric):

$$\mathcal{L}_X Y = -\mathcal{L}_Y X. \tag{A.1.15}$$

Note that "adjoint Leibniz" rule, if skew symmetry applies, is nothing but the Jacobi identity. The Lie derivative wrt a given vetor field X can also be defined to act on any tensor field $A \in \mathcal{T}^q_{r,p}$:

$$\mathcal{L}_X A = lim_{\epsilon\to 0}\left(\frac{\sigma_X(\epsilon)^* A|_{p'} - A|_p}{\epsilon}\right), \tag{A.1.16}$$

where $p' = \sigma_X(\epsilon)p$. It follows that for a function $f \in \mathcal{F}(M)$,

$$\mathcal{L}_X f = X[f], \tag{A.1.17}$$

where $X[f]$ is the directional derivative. Crucially, the Lie derivative acting on any object obeys the Leibniz rule, which is a requisite for a derivation[2]:

$$\mathcal{L}_X(fA) = X[f]A + f\mathcal{L}_X A, \tag{A.1.18}$$

$$\mathcal{L}_X(A \otimes B) = \mathcal{L}_X A \otimes B + A \otimes \mathcal{L}_X B, \tag{A.1.19}$$

for A, B tensor fields and f a function. The Lie derivative play a crucial role on a manifold: they generate the action of a diffeomorphism. Let us consider a diffeomorphism of the form

$$x^\mu \to x'^\mu = x^\mu + \epsilon X^\mu + O(\epsilon^2), \tag{A.1.20}$$

which is nothing but an active coordinate transformation. Consider now a generic tensor $A \in \mathcal{T}^q_{r,p}$. Let us denote with $T^{\mu_1 \dots \mu_q}_{\nu_1 \dots \nu_r}$ the components in the $\{x^\mu\}$ coordinates and with $T'^{\mu_1 \dots \mu_q}_{\nu_1 \dots \nu_r}$ the components in the $\{x'^\mu\}$ coordinates. Then,

$$\delta T^{\mu_1 \dots \mu_q}_{\nu_1 \dots \nu_r} = T'^{\mu_1 \dots \mu_q}_{\nu_1 \dots \nu_r} - T^{\mu_1 \dots \mu_q}_{\nu_1 \dots \nu_r} \tag{A.1.21}$$

are the components of a tensor given by

$$\delta T = -\epsilon \mathcal{L}_X T + O(\epsilon^2). \tag{A.1.22}$$

Thus the Lie derivative encodes the transformation of the objects on a manifold under a diffeomorphism.
The Lie bracket is a map taking two vector fields into another vector field, defined as:

$$\begin{aligned} [\cdot,\cdot] &: \mathcal{T}^1_0 \times \mathcal{T}^1_0 \to \mathcal{T}^1_0 \\ X, Y &\mapsto [X, Y]. \end{aligned} \tag{A.1.23}$$

Let U_i be some chart with coordinates $\{x^\mu\}$. Then, given two vector fields $X = X^\mu(x)\frac{\partial}{\partial x^\mu}$ and $Y = Y^\mu(x)\frac{\partial}{\partial x^\mu}$, their Lie bracket is defined as:

$$[X, Y]f = X[Y[f]] - Y[X[f]], \tag{A.1.24}$$

so that in components:

$$[X, Y] = \Big(X^\alpha(x) \frac{\partial}{\partial x^\alpha} Y^\mu(x) - Y^\alpha(x) \frac{\partial}{\partial x^\alpha} X^\mu(x) \Big) \frac{\partial}{\partial x^\mu}. \tag{A.1.25}$$

[2] A derivation can be thought a generalisation of the derivative operator, and the Leibniz rule is its defining property.

Thus, we have that

$$[X, Y] = \mathcal{L}_X Y. \tag{A.1.26}$$

However, this is a mere coincidence: the Lie derivative and the Lie bracket are two different objects a priori. The Lie bracket satisfies:

1. Bilinearity:

$$[aX + bY, Z] = [aX, Z] + [bY, Z] \quad \text{and} \quad [X, aY + bZ] = [X, aY] + [X, bZ]. \tag{A.1.27}$$

2. Skew symmetry:

$$[X, Y] = -[Y, X] \tag{A.1.28}$$

3. Jacobi Identity ("adjoint Leibniz" rule):

$$[[X, Y], Z] + [[Z, X], Y] + [[Y, Z], X] = 0 \qquad \Big([[X, Y], Z] = [[X, Y], Z] + [Y, [X, Z]]\Big). \tag{A.1.29}$$

The equivalence between Jacobi identity and "adjoint Liebniz" rule follows from 2.
The Lie derivative forms a closed algebra under the Lie bracket:

$$[\mathcal{L}_X, \mathcal{L}_Y] = \mathcal{L}_X\mathcal{L}_Y - \mathcal{L}_Y\mathcal{L}_X = \mathcal{L}_{[X,Y]}, \tag{A.1.30}$$

where the first step makes clear that the square brackets on the lhs denote a commutator.

Useful Formulae

Cartan's Magic Formula:

$$\mathcal{L}_v\alpha = (di_v + i_v d)\alpha, \tag{A.1.31}$$

with α being a k-form and v a vector. It immediately follows that

$$d\mathcal{L}_v\alpha = \mathcal{L}_v d\alpha. \tag{A.1.32}$$

$$i_{[v,w]}\alpha = (\mathcal{L}_v i_w - i_w \mathcal{L}_v)\alpha. \tag{A.1.33}$$

$$i_X(\alpha \wedge \beta) = i_X\alpha \wedge \beta + (-1)^k \alpha \wedge i_X\beta, \tag{A.1.34}$$

with α being a k-form.

$$\mathcal{L}_{fX}\alpha = f\mathcal{L}_X\alpha + df i_X\alpha \tag{A.1.35}$$

Diffeomorphisms on a Manifold

Flows on a manifold M define diffeomorphisms of M. Than, we can construct the push-forward and the pull-back, which satisfy:

$$d\left(f^*\alpha\right) = f^*(d\alpha) \tag{A.1.36}$$

$$i_{f^*X} f^*\alpha = f^*\left(i_X\alpha\right). \tag{A.1.37}$$

for a form α.

$$[f_*X, f_*Y] = f_*([X,Y]), \tag{A.1.38}$$

for two vectors X, Y.

Connection and Covariant Derivative

The connection is a map of the form:

$$\begin{aligned} \nabla : \mathscr{T}_0^1(M) \times \mathscr{T}_0^1(M) &\to \mathscr{T}_0^1(M) \\ X, Y &\mapsto \nabla_X Y. \end{aligned} \tag{A.1.39}$$

We require it to obey:

$$\begin{aligned} \nabla_X(Y+Z) &= \nabla_X Y + \nabla_X Z, \\ \nabla_{(X+Y)}Z &= \nabla_X Z + \nabla_Y Z, \\ \nabla_{fX}Y &= f\nabla_X Y, \\ \nabla_X(fY) &= X[f]Y + f\nabla_X Y, \end{aligned} \tag{A.1.40}$$

where $X, Y \in \mathscr{T}_0^1(M)$, $f \in \mathcal{F}(M)$. Given a coordinate basis $\{\partial_\mu\}$ for vectors in a patch U, according to the properties above, we define the covariant derivative explicitly as:

$$\nabla_\mu \partial_\nu = \Gamma_{\mu\nu}{}^\alpha \partial_\alpha, \tag{A.1.41}$$

where $\Gamma_{\mu\nu}{}^\alpha$ are the connection components. Given the vectors $X = X^\mu e_\mu$, $Y = Y^\mu e_\mu$, we can use the above formula together with the properties to obtain:

$$\nabla_X Y = X^\mu \underbrace{\left(\frac{\partial}{\partial x^\mu} Y^\nu + \Gamma_{\mu\alpha}{}^\nu Y^\alpha \right)}_{\nabla_\mu Y^\nu} \partial_\nu, \tag{A.1.42}$$

where the notation at the bottom is ubiquitous in the physics literature, and is used throughout this dissertation. By requiring that the covariant derivative of a contraction between a vector and a one form is the contraction of the covariant derivatives, it follows that

$$\nabla_\mu dx^\nu = -\Gamma_{\mu\alpha}{}^\nu dx^\alpha. \tag{A.1.43}$$

This implies that

$$\nabla_X \alpha = X^\mu \underbrace{\left(\frac{\partial}{\partial x^\mu} \alpha_\nu - \Gamma_{\mu\nu}{}^\beta \alpha_\beta \right)}_{\nabla_\mu \alpha_\nu} dx^\nu. \tag{A.1.44}$$

This can be easily extended to general tensors, for which the general expression takes the form:

$$\nabla_\mu T^{\nu_1\nu_2}{}_{\nu_3\nu_4} = \frac{\partial}{\partial x^\mu} T^{\nu_1\nu_2}{}_{\nu_3\nu_4} + \Gamma_{\mu\alpha}{}^{\nu_1} T^{\alpha\nu_2}{}_{\nu_3\nu_4} + \Gamma_{\mu\alpha}{}^{\nu_2} T^{\nu_1\alpha}{}_{\nu_3\nu_4} - \Gamma_{\mu\nu_3}{}^\alpha T^{\nu_1\nu_2}{}_{\alpha\nu_4} - \Gamma_{\mu\nu_4}{}^\alpha T^{\nu_1\nu_2}{}_{\nu_3\alpha}. \tag{A.1.45}$$

The *metric connection* satisfies:

$$(\nabla_\alpha g)_{\mu\nu} = 0. \tag{A.1.46}$$

This means that the inner product of any two vectors X, Y is constant under parallel transport along any arbitrary curve with tangent vector V:

$$\nabla_V [g(X,Y)] = V^\kappa \left[(\nabla_\kappa g)(X,Y) + g(\nabla_\kappa X, Y) + g(X, \nabla_\kappa Y) \right] = V^\kappa X^\mu Y^\nu (\nabla_\kappa g)_{\mu v} = 0. \tag{A.1.47}$$

Riemann Tensor

The Riemann tensor is defined as:

$$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{A.1.48}$$

for vectors X, Y, Z. It is commonly written as $R(X,Y)Z$. Its components are given by:

$$R^{\alpha}{}_{\rho\mu\nu} = \frac{\partial}{\partial x^{\mu}} \Gamma_{\nu\rho}{}^{\alpha} - \frac{\partial}{\partial x^{\nu}} \Gamma_{\mu\rho}{}^{\alpha} + \Gamma_{\nu\rho}{}^{\beta} \Gamma_{\mu\beta}{}^{\alpha} - \Gamma_{\mu\rho}{}^{\beta} \Gamma_{\nu\beta}{}^{\alpha}. \tag{A.1.49}$$

The Riemann tensor has a geometrical interpretation. Let us consider an infinitesimal parallelogram with sides ϵ^{μ} and δ^{μ}, i.e. defined by the coordinates (x^{μ}), $(x^{\mu} + \epsilon^{\mu})$, $(x^{\mu} + \epsilon^{\mu} + \delta^{\mu})$, $(x^{\mu} + \delta^{\mu})$. Let us consider a vector V at the point p corresponding to x^{μ}. Then, we can obtain two vectors at the point q with coordinates $(x^{\mu} + \epsilon^{\mu})$ by parallely transporting V along the following two paths (in coordinates):

1. $(x^{\mu}) \rightarrow (x^{\mu} + \epsilon^{\mu}) \rightarrow (x^{\mu} + \epsilon^{\mu} + \delta^{\mu})$.
2. $(x^{\mu}) \rightarrow (x^{\mu} + \delta^{\mu}) \rightarrow (x^{\mu} + \epsilon^{\mu} + \delta^{\mu})$.

Then, their difference is given by:

$$V_1^{\mu} - V_2^{\mu} = \delta^{\alpha} \epsilon^{\beta} R^{\mu}_{\rho\alpha\beta} V^{\rho}, \tag{A.1.50}$$

where the subscripts refer to the path. The Ricci tensor is defined as

$$R_{\rho\nu} = R^{\alpha}{}_{\rho\alpha\nu}. \tag{A.1.51}$$

The Ricci scalar is:

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{A.1.52}$$

Vielbeins

Let $\{x^{\mu}\}$ be the coordinates of a point in a chart, $\{\partial_{\mu}\}$ be the corresponding basis for the tangent space and $\{dx^{\mu}\}$ the corresponding basis for the cotangent space. Any other basis $\{\hat{e}_a\}$ $(a = 1, ..., m)$ is given by

$$\hat{e}_a = \hat{e}_a{}^{\mu} \partial_{\mu} \quad \text{with} \quad \hat{e}_a{}^{\mu} \in GL(m, \mathbb{R}). \tag{A.1.53}$$

The matrices $\hat{e}_a{}^\mu$ are the vielbein, and we choose them so that

$$\hat{e}_a{}^\mu \hat{e}_b{}^\nu g_{\mu\nu} = \eta_{ab} \iff g(\hat{e}_a, \hat{e}_b) = \eta_{ab}, \tag{A.1.54}$$

where η_{ab} is the pseudo-Riemannian metric (diagonal with arbitrary signature). We can extend this to all points in the chart. In words, the metric on a Riemannian (pseudo-Riemannian) manifold can be brought to the usual Euclidean (pseudo-Riemannian) metric via local coordinate transformations.

We define the following notation

$$e^a{}_\mu \hat{e}_a{}^\nu = \delta_\mu{}^\nu \quad \text{and} \quad e^a{}_\mu \hat{e}_b{}^\mu = \delta^a{}_b, \tag{A.1.55}$$

which immediately implies that

$$g_{\mu\nu} = e^a{}_\mu e^b{}_\nu \eta_{ab}. \tag{A.1.56}$$

We define the dual basis to $\{\hat{e}_a\}$ as $\{\bar{\theta}^a\}$, satisfying:

$$< \bar{\theta}^a, \hat{e}_b >= \delta^a_b. \tag{A.1.57}$$

The spin connection components are defined as:

$$\nabla_a \hat{e}_b = \nabla_{\hat{e}_a} \hat{e}_b = \omega_{ab}{}^c \hat{e}_c. \tag{A.1.58}$$

They are related to the connection components by:

$$\omega_{ab}{}^c = \hat{e}_a{}^\mu e^c{}_\nu \big(\frac{\partial}{\partial x^\mu} \hat{e}_b{}^\nu + \hat{e}_a{}^\beta \Gamma_{\mu\beta}{}^\nu \big). \tag{A.1.59}$$

A.2 Conventions and Facts about Spinors

Spinors are objects that are acted upon by objects that satisfy the Clifford algebra. A Clifford algebra in D dimensions consists of a set with D elements Γ_m $(m = 1, ..., D)$ and an identity element I such that:

$$\{\Gamma_m, \Gamma_n\} = \Gamma_m \Gamma_n + \Gamma_n \Gamma_m = 2\eta_{mn} \mathbb{1}_D, \tag{A.2.1}$$

where η_{mn} is (are the components of) the flat diagonal metric of arbitrary signature. With the above elements, it is possible to build a group under multiplication, with the following underlying set

$$C_D = \{\pm I, \pm\Gamma_m, \Gamma_{m_1 m_2}, ..., \Gamma_{m_1 \ldots m_D}\}, \tag{A.2.2}$$

where $\Gamma_{m_1 \dots m_D} = \Gamma_{[m_1} \dots \Gamma_{m_D]}$.
For *even* D, the group C_D has $2^D + 1$ inequivalent irreducible representations. Of these, 2^D are one-dimensional and the remaining has dimensions $2^{D/2}$. The latter is the only one which is a representation of the Clifford algebra. Hence, we represent Γ_m with $2^{D/2} \times 2^{D/2}$ matrices and $I = \mathbb{1}_D$. Hence, a generic (Dirac) spinor has $2^{D/2}$ complex components. The group $Spin(1, D-1)$ is, by definition, generated by $\frac{1}{2}\Gamma_{mn}$. A general element has the form $\Lambda = \exp\left(\frac{1}{4} w^{mn} \Gamma_{mn}\right)$. It is customary to introduce a $D+1$th gamma matrix as:

$$\Gamma^{D+1} = \Gamma_0 \Gamma_1 \cdots \Gamma_{D-1}, \tag{A.2.3}$$

which commutes with Γ_m. This means that Γ^{D+1} acted on a string of gamma matrices "counts" they are of odd or even number. Moreover, we have that $(\Gamma^{D+1})^2 = \mathbb{1}_{2^{D/2}}$[3], which means that spinors can be classified according to:

$$\Gamma^{D+1} \psi = \pm \psi. \tag{A.2.4}$$

This defines Weyl spinors and chirality. Moreover, Γ^{D+1} anticommutes with the generators $\frac{1}{4} w^{mn} \Gamma_{mn}$, which means that chirality is preserved by Lorentz transformations. Hence, Weyl spinors of different chiralities define different irreducible representations (modules).

[3]Strictly speaking this is true only in half of the cases. However, in the other half we can redefine Γ^{D+1} with a factor of i and achieve the same result.

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