The Massive Classical Double Copy in Three Dimensions

Cameron Beetar

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in the Theoretical Physics Group, Blackett Laboratory,
Imperial College, London SW7 2AZ, United Kingdom
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Supervised by Dr Mariana Carrillo Gonzalez

Abstract

Double copy theory is conventionally presented as a relationship between scattering amplitudes in gauge and gravitational theories that is informally captured by the statement: (Yang-Mills)^2 = (gravity). However, more recently, the double copy procedure has been shown to hold in relating the equations of motion of gravity to that of biadjoint scalar theory and Yang-Mills theory, respectively, following suitable choices of ansätze and using a Kerr-Schild form of the metric. This work explores the cases of classical topologically massive theories in three spacetime dimensions; introducing the appropriate Chern-Simons terms to provide the 3-dimensional theories with 'topological masses'. In particular, it is shown that the equations of motion of topologically massive gravity in (2+1)-dimensions are able to reproduce the time-dependent equations of motion of topologically massive Yang-Mills theory and of massive biadjoint scalar theory (also in (2+1)-dimensions), respectively.

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Contents

Conventions

A soon-to-be-encountered difficulty for anyone venturing into double copy literature is the varying conventions, especially in terms of the Lie algebra structure constants and the Minkowski metric sign choice.

In this work we always use the convention that $[T^a, T^b] = i f^{abc} T^c$, for structure constants f^{abc} and Lie algebra generators T^a. This will be reiterated in the text when appropriate, but we pre-empt this here to prepare the reader from the outset.

A similar ambiguity in the literature is for how one chooses the Minkowski metric. In this work, we choose the 'mostly plus' metric $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, ...)$ throughout, regardless of the dimension of spacetime under consideration.

Readers familiar with General Relativity will doubtless also be familiar with the Einstein-Hilbert action in d-dimensions

$$S_{\mathrm{EH}} = \int \mathrm{d}^d x \sqrt{|g|} \Big(\frac{1}{\kappa^2} R + L_{\mathrm{matter}} \Big)$$

and the corresponding equations of motion $G_{\mu\nu} = \frac{\kappa^2}{2} T_{\mu\nu}$. While the actual value of κ is not altogether important to understand the results presented in this work, we include it for completeness. The gravitational constant, κ, in d-dimensions is defined in terms of the reduced Planck mass in d-dimensions via [1]

$$\frac{\kappa^2}{2} = \frac{1}{\left(M_{\mathrm{Pl}}^{(d)}\right)^{d-2}} = \frac{(d-1)(d-2)\pi^{(d-1)/2}}{(d-3)\Gamma\left(\frac{d+1}{2}\right)} G^{(d)},$$

where $G^{(4)}$ is (the conventional) Newton's gravitational constant and $\Gamma(x)$ is the gamma function. In four dimensions this gives the usual $\kappa^2 = 16\pi G^{(4)}$, and in three dimensions the result is indeterminate and, while this is not discussed further in this work, we continue to use the form of the Einstein-Hilbert action above in three dimensions.

We allow for both the full metric, $g_{\mu\nu}$ and the base (Minkowski) metric $\eta_{\mu\nu}$ to raise/lower indices. We will use the convention that when an object has had an index raised by the full metric we will write a 'bar' above it so that

$$g^{\mu\alpha} T_{\alpha\nu} = \bar{T}^{\mu}_{\ \nu}, \quad \text{whereas}$$
$$\eta^{\mu\alpha} T_{\alpha\nu} = T^{\mu}_{\ \nu},$$

which is also reiterated later in the text when appropriate.

Finally, we use ε for the Levi-Civita *symbol*, and   for the Levi-Civita *tensor density*.

1 Introduction

"*A method is more important than a discovery, since the right method will lead to new and even more important discoveries.*"

– Lev Landau [2].

The scientific description of gravity has experienced dramatic changes – from Galileo's historical breakthrough in understanding (that gravitational acceleration is independent of the mass) to Einstein's General Relativistic theory (which tells us that Galileo was correct only in some approximation or in some limit; consider *e.g.* gravitational back-reaction). It is a testament to both the complexity and the elegance of General Relativity (GR) that, over a century after Einstein's original work [3], it is still very much a field of active research[1]. Gauge theory has, similarly, been under constant development, and it provides us with the fundamental tools to understand much of the world we experience; the Standard Model (arguably) being its crowning achievement.

While gravity and gauge theories, upon initial inspection, may appear to be entirely independent, there are clear hints at a similar underlying structure. Diffeomorphism invariance in GR and all the structures that accompany it (local symmetries, covariant derivatives, connections *etc.*) are reminiscent of their gauge theory counterparts [4]. Understanding gauge theories geometrically makes this relationship more precise. Considering gauge theories in the practical context of Quantum Field Theories (QFTs) and their associated observables, the principal measurable quantity to be found in conventional QFT calculations is the *scattering amplitude*, from which a *differential cross-section* is computed, which is finally turned into a 'cross-section', which, despite the nomenclature, is a measure of probability of some process occurring.

Double copy theory (otherwise terms 'the double copy procedure' or simply 'the double copy') is a statement about the relationship between scattering amplitudes in gravity and in gauge theory that makes extensive use of the *Colour-Kinematics (CK) duality*, commonly referred to as the *Bern-Carrasco-Johansson (BCJ) duality* in the context of scattering amplitudes, after their seminal papers proposing and then describing the duality [5, 6] (in fact, some have argued that the double copy will only work[2] when the BCJ duality holds [7]). The idea of

[1] See, for instance, https://arxiv.org/list/gr-qc/new

[2] At the time of writing, September 2022.

double copy theory is typically (informally) written as [8]

$$\text{gravity} = (\text{gauge theory})^2. \tag{1.1}$$

The informal equality (1.1), roughly corresponding to the principal conjecture[3] of double copy theory, has its origins in string theory. In particular, it derives from an observed relationship between open-string scattering amplitudes (called *Veneziano* scattering amplitudes) and closed-string scattering amplitudes (also known as *Virasoro-Shapiro* scattering amplitudes) [11–14]. This relationship is summarised by the relation [11, 15]

$$M(s,t,u) = \frac{\sin(\pi\alpha s)}{\pi\alpha} A(s,t)A(s,u), \tag{1.2}$$

where $M(...)$ is the Virasoro-Shapiro amplitude, $A(...)$ is the Veneziano amplitude, $s = (p_1+p_2)^2$, $t = (p_2 + p_3)^2$ and $u = (p_1 + p_3)^2$ are the conventional Mandelstam invariants, and $\alpha = T^{-1}$ where T is the string tension. Importantly, (1.2) also applies to the low-energy ($\alpha \to 0$ field theory) limit of these theories – corresponding to gluons on the open string side, and to gravitons, dilatons and a 2-form field on the closed string side – one finds the *Kawai-Lewellen-Tye (KLT) relation* for the 4-point amplitudes[4] [4, 11]

$$M_4^{\text{tree}}(1,2,3,4) = \left(\frac{\kappa}{2}\right)^2 s A_4^{\text{tree}}(1,2,3,4) A_4^{\text{tree}}(1,2,4,3), \tag{1.3}$$

where the $A_4^{\text{tree}}[...]$ are *partial colour-ordered amplitudes*, related to the full amplitudes through the expression [8]

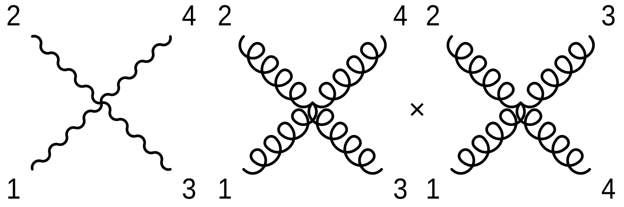
$$A_4^{\text{full,tree}} = \left(\frac{\kappa^2}{2}\right)\left(A_4^{\text{tree}}[1,2,3,4]\text{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4}) + \text{permutations of } (2,3,4)\right), \tag{1.4}$$

where the a_i label the colour charge generator on the i^{th} leg of the interaction (see below). The

[3] We say 'conjecture' as double copy theory is *suspected* to hold to all orders, but a proof does not yet exist beyond (for the BCJ double copy in particular) tree-level [9]. However, other works on the BCJ double copy are strong indicators that the double copy likely holds to loop-level [6, 10].

[4] See, in particular, chapter 7 of [4].

expression (1.3) may be diagrammatically represented as



(1.5)

where, on the left-hand side we have the Feynman diagram of the tree-level 4-graviton scattering amplitude, and on the right-hand side we have the Feynman diagram of the product of two tree-level colour-ordered 4-gluon partial[5] scattering amplitudes. As is noted in [11], these relations have been generalised in the context of string theory to higher-point tree-level amplitudes [15], and was appropriately generalised to arbitrary numbers of external particles in the low-energy field theory limit [17].

The utility of such a relation as (1.3) cannot be overemphasized – traditional approaches to gravity begin with a Lagrangian, from which appropriate Feynman rules/diagrams are deduced, and in GR we (conventionally[6]) also take perturbations about the Minkowski metric, $g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}$ to account for the graviton field, $h_{\mu\nu}$ [11]. Thereafter, one fixes the gauge, and determines the n-point functions (starting from $n = 2 \equiv$(propagators) up to the desired/allowed number) and, finally, finds the scattering amplitudes. Gravitational theories – or, at least, those gravitational theories that are common in the literature – have an incredibly disheartening drawback: they have an infinite number of vertices of all (n N) multiplicities [11]. This is in contrast to their gauge theory counterparts, that are limited to have only 3- and 4-point vertices. Thus, the relation (1.3) – and the equivalent relations for the n-point amplitudes (not shown here, see §2.1) – represents an important reduction in the complexity of the gravitational problem one might wish to solve.

While the double copy procedure is an inherently useful calculational tool for computing graviton scattering amplitudes, it is apparent that we do not properly understand the implications of it – we do not yet fully understand *all* the consequences of BCJ duality for a start [5, 7, 11].

The range of applicability of the double copy is not limited to simplifying graviton scattering amplitudes. Similar relationships – at the centre of which lies the CK duality – exist between

[5]Partial amplitudes are the gauge invariant part of the full scattering amplitude, see §2.1, chapter 13 of [8], and [16] (this final source provides useful relations for separating out the colour factors).

[6]The author was reminded that one could consider perturbations about a non-Minkowskian $g_{\mu\nu}$, but that this is significantly more complicated [18].

other theories, and there is a so-called *web of theories* connected through the double copy, and the best known of these are presented in table 1. While we do not investigate this web of theories,

FT FT	YM	$\mathcal{N}=4$ sYM	χPT	BAS
YM	gravity$^+$	$\mathcal{N}=4$ SG	BI	YM
$\mathcal{N}=4$ sYM	$\mathcal{N}=4$ SG	$\mathcal{N}=8$ SG	$\mathcal{N}=4$ sDBI	$\mathcal{N}=4$ sYM
χPT	BI	$\mathcal{N}=4$ sDBI	sGal	χ PT
BAS	YM	$\mathcal{N}=4$ sYM	χPT	BAS

Table 1: Table of the web of theories linked by double copy relations (taken from [19]). The first row and column represent which field theories we are using as 'single copies' for the double copy theory, and the remaining rows and columns detail what the outcome of forming the double copy from the field theory 'product'. The '+' for gravity is a reference to the additional dilaton and 2-form that we find in the Yang-Mills (YM) YM YM double copy. The abbreviations are: sYM - super Yang-Mills, χPT - chiral perturbation theory, BAS - biadjoint scalar, SG - supergravity, BI - Born-Infeld, sDBI - super-Dirac Born-Infeld, sGal - special Galileon.

table 1 should emphasise the vast range of applicability of the double copy and exemplify its use as a powerful simplification tool for theories whose scattering amplitude computations are intricate and/or tedious. Understanding *why* and *how* the double copy works may also provide deeper insights into the complicated (full) theories whose results the single-copy product-spaces are able to reproduce.

Motivated by both the insights the double copy affords, as well as the missing understanding of precisely how BCJ duality works, Monteiro *et al* investigated whether the double copy could still hold in certain classical contexts [7] – more recent work has shown that these classical results are closely related to the scattering amplitude approach to the double copy [20]. Remarkably, in [7] they found that a double copy relation can be found for spacetimes having a *Kerr-Schild* representation of the metric (see §2.2). More recent work on classical double copy theory has been motivated by the study of the double copy of massive gauge theories in the high-energy limit [21].

These massive gauge theories are of interest for several reasons. The most relevant of these for this work is highlighted by the findings of [21], whose results suggest that there may be a topologically massive double copy at all loop orders. More generally, the massive Yang-Mills amplitudes double copy to *de Rham-Gabadadze-Tolley (dRGT) massive gravity* in four dimensions [21, 22]. In particular, it is noted in [21, 23] that this makes use of a special choice of Wilson coefficients 4-point amplitudes, and also that the 5-point amplitude double copy suffers from spurious poles, avoidable by making use of the *spectral condition* [21, 23]. One can avoid the necessity for the spectral condition when considering the 5-point amplitudes by working with the massive double copy in three dimensions, provided the Yang-Mills amplitudes satisfy just one BCJ relation – as opposed to four BCJ relations in the massless case [21, 24]. Furthermore, a result relevant to this work is that it has been shown that topologically massive Yang-Mills (see §3.2) amplitudes double copy to topologically massive gravitational scattering amplitudes [21, 24, 25]. The inclusion of matter into these topologically massive theories has also been considered with varying degrees of success [26–28].

This work will consider many of the aspects of the *classical* double copy, with the goal of discussing the *time-dependent solutions* of topologically massive gravity and their relation to topologically massive Yang-Mills theory, as well as massive biadjoint scalar theory. The motivation for this is based on the aforementioned scattering amplitude results that indicate that the double copy of these topologically massive theories may hold to all loop orders [21], and is further justified by the previously discussed links between scattering amplitudes as the classical double copy [20]. To achieve this in a coherent way, this thesis is structured as follows: §2 briefly discusses the BCJ double copy before introducing the classical double copy and many of the known results in the literature; §3 considers topologically massive theories, intended to provide a reader new to the material with a basic understanding of these theories before, finally, showing that the classical double copy of topologically massive gravity reproduces certain time-dependent equations of motion of topologically massive Yang-Mills theory (the *single copy*) and of biadjoint scalar theory (the *zeroth copy*) – plane wave and shockwave solutions in particular; §4 contains a conclusion and potential avenues for future work. Appendices are included for completeness and should be utilised as necessary.

2 Overview of Basic Double Copy Theory

We introduce double copy theory in the more conventional context of (typically) 4-dimensional theories. We reserve the primary discussion of 3-dimensional topologically massive theories for later discussion in §3, as this will be accompanied by some additional considerations to those presented in this section.

2.1 The BCJ Double Copy

The idea of classical double copy theory would likely not exist were it not for the scattering amplitude programme of double copy theory. Hence, we briefly introduce the BCJ double copy in order to better understand the origins of classical double copy theory, as well as to provide some context for the double copy as a more general idea than it may be represented for the majority of this work. This discussion will not be thorough, but rather an exercise in 'hand-waving' in order to expediently arrive at the essential results, but the references herein provide a fuller perspective when/where it may be desired.

The BCJ double copy is based on the CK duality, which BCJ originally proposed in [5]. The key finding is that, for Yang-Mills theory[7], the colour factors and the kinematic factors satisfy the same algebra – implying that they satisfy the same Jacobi identity and have the same symmetry properties [11]. The usefulness of this becomes more apparent when one considers that, in Yang-Mills theory, for a full colour-dressed n-point amplitude[8], one may always factorise the amplitude in terms of cubic vertices/graphs/diagrams [1, 8, 11]. Critically, the l-loop n-point scattering amplitude in d-dimensions may be expressed as [1, 11]

$$A_n^l = i^{l-1} g^{n-2+2l} \sum_{G \quad G_3} \int \frac{d^{d\cdot l} q}{(2\pi)^{d\cdot l}} \frac{1}{S_G} \frac{C(G)N(G)}{D(G)} \tag{2.1}$$

where g is the coupling constant, the sum is over cubic graphs, G_3, and $C(G)$, $N(G)$, S_G and $D(G)$ are the colour factor, kinematic factor, symmetry factor and denominator, respectively. The denominator is a product of the denominators of the Feynman propagators of each of the

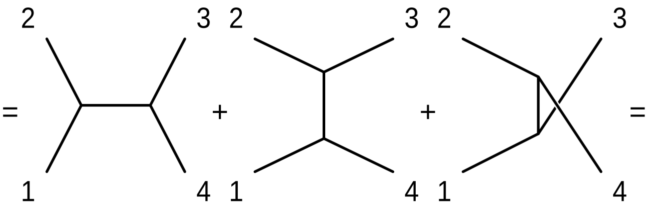
[7]CK duality is intended to be a more general statement that is not necessarily limited to Yang-Mills theory, and instead is encapsulated by the general idea that one may organise a perturbative expansion such that there exists a bijective map between (a) colour factor Lie-algebra *identities* that are associated to certain Feynman diagrams, and (b) the kinematic numerators of those same diagrams [11]. However, this generality is not necessary for the basic discussion we present here.

[8]While attempts have been made to also understand CK duality at the level of the Lagrangian, it remains poorly understood, and we do not discuss it here. One may refer to *e.g.* [9, 29] for further discussion on this point.

internal lines of the diagram so will be proportional to $(p^2 - m_i^2)$ where m_i is the mass of the propagator. The measure, $\mathrm{d}^{d \cdot l}q = \prod_{n=1}^{l} \mathrm{d}^d q_n$, is over the loop momenta, q_n. The factorisation of the numerator into a group-theoretic colour factor – $C(G)$, which is a polynomial of the *structure constants*, f^{abc} – and a kinematic factor – $N(G)$, composed of Lorentz-invariant contractions of polarisation vectors and momenta – is the essential observation to be made from the form of (2.1) [8]. In the case where a non-cubic vertex is present, one multiplies by a factor of 1 in the form of $(p^2 - m_i^2)/(p^2 - m_i^2)$ – $N(G)$ factor gains a new factor in its product, and similarly for $D(G)$ [1].

A useful tree-level example is of 2-2 scattering (4-point amplitude) is [8]

$$A_4^{\text{tree}} =$$



$$= \frac{C_s N_s}{s} + \frac{C_t N_t}{t} + \frac{C_u N_u}{u}, \tag{2.2}$$

where it should be understood that the lines in the above diagrams are the same as the curly gluon lines in the diagrammatic relation in (1.5), but we have represented these as lines above to prevent the diagrams (especially the u-channel diagram) from being unintelligible. The colour factors are

$$C_s \equiv 2f^{a_1 a_2 b} f^{b a_3 a_4}, \quad C_t \equiv 2f^{a_1 a_3 b} f^{b a_4 a_2}, \quad C_u \equiv 2f^{a_1 a_4 b} f^{b a_2 a_3} \tag{2.3}$$

where the a_i are indexed by the leg of the diagram to which they belong to and the b comes from the propagator between vertices. One can notice now that $C_s + C_t + C_u = 0$ (which is simply an expression of the Jacobi identity) which does not fix the numerators in (2.2). In fact, there are a set of transformations called *generalized gauge transformations* that leave the amplitude in (2.2) invariant for *any* set of trivalent diagrams whose colour factors obey a Jacobi identity [5, 8]

$$C_i + C_j + C_k = 0, \tag{2.4}$$

where the kinematic numerators may then transform as

$$N_i \to N_i + D_i \Delta, \quad N_j \to N_j + D_j \Delta, \quad N_k \to N_k + D_k \Delta, \tag{2.5}$$

where Δ is some arbitrary function, and where the D_σ are the propagators that appear *uniquely* in the σ diagram, otherwise termed ‘unshared’ propagators. The CK duality is then realised by the statement that there exists a generalised gauge choice for the N_σ, the *colour-dual gauge*, such that the kinematic factors obey the same algebra as the colour factors [1, 5, 8, 11]

$$\begin{aligned} C_i &= -C_j & N_i &= -N_j \\ C_i + C_j + C_k &= 0 & N_i + N_j + N_k &= 0. \end{aligned} \tag{2.6}$$

An interesting implication of the CK duality is manifested in a new set of relations between n-point colour-ordered amplitudes, known as *(fundamental) BCJ relations* [1, 5, 8]

$$\sum_{i=3}^{n} \left(\sum_{j=3}^{i} s_{2j} \right) A_n[1, 3, ..., i, 2, i+1, ..., n] = 0, \tag{2.7}$$

where we use as notation for the Mandelstam variable $s_{2j} = (p_2 \cdot p_j)^2$.

The most relevant implication that CK duality has, with respect to this work, is that once one has successfully found a generalised gauge transformation to the colour-dual gauge, then one may write the n-point l-loop amplitude with the replacement $C(G) \to N(G)$ in (2.1) [5, 8, 11]. The BCJ double copy relation is the statement that we may use this formula to find a gravitational scattering amplitude with certain substitutions [8, 11]. To make this statement precise (and closely following [11]), suppose we have two distinct l-loop n-point amplitudes, A^l_n and $\tilde{A}^l_n$, each in the form of (2.1). Additionally, let these amplitudes have the *same* colour factors, and allow for the kinematic factors to be different ($N(G)$ and $\tilde{N}(G)$). If we assume that at least one of the amplitudes, say $\tilde{A}^l_n$, exhibits CK duality then we may replace the colour factors of the other amplitude with the kinematic factors of $\tilde{A}^l_n$. Then we may obtain an n-point l-loop gravitational scattering amplitude [5, 6, 8, 11]

$$\boxed{M^l_n = A^l_n\Big|_{\substack{C \to \tilde{N} \\ g \to \kappa/2}} = i^{l-1}\left(\frac{\kappa}{2}\right)^{n-2+2l} \sum_{G\;G_3} \int \frac{\mathrm{d}^{d\cdot l}q}{(2\pi)^{d\cdot l}} \frac{1}{S_G} \frac{N(G)\tilde{N}(G)}{D(G)}}, \tag{2.8}$$

known as the *BCJ double copy relation*.

We conclude our overview of the BCJ double copy at this point. There is much more that can be said, but this would not provide much useful insight for the sections that follow. However, the references cited above provide more detailed discussions. In particular, the reader is pointed to any of [1, 8, 11] for an excellent introduction to double copy theory as a whole.

2.2 The Classical Double Copy

It is natural to consider what, if any, of the features of (inherently quantum) double copy theory may be observed in the classical case. It has been found that there are relationships that carry through to the level of classical description, provided there is a Kerr-Schild description of the (gravitational) metric [7]. That is, the metric can be expressed as [11, 21, 30–32],

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\varphi k_\mu k_\nu, \tag{2.9}$$

where κ is some constant, φ is a scalar function, and the covector k_μ satisfies

$$g^{\mu\nu} k_\mu k_\nu = 0 = \eta^{\mu\nu} k_\mu k_\nu, \tag{2.10}$$

that is, k_μ is null with respect to the full metric ($g_{\mu\nu}$) and the flat (Minkowski) metric ($\eta_{\mu\nu}$). As well as the null condition, k_μ is also geodetic [11, 21], so

$$(k \cdot \quad)k_\mu = 0. \tag{2.11}$$

Enforcing $g_{\mu\nu} g^{\nu\rho} = \delta^\rho_\mu$ (and using this for $\eta_{\mu\nu}$, too) one finds that $\eta^{\mu\nu} = g^{\mu\nu}$ trivially satisfies this condition, however, a more general solution for $g^{\mu\nu}$ is found by considering

$$\begin{aligned} g_{\mu\nu} g^{\nu\rho} &= (\eta_{\mu\nu} + \kappa\varphi k_\mu k_\nu)(\eta^{\nu\rho} - \kappa\varphi \eta^{\nu\alpha} k_\alpha \eta^{\rho\beta} k_\beta) \\ &= \delta^\rho_\mu - \kappa\varphi \delta^\alpha_\mu k_\alpha \eta^{\rho\beta} k_\beta + \kappa\varphi k_\mu k_\nu \eta^{\nu\rho} - \kappa^2 \varphi^2 k_\mu (k_\nu \eta^{\nu\alpha} k_\alpha) \eta^{\rho\beta} k_\beta \\ &= \delta^\nu_\mu + \kappa\varphi (k_\mu k^\rho - k_\mu k^\rho) \\ &= \delta^\rho_\mu \quad = \quad \boxed{g^{\mu\nu} = \eta^{\mu\nu} - \kappa k^\mu k^\nu \varphi,} \end{aligned} \tag{2.12}$$

where in the first line we have emphasized that the indices on k_μ are raised using the Minkowski metric[9], and in the second line we used the null property of k_μ. At this point, we re-emphasise that we will principally be concerned with the 3-dimensional case, however, in this section calculations are performed on a 4-dimensional manifold, as this is more familiar and better suited to the purpose of introducing the material.

[9] In order to make it clear which upper-indexed objects have had their indices raised with the full inverse metric, $g^{\mu\nu}$, we will adopt the convention (suggested by M. Carrillo Gonzalez) that objects whose indices have been raised by the full metric will be written with a 'bar' above them; e.g. $g^{\mu\nu} k_\nu = \bar{k}^\mu$.

Assuming that a Kerr-Schild form of the metric exists, we can make use of its properties to find the form of the Ricci tensor and Ricci scalar (and therefore the Einstein Tensor). To do this, let

$$h_{\mu\nu} \equiv \varphi k_\mu k_\nu = \quad g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}. \tag{2.13}$$

We will treat $h_{\mu\nu}$ as a 'perturbation' about the Minkowski metric. Of course, the $h_{\mu\nu}$ is *not* a perturbation, but it is trivial that $(h_{\mu\nu})^2 \equiv h_\mu^{\ \alpha} h_{\alpha\nu} = 0$, since $h_{\mu\nu}$ acquires its indices from the null vector k_μ, so any contraction of indices in $h_{\mu\nu}$ (or contracted product thereof) is vanishing. Thus, we can simply contract any 'perturbative' expansion to first order in $h_{\mu\nu}$ and the result is *exact*. Solving for the first order form of the Ricci tensor, one finds[10],

$$\begin{aligned} g^{\mu\alpha} R_{\alpha\nu} &= \bar{R}^\mu_{\ \nu} \\ &= R[\eta]^\mu_{\ \nu} - \tfrac{1}{2}\kappa(2h^{\mu\alpha} R[\eta]_{\nu\alpha} + h^\mu_{\ \nu;\beta}{}^{;\beta} - h_\nu^{\ \beta;\mu}{}_{;\beta} - h^{\mu\beta}{}_{;\nu;\beta} + h^\beta_{\ \beta;\nu}{}^{;\mu}) \end{aligned} \tag{2.14}$$

where we have let κ behave as a 'perturbation' parameter for $h_{\mu\nu}$, and $R[\eta]_{\mu\nu}$ is the Ricci tensor associated to the 'unperturbed' (Minkowski) metric. Numerous terms now vanish: any $R[\eta]_{\mu\nu}$ *etc*, will vanish since M^4 (Minkowski 4-space) is Ricci-flat; any internal contraction in $h = h^\alpha_{\ \alpha}$ *etc*, is zero since k_α is null. Removing these terms, contracting indices, and noting that the covariant derivatives reduce to partial derivatives in flat space, we find,

$$\bar{R}^\mu_{\ \nu} = -\frac{\kappa}{2}\Big[\partial^2(h^\mu_{\ \nu}) - \partial^\mu\partial_\beta(h^\beta_{\ \nu}) - \partial_\nu\partial_\beta(h^{\mu\beta})\Big], \tag{2.15}$$

in agreement with [35]. Using the same idea, we could calculate the Ricci scalar, however this is calculation is simply the trace over $\bar{R}^\mu_{\ \nu}$,

$$\begin{aligned} \bar{R}^\mu_{\ \mu} &= -\frac{\kappa}{2}\Big[- \partial^\mu\partial_\beta(h^\beta_{\ \mu}) - \partial_\mu\partial_\beta(h^{\mu\beta})\Big] \\ &= \kappa\partial^\mu\partial^\beta(h_{\mu\beta}), \end{aligned} \tag{2.16}$$

[10]Use of the xAct (tensor computational algebra) package in Mathematica was used to expand the results [33, 34].

where we have, again, used the properties of k_μ to simplify (2.16), and we may now deduce the form of the Einstein tensor

$$\begin{aligned} \bar{G}^{\mu}_{\ \nu} &:= \bar{R}^{\mu}_{\ \nu} - \tfrac{1}{2}\delta^{\mu}_{\ \nu}\bar{R} \\ &= -\frac{\kappa}{2}\Big[\partial^2(h^{\mu}_{\ \nu}) - \partial^{\mu}\partial_{\beta}(h^{\beta}_{\ \nu}) - \partial_{\nu}\partial_{\beta}(h^{\mu\beta}) + \delta^{\mu}_{\ \nu}\partial^{\alpha}\partial^{\beta}(h_{\alpha\beta})\Big]. \end{aligned} \tag{2.17}$$

Another quantity that will be useful in later discussions is the *Cotton tensor*, $C^{\mu\nu}$; the 3-dimensional analogue of the Weyl tensor (that is, it is invariant under conformal transformations and is, therefore, zero for conformally-flat spacetimes), defined as

$$C^{\mu\nu} := \varepsilon^{\mu\alpha\beta} \quad {}_{\alpha}(R^{\nu}_{\ \beta} - \tfrac{1}{4}g^{\nu}_{\ \beta}R), \tag{2.18}$$

where $\varepsilon^{\mu\alpha\beta}$ is the Levi-Civita symbol and $\quad_{\alpha}$ is the covariant derivative. This will be relevant for §3.3 in particular.

2.3 The Single and Zeroth Copies

Before demonstrating that the double copy does reproduce the results of the so-called *single* and *zeroth* copy, it is important to understand what it is that we hope to reproduce. We expect that the double copy we find in our classical Kerr-Schild gravity solution should have equations of motion that correspond to the equations of motion of *linearised Yang-Mills theory* and of *linearised biadjoint scalar theory*, respectively. This should not be surprising – as we saw in 2.2, a Kerr-Schild form of the metric *linearises* gravity. The linearised Yang-Mills and linearised biadjoint scalar theories are briefly introduced in this section.

2.3.1 Biadjoint Scalar Theory or The Zeroth Copy

Biadjoint scalar theory is described by the Lagrangian [27]

$$L_{\mathrm{BAS}} = \frac{1}{2}\partial_{\mu}\Phi^{a\tilde{a}}\partial^{\mu}\Phi^{a\tilde{a}} - \frac{1}{3!}f^{abc}f^{\tilde{a}\tilde{b}\tilde{c}}\Phi^{a\tilde{a}}\Phi^{b\tilde{b}}\Phi^{c\tilde{c}} + \frac{1}{2}m^2\Phi^{a\tilde{a}}\Phi^{a\tilde{a}} + g\Phi^{a\tilde{a}}J^{a\tilde{a}}, \tag{2.19}$$

where g is a coupling constant, f^{abc} ($f^{\tilde{a}\tilde{b}\tilde{c}}$) are the structure constants of the (potentially distinct) Lie algebras generated by T^a ($T^{\tilde{a}}$)

$$[T^a, T^b] = if^{abc}T^c \quad \text{(similarly for } T^{\tilde{a}}). \tag{2.20}$$

It is important to note that the mass term in (2.19) will not be present for the 4-dimensional theories we consider, however it will be present in the 3-dimensional case. One can note [27, 36] that this result can be gauged by minimal coupling, sometimes referred to as Yang-Mills+Biadjoint Scalar (YM+BAS) theory, ($\partial_\mu \Phi^{a\tilde{a}} \to D_\mu = \partial_\mu \Phi^{a\tilde{a}} + f^{abc} A^b_\mu \Phi^{c\tilde{a}}$, where D_μ is the covariant derivative based on gauging *one* of the Lie algebra components) [27], however, as we will only be concerned with the linearised theory we spare ourselves the additional complication of gauging this theory. The equations of motion of the (full, not necessarily linear) theory are

$$\partial^2 \Phi^{a\tilde{a}} + \frac{1}{2} f^{abc} f^{\tilde{a}\tilde{b}\tilde{c}} \Phi^{b\tilde{b}} \Phi^{c\tilde{c}} - m^2 \Phi^{a\tilde{a}} = g J^{a\tilde{a}}, \tag{2.21}$$

where the difference between this and the gauged equations of motion is simply the replacement $D_\mu \leftrightarrow \partial_\mu$. Linearising this theory is done by applying the separation of the scalar field from its Lie algebra indices by choosing

$$\Phi^{a\tilde{a}} \equiv c^a c^{\tilde{a}} \Phi, \quad J^{a\tilde{a}} \equiv c^a c^{\tilde{a}} J, \tag{2.22}$$

where the c^a and $c^{\tilde{a}}$ are constant colour factors indexed by the Lie algebra generator for which they are coefficients, and potentially coming from distinct Lie algebras. In the gauged scenario this would also require the choice $A^a_\mu \equiv c^a A_\mu$. Applying this linearisation to the equations of motion (2.21) we find

$$\begin{gathered} \partial^2 \Phi^{a\tilde{a}} - m^2 \Phi^{a\tilde{a}} = g J^{a\tilde{a}}, \\ \boxed{c^a c^{\tilde{a}} \left(\partial^2 \Phi - m^2 \Phi \right) = c^a c^{\tilde{a}} g J}, \end{gathered} \tag{2.23}$$

(the same is true for the gauge equations of motion) where we have used that $c^a c^b$ is symmetric and f^{abc} is antisymmetric to remove the second term of (2.21) (and in the gauged case this also causes the covariant derivative to reduce to a partial derivative). Equation (2.23) will be what we hope to recover as the zeroth copy from the gravity theory.

2.3.2 Yang-Mills Theory or The Single Copy

We now repeat the process of the previous section, this time for Yang-Mills theory. Yang-Mills theory is described by the Lagrangian (in component form)

$$L_{\mathrm{YM}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu} + gA^{a\mu}J^a_\mu, \tag{2.24}$$

where a is the index of the Lie algebra, g is the coupling constant, and the field strength $F^a_{\mu\nu}$ is defined by[11]

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + if^{abc}A^b_\mu A^c_\nu. \tag{2.25}$$

The well known equations of motion of (2.24) are given by [21, 27]

$$D_\alpha F^{\alpha\mu} = gJ^\mu, \tag{2.26}$$

where the covariant derivative is $D_\mu = \partial_\mu + [A_\mu, \cdot]$. Linearising the equation of motion means making use of the separation of the gauge vector field as

$$A^a_\mu \equiv c^a A_\mu, \tag{2.27}$$

where, as before, the c^a are constant colour factors whose index is associated to the Lie algebra. With this separation, the equation of motion are reduced to

$$\boxed{c^a\partial_\alpha F^{\alpha\mu} = c^a gJ^\mu}, \tag{2.28}$$

where $F^{\alpha\mu} := \partial^\alpha A^\mu - \partial^\mu A^\alpha$. One should note that this is just an algebra-indexed form of the sourced Maxwell equations. Equation (2.28) is the equation of motion for the single copy that we expect our gravity theory will reproduce.

2.4 Stationary Kerr-Schild Solutions

Restricting the analysis to the stationary case (where time derivates are all vanishing) we can simplify the expressions in (2.15) and get an overview of the (stationary) Ricci tensor's structure

[11] As §3 is of primary interest in this work, it provides a more thorough examination of the Lie algebra. Yang-Mills theory is also discussed in many textbooks and articles. The reader is directed to, *e.g.* [37] for additional details, although it is noted that conventions for where one inserts the coupling constant g are di erent.

after expanding $h^{\mu}_{\ \nu}$ in terms of the scalar field φ and the null vector(s) k^{μ}, where we additionally assume that all the functional dependence of k^0 is absorbed into the scalar field and set $k^0 = 1$, without loss of generality, as was done in [38], such that $k^{\mu} = (1, \hat{k})$ where $\hat{k}$ is a unit spatial vector so that k^{μ} remains null, thus

$$\bar{R}^0_{\ 0} = \frac{\kappa}{2}\partial_i^2\varphi, \tag{2.29}$$

$$\bar{R}^i_{\ 0} = -\frac{\kappa}{2}\partial_j\left[\partial^i(\varphi k^j) - \partial^j(\varphi k^i)\right], \tag{2.30}$$

$$\bar{R}^i_{\ j} = -\frac{\kappa}{2}\partial_l\left[\partial^l(\varphi k^i k_j) - \partial^i(\varphi k^l k_j) - \partial_j(\varphi k^i k^l)\right], \tag{2.31}$$

and using the same assumptions in (2.16)

$$\bar{R} = \kappa\partial^i\partial^j(\varphi k_i k_j), \tag{2.32}$$

all of which agree with the results (up to factors of κ) found in [7] and [21].

Now, the key step in the case of classical double copy theory is appropriately identifying a vector field that will produce Maxwell's equations. The conventional identification for the vector field that we make is taken from [21] as

$$A^a_{\mu} \equiv c^a A_{\mu} := c^a\varphi k_{\mu}, \tag{2.33}$$

which is known as the *Kerr-Schild ansatz* and is also referred to in the literature as the *single copy*, and where the c^a are constant colour factors. The superscript index a is used in generalising to non-Abelian gauge vector fields, however in the example calculation we have thus far we are considering an Abelian case and so there is no need for this index, and we can simply absorb the single c into φ (such that $A^{\mu} = \varphi k^{\mu}$ for this discussion). Restricting to the (stationary) Ricci-flat case ($\bar{R}_{\mu\nu} = 0$) we can re-cast (2.29) and (2.30) in terms of the vector field as

$$\begin{aligned}\bar{R}^{\mu}_{\ 0} = 0 &= -\frac{\kappa}{2}\partial_{\nu}\left[\partial^{\mu}(\varphi k^{\nu}) - \partial^{\nu}(\varphi k^{\mu})\right]\\ &= -\frac{\kappa}{2}\partial_{\nu}\left[\partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}\right]\\ &= -\frac{\kappa}{2}\partial_{\nu}F^{\mu\nu}\\ &\boxed{0 = \partial_{\nu}F^{\nu\mu},}\end{aligned} \tag{2.34}$$

where we have used the conventional (Abelian) electromagnetic field tensor $F^{\nu\mu} = \partial^\nu A^\mu - \partial^\mu A^\nu$. Thus, we recover Maxwell's vacuum equations with this identification of the vector field.

Of course, we would like the analogy to hold in the case where there is a non-zero energy-momentum tensor, that is, for situations more general than that of the vacuum Einstein equations. To do this, we start from the non-vacuum Einstein equations

$$\begin{aligned} G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R &= \frac{\kappa^2}{2}T_{\mu\nu} \\ 2R - nR &= \kappa^2 T \\ R &= -\frac{\kappa^2}{(n-2)}T, \end{aligned} \tag{2.35}$$

where we have implicitly acted with $g^{\mu\nu}$ on the second equation to find the third, assuming an n-dimensional spacetime. Using this relationship between the Ricci scalar and the trace of the energy-momentum tensor, we can write the Ricci tensor in terms of the energy-momentum tensor and the trace of the energy-momentum tensor, called the *trace-reversed equations of motion*

$$R_{\mu\nu} = \frac{\kappa^2}{2}\left(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T\right). \tag{2.36}$$

Working with the Ricci tensor (2.15) in its Kerr-Schild form, and applying the Kerr-Schild ansatz, one finds

$$\begin{aligned} -2\bar{R}^{\mu}_{\ \nu} &= \kappa\left[(\partial_\alpha F^{\alpha\mu})k_\nu + X^{\mu}_{\ \nu}\right], \quad \text{where} \\ X^{\mu}_{\ \nu} &= A^\mu(\partial^2 k_\nu) - (\partial_\alpha A^\alpha)(\partial^\mu k_\nu) - (\partial_\nu\partial_\alpha A^\mu)k^\alpha \\ &\quad + (\partial_\alpha A^\mu)(\partial_\nu k^\alpha) - (\partial_\nu A^\mu)(\partial_\alpha k^\alpha) - A^\mu(\partial_\nu\partial_\alpha k^\alpha) \end{aligned} \tag{2.37}$$

where again we use (2.29) and (2.30), in particular. We can now make use of a timelike Killing vector, V^ν, where $k \cdot V = 1$ (given the previous choice of $k^\mu = (1, \hat{k})$, by specifying that V^μ is timelike $k \cdot V = 1 \quad V_\mu = (1, \mathbf{0})$, where $\mathbf{0}$ is the zero 3-vector). This choice of Killing vector, in conjunction with the form of k^μ as well as the fact that we are assuming stationarity, means that

$$
\begin{aligned}
X^{\mu}_{\ \nu} V^{\nu} &= X^{\mu}_{\ 0} \\
&= -A^{\mu}(\partial^2 k_0) - (\partial_\alpha A^{\alpha})(\partial^{\mu} k_0) - (\partial_0 \partial_\alpha A^{\mu}) k^{\alpha} \\
&- (\partial_\alpha A^{\mu})(\partial_0 k^{\beta}) - (\partial_0 A^{\mu})(\partial_\alpha k^{\alpha}) - A^{\mu}(\partial_0 \partial_\alpha k^{\alpha}) \\
&= 0,
\end{aligned}
\tag{2.38}
$$

and therefore we can use $X^{\mu}_{\ \nu} V^{\nu} = 0$, to contract (2.37)

$$
\begin{aligned}
-2\bar{R}^{\mu}_{\ \nu} V^{\nu} &= -2\bar{R}^{\mu}_{\ 0} = \kappa \partial_\alpha (F^{\alpha\mu}) k_\nu V^{\nu} = -\kappa^2 \left(\bar{T}^{\mu}_{\ \nu} - \tfrac{1}{2} T \delta^{\mu}_{\ \nu} \right) V^{\nu} \\
&\boxed{\partial_\alpha F^{\alpha\mu} = g J^{\mu} := -\frac{\kappa}{k \cdot V} \left(\bar{T}^{\mu}_{\ \nu} - \tfrac{1}{2} T \delta^{\mu}_{\ \nu} \right) V^{\nu}} \\
&\partial_\alpha F^{\alpha\mu} = -\kappa \left(\bar{T}^{\mu}_{\ 0} - \tfrac{1}{2} T \delta^{\mu}_{\ 0} \right),
\end{aligned}
\tag{2.39}
$$

and hence we arrive at Maxwell's (Abelian) sourced equations, and it is trivially noted that inclusion of a factor of c^a on either side of the equation reproduces the result (2.28) from the linearised Yang-Mills theory. If we use the same 'Killing vector trick' (with V_μ) again, we will be left only with the $\bar{R}^{0}_{\ 0}$ equation,

$$
\begin{aligned}
-2\bar{R}^{0}_{\ 0} &= -2\bar{R}^{\mu}_{\ \nu} V^{\nu} V_\mu \\
&= \kappa \partial_\alpha F^{\alpha\mu} V_\mu \\
&= \partial_\alpha (\partial^{\alpha} k^{\mu} \varphi - \partial^{\mu} k^{\alpha} \varphi) V_\mu \\
&= \partial^2 \varphi \\
&\boxed{\partial^2 \varphi = \rho := \frac{\kappa}{2} \bar{T}^{0}_{\ 0}},
\end{aligned}
\tag{2.40}
$$

which is the expression that relates to the *zeroth copy*; a relationship that provides a link between biadjoint scalar theory, where, in this instance, the ansatz chosen is [21]

$$
\varphi^{a\tilde{a}} \equiv c^{a} c^{\tilde{a}} \varphi,
\tag{2.41}
$$

where the indices $a, \tilde{a}$ are two Lie-algebra indices (that are not necessarily the same Lie-algebra). As with the linearised Yang-Mills case, we now have an expression that matches the results for the biadjoint scalar theory (in four dimensions, so $m = 0$) provided one multiplies both sides of the equation with two constant colour factors, $c^a c^{\tilde{a}}$.

Theory\Result	Single Copy/LYM-EOM	Zeroth Copy/LBAS-EOM
Gravity	$c^a \partial_\alpha F^{\alpha\mu} = -c^a \kappa \left(\bar{T}^\mu_0 - \frac{1}{2} T \delta^\mu_0 \right)$	$c^a c^{\tilde{a}} \partial^2 \varphi = \frac{\kappa}{2} \bar{T}^0_0$
LYM/LBAS	$c^a \partial_\alpha F^{\alpha\mu} = c^a g J^\mu$	$c^a c^{\tilde{a}} \partial^2 \Phi = c^a c^{\tilde{a}} g J$

Table 2: Table showing a summary of the results for the Schwarzchild black hole double copy. In particular, we show the equations of motion of each (linearised) theory and the analogous 'copied' results from gravity. We have left the sources of gravity untransformed to make the relationship between the sources manifest.

Before we proceed, we briefly reflect on a potentially insidious subtlety that occurs when performing these calculations (and appears to only be mentioned in [39]). Recall, in the above (static solution) derivations, we chose $k^\mu = (1, \hat{k})$. This choice seems a valid one – it is trivial to see that any Kerr-Schild metric (2.9) is invariant under

$$k_\mu \to f k_\mu, \quad \varphi \to \varphi / f^2, \tag{2.42}$$

where f is some arbitrary function of the spacetime coordinates. Imposing the geodetic property (2.11) partially restricts the form of f but does not *determine* f. Since the metric is invariant against such transformations, so too are the Riemann curvature tensor, the Ricci tensor and the Ricci scalar. However, despite the fact that the gravitational theory is 'well-behaved' under transformations (2.42), the same cannot be said of the single and zeroth copies that follow from the particular choice[12] of f. Critically, we would like the double copy to reproduce single and zeroth copy results in an intuitive way – (un)sourced equations of motion in gravitational theories should be related to (un)sourced equations of motion in both the gauge theory and scalar field theory. To that end, one finds that there are two predominant choices that can be made as a general recipe for doing classical double copy theory [39]:

- **Stationary Solutions Choice:**

  As was done to simplify (2.37); choose $k \cdot V$ such that the single copy satisfies Maxwell's equations of motion; this determines f.

- **Time-Dependent (Wave) Solutions Choice:**

  As will be shown in §2.6; enforce $\quad {}_\mu k_\nu = 0$ and work in light-cone coordinates. Notably,

[12]See, in particular, [39] §IV for further elucidation of this point.

this does not completely remove freedom in one's choice of f, however it does restrict it to being dependent on $u = t - z$ (see §2.6). Interestingly, when one generalises to a curved base space ($g_{\mu\nu} = \tilde{g}_{\mu\nu} + \kappa\varphi k_\mu k_\nu$) there does not appear to be a choice of $k \cdot V$ such that the gauge symmetry is not broken – it is found that there is non-minimal coupling to the curved base space [39].

The choices above are those that will be used hereafter in this work, with the only notable exception being §2.5, which follows an altogether different approach, distinct from the computational/derivational method than that used in the rest of this thesis.

2.5 Self-Dual Solutions

An interesting example, that makes use of a slightly different approach, is that of the self-dual Yang-Mills equations of motion

$$F_{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \tag{2.43}$$

which can be obtained as the single copy of 4-dimensional Kerr-Schild-like metrics in light-cone coordinates (we include this only for interest's sake, and a more thorough discussion can be found in [40], whose calculations we follow in this discussion) for self-dual gravitational theories

$$R_{\mu\nu\rho\sigma} = \frac{i}{2}\varepsilon_{\mu\nu\alpha\beta}R^{\alpha\beta}{}_{\rho\sigma}. \tag{2.44}$$

Thus far, we have been using k^μ as some function of spacetime coordinates in position space; we can suppose that the Kerr-Schild form of the metric exists in momentum space as well, where k^μ is replaced by a differential operator, $\hat{k}^\mu$, when written in position space[13]. Then the Kerr-Schild form of the metric takes on a slightly modified form

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\hat{k}_\mu\hat{k}_\nu(\varphi) =: \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{2.45}$$

where the $\hat{k}_\mu$'s are commuting[14] (arbitrary) linear differential operators, of the form $\hat{k}_\mu = A_\mu^{\ \nu}\partial_\nu$, for A a constant matrix, that act on the scalar field φ. We apply the same restrictions as [7], namely:

[13] Note this $\hat{k}$ is a differential operator whereas in an earlier discussion $\hat{k}$ was used to denote a unit spatial vector.

[14] The differential operators *must* commute in order for the metric to remain symmetric.

1. We do not consider a double copy with a dilaton $\quad \mathrm{tr}(h_{\mu\nu}) = 0$.

2. Relating to the previous condition, $\mathrm{tr}(h_{\mu\nu}) \quad \eta^{\mu\nu}\hat{k}_\mu\hat{k}_\nu(\varphi) \quad \eta^{\mu\nu}\hat{k}_\mu\hat{k}_\nu(\varphi) = 0$.

3. We assume $\eta^{\mu\nu}\hat{k}_\mu(\varphi)\hat{k}_\nu(\phi) = 0$.

These conditions mean that the inverse metric is of an identical form to (2.12), with $k \to \hat{k}$ and these *act on* φ instead of multiplying with it. In particular, we consider a 4-dimensional spacetime in light-cone coordinates

$$u = t - z, \quad v = t + z, \quad w = x + iy, \quad w \ \ = x - iy, \tag{2.46}$$

such that

$$\begin{aligned} \mathrm{d}s^2 &= -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ &= -\mathrm{d}(t-z)\,\mathrm{d}(t+z) + \mathrm{d}(x+iy)\,\mathrm{d}(x-iy) \\ &= -\mathrm{d}u\,\mathrm{d}v + \mathrm{d}w\,\mathrm{d}w \ \ , \end{aligned} \tag{2.47}$$

is the Minkowski line element in these coordinates. An “inspired” [40] choice[15] for the values of our $\hat{k}$’s are

$$\hat{k}_u = 0, \quad \hat{k}_v = \frac{1}{4}\partial_w, \quad \hat{k}_w = 0, \quad \hat{k}_{w} \ \ = \frac{1}{4}\partial_u, \tag{2.48}$$

whereafter, using (2.47) and (2.48) to infer the light-cone metric structure it becomes apparent that $\hat{k}_\mu\partial^\mu = -\frac{1}{4}\partial_w\partial_u + \frac{1}{4}\partial_u\partial_w \equiv 0$. The Christoffel symbols are (using xAct and treating $h_{\mu\nu}$ as a perturbation once again [34])

$$\begin{aligned} \Gamma^\rho_{\mu\nu} &= \tfrac{1}{2}\kappa\big(h^\rho_{\nu\,,\mu} + \kappa h^{\rho\alpha}(h_{\mu\nu,\alpha} - h_{\nu\alpha,\mu} - h_{\mu\alpha,\nu}) + h^\rho_{\mu\,,\nu} - h_{\mu\nu}{}^{,\rho}\big) \\ &= \tfrac{1}{2}\kappa\big(h^\rho_{\nu\,,\mu} + \kappa h^{\rho\alpha}h_{\mu\nu,\alpha} + h^\rho_{\mu\,,\nu} - h_{\mu\nu}{}^{,\rho}\big) \\ &= \frac{\kappa}{2}\Big(\partial_\mu\hat{k}_\nu\hat{k}^\rho(\varphi) + \partial_\nu\hat{k}_\mu\hat{k}^\rho(\varphi) - \partial^\rho\hat{k}_\mu\hat{k}_\nu(\varphi) + \kappa(\hat{k}^\rho\hat{k}^\alpha(\varphi))(\partial_\alpha\hat{k}_\mu\hat{k}_\nu(\varphi))\Big), \end{aligned} \tag{2.49}$$

[15] This choice is not as random as it appears at first glance, and comes about from considering a self-dual Yang-Mills equations in Minkowski spacetime, $F_{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$, with complexified gauge field A_μ, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\nu - ig[A_\mu, A_\nu]$ is the Yang-Mills field strength tensor and $[T^a, T^b] = if^{abc}T^c$ defines the Lie algebra. In the light-cone gauge $A_u = 0$ and the self-dual Yang-Mills equations suggest that $A_w = 0$, $A_v = \frac{1}{4}\partial_w\Phi$, and $A_{w^*} = -\frac{1}{4}\partial_u\Phi$, where Φ is a Lie-algebra valued scalar field. See [40] for an explicit and thorough discussion of the analogy between the self-dual Yang Mills equations and the self-dual gravity equations, $R_{\mu\nu\rho\sigma} = \frac{i}{2}\varepsilon_{\mu\nu\alpha\beta}R^{\alpha\beta}{}_{\rho\sigma}$, although the sign conventions in this work are different from those used in [40].

where we have used the third restriction ($\eta^{\mu\nu}\hat{k}_\mu(\varphi)\hat{k}_\nu(\phi) = 0$) to simplify the result in going from the first to the second line, all indices on the right are raised using the Minkowski metric, and the result agrees with that found in [7]. We can substitute this result into the definition of the Riemann curvature tensor and trace over this to find the Ricci curvature tensor as

$$\begin{aligned} R_{\mu\nu} &= \frac{\kappa}{2}\Big[-\partial^2 h_{\mu\nu} + \kappa\big((\partial_\alpha\partial_\beta h_{\mu\nu})h^{\alpha\beta} - (\partial^\alpha h_{\mu\beta})(\partial^\beta h_{\alpha\nu})\big)\Big] \\ &= \frac{\kappa}{2}\Big[-\partial^2 \hat{k}_\mu\hat{k}_\nu\varphi + \kappa\Big((\partial_\alpha\partial_\beta\hat{k}_\mu\hat{k}_\nu\varphi)(\hat{k}^\alpha\hat{k}^\beta\varphi) - (\partial^\alpha\hat{k}_\mu\hat{k}_\beta\varphi)(\partial^\beta\hat{k}_\alpha\hat{k}_\nu\varphi)\Big)\Big], \end{aligned} \tag{2.50}$$

where extensive use of the restrictions has been made, and from which it is apparent that a trace over the Ricci tensor (to find the Ricci scalar) is clearly zero. Einstein's equations in vacuum thus read $R_{\mu\nu} = 0$. Given the choice of light-cone coordinates, we know the precise form of $h_{\mu\nu}$, and the non-vanishing components are

$$h_{vv} = \frac{1}{16}\partial_w^2\varphi, \quad h_{w\,w} = \frac{1}{16}\partial_u^2\varphi, \quad h_{vw} = h_{w\,v} = \frac{1}{16}\partial_u\partial_w\varphi. \tag{2.51}$$

Using these values in conjunction with the self-dual gravity equations (2.44) as well as the first line of (2.50), we find that the vacuum Einstein equations may be reduced to

$$\begin{aligned} 0 &= \partial^2\varphi - \frac{\kappa}{16}\Big((\partial_u^2\varphi)(\partial_w^2\varphi) - (\partial_u\partial_w\varphi)^2\Big) \\ &= \partial^2\varphi - \frac{\kappa}{2}(\partial_\mu\partial_\nu\varphi)(h^{\mu\nu}) \\ &= \partial^2\varphi - \frac{\kappa}{2}\{\partial_w\varphi, \partial_u\varphi\}, \end{aligned} \tag{2.52}$$

which is a single equation for the scalar function, φ – a result first derived by Plebanski [41] – and we have defined the Poisson brackets as

$$\{a, b\} := (\partial_w a)(\partial_u b) - (\partial_w b)(\partial_u a). \tag{2.53}$$

The final line of (2.52) has a parallel with the result for the self-dual Yang-Mills equations of motion (in Minkowski spacetime using light-cone coordinates) for the gauge field with the modified Kerr-Schild ansatz $A^a_\mu = \hat{k}_\mu\varphi^a$, and writing $\Phi = \varphi^a T^a$, [40]

$$\hat{k}_\mu\Big(\partial^2\Phi + ig[\partial_w\Phi, \partial_u\Phi]\Big) = 0. \tag{2.54}$$

Thus, we see that, up to the change of $\{\cdot,\cdot\} \to [\cdot,\cdot]$, the self-dual Yang-Mills solution (2.54) is captured by the self-dual gravity relation (2.52).

For further, in-depth discussion of the relationship between self-dual Gravity and self-dual Yang-Mills, and for a derivation of the solutions for φ we direct the reader to [40]. We do not extend the discussion further, as this relationship between self-dual Yang-Mills and self-dual gravity is an example that is quite distinct from the rest of this work.

Theory\Result	Defining Equations	DE[φ]
SDYM	$F_{\mu\nu} = \frac{i}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}$	$\partial^2\Phi + ig[\partial_w\Phi, \partial_u\Phi] = 0$
SDG	$R_{\mu\nu\rho\sigma} = \frac{i}{2}\varepsilon_{\mu\nu\alpha\beta}R^{\alpha\beta}{}_{\rho\sigma}$	$\partial^2\varphi - \frac{\kappa}{2}\{\partial_w\varphi, \partial_u\varphi\} = 0$

Table 3: Table showing a summary of the results for the self-dual double copy. In particular, we show the defining equations of each theory and the associated differential equations (DE) for the scalar fields for each theory in the light-cone coordinates described in the text.

2.5.1 Schwarzchild Black Holes

The simplest example of a sourced time-independent classical double copy relation is found when considering a static, pointlike (Dirac-delta) source in a spherically symmetric space. The Schwarzchild solution [42] is the most general of these spacetimes, and, by Birkhoff's theorem [43], we know it is time-independent and asymptotically flat – as $r \to \infty$ one finds $g_{\mu\nu} \to \eta_{\mu\nu}$.

Most individuals are introduced to the Schwarzchild metric in a form that makes the Schwarzchild radius, $r_s = 2GM$, and the spherical symmetry manifest in Schwarzchild coordinates (t, r, θ, ϕ) via the line element expression [44]

$$\mathrm{d}s^2 = -\left(1 - \frac{2GM}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2GM}{r}\right)^{-1}\mathrm{d}r^2 + r^2\mathrm{d}\Omega_2^2, \tag{2.55}$$

where t is time, r is the (spatial) radial distance from the origin, θ and φ are the usual spherical coordinates, $\mathrm{d}\Omega_2^2 = \mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2$ is the metric on the unit-radius 2-sphere, M is the mass, and $G \equiv G^{(4)}$ is Newton's gravitational constant in four dimensions. To make use of our calculations at the start of §2.4, however, we must express the Schwarzchild metric in Kerr-Schild form. This is a known result [7]

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{2GM}{r}k_\mu k_\nu, \quad \text{where}$$
$$k^\mu = \left(1, \frac{x^i}{r}\right); \quad r^2 = x_i x^i, \quad i = 1, 2, 3. \tag{2.56}$$

For the pointlike source we choose

$$T^{\mu\nu} = M v^\mu v^\nu \delta^{(3)}(\boldsymbol{x}), \tag{2.57}$$

where M we choose the pure timelike $v^\mu = (1, 0, 0, 0)$. Using that $\kappa^2 = 16\pi G$, the form of the graviton in (2.13), and the Kerr-Schild form of the Schwarzchild metric in (2.56), we may infer the form of the scalar field

$$\frac{2GM}{r} = \kappa\varphi \qquad \varphi \overset{!}{=} \frac{\kappa M}{8\pi r}. \tag{2.58}$$

This form of φ is slightly inconvenient for our purposes, and so we make use of the normalisation suggested in [7], which amounts to redefining (for this discussion only)

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \quad \boxed{h_{\mu\nu} = \frac{\kappa}{2}\varphi k_\mu k_\nu \overset{(2.58)}{=} \varphi = \frac{M}{4\pi r}}. \tag{2.59}$$

Then applying the Kerr-Schild ansatz (2.33) and $\kappa/2 \to g$ one finds

$$A^\mu = g\frac{M}{4\pi r}\left(1, \frac{\boldsymbol{x}}{r}\right). \tag{2.60}$$

If one additionally makes the transformation $M \to c^a T^a$ where c^a is a constant colour charge and T^a is the associated Lie algebra generator, then one readily obtains

$$A^\mu = c^a T^a \frac{g}{4\pi r}\left(1, \frac{\boldsymbol{x}}{r}\right). \tag{2.61}$$

The latter transformation identifies mass 'charge' in gravity with colour charge in the gauge theory. Relating this to our earlier discussions, consider now substituting the solution (2.61) into the boxed equation in (2.39) – we still need to compute the source term, gJ^μ, which we find by computing the right-hand side of (2.39). First, note that the v^μ chosen for (2.57) may be identified with the V^μ used in deriving (2.39); $v \cdot k = 1$ and v^μ is a timelike Killing vector for Schwarzchild spacetime, so we may use it without fear of contradiction in the results of the derivation. Then substituting in the form of (2.57) and performing all the transformations used in producing (2.61), namely

$$\frac{\kappa}{2} \to g, \quad \varphi k^\mu \to A^\mu, \quad M \to c^a T^a, \tag{2.62}$$

we find, for the single copy

$$\partial_\alpha F^{\alpha\mu} = gJ^\mu, \quad J^\mu = -c^a T^a v^\mu \delta^{(3)}(\boldsymbol{x}). \tag{2.63}$$

The result precisely describes a point (colour) source spatially fixed at the origin and unchanging in time (since $v^\mu = (1, \mathbf{0})$), however the form of the gauge field A^μ does not make this entirely obvious.

A gauge transformation of the field A^a_μ provides greater insight into the solution, and we will use the gauge transformation described in [7], which notes that the equations of motion found are those of sourced Maxwell equations, so a valid gauge transformation to consider would be [7]

$$\begin{gathered} A^a_\mu \to A^a_\mu + \partial_\mu \chi^a(x), \quad \chi^a(x) = \chi^a(r) = -g\frac{c^a}{4\pi}\log(r/r_0) \\ \boxed{A_\mu = \left(g\frac{c^a T^a}{4\pi r}, 0, 0, 0\right)}, \end{gathered} \tag{2.64}$$

where we have introduced a length scale r_0 to ensure the object in the logarithm is dimensionless. That this is the Coulomb solution for static (colour) charge is now apparent – that it is the electromagnetic Coulomb solution is verified by a trivial substitution $c^a T^a \to Q$, where Q is the charge of the source.

If one chose to evaluate the zeroth copy, following (2.40), one would find the sourced Poisson equation solution

$$\partial^2 \varphi = \rho, \quad \rho = -gc^{\tilde{a}} T^{\tilde{a}} c^a T^a \delta^{(3)}(\boldsymbol{x}). \tag{2.65}$$

which is a Poisson equation for a static (since φ is independent of t) point (colour) charge density, and we have used now $M \to c^{\tilde{a}} T^{\tilde{a}} c^a T^a$ to ensure consistency with the results of biadjoint scalar theory. Once again, using $c^{\tilde{a}} T^{\tilde{a}} c^a T^a \to Q$ will produce the equivalent equation for a static point source charge in Maxwell theory.

The Schwarzchild double copy example presented in this section having a single copy corresponding to a Coulomb solution(s) should convey the essence of the link between gravity and gauge theories – from a Schwarzchild black hole solution in gravity, we have found a point charge solution in linearised Yang-Mills theory and Maxwell theory, as well as the point charge Poisson equations. Interestingly, these solutions may be extended to higher dimensions, but we do not investigate these solutions in this work[16].

2.5.2 Kerr Black Holes

Motivated by the findings when applying classical double copy theory to the Schwarzchild solution, one may wish to consider how these results may extend to a slightly more conceptually complex object such as the Kerr black holes solution. Indeed, the double copy may be used, naturally with slightly different single and zeroth copies. We use this section to provide a succinct report of the findings of [7], to further motivate the usefulness of the classical double copy.

Kerr black holes are normally introduced in the context of the *Kerr-Newman family* of black hole solutions (charged and rotating solutions) [45, 46] in *Boyer-Lindquist coordinates* (t, r, θ, ϕ) [47, 48]

$$\begin{aligned} \mathrm{d}s^2 = &- \frac{\Delta - a^2\sin^2(\theta)}{\Sigma}\mathrm{d}t^2 - 2a\sin^2(\theta)\frac{r^2 + a^2 - \Delta}{\Sigma}\mathrm{d}t\mathrm{d}\phi \\ &+ \frac{(r^2 + a^2)^2 - \Delta a^2\sin^2(\theta)}{\Sigma}\sin^2(\theta)\mathrm{d}\phi^2 + \frac{\Sigma}{\Delta}\mathrm{d}r^2 + \Sigma\mathrm{d}\theta^2, \end{aligned} \tag{2.66}$$

where the conventions used are

$$\Sigma = r^2 + a^2\cos^2(\theta), \quad \Delta = r^2 - 2Mr + a^2 + e^2, \quad e = \sqrt{Q^2 + P^2}, \tag{2.67}$$

and we interpret Q as the electric charge, P as the magnetic charge, and a as the angular momentum per unit mass. We will focus on the uncharged case ($Q, P = 0 \quad e = 0$).

We revert to our graviton conventions as used before the Schwarzchild discussion (still maintaining the time dependence of φ): $h_{\mu\nu} = \varphi(r)k_\mu k_\nu$. In this case, the Kerr-Schild form of

[16] See [7] for further discussion of this topic.

the metric is found by defining [7]

$$
\begin{aligned}
g_{\mu\nu} &= \eta_{\mu\nu} + \kappa\frac{2GM\,r^3}{r^4+a^2z^2}k_\mu k_\nu \qquad \varphi(r,z) = \frac{2GM\,r^3}{r^4+a^2z^2}, \quad \text{where} \\
k_\mu &= \left(1, \frac{rx+ay}{r^2+a^2}, \frac{ry-ax}{r^2+a^2}, \frac{z}{r}\right), \quad \text{and} \\
1 &= \frac{x^2+y^2}{r^2+a^2} + \frac{z^2}{r^2} \quad \text{implicitly defines } r.
\end{aligned}
\tag{2.68}
$$

To make r defined over the full space, we also define [7]

$$
r = 0, \quad \text{for all values} \quad (x,y,z) \quad \{x^2+y^2 \le a^2, \quad z = 0\}. \tag{2.69}
$$

Note that this solution still has a pure timelike Killing vector $V^\mu = (1,0,0,0)$, so our previous discussions are still applicable. Applying the modified Kerr-Schild ansatz

$$
A^a_\mu = c^a\frac{\kappa}{8\pi}\varphi k_\mu \xrightarrow{\frac{\kappa}{2}\to g} A^a_\mu = c^a\frac{g}{4\pi}\varphi k_\mu \tag{2.70}
$$

reproduces the equations of motion in §2.5.1 with non-zero magnetic field contributions. The source in this case is a disk with a ring singularity at $x^2+y^2=a^2$ [7, 49, 50]. Making use of the results[17] of [7, 49, 50], we make the 'spheroidal' coordinate transformation

$$
\begin{aligned}
x &= \sqrt{r^2+a^2}\sin(\theta)\cos(\phi), \\
y &= \sqrt{r^2+a^2}\sin(\theta)\sin(\phi), \\
z &= r\cos(\theta),
\end{aligned}
\tag{2.71}
$$

whereafter the energy-momentum tensor is defined by [7, 49]

$$
\begin{aligned}
T^{\mu\nu} &= (\omega^\mu\omega^\nu + \zeta^\mu\zeta^\nu)S(\theta,z), \quad \text{where} \\
\omega^\mu &= \tan(\theta)\big(1, 0, 1/(a\sin^2(\theta)), 0\big) \\
\zeta^\mu &= \big(0, 1/(a\cos(\theta)), 0, 0\big), \quad \text{and} \\
S(\theta,z) &= -\frac{M}{8\pi^2 a\cos(\theta)}\delta(z)\Theta(a-\rho), \quad \rho = a\sin(\theta).
\end{aligned}
\tag{2.72}
$$

The first term in this form of $T^{\mu\nu}$ corresponds to a negative (proper) surface density rotating

[17] The discussions of the source of the Kerr(-Newman) metric is discussed in depth in [49, 50], and the reader is directed to these sources for more comprehensive discussion of the topic.

about the z-axis with superluminal (faster-than-light) velocity, whereas the second term corresponds to a radial pressure term [7, 49]. Now that we now have an energy-momentum tensor describing an appropriate source, we consider again (2.39). The source term now takes on the form

$$J^{\mu} = -c^a T^a \delta(z)\Theta(a-\rho)\frac{\sec^3(\theta)}{4\pi a^2}\Big(1, 0, 1/a, 0\Big), \tag{2.73}$$

which is the form of a (colour) charge distribution rotating about the z-axis [7]. The equivalent Maxwell solution is found by again relabelling $c^a T^a \to Q$ – although it is emphasized that this Q is *not* the same as that appearing in (2.67).

One of the details we have ignored[18] in this calculation is the coordinate transformations (from Boyer-Lindquist → Kerr-Schild → spheroidal). In doing so, we have masked the double copy structure, especially in the final transformation to spheroidal coordinates. We do not attempt to alleviate this at present as it constitutes many equations without much gain in intuition[19].

Once again, this solution is generalisable to higher dimensional (*Myers-Perry*) black holes [7, 51], although we do not address these solutions here. Instead, we now leave the realm of static solutions and consider the results of some simple time-dependent solutions.

2.6 Time Dependent Kerr-Schild Solutions

In §2.4 we made use of stationarity to simplify many calculations. We now remove this assumption and discuss two solutions to time-dependent Kerr-Schild systems. All of the analysis in §2.2 still applies.

2.6.1 Kerr-Schild Plane Wave Solutions

As a first example of a time-dependent Kerr-Schild solution, we consider a Kerr-Schild spacetime for which there exists a null, covariantly constant vector field k_μ,

$${}_{\mu}k_{\nu} = 0, \tag{2.74}$$

[18] As is also done in the presentation of [7].

[19] In [7] they do o□er an alternative way to perceive the double copy structure, however this is not illuminating without going through the coordinate transformations, so we do not present it here.

and any spacetimes admitting such a null vector are called *plane-fronted gravitational waves with parallel rays*, abbreviated in the literature as *pp*-waves [32]. Initially, *pp*-waves were described by spacetimes having line elements (*i.e.* metrics) of a form that could be written in *Brinkmann coordinates* [32, 52, 53]

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + 2\mathrm{d}u\mathrm{d}v + 2H(x, y, u)\mathrm{d}u^2, \tag{2.75}$$

where $H(x, y, u)$ is a scalar function. It is easy to see that this expression is almost identical to the light-cone coordinate expression of the Minkowski line element with an additional term (the final term in (2.75)). Thus, if we let

$$\begin{aligned} &u = \tfrac{1}{\sqrt{2}}(z - t), \quad v = \tfrac{1}{\sqrt{2}}(t + z), \\ &k_\mu \mathrm{d}x^\mu = \mathrm{d}u, \quad \kappa\varphi = \kappa\varphi(x, y, u) \equiv 2H(x, y, u) \\ &\boxed{\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + 2\mathrm{d}u\mathrm{d}v + \kappa\varphi k_\mu k_\nu \mathrm{d}x^\mu \mathrm{d}x^\nu}, \end{aligned} \tag{2.76}$$

then we are able to explicitly write the metric in Kerr-Schild form (2.9). The choice of u as being oppositely-oriented to conventional light-cone coordinates is useful for direct comparison of (2.75) and (2.76), however beyond that it is simply inconvenient (especially for comparison to other works), and so we redefine

$$\begin{aligned} &u = \tfrac{1}{\sqrt{2}}(t - z) \qquad k_\mu \mathrm{d}x^\mu = -\mathrm{d}u, \\ &\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 - 2\mathrm{d}u\mathrm{d}v + \kappa\varphi k_\mu k_\nu \mathrm{d}x^\mu \mathrm{d}x^\nu \end{aligned} \tag{2.77}$$

We already have the Einstein tensor for a generic Kerr-Schild metric from (2.17), and we can use that the Brinkmann form of the metric has a Kerr-Schild representation to find the Einstein tensor and the corresponding vacuum equations of motion. Given the light-cone coordinates specified above, we know $k^\mu = (0, 1, 0, 0)$ (recall that we raise the indices on k_μ with $\eta^{\mu\nu}_{(\mathrm{LC})}$) where $\eta_{(\mathrm{LC})\mu\nu}$ is the Minkowski metric in the Light-Cone coordinates and $\eta^{\mu\nu}_{(\mathrm{LC})} = \eta_{(\mathrm{LC})\mu\nu}$, and where we have ordered the coordinates as $r^\mu = (u, v, x, y)$. First, notice from (2.16) that the Ricci scalar is zero since φ depends on u and not v, and only h_{00} is non-zero, and $\partial^\mu = (-\partial_v, -\partial_u, \partial_x, \partial_y)$ where, again, we have raised the index using $\eta^{\mu\nu}_{\mathrm{LC}}$. Thus, the vacuum Einstein equations reduce to $\bar{R}^\mu_\nu = 0$. The relevant non-zero components for the calculation of the Ricci tensor are

$$h^1_{\ 0} = \varphi(u,x,y), \quad h^{11} = \varphi(u,x,y), \tag{2.78}$$

the result of which is that only the first term in (2.17) produces a non-zero result, and the Einstein equations are reduced to (following the preceding discussion) as

$$\bar{G}^\mu_{\ \nu} = 0, \quad \bar{R} = 0 \overset{\text{EOM}}{=} \bar{R}^\mu_{\ \nu} = 0 \implies \partial^2\varphi = \partial_j^2\varphi = 0, \tag{2.79}$$

where $j = \{x, y\}$, in agreement with [7, 53]. Independence of φ from v means that only the r_j derivatives of φ are non-trivial. The solution for the j-coordinates is the familiar

$$\partial_j^2\varphi(x,y;u) = 0 \implies \varphi(x,y;u) = a_0(u)\exp(\pm i\tilde{r}.p) \tag{2.80}$$

where $\tilde{r} = (0,0,x,y)$, $a_0(u)$ is a scalar function, and by writing $\varphi(x,y;u)$ we mean that this result should be considered at a particular value of u. This solution corresponds to a plane-fronted wave; the ‘plane’ is the xy-plane, the dependence of φ on u means that the wave is propagating at the speed of light, and we may interpret the four vector, p^μ, as a momentum with $p^2 = 0$ (*i.e.* φ is massless). We can now consider the single and zeroth copies of this result by substituting in our expression for φ into the Kerr-Schild ansatz for the gauge field. Considering (2.39) and (2.40), one trivially sees that the single copy solutions are

$$A^a_\mu = c^a A_0(u) e^{\pm i\tilde{r}\cdot p} k_\mu \tag{2.81}$$

where, as before, c^a is a constant colour factor, and we use $a_0 \to A_0$ to indicate that the field the wave represents has different units. The result corresponds to a (linearised) Yang-Mills plane wave propagating in the z-direction with polarization vector $\epsilon_\mu = k_\mu$ [54, 55]. The same can be said of the zeroth copy solution

$$\Phi^{a\tilde{a}} = c^a c^{\tilde{a}} y_0(u) e^{\pm i\tilde{r}\cdot p}. \tag{2.82}$$

which is just a plane wave in biadjoint scalar theory, and we have transformed $A_0 \to y_0$ again to highlight that the field may have different units. Both (2.81) and (2.82) are solutions to their respective *sourceless* equations of motion.

2.6.2 Kerr-Schild Shockwave

Adding a source does not alter the left-hand side of Einstein's equations, and we would instead find the only non-zero equation to be

$$\begin{aligned} \bar{G}^1_{\ 0} &= \frac{\kappa^2}{2}\bar{T}^0_{\ 1} \\ -\partial_j^2\varphi &= \kappa\bar{T}^0_{\ 1} =: gJ(u,x,y), \end{aligned} \tag{2.83}$$

where we have defined J to be the source and, since φ is only dependent on $x,\ y,$ and u so too is the source. The solutions to this equation are dependent on the source. A common form for the (non-zero components of the) energy-momentum tensor is a δ-function source, discussed in the following section.

As before, we could have gone about this derivation in a slightly different manner, making use of the knowledge that in chosen coordinates $-\partial_u$ is a Killing vector, so that we can write $V_\mu \mathrm{d}x^\mu = \mathrm{d}v$, so we have $k\cdot V = k^\mu V_\mu = 1$. Then the reduction of the equations is much faster than before (where we had to consider the various components separately)

$$\bar{G}^\mu_{\ \nu}V^\nu V_\mu = \frac{\kappa^2}{2}\bar{T}^\mu_{\ \nu}V^\nu V_\mu = \quad -\bar{G}^1_{\ 0} = -\frac{\kappa^2}{2}\bar{T}^1_{\ 0}, \tag{2.84}$$

and hence yields 2.83 more efficiently.

We now consider a specific form of the source – a point particle moving at the speed of light represented by a δ-function – that has come to be known as a *shockwave solution*. The results were initially found by Aichelburg and Sexl by considering the gravitational field of a massless particle, and the results stated hereafter in this section are primarily based upon their work [56][20]. We begin by finding the form of $h_{\mu\nu}$, from which we find the form of φ.

In [56], the shockwave solution found in their first approach is found using the *linearised approximation* to the metric[21]. We emphasise again that, while the solutions found in this thesis are, indeed, for a linearised form of the metric, the choice of $h_\mu\nu = \varphi k_\mu k_\nu$ and the null and geodetic properties of k_μ mean that the linear form of the metric is *exact* (since $(h_{\mu\nu})^2 = 0$). The set-up of the problem begins by stating that the energy-momentum tensor of a point particle of

[20] Shockwaves are inherently interesting solutions in-and-of themselves. A brief discussion of shockwave solutions in a non-linear form of the metric is included in the appendices §A.

[21] However, the results from the linear approximation agree with the results found for the full metric [56]; compare the results found in this section with those of appendices §A.

mass, m, and having velocity, v, in the z-direction (we are no longer working in light-cone/light-cone-like coordinates, but instead in traditional Minkowski coordinates, and we shall shortly take the massless, ultra-relativistic limit, $m \to 0$, $v \to 1$) is [56]

$$T_{\mu\nu}(x) = \frac{m}{\sqrt{1 - v^2}}\delta(x)\delta(y)\delta(z - vt)s_\mu s_\nu \tag{2.85}$$

where $s_\mu = \delta^0_\mu + v\delta^3_\mu$. Einstein's equations are as in (2.83), however, we need not have simplified as much as was done – the statement

$$\partial_\alpha\partial^\alpha(\varphi k_\mu k_\nu) \equiv \quad (\varphi k_\mu k_\nu) = \quad (h_{\mu\nu}) = \kappa T_{\mu\nu}, \tag{2.86}$$

is the same statement as would be found had we not reduced the equations of motion to the non-trivial components. The Dirac-delta form of the energy-momentum tensor hints at the use of a Green function to solve for the 'graviton' $h_{\mu\nu}$. We use the retarded Green function and find

$$h_{\mu\nu}(x) = \frac{m s_\mu s_\nu}{\sqrt{\left[(1 - v^2)(x^2 + y^2) + (z - vt)^2\right](1 - v^2)}}. \tag{2.87}$$

We now take the ultra-relativistic limit $v \to 1$ and defining $p := m(1 - v^2)^{-1/2}$ and requiring that p remain constant as we take the limit (which, itself, enforces that $m \to 0$, the massless limit). This limit is di□cult to take for the left-hand side of (2.86) given the form of the solution (2.87). However, applying the limit to the energy-momentum tensor is trivial, and it becomes

$$\lim_{\substack{v \to 1 \\ p=\text{const.}}} T_{\mu\nu} = p\delta(x)\delta(y)\delta(z - t)s_\mu s_\nu \tag{2.88}$$

where now it is understood that $s_\mu \overset{v \to 1}{=} \delta^0_\mu + \delta^3_\mu$. It is naturally tempting to use a Green function to solve for $h_{\mu\nu}$ in this limit, but this cannot be used – the solution is found by an ansatz that separates the non-trivial δ-function from the rest of the solution [56, 57], which yields

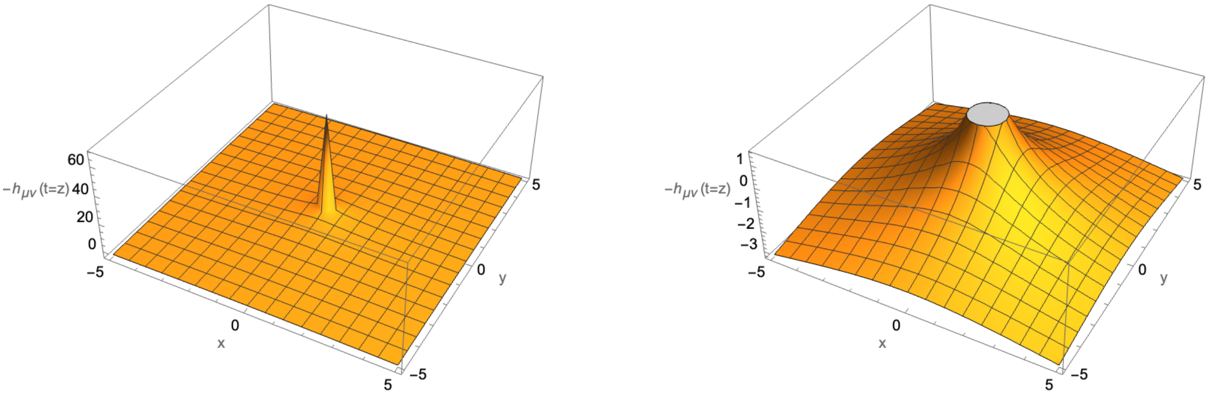
$$h_{\mu\nu}(x) = \frac{1}{2}p\delta(z - t)\ln(x^2 + y^2)s_\mu s_\nu + h^H_{\mu\nu}, \tag{2.89}$$

where $h^H_{\mu\nu}$ is the homogeneous solution to (2.86), and we have made use of the 2-dimensional

Poisson equation Green function

$$
\begin{aligned}
(\partial_x^2 + \partial_y^2)G(x,y) &= 2\pi\delta(x)\delta(y),\\
G(x,y) &= \frac{1}{2}\ln(x^2+y^2).
\end{aligned}
\tag{2.90}
$$

Thus, $h_{\mu\nu}$ is only non-zero on the hyperplane $z = t$ and the metric elsewhere reduces to the Minkowski metric. It is interesting to visualise how $h_{\mu\nu}$ causes $g_{\mu\nu}$ to deviate from Minkowski space when on the hyperplane $z = t$; this is shown in figure 1. Relating this back to our original



(a) (b)

Figure 1: Graphics showing the values of $-h_{\mu\nu}$ in the $z = t$ hyperplane (*i.e.* the non-Minkowskian contribution to the metric). The homogeneous contribution is ignored, and we assume we are on a non-zero value of $h_{\mu\nu}$; that is h_{00}, h_{33}, h_{13} and h_{31}. Thus, these are effectively plots of $-\ln(\frac{x^2+y^2}{r_0^2})$, where we have chosen $r_0^2 = 2$. Note: the negative values are used, as otherwise the 'spike' is not visible. (a) Graphic showing the divergence at the particle's location. (b) Graphic showing the effect of $h_{\mu\nu}$ in a region near the particle's location.

purpose (finding the form of φ), we see that (2.89) suggests

$$
\varphi(x) \quad \delta(z-t)\ln\left(\sqrt{x^2+y^2}\right), \tag{2.91}
$$

in agreement with [7, 58].

The form of the graviton in (2.89) is missing one critical element – a scale factor in the logarithm to ensure its argument is unitless. Re-expressing this in light-cone coordinates, we have

$$
h_{\mu\nu} = p\delta(u)\ln\left(\frac{\tilde{r}}{r_0}\right)k_\mu k_\nu, \tag{2.92}
$$

where we have ignored the homogeneous contribution, r_0 is an arbitrary scale factor, and we have used $\tilde{r}^\mu = (0, 0, x, y)$. With this choice, we have an expression for the single copy

$$A^a_\mu = -c^a \frac{Q}{4\pi} \delta(u) \ln\left(\frac{\tilde{r}}{r_0}\right) k_\mu \tag{2.93}$$

where we have used $p \to \frac{Q}{4\pi}$ to highlight that this is a charged (Yang-Mills) shockwave [54]. The zeroth copy is then

$$\Phi^{a\tilde{a}} = c^a c^{\tilde{a}} \frac{y}{4\pi} \delta(u) \ln\left(\frac{\tilde{r}}{r_0}\right), \tag{2.94}$$

producing what could be called a biadjoint scalar shockwave, and making the choice $Q \to y$ to highlight that these charges may be distinct.

The discussions presented in this section may be expanded upon in several instances – we have not given much attention to the perspective gained from a *purely* Yang-Mills or biadjoint scalar approach. There is, however, little to be added – one could solve for these solutions explicitly and then arrive at the results presented in this section, and we avoid this for brevity. The reader is directed to [59] for a more expansive discussion of the methods introduced in this section (and discussions of monopoles).

3 Topologically Massive Double Copy in 3D

We shall now investigate the primary theory we wish to elucidate in this work – the classical double copy of TMG in 3D. The approach to this discussion is pedagogical; topologically massive theories are introduced before we broach the principal subject, and it is the hope of the author that this methodology best articulates the most interesting and important aspects of the theories.

An initial deliberation that one may have is why these theories are *topologically* massive. The generic answer is that the additional term that are included in the Lagrangians are linked to the Chern-Simons secondary characteristic classes, and the action $S_{\text{top.}} = \int d^3x L_{\text{top.}}$ is proportional to these classes[22] [60, 61].

We will introduce Topologically Massive Spinor Electrodynamics (TMSE), followed by Topologically Massive Yang-Mills (TMYM) theory, before we introduce Topologically Massive Gravity (TMG). In our discussion of TMG, we will assume a Kerr-Schild form of the metric and aim to show that the classical double copy will (applying the single/zeroth copy ansatzes) will produce the results expected for the TMYM and biadjoint scalar theories, respectively. Our discussion will often follow that of [61], however we note that the sign conventions di□er – we always use $\eta_{\mu\nu} = \text{diag}(-1, 1, 1)$.

Finally, many of the details in the discussions of the topologically massive gauge theories are superfluous – all that we need for our double copy analysis is the equations of motion. However, this section should serve the partial purpose of introducing the reader to topologically massive gauge theories, and in this vain many further calculations are included. Additionally, many of these results may be found in [61]. Where the results are simply presented in [61], we aim to fill in the missing details – several of which are exceptionally tedious calculations which appear explicitly in the coming sections and are not found in any of the conventional literature[23].

3.1 Abelian Gauge Theory

Our 'elementary' example of a topologically massive Abelian vector theory is TMSE, whose action is built up from a gauge term, a fermion term, an interaction term and a Chern-Simons

[22] More on the link to these classes is discussed in the appendices; see §B.
[23] Based on the author's experience in producing this work.

term [61]

$$
\begin{aligned}
S_{\mathrm{TMSE}} &= L_G + L_F + L_I + L_{CS} \\
&= \int \mathrm{d}^3x \Big[\Big(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \Big) + \Big(i\bar{\psi} \partial\!\!\!/ \psi - m\bar{\psi}\psi \Big) + \Big(e\bar{\psi}\gamma^{\mu}\psi A_{\mu} \Big) + \Big(\frac{\mu}{4} \varepsilon^{\mu\nu\alpha} F_{\mu\nu} A_{\alpha} \Big) \Big],
\end{aligned}
\tag{3.1}
$$

where $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)$, the γ^μ are a 2-dimensional realisation of the Dirac algebra γ-matrices from spinor electrodynamics ($\gamma^0 = \sigma^3$, $\gamma^1 = i\sigma^1$, $\gamma^2 = i\sigma^2$; σ^μ are the Pauli matrices), $\bar{\psi} \equiv \psi^\dagger \gamma^0$ (the Dirac adjoint), e is a dimensionful[24] coupling constant $[e] = 1/2$, μ is the (topological) mass given to the gauge field, and we have adopted the Feynman 'slash' convention $\gamma^\mu \partial_\mu \equiv \partial\!\!\!/$. One can easily find the equations of motion for ψ by solving the Euler-Lagrange equations for $\bar{\psi}$

$$
(i\partial\!\!\!/ - m + eA\!\!\!/)\psi = 0,
\tag{3.2}
$$

and, similarly, solving the Euler-Lagrange equations for the gauge field A_μ one finds the normal Maxwell term as well as an additional contribution from the Chern-Simons term in the action

$$
\frac{\mu}{2} \varepsilon^{\mu\alpha\beta} F_{\alpha\beta} + \partial_\alpha F^{\alpha\mu} = -e\bar{\psi}\gamma^\mu\psi = J^\mu.
\tag{3.3}
$$

Critically, performing the (standard) gauge transformation

$$
\begin{aligned}
A_\mu &\to A_\mu = A_\mu + \frac{1}{e}\partial_\mu\theta, \\
\psi &\to \psi \ = \exp(i\theta)\psi,
\end{aligned}
\tag{3.4}
$$

then we know the only term that needs to be checked for gauge invariance is the (new) Chern-Simons contribution, which transforms as

$$
\begin{aligned}
L_{CS} \to L_{CS} &= \frac{\mu}{4}\varepsilon^{\mu\nu\alpha} F_{\mu\nu} A_\alpha \\
&= \frac{\mu}{4}\varepsilon^{\mu\nu\alpha} \Big[(A_\alpha + \tfrac{1}{e}\partial_\alpha\theta)\partial_\mu(A_\nu + \tfrac{1}{e}\partial_\nu\theta) - (A_\alpha + \tfrac{1}{e}\partial_\alpha\theta)\partial_\nu(A_\mu + \tfrac{1}{e}\partial_\mu\theta) \Big] \\
&= \frac{\mu}{4}\varepsilon^{\mu\nu\alpha} \Big[F_{\mu\nu} A_\alpha + \tfrac{1}{e} F_{\mu\nu} \partial_\alpha\theta \Big] \\
&= L_{CS} + \partial_\alpha \Big(\frac{\mu}{4e} \varepsilon^{\mu\nu\alpha} F_{\mu\nu} \Big),
\end{aligned}
\tag{3.5}
$$

[24] A dimensionless action being the guiding principle, in (natural) mass units: $[\int \mathrm{d}^3x] = -3$, $[\partial_\mu] = -1$ $\quad [A] = 1/2$, and $[\psi] = 1$ $\quad [e] = 1/2$. Note that this differs from the 4-dimensional case where $[e] = 0$.

where we have used the antisymmetry of the Levi-Civita symbol (and the symmetry of derivative orderings) to rewrite the second term as a total derivative. Thus, the equations of motion are invariant under gauge transformations of the form of (3.4). We briefly remark that discrete transformations have the following effects [61]:

- Charge Conjugation: Equations of motion are invariant.
- Parity Transformation: Mass terms in equations of motion change sign.
- Time Reversal: Mass terms in equations of motion change sign.
- Time-Parity Transformation: Equations of motion invariant.

Hence, we do have CPT-symmetry.

3.1.1 Abelian Gauge Field Solutions

We can use the equations of motion (3.3) to solve for the vector potential, A^μ. First, notice that the dual of $F_{\mu\nu}$ is

$$F^\alpha := \frac{1}{2}\varepsilon^{\alpha\mu\nu}F_{\mu\nu}; \quad F^{\alpha\beta} := \varepsilon^{\alpha\beta\mu} \quad F_\mu. \tag{3.6}$$

Taking the divergence of the equations of motion (assuming that the current is conserved) 3.3 we find that

$$\begin{aligned} \partial_\mu\Big(\frac{\mu}{2}\varepsilon^{\mu\alpha\beta}F_{\alpha\beta} + \partial_\alpha F^{\alpha\mu}\Big) &= \partial_\mu J^\mu \\ \partial_\mu(\ \ F^\mu) &= 0, \end{aligned} \tag{3.7}$$

where the second term vanishes due to the symmetric-antisymmetric product. This amounts to a statement of the *Bianchi identity* of the dual field, $\ \ F^\mu$. The dual form of (3.3) is

$$\begin{aligned} -\varepsilon_{\rho\sigma\mu}\Big(\frac{\mu}{2}\varepsilon^{\mu\alpha\beta}F_{\alpha\beta} + \partial_\alpha F^{\alpha\mu}\Big) &= -\varepsilon_{\rho\sigma\mu}J^\mu, \\ -\varepsilon_{\rho\sigma\mu}\partial_\alpha F^{\alpha\mu} - \frac{\mu}{2}\varepsilon_{\rho\sigma\mu}\varepsilon^{\mu\alpha\beta}F_{\alpha\beta} &= -\varepsilon_{\rho\sigma\mu}J^\mu, \\ -\partial_\sigma \quad F_\rho + \partial_\rho \quad F_\sigma - \frac{\mu}{2}(\delta^\alpha_\rho\delta^\beta_\sigma - \delta^\alpha_\sigma\delta^\beta_\rho)F_{\alpha\beta} &= -\varepsilon_{\rho\sigma\mu}J^\mu, \\ \boxed{-\partial_\sigma \quad F_\rho + \partial_\rho \quad F_\sigma - \mu F_{\rho\sigma} = -\varepsilon_{\rho\sigma\mu}J^\mu}&, \end{aligned} \tag{3.8}$$

where we have used the 3-dimensional Levi-Civita product identity in going from the second to the third line. Taking the divergence of the final (boxed) equation above yields a Klein-Gordon-like equation for the dual field strength

$$
\begin{aligned}
\partial^\rho\Big(- \partial_\sigma \quad F_\rho + \partial_\rho \quad F_\sigma - \mu F_{\rho\sigma}\Big) &= -\partial^\rho \varepsilon_{\rho\sigma\mu} J^\mu, \\
F_\sigma - \partial^\rho\partial_\sigma \quad F_\rho - \mu\partial^\rho F_{\rho\sigma} &= -\partial^\rho \varepsilon_{\rho\sigma\mu} J^\mu, \\
F_\sigma - \mu^2 \quad F_\sigma + \mu J_\sigma &= -\partial^\rho \varepsilon_{\rho\sigma\mu} J^\mu, \\
\boxed{(\quad + \mu^2) \quad F^\sigma = \mu\Big(\eta^{\sigma\mu} - \varepsilon^{\sigma\mu\rho}\frac{\partial_\rho}{\mu}\Big)J_\mu},
\end{aligned}
\tag{3.9}
$$

where we have used (3.7) and (3.3) to go from line 2 to 3, and we have raised/lowered all indices to produce the boxed equation. Hence, we can see that the excitations of the gauge field are massive.

One can now make an interesting observation that (3.9) can be factorised into two operations [61]

$$
\begin{aligned}
\Big(\eta_{\alpha\sigma} + \varepsilon_{\alpha\sigma\beta}\frac{\partial^\beta}{\mu}\Big)\Big(\eta^{\sigma\mu} - \varepsilon^{\sigma\mu\rho}\frac{\partial_\rho}{\mu}\Big) \quad F_\mu &= \mu\Big(\eta^{\sigma\mu} - \varepsilon^{\sigma\mu\rho}\frac{\partial_\rho}{\mu}\Big)J_\mu, \\
\boxed{\Big(\eta_{\alpha\sigma} + \varepsilon_{\alpha\sigma\beta}\frac{\partial^\beta}{\mu}\Big) \quad F^\sigma = \frac{1}{\mu}J^\sigma},
\end{aligned}
\tag{3.10}
$$

which, using the boxed equations in (3.9) and (3.10), allows us to immediately write down the form of a solution for $\quad F^\mu$

$$
\begin{aligned}
F^\mu &= \frac{\mu}{\quad + \mu^2}\Big(\eta^{\mu\sigma} - \varepsilon^{\mu\sigma\rho}\frac{\partial_\rho}{\mu}\Big)J_\sigma \\
&= \frac{1}{\mu}\Big(\eta^{\mu\sigma} + \varepsilon^{\mu\sigma\rho}\frac{\partial_\rho}{\mu}\Big)^{-1}J_\sigma.
\end{aligned}
\tag{3.11}
$$

We have performed an abuse of notation above, and the $1/(\quad + \mu^2)$ factor should be understood as the *inverse operator* $(\quad + \mu^2)^{-1}$ that, when acting in conjunction with $(\quad + \mu^2)$ is just the identity operator. We will continue to make similar use of this notation throughout this section. Using the solution for the dual field strength to obtain a solution for the gauge vector field is a ‘finicky’ procedure that makes use of a subtle trick [18]. The procedure amounts to acting on both sides of (3.11) with a Levi-Civita symbol and a derivative. Explicitly, we find

$$
\begin{aligned}
&\text{LHS} \to \varepsilon_{\mu\alpha\beta}\partial^{\beta}(\ F^{\mu}) = -\ \ A_{\alpha} + \partial^{\rho}\partial_{\alpha}A_{\rho}, \\
&\text{RHS} \to \varepsilon_{\mu\alpha\beta}\partial^{\beta}\Big[\frac{\mu}{\ + \mu^2}\Big(\eta^{\mu\sigma} - \varepsilon^{\mu\sigma\rho}\frac{\partial_{\rho}}{\mu}\Big)J_{\sigma}\Big] = \frac{\mu}{\ + \mu^2}\Big(\varepsilon_{\mu\alpha\beta}\partial^{\beta}J^{\mu} - \frac{1}{\mu}\ \ J_{\alpha}\Big), \\
&\boxed{A_{\alpha} = \frac{\mu}{\ + \mu^2}\Big(\frac{1}{\mu}J_{\alpha} - \frac{1}{\ }\varepsilon_{\mu\alpha\beta}\partial^{\beta}J^{\mu}\Big) + [\text{gauge term(s)}]}.
\end{aligned}
\tag{3.12}
$$

The final gauge term(s) in the boxed equation of (3.12) has come from the second term, ($\partial^{\rho}\partial_{\alpha}A_{\rho}$) from the LHS transformation. Additionally, we have again made use of the notion of inverse operators and the commutativity of differential operators to obtain the boxed result.

We conclude the discussion of topologically massive Abelian gauge theory at this juncture – content with the knowledge that, the gauge field excitations are, indeed, massive. One could consider the spinor field and/or one could continue by examining the spin content of the theory. We shall do neither here, as it would be extraneous and would not provide any significant insights into the theory that relate to the overarching discussion of classical double copy theory, but we direct the reader to [61], where the spin content of the theory is examined in detail, and to [62] for a more general discussion of spin in three dimensions. The key takeaways from the discussion of the spin of the gauge field in [61] are:

1. The spin content is dependent on the Lorentz group (not just the rotation group $O(2)$);
2. The massless theory is spinless;
3. If one wants a parity invariant version of the theory the spin must be zero or one must introduce an additional degree of freedom; and
4. The field excitations described by (3.3) correspond to a (massive) spin-1 particle.

Having introduced topologically massive Abelian gauge theory in three dimensions, we extend this analysis to the non-Abelian case of topologically massive Yang-Mills in the following section.

3.2 Yang-Mills Gauge Theory

The generalisation of the discussion in the previous section to the case of a non-Abelian gauge field (now considering *only* the gauge contributions[25] in the Lagrangian) begins with the Yang-

[25] We will still include a coupling to matter through an $A_{\mu}J^{\mu}$ term, but this J^{μ} has no explicit dynamics and so can be considered an external current [18].

Mills form of the Lagrangian with a Chern-Simons term [61]

$$L_{\mathrm{TMYM}} = -\frac{1}{2g^2}\mathrm{tr}\Big(F^{\mu\nu}F_{\mu\nu}\Big) + \frac{\mu}{2g^2}\varepsilon^{\mu\nu\alpha}\mathrm{tr}\Big(F_{\mu\nu}A_\alpha - \frac{2}{3}A_\mu A_\nu A_\alpha\Big) + 2\mathrm{tr}\Big(A_\mu J^\mu\Big), \tag{3.13}$$

where the trace is over the Lie algebra indices (suppressed), we have again chosen μ to be the (topological) mass of the gauge field, and where g is the coupling constant satisfying $[\mu/g^2] = 0$ (μ/g^2 is dimensionless). The factor of 2 in front of the source term is chosen to cancel the normalisation from taking the trace. We will now assume we are working in a representation of $\mathfrak{su}(N)$, and we will make use of the conventions used in [63], so that we now write the gauge field and field strength tensors in terms of the generators of the Lie algebra, T^a, as

$$\begin{aligned} A_\mu &\equiv gT^a A^a_\mu, \\ F_{\mu\nu} &\equiv gT^a F^a_{\mu\nu} \equiv gT^a\Big(\partial_\mu A^a_\nu - \partial_\nu, A^a_\mu + igf^{abc}A^b_\mu A^c_\nu\Big). \end{aligned} \tag{3.14}$$

The conventions (and some useful results) based on our use of $\mathfrak{su}(N)$ are [61, 63]

$$\begin{aligned} D_\mu &\equiv \partial_\mu + [A_\mu, \cdot], \\ [T^a, T^b] &\equiv if^{abc}T^c, \\ \mathrm{tr}(T^a) &= 0, \\ \mathrm{tr}(T^aT^b) &\equiv \frac{1}{2}\delta^{ab}, \\ \mathrm{tr}(T^aT^bT^c) &= \frac{1}{4}\Big(d^{abc} + if^{abc}\Big), \quad \text{where} \\ d^{abc} &= 2\mathrm{tr}\Big(\{T^a, T^b\}T^c\Big), \end{aligned} \tag{3.15}$$

where D_μ is the covariant derivative, f^{abc} are the structure constants, $[\cdot,\cdot]$ is the commutator and $\{\cdot,\cdot\}$ is the anti-commutator. One should note that the object d^{abc} is a totally symmetric object due to the cyclic property of the trace

$$\mathrm{tr}(T^aT^bT^c) = \mathrm{tr}(T^bT^cT^a) = \mathrm{tr}(T^cT^aT^b). \tag{3.16}$$

As in §3.1, we wish to find the equations of motion for the gauge field. The first term in (3.13) produces the usual contribution to the equations of motion, $D_\mu F^{\mu\nu}$, so we turn our attention to the contribution to the equations of motion coming from the Chern-Simons term.

To do this, we will expand the second term and we will solve for the Euler-Lagrange equation contributions. The Chern-Simons term is expanded as

$$
\begin{aligned}
&\frac{\mu}{2g^2}\varepsilon^{\mu\nu\alpha}\mathrm{tr}\Big(F_{\mu\nu}A_\alpha - \frac{2}{3}A_\mu A_\nu A_\alpha\Big)\\
&\quad = \frac{\mu}{2}\varepsilon^{\mu\nu\alpha}\mathrm{tr}\Big((\partial_\mu A^a_\nu)A^b_\alpha T^a T^b - (\partial_\nu A^a_\mu)A^b_\alpha T^a T^b + igf^{acd}A^c_\mu A^d_\nu A^b_\alpha T^a T^b - \frac{2g}{3}A^a_\mu A^b_\nu A^c_\alpha T^a T^b T^c\Big)\\
&\quad = \frac{\mu}{4}\varepsilon^{\mu\nu\alpha}\Big((\partial_\mu A^a_\nu)A^a_\alpha - (\partial_\nu A^a_\mu)A^a_\alpha + igf^{acd}A^c_\mu A^d_\nu A^a_\alpha - \frac{g}{3}if^{abc}A^a_\mu A^b_\nu A^c_\alpha\Big)\\
&\quad = \frac{\mu}{4}\varepsilon^{\mu\nu\alpha}\Big((\partial_\mu A^a_\nu)A^a_\alpha - (\partial_\nu A^a_\mu)A^a_\alpha + igf^{abc}A^a_\mu A^b_\nu A^c_\alpha - \frac{g}{3}if^{abc}A^a_\mu A^b_\nu A^c_\alpha\Big)\\
&\quad = \frac{\mu}{4}\varepsilon^{\mu\nu\alpha}\Big((\partial_\mu A^a_\nu)A^a_\alpha - (\partial_\nu A^a_\mu)A^a_\alpha + \frac{2g}{3}if^{abc}A^a_\mu A^b_\nu A^c_\alpha\Big),
\end{aligned}
\tag{3.17}
$$

where we first expanded in the generators, then explicitly took the traces (and used that the symmetric-antisymmetric combination in the term $\varepsilon^{\mu\nu\alpha}d^{abc}A^a_\mu A^b_\nu A^c_\alpha = 0$), then rearranged the indices to simplify the final result. Using the form of (3.17), one of the Chern-Simons contributions to the Euler-Lagrange equation is trivial

$$
\begin{aligned}
-\partial_\sigma\frac{\partial L_{\mathrm{CS}}}{\partial(\partial_\sigma A^g_\rho)} &= -\frac{\mu}{4}\varepsilon^{\mu\nu\alpha}\partial_\sigma\Big(\delta^\sigma_\mu\delta^\rho_\nu\delta^{ag}A^a_\alpha - \delta^\sigma_\nu\delta^\rho_\mu\delta^{ag}A^a_\alpha\Big)\\
&= -\frac{\mu}{4}\partial_\sigma\Big(\varepsilon^{\sigma\rho\alpha}A^g_\alpha - \varepsilon^{\rho\sigma\alpha}A^g_\alpha\Big)\\
&= \frac{\mu}{4}\varepsilon^{\rho\sigma\alpha}\Big(\partial_\sigma A^g_\alpha - \partial_\alpha A^g_\sigma\Big),
\end{aligned}
\tag{3.18}
$$

while the other is slightly more tedious

$$
\begin{aligned}
\frac{\partial L_{\mathrm{CS}}}{\partial A^g_\rho} &= \frac{\mu}{4}\varepsilon^{\mu\nu\alpha}\Big(\partial_\mu A^a_\nu\delta^{ag}\delta^\rho_\alpha - \partial_\nu A^a_\mu\delta^{ag}\delta^\rho_\alpha + \frac{2g}{3}if^{abc}(\delta^{ag}\delta^\rho_\mu A^b_\nu A^c_\alpha + \delta^{bg}\delta^\rho_\nu A^a_\mu A^c_\alpha + \delta^{cg}\delta^\rho_\alpha A^a_\mu A^b_\nu)\Big)\\
&= \frac{\mu}{4}\Big(\varepsilon^{\mu\nu\rho}\partial_\mu A^g_\nu - \varepsilon^{\mu\nu\rho}\partial_\nu A^g_\mu + \frac{2g}{3}i(\varepsilon^{\rho\nu\alpha}f^{gbc}A^b_\nu A^c_\alpha + \varepsilon^{\mu\rho\alpha}f^{agc}A^a_\mu A^c_\alpha + \varepsilon^{\mu\nu\rho}f^{abg}A^a_\mu A^b_\nu)\Big)\\
&= \frac{\mu}{4}\varepsilon^{\mu\nu\rho}\Big((\partial_\mu A^g_\nu - \partial_\nu A^g_\mu) + \frac{2g}{3}i(-f^{gbc}A^b_\nu A^c_\mu - f^{agc}A^a_\mu A^c_\nu + f^{abg}A^a_\mu A^b_\nu)\Big)\\
&= \frac{\mu}{4}\varepsilon^{\mu\nu\rho}\Big((\partial_\mu A^g_\nu - \partial_\nu A^g_\mu) - \frac{2g}{3}if^{gbc}(A^b_\nu A^c_\mu + A^c_\mu A^b_\nu + A^c_\mu A^b_\nu)\Big)\\
&= \frac{\mu}{4}\varepsilon^{\mu\nu\rho}\Big((\partial_\mu A^g_\nu - \partial_\nu A^g_\mu) + 2gif^{gbc}A^b_\mu A^c_\nu\Big)
\end{aligned}
\tag{3.19}
$$

where we have taken the derivatives, contracted all Kronecker-δ's, factorised-out the Levi-Civita symbol $\varepsilon^{\mu\nu\rho}$ making all necessary index changes, before finally factoring the structure constants out of the relevant terms relabelling summed indices where necessary and using the antisymmetry of f^{gbc}. The Chern-Simons term contribution to the Euler-Lagrange equations may thus be expressed as

$$\begin{aligned}
&\frac{\partial L_{\text{CS}}}{\partial A^g_\rho} - \partial_\sigma \frac{\partial L_{\text{CS}}}{\partial(\partial_\sigma A^g_\rho)} = gJ^\mu \\
&\frac{\mu}{2}\varepsilon^{\mu\nu\rho}(\partial_\mu A^g_\nu - \partial_\nu A^g_\mu + igf^{gbc}A^b_\nu A^c_\mu) = \frac{\mu}{2}\varepsilon^{\rho\mu\nu}F_{\mu\nu} = gJ^\mu.
\end{aligned} \tag{3.20}$$

Now, including the contribution associated to the Yang-Mills term and the source term, the equations of motion may simply be expressed as

$$\boxed{D_\alpha F^{\alpha\mu} + \frac{\mu}{2}\varepsilon^{\mu\alpha\beta}F_{\alpha\beta} = gJ^\mu}. \tag{3.21}$$

Notice now that the equations of motion are *gauge covariant*.

The source-free form of the Lagrangian in (3.13) changes (as in the Abelian case) by a total derivative. This can be shown explicitly by considering a gauge transformation (one should recall that the transformation rule for A_μ is found by enforcing gauge covariance of the Yang-Mills Lagrangian, so we need only consider the Chern-Simons term in the topologically massive Yang-Mills Lagrangian) of the form [61, 64]

$$\begin{aligned}
A_\mu &\to A_\mu = U^{-1}A_\mu U + U^{-1}\partial_\mu U, \quad U \equiv U(x) \quad \mathrm{SU}(n) \\
F_{\mu\nu} &\to F_{\mu\nu} = U^{-1}F_{\mu\nu}U,
\end{aligned} \tag{3.22}$$

where U is a local transformation and is an element of the symmetry group ($\mathrm{SU}(n)$) of the original Yang-Mills theory. One sees that the Chern-Simons term transforms as (taking care to note that we are transforming at the level of the Lagrangian and so do not discard any total derivatives as yet)

$$
\begin{aligned}
L_{\mathrm{CS}} \to L_{\mathrm{CS}} &= \frac{\mu}{2g^2}\varepsilon^{\mu\nu\alpha}\mathrm{tr}\Big(F_{\mu\nu}A_\alpha - \frac{2}{3}A_\mu A_\nu A_\alpha\Big) \\
&= \frac{\mu}{2g^2}\varepsilon^{\mu\nu\alpha}\mathrm{tr}\Big[U^{-1}F_{\mu\nu}U(U^{-1}A_\alpha U + U^{-1}\partial_\alpha U) \\
&- \frac{2}{3}(U^{-1}A_\mu U + U^{-1}\partial_\mu U)(U^{-1}A_\nu U + U^{-1}\partial_\nu U)(U^{-1}A_\alpha U + U^{-1}\partial_\alpha U)\Big] \\
&= L_{\mathrm{CS}} + \frac{\mu}{2g^2}\varepsilon^{\mu\nu\alpha}\mathrm{tr}\Big[U^{-1}F_{\mu\nu}\partial_\alpha U - \frac{2}{3}U^{-1}\{A_\mu A_\nu\partial_\alpha U + A_\mu(\partial_\nu U)U^{-1}A_\alpha U \\
&+ (\partial_\mu U)U^{-1}A_\nu A_\alpha U + A_\mu(\partial_\nu U)U^{-1}(\partial_\alpha U) + (\partial_\mu U)U^{-1}A_\nu(\partial_\alpha U) \\
&+ (\partial_\mu U)U^{-1}(\partial_\nu U)U^{-1}A_\alpha U + (\partial_\mu U)U^{-1}(\partial_\nu U)U^{-1}(\partial_\alpha U)\}\Big] \\
&= L_{\mathrm{CS}} + \frac{\mu}{2g^2}\varepsilon^{\mu\nu\alpha}\mathrm{tr}\Big[U^{-1}(\partial_\mu A_\nu - \partial_\nu A_\mu)\partial_\alpha U - \frac{2}{3}U^{-1}\{A_\mu(\partial_\nu U)U^{-1}(\partial_\alpha U) \\
&+ (\partial_\mu U)U^{-1}A_\nu(\partial_\alpha U) + (\partial_\mu U)U^{-1}(\partial_\nu U)U^{-1}A_\alpha U + (\partial_\mu U)U^{-1}(\partial_\nu U)U^{-1}(\partial_\alpha U)\}\Big] \\
&= L_{\mathrm{CS}} - \frac{\mu}{g^2}\varepsilon^{\mu\nu\alpha}\mathrm{tr}\Big[\partial_\mu\big(A_\nu(\partial_\alpha U)U^{-1}\big) + \frac{1}{3}(\partial_\mu U)U^{-1}(\partial_\nu U)U^{-1}(\partial_\alpha U)U^{-1}\Big],
\end{aligned}
\tag{3.23}
$$

where we have used the commutator contribution of $F_{\mu\nu}$ to cancel with the $O(A^2)$ terms to obtain the fourth equality and have arrived at the final equality by writing terms as total derivatives. Following [61], we assume that the transformation is (spatially and temporally) asymptotically the identity ($U(x) \to \mathrm{I}$ as $x \to \infty$), then the contribution to the action $S_{\mathrm{CS}} = \int \mathrm{d}x L_{\mathrm{CS}}$ is just a total derivative. We have not expressed the final equality in (3.23) making use of this relationship – applying the condition $U(x) \to \mathrm{I}$ as $x \to \infty$ – causes the first additional term (a surface integral) to vanish

$$\int \mathrm{d}^3x\frac{\mu}{g^2}\varepsilon^{\mu\nu\alpha}\mathrm{tr}\Big[\partial_\mu\big(A_\nu(\partial_\alpha U)U^{-1}\big)\Big] = 0. \tag{3.24}$$

However, this does not address the final term. Choosing $U \quad \mathrm{SU}(M)$, $M \le N$ so that SU(M) is a subgroup of the full gauge group $\mathrm{SU}(N)$, then the remaining (additional) contribution to the action is proportional to the *winding number*, $w(U)$, of the gauge transformation, related to the *homotopy class* of U [61, 65] and defined by[61]

$$w(U) := \frac{1}{24\pi^2}\int d^3x\varepsilon^{\alpha\beta\gamma}\mathrm{tr}\Big[\partial_\mu U)U^{-1}(\partial_\nu U)U^{-1}(\partial_\alpha U)U^{-1}\Big]. \tag{3.25}$$

Indeed, we may write the transformed Lagrangian as [61]

$$\int \mathrm{d}x L_{\mathrm{CS}} \to \int \mathrm{d}x L_{\mathrm{CS}} + \frac{\mu 8\pi^2}{g^2} w(U). \tag{3.26}$$

The significance of this is that the Chern-Simons term's contribution action *is not* zero, instead, it takes an integer-multiple of $\mu 8\pi^2/g^2$, where the integer tells us to which homotopy class the gauge transformation, U, belongs[26]. The action itself need not be invariant, rather it is the contribution to the *path integral*, $\exp(i \int \mathrm{d}^3 x L)$, that we hope to leave invariant to preserve the expectation values of gauge invariant operators [18, 61]. The implications of this requirement, for an arbitrary gauge invariant operator, $\hat{O}$, are [61]

$$\begin{aligned} e^{i \frac{8\pi^2 \mu}{g^2} w(U)} \; \hat{O} \; &\stackrel{!}{=} \; \hat{O} \\ \boxed{4\pi \frac{\mu}{g^2} = c, \quad \mathbb{Z}}, \end{aligned} \tag{3.27}$$

which is a *mass quantization condition*. We turn now to finding a Klein-Gordon-like equation in this non-Abelian case, using a similar approach to that used in §3.1.1.

3.2.1 'Klein-Gordon' Non-Abelian Gauge Field Strength Equation

We now wish to find a differential equation that will encapsulate the solutions for the non-Abelian gauge field, as an analogue to what was done in §3.1. We are interested in showing that the gauge field excitations are massive, and this should be true regardless of the presence of a source. To that end (and to simplify the calculations), in precisely the same manner as before, we dualise the source-free form of (3.21) – where we will still make use of (3.6) as the definition of the dual, but it should be obvious that the identity is now being applied to the Yang-Mills (non-Abelian) field strength and not just the Maxwell terms – by making use of the covariant (dual) Bianchi identity. Proving this identity makes use of an analogous approach to that shown in (3.7), but now we take the covariant derivative of the equations of motion in (3.21). We wish to find $D_\mu \quad F^\mu = 0$, so it is sufficient to show that the covariant divergence of the first (Yang-Mills) term is vanishing. The result of trying to analytically compute this result

[26] The discussion of winding numbers in this work will not exceed the qualitative one presented. However, [65] provides a thorough discussion of the winding number as it is used in this context. Winding numbers also appear in alternative contexts – which is not surprising as they are topological objects – such as superconductivity and string theory. See, for instance, [66] and [67].

is quite long, and we attempt to organize the twelve terms systematically:

$$\begin{aligned}
&D_\nu(D_\mu F^{\mu\nu}) \\
&= (\underbrace{\partial_\nu}_{A} + \underbrace{gA_\nu^a[T^a,\cdot]}_{B}) \circ (\underbrace{\partial_\mu}_{P} + \underbrace{gA_\mu^b[T^b,\cdot]}_{Q}) \circ (\underbrace{g\partial^\mu A^{\nu c}}_{X} - \underbrace{g\partial^\mu A^{\nu c}}_{Y} + \underbrace{ig^2 f^{cde} A^{\mu d} A^{\nu e}}_{Z}) T^c \\
&\stackrel{?}{=} 0,
\end{aligned} \tag{3.28}$$

so that the twelve terms may be expressed in tabular form shown in table 4. Then the covariant

APX-1	$g\partial_\nu \partial^2 A^{\nu c} T^c$
APY-1	$-g\partial_\mu \partial^2 A^{\mu c} T^c$
APZ-2	$ig^2 f^{cde}\left[(\partial_\nu\partial_\mu A^{\nu d})A^{\mu e} + A^{\nu d}(\partial_\nu\partial_\mu A^{\mu e}) + (\partial_\mu A^{\nu d})(\partial_\nu A^{\mu e}) + (\partial_\nu A^{\nu d})(\partial_\mu A^{\mu e})\right]T^c$
AQX-2	$ig^2 f^{cde}\left[(\partial_\nu A_\mu^d)(\partial^\mu A^{\nu e}) + A_\mu^d \partial_\nu \partial^\alpha A^{\mu e}\right]T^c$
AQY-2	$-ig^2 f^{cde}\left[(\partial_\nu A_\mu^d)(\partial^\nu A^{\mu e}) + A_\mu^d \partial^2 A^{\mu e}\right]T^c$
AQZ-3	$-g^3 f^{gde} f^{bgc}\left[(\partial_\nu A_\mu^b)A^{\nu d}A^{\mu e} + A_\mu^b(\partial_\nu A^{\nu d})A^{\mu e} + A_\mu^b A^{\nu d}(\partial_\nu A^{\mu e})\right]T^c$
BPX-2	$ig^2 f^{cde} A_\nu^d \partial^2 A^{\nu e} T^c$
BPY-2	$-ig^2 f^{cde} A_\nu^d \partial_\mu \partial^\nu A^{\mu e} T^c$
BPZ-3	$-g^3 f^{gde} f^{bgc} A_\nu^b\left[(\partial_\mu A^{\nu d})A^{\mu e} + A^{\nu d}(\partial_\mu A^{\mu e})\right]T^c$
BQX-3	$-g^3 f^{gde} f^{bgc} A_\nu^b A_\mu^d \partial^\nu A^{\nu e} T^c$
BQY-3	$g^3 f^{deg} f^{bgc} A_\nu^b A_\mu^d \partial^\nu A^{\mu e} T^c$
BQZ-4	$-ig^4 f^{kde} f^{bkg} f^{agc} A_\nu^a A_\mu^b A^{\nu d} A^{\mu e} T^c$

Table 4: Table of systematically organized products from (3.28). Note that in all cases Lie algebra indices have been transformed for easier comparison of the terms. The notation 'AQZ-2' indicates that the term is second order in the coupling constant g, and should be matched to the same order of terms. Additionally, Real(Complex) terms should only cancel with other Real(Complex) terms. Terms are colour coded to show which combinations are vanishing.

divergence $D_\nu(D_\mu F^{\mu\nu})$ is, indeed, vanishing. We see this by noting that all the 'grouped' terms vanish when combined. The terms of orders 0 and 1 in the structure constants vanish by a simple term-matching procedure. Additionally, $BQX + BQY = 0$, trivially. The first term of BPZ cancels with the last term of AQZ, the second term of BPZ cancels with the second term of AQZ, and the remaining term cancels since it is a product of a symmetric and antisymmetric object (in the Lie algebra indices d, e). Finally, BQZ is also vanishing due to the symmetric/antisymmetric product in d, e. Thus, we may conclude that the covariant divergence of the Yang-Mills term in the equation of motion is vanishing. The implications of this are, using the equations of motion (3.21) and the now-proved result (3.28), that we are forced to conclude that

$$\boxed{D_\nu \quad F^{\nu} = 0}, \tag{3.29}$$

which is the *Yang-Mills dual field Bianchi identity*. Dualising the source-free equations of motion produces an equation of motion for the dual field (using precisely the same method shown in (3.8))

$$\begin{gathered}\varepsilon_{\rho\sigma\mu}\left(D_\alpha F^{\alpha\mu} + \frac{\mu}{2}\varepsilon^{\mu\alpha\beta}F_{\alpha\beta}\right) = 0 \\ \boxed{D_\rho \quad F_\sigma - D_\sigma \quad F_\rho - \mu F_{\rho\sigma} = 0}.\end{gathered} \tag{3.30}$$

Taking the covariant divergence of (3.30) one finds

$$\begin{aligned}&D_\rho\left(D^\rho \quad F_\sigma - D_\sigma \quad F^\rho - \mu F^{\rho}_{\sigma}\right) = 0 \\ &(D_\rho D^\rho + \mu^2) \quad F_\sigma = D_\rho D_\sigma \quad F^\rho,\end{aligned} \tag{3.31}$$

where we have made use of the equations of motion (3.21) to find this non-Abelian covariant analogue of (3.9). Thus, the source-free equation has massive degrees of freedom. One cannot proceed as was done in the Abelian case, making use of inverse operations such as $\quad^{-1}$ because the non-Abelian analogue, $D_\mu D^\mu$ is no longer gauge invariant. One could pick a gauge and solve for the field in *that* particular gauge, but this does not add significant value to the principal ideas of this work. We thus do not proceed as before, and instead conclude the discussion of the solutions at this point.

3.2.2 Linearised Yang-Mills Equations of Motion

Before discussing the gravitational theory, it is essential that we present the TMYM equations which we hope shall be derivable when applying the Kerr-Schild Ansatz. In particular, we wish to show that the *linearised TMYM equations of motion* are reproduced.

We linearise our TMYM equations of motion by applying the following ansatz to (3.14):

$$A_\mu \equiv gT^a A^a_\mu \equiv gT^a c^a A_\mu, \tag{3.32}$$

that is, we make the ansatz that A^a_μ is separable into its gauge component, the constant colour factor c^a, and its Lorentz component, A_μ. The critical implications of this choice are that

$$[A_\mu, A_\nu] = g^2 A^a_\mu A^b_\nu [T^a, T^b] = g^2 A_\mu A_\nu c^a c^b (i f^{abc}) T^c = 0, \tag{3.33}$$

where the final equality is due to the antisymmetry of f^{abc} multiplying with the symmetric colour factor product $c^a c^b$. The has further implications for the equations of motion for the equations of motion

$$\begin{aligned} gJ^\mu &= D_\alpha F^{\alpha\mu} + \frac{\mu}{2}\varepsilon^{\mu\alpha\beta}F_{\alpha\beta} \\ &= c^a \partial_\alpha (\partial^\alpha A^\mu - \partial^\mu A^\alpha) T^a + \frac{\mu}{2}\varepsilon^{\mu\alpha\beta} c^a (\partial_\alpha A_\beta - \partial_\beta A_\alpha) T^a, \\ &\boxed{c^a \left[\partial_\alpha F^{\alpha\mu} + \frac{\mu}{2}\varepsilon^{\mu\alpha\beta}F_{\alpha\beta}\right] = c^a g J^\mu}, \end{aligned} \tag{3.34}$$

where we have introduced $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$ as a Maxwell-like field strength tensor, and have similarly separated $J^{\mu a} \equiv c^a J^\mu$. One should note that this equation of motion is in the Lie-algebra indexed form of (3.3). The boxed equation in (3.34) is the result that we hope to reproduce when applying double copy theory to the Kerr-Schild metric in TMG. It should also be observed, then, that if the application of the ansatz (2.33) is successfully applied with c^a included (that is, the ansatz reproduces the linearised TMYM equations of motion), then it will automatically satisfy the TMSE equations of motion. All that now remains is to take on TMG and to show that, with a Kerr-Schild ansatz, it reproduces the linearised TMYM equations of motion; this will be the subject of the following sections.

3.3 Gravitational Theory

Finally, we consider Topologically Massive Gravity (TMG). As in the previous sections, we begin with the action (without a cosmological constant) [21, 61, 68],

$$S = \frac{1}{\kappa^2}\int \mathrm{d}^3x \sqrt{-g}\Big[- R - \frac{1}{2m}\ {}^{\mu\nu\rho}\Big(\Gamma^\alpha_{\mu\beta}\partial_\nu\Gamma^\beta_{\alpha\rho} + \frac{2}{3}\Gamma^\alpha_{\mu\beta}\Gamma^\beta_{\nu\gamma}\Gamma^\gamma_{\rho\alpha}\Big)\Big] + S_{\text{Matter}}. \tag{3.35}$$

A direct computation of the equations of motion yields

$$
\begin{aligned}
-\frac{\kappa^2}{2}T_{\mu\nu} = G_{\mu\nu} - \frac{1}{8m}\Bigg[& {}_{\nu\beta\gamma}(\Gamma^{\alpha}{}_{\mu}{}^{\beta;\gamma}{}_{;\alpha} - \Gamma_{\mu}{}^{\alpha\beta;\gamma}{}_{;\alpha}) - {}_{\mu\beta\gamma}(\Gamma^{\alpha}{}_{\nu}{}^{\beta;\gamma}{}_{;\alpha} - \Gamma_{\nu}{}^{\alpha\beta;\gamma}{}_{;\alpha}) \\
& - {}_{\nu}{}^{\beta\gamma}\Gamma^{\alpha}{}_{\mu\beta,\gamma;\alpha} - {}_{\mu}{}^{\beta\gamma}\Gamma^{\alpha}{}_{\nu\beta,\gamma;\alpha} - {}_{\alpha\beta}{}^{\gamma}(\Gamma_{\mu\nu}{}^{\alpha;\beta}{}_{;\gamma} + \Gamma_{\nu\mu}{}^{\alpha;\beta}{}_{;\gamma}) + {}_{\nu}{}^{\lambda\delta}g^{\beta\gamma}g_{\mu\alpha}\Gamma^{\alpha}{}_{\beta\lambda,\delta;\gamma} \\
& - g_{\nu\alpha}(- {}_{\mu}{}^{\lambda\delta}g^{\beta\gamma}\Gamma^{\alpha}{}_{\beta\lambda,\delta;\gamma} + {}^{\beta\gamma\lambda}\Gamma^{\alpha}{}_{\mu\beta,\gamma;\lambda}) - {}^{\beta\gamma\lambda}g_{\mu\alpha}\Gamma^{\alpha}{}_{\nu\beta,\gamma;\lambda}\Bigg].
\end{aligned}
\tag{3.36}
$$

While it is not immediately apparent looking at (3.36), the terms in square brackets reduce[27] to $8C_{\mu\nu}$, where $C_{\mu\nu}$ is the Cotton tensor introduced in (2.18). Thus, the resulting Einstein equations are,

$$\bar{G}^{\mu}_{\nu} + \frac{1}{m}\bar{C}^{\mu}_{\nu} = -\frac{\kappa^2}{2}\bar{T}^{\mu}_{\nu}, \tag{3.37}$$

in agreement with results from [21, 69]. Critically, the Cotton tensor is traceless, which we utilise to find

$$
\begin{aligned}
-\frac{\kappa^2}{2}\bar{T} = \bar{G}^{\mu}_{\mu} &= \bar{R}^{\mu}_{\mu} - \frac{1}{2}\delta^{\mu}_{\mu}\bar{R} \\
&= -\frac{1}{2}\bar{R} \\
\kappa^2\bar{T} &= \bar{R},
\end{aligned}
\tag{3.38}
$$

whence we rewrite the equations of motion (3.37) in the (partially) trace-reversed form

$$\bar{R}^{\mu}_{\nu} + \frac{1}{m}\bar{C}^{\mu}_{\nu} = -\frac{\kappa^2}{2}\left(\bar{T}^{\mu}_{\nu} - \bar{T}\delta^{\mu}_{\nu}\right). \tag{3.39}$$

We can make use of the Killing vector trick employed several times in §2, where we hold-off on making use of a specific coordinate system for now. Before we perform the contraction, we modify (3.39), writing it in a *fully* trace-reversed form, by noting

$$
\begin{aligned}
\bar{R}^{\mu}_{\nu} + \frac{1}{m}\bar{C}^{\mu}_{\nu} &= -\frac{\kappa^2}{2}\left(\bar{T}^{\mu}_{\nu} - \bar{T}\delta^{\mu}_{\nu}\right) \\
\bar{R}^{\mu}_{\nu} + \frac{1}{m}{}^{\mu\alpha\beta}{}_{\alpha}(\bar{R}_{\nu\beta} - \tfrac{1}{4}g_{\nu\beta}\bar{R}) &= -\frac{\kappa^2}{2}\left(\bar{T}^{\mu}_{\nu} - \bar{T}\delta^{\mu}_{\nu}\right) \\
\bar{R}^{\mu}_{\nu} + \frac{1}{m}{}^{\mu\alpha\beta}{}_{\alpha}\bar{R}_{\nu\beta} &= -\frac{\kappa^2}{2}\left(\bar{T}^{\mu}_{\nu} - \bar{T}\delta^{\mu}_{\nu} - \frac{1}{2m}{}^{\mu\alpha\beta}{}_{\alpha}g_{\nu\beta}\bar{T}\right),
\end{aligned}
\tag{3.40}
$$

[27] Showing that this is the case from the form of the equations of motion (3.36) is tedious, and a much more direct method makes use of the *dreibein* formalism to produce this result. We can transform back from the dreibein formalism to verify the claim that the terms in square brackets produces $8C_{\mu\nu}$. See §3.3.1

where we make use of (2.18) to expand the Cotton tensor, followed by (3.38). We may now utilise the Kerr-Schild form of the Ricci tensor from (2.15). The left-hand side of (3.40) becomes

$$\mathrm{LHS} = -\frac{\kappa}{2}\Big[\partial^2(h^{\mu}_{\ \nu}) - \partial^{\mu}\partial_{\beta}(h^{\beta}_{\ \nu}) - \partial_{\nu}\partial_{\beta}(h^{\mu\beta})\Big] - \frac{\kappa}{2m}\ {}^{\mu\alpha\beta}\ {}_{\alpha}\Big[\partial^2(h_{\nu\beta}) - \partial_{\nu}\partial_{\gamma}(h^{\gamma}_{\ \beta}) - \partial_{\beta}\partial_{\gamma}(h^{\gamma}_{\nu})\Big], \tag{3.41}$$

and to this form we now contract with a Killing vector V^{ν} and apply the Kerr-Schild single copy ansatz (2.33).

Before we continue, we briefly show that the gravitational excitations are also, themselves, massive. As in the previous sections, these massive excitations should be present regardless of the presence of any source terms, so we consider only the source-free equations ($T_{\mu\nu} = 0$). Showing the gravitational excitations are massive is achieved by recasting the equations of motions as [61]

$$\begin{aligned} &O(m)_{\mu\nu}^{\ \ \rho\sigma}R_{\rho\sigma} = 0, \quad \text{where} \\ &O(m)_{\mu\nu}^{\ \ \rho\sigma} := \left(\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu} - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\right) + \frac{1}{m\sqrt{|g|}}\varepsilon_{\mu}^{\ \alpha\beta}\left(\delta^{\rho}_{\beta}\delta^{\sigma}_{\nu} - \frac{1}{4}g^{\rho\sigma}g_{\nu\beta}\right) \quad {}_{\alpha}. \end{aligned} \tag{3.42}$$

This operator is analogous to the one discussed in the case of TMSE (see in particular (3.10)). Assuming also that the Ricci scalar is vanishing, $R = 0$, one finds [61]

$$\begin{aligned} &m^2 O(-m)_{\mu\nu}^{\ \ \alpha\beta}O(m)_{\alpha\beta}^{\ \ \rho\sigma}R_{\rho\sigma} = 0 \\ &\boxed{(\ \ ^2 + m^2)R_{\mu\nu} = -g_{\mu\nu}R^{\alpha\beta}R_{\alpha\beta} + 3R^{\alpha}_{\mu}R_{\nu\alpha}}, \end{aligned} \tag{3.43}$$

which is the expression analogous to the Abelian gauge theory case (3.9) and the non-Abelian gauge theory case (3.31); showing that the gravitational excitations are, indeed, massive.

Recall now the strategies described at the end of §2.4; we can consider stationary/wave solutions most conveniently by a convenient choice of coordinates. Before we do this, however, we address the issue of the simplification of the equations of motion discussed earlier in this section.

3.3.1 Diversion: Dreibeins and TMG

As was previously noted, the jump from (3.36) to (3.37) is more akin to a long-jump than an obvious logical step. The discussion leading to this section sought to avoid introducing new conventions in the middle of what was (apart from the 'on-faith' simplification of the equation of motion) otherwise a logically consistent introduction to TMG. However, as §3 is supposed to serve not only as an introduction to the 3-dimensional topologically massive double copy theory, but also as an introduction to topologically massive theories in three dimensions in general, we introduce the more conventional dreibein formalism here. Some familiarity with differential geometry (especially *forms* and *exterior calculus*) is assumed[28].

At the most basic level, the dreibein formalism[29] is captured by the statement that we should typically be able to pick a local coordinate frame which is Minkowskian (flat). The *dreibeins*, e^a, are defined by

$$\begin{aligned} g = g_{\mu\nu}\mathrm{d}x^{\mu} \quad \mathrm{d}x^{\nu} = \eta_{ab}e^{a} \quad e^{b}, \quad \text{, where} \\ e^{a} = e^{a}_{\mu}\mathrm{d}x^{\mu} \qquad g_{\mu\nu} = \eta_{ab}e^{a}_{\mu}e^{b}_{\nu}, \end{aligned} \tag{3.44}$$

where the Greek indices are spacetime indices and the Latin indices are the dreibein indices. We additionally require $\det(e^{a}_{\mu}) = 0$ so that we may define the inverse of dreibeins or a dual basis for vectors ($e^{\mu}_{a}\partial_{\mu}$). So that the inverse metrics are defined in the anticipated way

$$g^{\mu\nu} = \eta^{ab}e^{\mu}_{a}e^{\nu}_{b}; \quad \eta^{ab} = g^{\mu\nu}e^{a}_{\mu}e^{b}_{\nu}. \tag{3.45}$$

All various index locations are defined in the intuitive way and are only acted upon by the appropriate metric object – Latin indices are only ever acted upon by η and Greek indices by g, so there is never any ambiguity (*e.g.* $e_{a\mu} = g_{\mu\alpha}e^{\alpha}_{a} = \eta_{ab}e^{b}_{\mu}$, and similarly for raising indices). Much more can be said about dreibeins, but we refrain from delving into unnecessary detail – we direct the reader to [72, 73] (or any comprehensive introduction to general relativity and/or gravity as a gauge theory).

Vector coefficients may be written in terms of dreibein indices as $V^{a} = e^{a}_{\mu}V^{\mu}$. The covariant

[28] The reader looking for a thorough introduction (and beyond) to differential geometry is directed to [70]; the reader aiming for a more rapid introduction is directed to the relevant sections in [71]. Most textbooks on gravity as a gauge theory are likely to include introductions to the relevant differential geometry as well. The discussions in [72] are likely the shortest and most rapid introduction to *all* the appropriate material in this section.

[29] Throughout this discussion we will refer only to 'dreibeins', although the experienced general relativist will be aware that many of the introductory explanations are generic, and one could equally substitute 'vielbeins' in place of 'dreibeins' – we do this to avoid additional unnecessary explanations.

derivative then also transforms to accommodate this

$$D_\mu V^a = \partial_\mu V^a + \omega^a_{\mu b} V^b, \tag{3.46}$$

where the new object, $\omega^a_{\mu b}$ is called the *spin connection* (a one-form), whose components are fixed by requiring that the parallel transport of the dreibein components, e^a_μ is zero [74, 75]

$$0 = D_\mu e^a_\nu = \partial_\mu e^a_\nu - \Gamma^\alpha_{\mu\nu} e^a_\alpha + \omega^a_{\mu b} e^b_\nu = 0. \tag{3.47}$$

Typically, one now introduces the *torsion forms* [72] and makes the assumption that the original spacetime connection $\Gamma^\alpha_{\mu\nu}$ is the torsion-free *Levi-Civita connection*. Then an expression of torsion (in the language of *forms*) is [72]

$$T^a := De^a = de^a + \omega^a_{\ b} \quad e^b \tag{3.48}$$

where d is the exterior derivative, ‘ ’ is the wedge product, and $\omega^a_{\ b} = \omega^a_{\mu b} \mathrm{d}x^\mu$. The statement that $T^a = 0$ is called *Cartan's first structure equation* [72]. One can also now introduce the *curvature forms* (whose definition[30] is also called *Cartan's second structure equation*) [72]

$$R^a_{\ b} := d\omega^a_{\ b} + \omega^a_{\ c} \quad \omega^c_{\ b} = R^a_{\ \mu\nu b} \mathrm{d}x^\mu \quad \mathrm{d}x^\nu, \tag{3.49}$$

whose form is analogous to the (form) expression of the Yang-Mills gauge field $F = dA + A \quad A$.

Using this formalism, the expression of the Chern-Simons contribution to the action (3.35) is [61]

$$S_{\mathrm{CS}} = -\frac{1}{4\kappa^2 m} \int \mathrm{d}^3x \ ^{\mu\nu\rho}\Big(R_{\mu\nu ab}\omega_\rho^{\ ab} + \frac{2}{3}\omega_{\mu a}^{\ \ b}\omega_{\nu b}^{\ \ c}\omega_{\rho c}^{\ \ a}\Big). \tag{3.50}$$

The approach we use to find the equations of motion is guided by comments in [61]. Varying this action with respect to the spin connection yields

$$\delta S_{\mathrm{CS}} = -\frac{1}{4\kappa^2 m} \int \mathrm{d}^3x \ ^{\mu\nu\rho}\Big(R_{\mu\nu}^{\ \ ab}\Big)\delta\omega_{\rho ab} \tag{3.51}$$

[30] Although it should be noted that we treat Cartan's second structure equation as a definition, a method that is more easily linked to the Riemann curvature is $R^a_{\ \mu\nu b} V^b := [D_\mu, D_\nu] V^a$. This also enhances the analogy between curvature in gravity and field strength in Yang-Mills since $F_{\mu\nu} \quad [\tilde{D}_\mu, \tilde{D}_\nu]$, where we have used $\tilde{D}_\mu$ to distinguish these Yang-Mills covariant derivatives ($\tilde{D}_\mu$) from their gravitational analogues (D_μ).

What we would like is an expression in terms of the dreibeins, and we may obtain one by considering the variation of (3.47) [61], from which one obtains (we leave out indices for readability)

$$
\begin{aligned}
(3.47) \quad 0 &= \partial(\delta e) + (\delta\omega)e + \omega(\delta e) - (\delta\Gamma)e - \Gamma(\delta e) \\
(\delta\omega)e &= \Big(- D(\delta e) + (\delta\Gamma)e\Big) \\
&\boxed{\delta\omega_{\mu ab} = e_b^{\nu}\Big(\delta\Gamma^{\rho}{}_{\mu\nu}e_{\rho a} - D_\mu \delta e_{\nu a}\Big)}.
\end{aligned}
\tag{3.52}
$$

We can substitute this finding in (3.51)

$$
\delta S_{\mathrm{CS}} = -\frac{1}{4\kappa^2 m}\int \mathrm{d}^3x \;{}^{\mu\nu\rho}\Big(R_{\mu\nu}{}^{ab}\Big)e_b^{\alpha}\Big(\delta\Gamma^{\beta}{}_{\rho\alpha}e_{\beta a} - D_\rho \delta e_{\alpha a}\Big). \tag{3.53}
$$

Focusing on the second term only, the only term that survives after performing a covariant 'integration by parts' is proportional to the expression for the second (differential) Bianchi identity [61, 72]

$$
\text{Second term} \qquad {}^{\mu\nu\rho}D_\mu R_{\nu\rho}{}^{ab} = 0. \tag{3.54}
$$

This means that only the first term contributes non-trivially to the variation:

$$
\delta S_{\mathrm{CS}} = -\frac{1}{4\kappa^2 m}\int \mathrm{d}^3x \;{}^{\mu\nu\rho}R_{\mu\nu\beta}{}^{\alpha}\delta\Gamma^{\beta}{}_{\rho\alpha} \tag{3.55}
$$

We would now like an expression of $\delta\Gamma$ in terms of the variation of the metric. One can make use of Riemannian normal coordinates to write this variation as [76]

$$
\delta\Gamma^{\sigma}_{\mu\nu} = -\frac{1}{2}\Big(g_{\lambda\mu} \;\; {}_{\nu}(\delta g^{\lambda\sigma}) + g_{\lambda\nu} \;\; {}_{\mu}(\delta g^{\lambda\sigma}) - g_{\mu\alpha}g_{\nu\beta} \;\; {}^{\sigma}(\delta g^{\alpha\beta})\Big). \tag{3.56}
$$

The results of inserting this expression into the variation of the action is

$$
\begin{aligned}
\delta S_{\mathrm{CS}} &= -\frac{1}{4\kappa^2 m}\int \mathrm{d}^3x \;{}^{\mu\nu\rho}R_{\mu\nu\beta}{}^{\alpha}\delta\Gamma^{\beta}{}_{\rho\alpha} \\
&= \frac{1}{8\kappa^2 m}\int \mathrm{d}^3x \;{}^{\mu\nu\rho}\Big[R_{\mu\nu\beta}{}^{\alpha}g_{\lambda\rho} \;\; {}_{\alpha}(\delta g^{\lambda\beta}) + R_{\mu\nu\beta\lambda} \;\; {}_{\rho}(\delta g^{\lambda\beta}) - R_{\mu\nu\beta\lambda}g_{\rho\sigma} \;\; {}^{\beta}(\delta g^{\sigma\lambda})\Big].
\end{aligned}
\tag{3.57}
$$

Making use of integration by parts and the relationship between covariant derivatives of the Riemann tensor and the Ricci tensor

$$
{}^{\alpha}R_{\mu\nu\alpha\beta} = \;\; {}_{\mu}R_{\beta\nu} - \;\; {}_{\nu}R_{\beta\mu}, \tag{3.58}
$$

we finally arrive at the expression of the variation as

$$
\begin{aligned}
\delta S_{\mathrm{CS}} &= \frac{1}{\kappa^2 m}\int \mathrm{d}^3x \; {}_{\rho}^{\;\alpha\sigma} \quad {}_{\alpha}\Big(R_{\sigma\beta} - \frac{1}{2}g_{\sigma\beta}R\Big)\delta g^{\rho\beta} \\
&\boxed{\delta S_{\mathrm{CS}} = \frac{1}{\kappa^2 m}\int \mathrm{d}^3x\sqrt{|g|}C^{\rho\beta}\delta g_{\rho\beta}}\,,
\end{aligned}
\tag{3.59}
$$

where $C^{\rho\beta}$ is the Cotton tensor defined in (2.18).

This concludes the discussion of dreibeins – the derivation of the Cotton tensor term from the Chern-Simons action was the primary goal of this section – justifying (3.37) – and the other goal was to provide a brief introduction to the dreibein formalism, conventionally used in discussions of gravity as a gauge theory.

We now return from this diversion and consider some time-dependent solutions arising from TMG using Kerr-Schild metrics.

3.4 3D TMG Time-Dependent Solutions

First, we consider wave solutions by choosing 3-dimensional light-cone coordinates, where the relevant details are summarised as

$$
\begin{aligned}
&u = t - z, \quad v = t + z, \quad x = x, \quad x^\mu = (u, v, x), \quad \partial_\mu = (\partial_u, \partial_v, \partial_x), \\
&\mathrm{d}s^2 = \mathrm{d}x^2 - 2\mathrm{d}u\mathrm{d}v + \kappa\varphi k_\mu k_\nu \mathrm{d}x^\mu \mathrm{d}x^\nu = \mathrm{d}x^2 - 2\mathrm{d}u\mathrm{d}v + \kappa\varphi \mathrm{d}u^2, \\
&k_\mu = (-1, 0, 0), \quad k^\mu = (0, 1, 0), \quad V_\mu = (0, 1, 0), \quad V^\mu = (-1, 0, 0),
\end{aligned}
\tag{3.60}
$$

which satisfies that k_μ is null and additionally[31] satisfies $\;{}_\mu k_\nu = 0$, while V^μ is easily verified to be a Killing vector (the metric is v-independent, so $L_V[g] = 0$ by quick calculation, where $L_V[\cdot]$ is the Lie derivative with respect to V), such that $k \cdot V = 1$. Additionally, with this choice of coordinates we find that $\bar{R}^\mu_\mu = 0$, which, looking at (3.39), implies that the energy-momentum tensor is traceless; $\bar{T} = 0$. Additionally, we will consider $\varphi \equiv \varphi(u, x)$, that is, φ is v-independent. Importantly, the fact that $\;{}_\mu k_\nu = 0$ allows us to replace the covariant derivative in (3.41) with a partial derivative, whereafter the contraction of the equations of motion with V^ν is simpler,

[31] These calculations can be (and were) done by hand, however it may also be done using xCoba from xAct [34] and/or the “diffgeo” package for Mathematica [77], which was found by the author to be better for quick Christoffel symbol calculations/verifications for a metric in a given coordinate basis. See appendices §C.

and applying the ansatz $A^\mu = k^\mu \varphi$, they reduce to

$$\begin{aligned} -\kappa \bar{T}^\mu_\nu V^\nu &= -\Big\{\partial_\alpha F^{\alpha\mu}k_\nu - \partial_\alpha\partial_\nu(\varphi k^\alpha k^\mu) + \frac{1}{m}\ {}^{\mu\alpha\beta}\partial_\alpha[\partial^\gamma F_{\gamma\beta}k_\nu - \partial_\gamma\partial_\nu(\varphi k^\gamma k_\beta)]\Big\}V^\nu \\ &\boxed{\partial_\alpha F^{\alpha\mu} + \frac{1}{m}\ {}^{\mu\alpha\beta}\partial_\alpha(\partial^\gamma F_{\gamma\beta}) = gJ^\mu := 2g\bar{T}^\mu_\nu V^\nu} \end{aligned} \tag{3.61}$$

where $\kappa/2 \to g$, and we have removed the additional terms by using that φ is independent of v so that $\partial_\mu(k^\mu\varphi) = \partial_v(\varphi) = 0$. Comparing the result obtained to that of the linearised TMYM equations of motion (3.34), we see that for the result found in (3.61) to be the correct single copy, we must require

$$\frac{1}{m}\ {}^{\mu\alpha\beta}\partial_\alpha(\partial^\gamma F_{\gamma\beta}) \overset{!}{=} \frac{\mu}{2}\varepsilon^{\mu\alpha\beta}F_{\alpha\beta}, \tag{3.62}$$

where we have left the Yang-Mills form on the RHS as $F_{\alpha\beta}$ and in terms of mass term μ to make it clear that this is from (3.34). We can consider what the implications are for the scalar field based on this result – in doing so we equate $F_{\mu\nu} \equiv F_{\mu\nu}$ and $\mu \equiv m$ – from which we find

$$\begin{aligned} (3.62) \quad \frac{1}{m^2}\ {}^{\mu\alpha\beta}\partial_\alpha(\partial^2\varphi k_\beta) &= \frac{1}{2}\varepsilon^{\mu\alpha\beta}(\partial_\alpha\varphi k_\beta - \partial_\beta\varphi k_\alpha) \\ {}^{\mu\alpha\beta}\partial_\alpha[\partial^2\varphi]k_\beta &= \varepsilon^{\mu\alpha\beta}\partial_\alpha[m^2\varphi]k_\beta \\ &\boxed{(\partial^2 - m^2)\varphi = 0}, \end{aligned} \tag{3.63}$$

where to obtain the final implication we have used that $\sqrt{|g|} = 1$ so that the epsilon symbol and the epsilon tensor may be numerically identified. Note that the boxed equation is a Klein-Gordon equation (more easily seen when considering the derivative in (t, x, y) coordinates), and so it is a differential equation for φ. It is interesting also interesting to observe that this solution pre-empts the massive biadjoint scalar solution as seen in (2.23). It has the known solution (noting that $\partial^2\varphi = \partial_x^2\varphi$ since φ is v-independent)

$$\varphi(x, u) = P(u)e^{mx} + Q(u)e^{-mx}, \tag{3.64}$$

where P and Q are arbitrary functions of u. We shall keep this in mind when we proceed to consider the zeroth copy. First, to fully recover the linearised TMYM (assuming that the relation (3.63) is satisfied) we simply include the missing colour factor in front of each side of the equations of motion

We can contract the single copy equation of motion (3.61) with another Killing vector to find a zeroth copy equation of motion

$$\begin{aligned} 2g\bar{T}^{\mu}_{\nu}V^{\nu}V_{\mu} &= \left[\partial_{\alpha}F^{\alpha\mu} + \frac{1}{m}\ {}^{\mu\alpha\beta}\partial_{\alpha}(\partial^{\gamma}F_{\gamma\beta})\right]V_{\mu} \\ &= \partial^2\varphi + \frac{1}{m}V_{\mu}\ {}^{\mu\alpha\beta}\partial_{\alpha}\partial^2\varphi k_{\beta} \\ &= \partial^2\varphi + mV_{\mu}\ {}^{\mu\alpha\beta}\partial_{\alpha}\varphi k_{\beta} \\ &\boxed{\partial^2\varphi + mV_{\mu}\ {}^{\mu\alpha\beta}\partial_{\alpha}\varphi k_{\beta} = 2g\bar{T}^{\mu}_{\nu}V^{\nu}V_{\mu} =: j}, \end{aligned} \tag{3.65}$$

where we have used the condition derived in (3.63) in going from the second to the third equality. In the chosen coordinates (3.60) this may be explicitly restated as

$$\begin{aligned} \partial^2\varphi + m\partial_x\varphi &= j \\ \partial_x^2\varphi + m\partial_x\varphi &= j, \end{aligned} \tag{3.66}$$

where the sign of the mass terms comes from defining ${}_{uvx} = 1$ ${}^{uvx} = -1$, as well as from the form of k_{μ}. If we 'turn o□' the source and apply (3.64) we find

$$\begin{aligned} m^2P(u)e^{mx} &= 0 \\ &\boxed{\varphi(u,x) = Q(u)e^{-mx}}, \end{aligned} \tag{3.67}$$

in agreement with results found in [21]. Thus, the consistency of the double copy throughout the zeroth and single copies implies the boxed solution for φ in (3.67). The scalar field solution (3.67) satisfies the same equations of motion as the massive biadjoint scalar theory (2.23), where all that is required is to insert colour pairs $c^a c^{\tilde{a}}$ on either side of the equation and to identify $j = J$ and $\varphi \equiv \Phi$.

It is easy to verify that a plane wave solution

$$\varphi(u,x) \equiv \varphi(t-z,x) = Q_0 e^{\pm ik(t-z)-mx}, \tag{3.68}$$

satisfies the source-free equations of motion. This solution corresponds to a plane wave that exponentially decays along the x-direction. We now proceed to consider the simplest form of the sourced equations found in this section – a point-source – otherwise known as a shockwave solution.

The shockwave solution in TMG is given by an energy-momentum tensor that takes the form (preserving our light-cone coordinates from the previous section)[32] [21]

$$T_{\mu\nu} = p\delta(x)\delta(u)\delta^v_\mu\delta^v_\nu. \tag{3.69}$$

Then we should consider the only non-zero component of the energy-momentum tensor, T_{vv}, or, in the form of the equations of the previous section, $\bar{T}^u_{\ v}$. Given the generality with which we solved for the solutions of φ in (3.60)-(3.66), one may be tempted to assume that we can choose to solve the differential equations for φ at any step in the process, and 'propagate' the solution to the different levels, and thus expect the consistency of the solutions will be maintained by the fact that we have been consistent in our derivation thereof. We begin with the simpler case of biadjoint scalar theory. This is *not* the case. The issue is two-fold [18, 21]:

1. The differential equation to be solved for *phi* in the case of TMG is third order in the derivative and the differential equation for φ in the TMYM case is second order in the derivative, so that solutions are lost when starting in the TMYM case.

2. The second issue is closely related to the first and concerns the boundary conditions. The choice of boundary conditions need to ensure the double copy relation holds, and given the different order of derivatives of φ in the TMG and TMYM cases, these will necessarily be different.

In essence, the statement is that the φ's are not the same when translating between TMG and TMYM. The boundary conditions that ensure the double copy may be used are known [21], and we will systematically discuss the solutions now.

For TMG, the boundary conditions are that for $x > 0$, the metric is asymptotically flat, and that the metric is flat *for all* $x < 0$ [21]

$$\begin{aligned} &1.\ x > 0: \quad h_{\mu\nu} \to 0 \text{ as } x \to \infty, \\ &2.\ x < 0: \quad h_{\mu\nu} = 0. \end{aligned} \tag{3.70}$$

Using the ansatz $\varphi(u, x) = \delta(u)f(x)$ and applying the source (3.69) to (3.61) (after restating it

[32]See §2.6.2 and §A. We lose the y-coordinate and our s_μ becomes $s_\mu = \delta^t_\mu + \delta^z_\mu \to \delta^v_\mu$.

in terms of φ and k_μ), we arrive at the following differential equation for f (after multiplying through by m)

$$\partial_x^3 f(x) + m\partial_x^2 f(x) = \kappa p m \delta(x), \tag{3.71}$$

whose solution is known (and was verified with Wolfram Mathematica [33]) to be

$$f(x) = C_1 + C_2 x + \kappa p x \Theta(-x) - \frac{1}{m}\kappa p \Theta(-x) + \frac{e^{-mx}}{m^2}\Big(C_3 + \kappa m p \Theta(x)\Big), \tag{3.72}$$

for constants C_1, C_2 and C_3 and where $\Theta(x)$ is the Heaviside function. Enforcing that the boundary conditions hold, we arrive at the solution

$$\boxed{f(x) = \kappa p x \Theta(-x) - \frac{1}{m}\kappa p \Theta(-x) + \frac{e^{-mx}}{m}\kappa p \Theta(x)}, \tag{3.73}$$

.

In the case of TMYM, the appropriate boundary conditions correspond to a vanishing field strength when $x < 0$ and to an asymptotically vanishing gauge field for $x > 0$ [21]

$$\begin{aligned} &1.\ x > 0: \quad A^\mu \to 0 \text{ as } x \to \infty, \\ &2.\ x < 0: \quad F_{\mu\nu} = 0. \end{aligned} \tag{3.74}$$

An appropriate choice for the source in this case is

$$J^\mu = Q\delta(u)\delta(x)\delta^\mu_v, \tag{3.75}$$

where we have exchanged $\kappa p \to Q$ to highlight the fact that this should now be a (colour) charged source, not a massively charged one. The ansatz for the gauge field (referring to (3.60)) is

$$A^a = -c^a \delta(u)\tilde{f}(x)\mathrm{d}u. \tag{3.76}$$

where $\tilde{f}$ is not, as yet, assumed to share any relationship with the f of TMG. Via substitution of this ansatz into (3.61), one finds the following differential equation

$$\partial_x^2 \tilde{f}(x) + m\partial_x \tilde{f}(x) = -gQ\delta(x), \tag{3.77}$$

whose solution is easily computed to be

$$\tilde{f}(x) = C_1 + \frac{e^{-mx}}{m}\Big(C_2 - (e^{mx} - 1)gQ\Theta(-x)\Big), \tag{3.78}$$

and, again, enforcing the boundary conditions yields

$$\boxed{\tilde{f}(x) = g\frac{Q}{m}\Big(e^{-mx} + \Theta(-x)\Big)}. \tag{3.79}$$

Finally, we consider the massive biadjoint scalar solutions. In this case the boundary conditions are (asymptotically) symmetric [21]

$$\varphi(x) \to 0 \text{ as } |x| \to \infty. \tag{3.80}$$

The ansatz for the biadjoint scalar is

$$\Phi^{a\tilde{a}} = c^a c^{\tilde{a}}\varphi. \tag{3.81}$$

The differential equation arising from (3.65) is just a sourced Klein-Gordon equation

$$(\partial_x^2 - m^2)\varphi = -y\delta(u)\delta(x), \tag{3.82}$$

where we have used $\kappa p \to y$ to highlight again that the charge may be of different kind. The solution in this case, applying the boundary conditions (3.80) is known to be [21]

$$\boxed{\varphi(x,u) = \frac{y}{2m}\delta(u)\Big(e^{mx}\Theta(-x) + e^{-mx}\Theta(x)\Big)}. \tag{3.83}$$

Thus, it should be clear that the solutions (3.73), (3.79) and (3.83) are now all *consistent*, but they each require their own consideration and an appropriate application of boundary conditions at each level. This is distinct from the 4-dimensional case; the derivatives in the gravitational and Yang-Mills theories were both second order, and consistency of φ across the double, single and zeroth copies was trivial.

This concludes our discussion of the time-dependent solutions of the classical 3-dimensional topologically massive double copy. Contrary to what one might expect, the stationary solutions (corresponding to the double copy of anyons[28]) is more complicated than the time-dependent solutions, and discussions of these solutions is to be considered by the author in future work.

4 Concluding Remarks

4.1 Review

The primary goal of this work has been to introduce the massive classical double copy in (2+1)-dimensions. The approach to doing this that was adopted in this work was to systematically (and, at times, pedagogically) build up the theory from the 'elementary' cases to the more challenging ones.

In §1, we provided contextual and historical placement for double copy theory, with a focus on the origins, limitations, and motivations for why the double copy is worth investigation.

The focus of §2 was on introducing the conventional and necessary tools used to tackle the classical double copy in particular; with a thorough introduction to Kerr-Schild metrics and their associated properties. This discussion was guided by some archetypal examples of successful applications of the classical double copy for relatively simple time-independent (Schwarzchild §2.5.1, Kerr §2.5.2) and time-dependent (plane waves §2.6.1, shockwaves §2.6.2) systems.

The principal investigation of this work was presented in §3. The goal of §3 was two-fold: to rigorously introduce the idea of topologically massive theories in 3-dimensions (TMSE §3.1, TMYM §3.2, TMG §3.3), and to show that the classical double copy in these topologically massive cases may successfully reproduce the linearised TMYM (single copy) and massive linearised biadjoint scalar theory (zeroth copy) time-dependent solutions – in particular, plane wave shockwave solutions (§3.4).

4.2 Discussion and Future Directions

The topologically massive classical double copy has been shown to successfully reproduce the time-dependent plane wave and shockwave equations of motion for linearised TMYM theory (and by virtue of this, topologically massive electrodynamics[33] (TME) also), as well as massive biadjoint scalar theory. The success of massive classical double copy in reproducing the time-dependent solutions was non-trivial – the boundary conditions are specifically chosen in order to reproduce the single and zeroth copy solutions. Nonetheless, it has been suggested in [21] that the implications of these double copy relations may suggest possible similar (potentially simpler) relations for the so-called *Weyl double copy* [55]; a double copy relation exhibited by

[33] We say TME instead of the full TMSE that was considered as the spinor equations of motion did not appear in the single copy.

the Weyl tensor (double copy) and Yang-Mills field strength (single copy). As was mentioned in the text, the Weyl tensor is vanishing in three dimensions, however, the existence of the analogous Cotton tensor means that these relations could be (and have been) examined in the context of a *Cotton double copy* [78].

A result that is discussed in the literature, but that is not presented here, is the 3-dimensional double copy of the topologically massive time-*independent* equations of motion; the double copy of *anyons* [28]. The approach to this static case used in [28] makes use of amplitude techniques, and claims that the double copy of anyons is to be realised when one takes the single copy generated by TMG with a massless spin-2 ghost field; whether this is the correct approach/interpretation is still under investigation. An obvious and, as yet, unexplored approach to the double copy of anyons would be to use the standard approach of this work – making use of the Kerr-Schild form of metrics and cleverly using Killing vectors to find the time-independent equations of motion.

Another avenue of research worth investigation is to consider using modified (generalised) Kerr-Schild forms of the metric. This investigation has various potential benefits; it could bring us closer to an understanding of between which theories a double copy procedure is possible and, if we suppose that the most basic construction of this idea might entail a generalised form [79]

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa\varphi k_\mu k_\nu + K_{\mu\nu}, \tag{4.1}$$

for an additional term, $K_{\mu\nu}$. This form of the metric is referred to as an *extended Kerr-Schild* metric (xKS) [79, 80], and we see that these xKS metrics would encapsulate the Kerr-Schild double copy and enlarge the solution space (and applicability) of the double copy further, provided solutions for non-trivial $K_{\mu\nu}$ were found. Furthermore, understanding and pushing further the limits of the *classical* double copy may lead to a better understanding of the CK duality, a much sought-after and elusive goal of the double copy community.

Investigations into Kerr-Schild spacetimes with curved background spacetimes, taking the form

$$g_{\mu\nu} = \tilde{g}_{\mu\nu} + \kappa\varphi k_\mu k_\nu, \tag{4.2}$$

have been done for maximally symmetric (de Sitter and Anti-de Sitter) background spacetimes ($\tilde{g}_{\mu\nu}$) [39]. One might consider extending this work using xKS metrics or by working with spaces that are *not* maximally symmetric, although this will likely present numerous difficulties. Other work on Kerr-Schild metrics in curved spacetimes was done in [81], where it was shown that the double copy may be used for certain conformally flat metrics.

Perhaps the most obvious extension of this work would be to simply work in other dimensions. Topological terms may be found in arbitrary dimensions (see §B) and an investigation into the topologically massive higher-dimensional ($d > 3$) double copy of these theories has not yet been done. Work on higher-dimensional objects (*e.g.* black branes) has been done [7], and extensions of this would be of interest.

Of interest to cosmologists would be to find a double copy (if it exists) of the *Friedmann-Lemaître-Robertson-Walker (FLRW) metric* (a spatially homogeneous and isotropic metric with time dependence, typically used to describe an expanding/contracting universe). Understanding what/which gauge theory corresponds to the single copy of the FLRW metric would potentially provide additional insight into the gravitational theory. Of course, this metric departs from the Kerr-Schild metrics discussed in this work. However, there may be a way to represent the FLRW metric using either an xKS metric or perhaps making use of the *generalised Gordon Ansatz* [82], which may make the research avenue tractable.

At this stage, it should be evident that there are numerous potential areas of investigation into classical double copy theory that are all worthy of investigation. The same is true of the scattering amplitudes programme [19]. Double copy theory is a fast-progressing, active field of research whose implications are not yet fully understood. While this work by no means introduces the reader to *all* the aspects of classical double copy theory, it should serve as a useful resource for those wishing to be introduced to the fundamental ideas governing the theory, and the fundamental examples of the theory at work. We expect that future work on the classical and scattering amplitude programmes of the double copy, respectively, will produce enlightening findings whose implications/effects will extend outside the context of double copy theory, itself.

A A Brief Word on Shockwaves

Gravitational shockwave solutions to Einstein's equations, derived in [56] are interesting solutions to Einstein's equations. In this section, we follow the original derivation by Aichelburg and Sexl [56]. The original work [56] is thorough, but we direct the reader to [58] for a discussion that is intended for those interested in classical double copy theory.

Aichelburg and Sexl sought to solve Einstein's equations for a gravitational field whose source was a massless particle. To do this, they applied two approaches:

1. In their first approach, the linearised Einstein equations are solved for a particle having mass m and velocity v. The relativistic limit ($v \to 1$) and massless limit ($m \to 0$) are taken, and it is found that in this limit the Einstein equations are the same as those of the full (non-linearised) solution for an energy-momentum tensor having a massless particle as its source.

2. The second approach is the 'full' solution approach; it begins with the Schwarzchild metric (that is, the metric that precisely describes a particle at rest), on which a Lorentz boost is performed such that the velocity approaches the speed of light ($v \to 1$) and take the massless limit is taken once again ($m \to 0$).

In this section, we consider the second approach (the first approach is essentially done in-text in §2.6.2), closely following [58]. We begin from the isotropic form of the Schwarzchild metric, whose line element is [58, 83],

$$\begin{aligned} \mathrm{d}s^2 &= -\frac{(1-Q)^2}{(1+Q)^2}\mathrm{d}t^2 + (1+Q)^4(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2), \quad \text{where} \\ Q &= \frac{m}{2\sqrt{x^2+y^2+z^2}}. \end{aligned} \tag{A.1}$$

We choose not to use polar coordinates as we will soon wish to perform a Lorentz boost in a particular direction, and this is most easily achieved with the specified coordinates. The line element (A.1) can be rewritten as

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 + \left[1 - \frac{(1-Q)^2}{(1+Q)^2}\right]\mathrm{d}t^2 - \left[1 - (1+Q)^4\right](\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2). \tag{A.2}$$

We now perform a Lorentz boost in the z-direction, thus

$$t = \frac{t - vz}{\sqrt{1 - v^2}}, \quad x = x, \quad y = y, \quad z = \frac{z - vt}{\sqrt{1 - v^2}}. \tag{A.3}$$

In terms of the boosted (primed) coordinates, the line element is

$$\begin{aligned} ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + \left[1 - \frac{(1 - Q)^2}{(1 + Q)^2}\right] \frac{(dt - vdz)^2}{1 - v^2} \\ - \left[1 - (1 + Q)^4\right] \left[dx^2 + dy^2 + \frac{(dz - vdt)^2}{1 - v^2}\right], \end{aligned} \tag{A.4}$$

where Q is expressed in terms of the boosted coordinates as

$$Q = \frac{m\sqrt{1 - v^2}}{2\sqrt{(1 - v^2)(x^2 + y^2) + (z - vt)^2}}. \tag{A.5}$$

In the $v \to 1$ limit (letting $m = p\sqrt{1 - v^2}$) we find that [56]

$$\begin{aligned} \lim_{v\to 1} Q &= \lim_{v \to 1} \frac{p(1 - v^2)}{2\sqrt{(1 - v^2)(x^2 + y^2) + (z - vt)^2}} \\ &= \begin{cases} \frac{p}{2|z - t|}, & z = t, \\ \infty, & z = t, \end{cases} \end{aligned} \tag{A.6}$$

where, therefore, the spacetime has a singularity at $z = t$. For all points having $z = t$, the line element takes on the convenient (Brinkmann-like form)

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + \frac{4p}{|t - z|}(dt - dz)^2, \tag{A.7}$$

and will have vanishing Riemann curvature (in the primed coordinates and ‘o’ of $z = t$). We now make use of a non-trivial relation, proved by Aichelburg and Sexl in the appendix of [56]; it states:

$$\lim_{v\to 1}\left[\{(z - vt)^2 + (1 - v^2)r^2\}^{-1/2} - \{(z - vt)^2 + (1 - v^2)\}^{-1/2}\right] = -2\delta(z - t)\ln(r), \tag{A.8}$$

where $r^2 = x^2 + y^2$. This result could be used in (A.5) if we could find an appropriate coordinate transformation, which exists and is presented in [56]

$$
\begin{aligned}
T(v): \quad & \tilde{z} - v\tilde{t} = z - vt \\
& \tilde{z} + v\tilde{t} = z + vt - 4p \ln\left[\sqrt{(z - vt)^2 + (1 - v^2)} - (z - t)\right],
\end{aligned}
\tag{A.9}
$$

which leaves x, y, and Q invariant. Applying this transformation to the metric and expanding around Q the line element is written as [56, 58]

$$
\mathrm{d}s^2 = -\mathrm{d}\tilde{t}^2 + \mathrm{d}\tilde{x}^2 + \mathrm{d}\tilde{y}^2 + \mathrm{d}\tilde{z}^2 - 4p\delta(\tilde{z} - \tilde{t}) \ln(\tilde{x}^2 + \tilde{y}^2)(\mathrm{d}\tilde{z} - \mathrm{d}\tilde{t})^2, \tag{A.10}
$$

which reduces to (A.7) for $t = z$ after performing the inverse coordinate transformation $T^{-1}(v = 1)$. Quite interestingly, we can easily see that (A.10) is of the form of a *pp*-wave metric, introduced in §2.6.1. Following a similar calculation as for that shown in the text, one finds that the Einstein equations reduce to [53, 56]

$$
\begin{gathered}
R_{00} = R_{11} = -R_{01} = -(H_{,xx} + H_{,yy}) = 8\pi p\delta(x)\delta(y)\delta(t - z) \\
\boxed{T_{\mu\nu} = p\delta(x)\delta(y)\delta(t - z)s_\mu s_\nu,}
\end{gathered}
\tag{A.11}
$$

where $s_\mu = \delta^0_\mu + \delta^3_\mu$. Thus, the gravitational field of a massless (point) particle only has non-vanishing Riemann curvature on the hypersurface $x = y = 0$, $z = t$. Further analysis (not presented here, see [56]) shows that the gravitational field behaves similarly to an electromagnetic field, being dilated orthogonal to the boost direction and compressed parallel to the boost direction, reducing to the hyperplane in the $v \to 1$ limit.

B Why 'Topologically' Massive?

As is mentioned at the start of §3, the topological contribution to the action is proportional to the secondary Chern-Simons class; a topological entity. We briefly elaborate on that statement now, following the discussion given in [61].

B.1 Even-Dimensional Chern-Simons Terms

In $d = 2n, n \quad Z^+$ (even dimensions), it is possible to use the existing gauge fields to construct an additional gauge invariant quantity, the *Pontryagin density*, P_d, which, when integrated over the total $2n$-dimensional space, produces a topological invariant of the space. In the lowest non-trivial dimensions, these are [61]

$$\begin{aligned} P_2 &\equiv \frac{1}{4\pi}\varepsilon^{\mu\nu}F_{\mu\nu} \\ P_4 &\equiv -\frac{1}{16\pi^2}\mathrm{tr}\Big(\frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}F_{\mu\nu}\Big), \end{aligned} \tag{B.1}$$

where the trace is over the Lie-algebra indices in the chosen representation ($F_{\mu\nu} = T^a F^a_{\mu\nu}$ for a chosen set of Lie-algebra generators T^a), $F_{\mu\nu}$ is the usual (potentially) non-Abelian gauge field tensor, and where the trace normalisation depends on both which Lie algebra is being considered as well as chosen conventions *e.g.* Srednicki chooses to normalise $\mathfrak{su}(N)$ by the condition $\mathrm{tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$ [84]. It is worth noting that the values P_2 and P_4 are covariant densities, and, due to their independence from the metric they are topological entities, independent of the local properties of the manifold where they exist [85]. Importantly, these gauge invariant Pontryagin densities can be written as total derivatives of some gauge variant objects, X^μ_d as [61]

$$\begin{aligned} P_d &= \partial_\mu X^\mu_d \\ X^\mu_2 &= \frac{1}{2\pi}\varepsilon^{\mu\nu}A_\nu \\ X^\mu_4 &= -\frac{1}{16\pi^2}\varepsilon^{\mu\alpha\beta\gamma}\mathrm{tr}\Big(A_\alpha F_{\beta\gamma} - \frac{2}{3}A_\alpha A_\beta A_\gamma\Big). \end{aligned} \tag{B.2}$$

The vectors, X^μ, are called *Chern-Simons currents* or *anomaly currents* [85]. One recovers the second characteristic Chern-Simons class (also called the *Chern-Pontryagin gauge field invariant* [85]) by integrating X^μ over the $(d-1)$-dimensions $x^\nu, \nu = \mu$. The Chern-Simons second characteristic class is gauge invariant up to additive factors of (the discrete) winding number of a gauge transformation.

B.2 Odd-Dimensional Chern-Simons Currents

It is anticipated that, throughout the reading of the preceding paragraphs and equations, the reader may not be satisfied – this work is principally concerned with TMG in *three* dimensions, whereas the preceding discussion has been limited to *even*-dimensional spaces. The link between the even- and odd-dimensional cases may be described following discussions taken from [85]. One notes that in the Chern-Simons currents, there is one free index that appears in the respective Levi-Civita symbols, ε. Given that this is a totally antisymmetric object, none of the indices are repeating, and so for any summed index α we know $\alpha = \mu$ (where μ is the free index). Thus, for a particular value of the index μ, this index will not occur in any of the gauge field objects (A_α, $F_{\alpha\beta}$) and, since these are the objects from which the Chern-Simons current is comprised, we can construct odd, $(d-1)$-dimensional, Chern-Simons terms from the even, d-dimensional, Chern-Simons currents,

$$\begin{aligned} X_2^\mu &= \frac{1}{2\pi}\varepsilon^{\mu\nu}A_\nu & X_1 &= \frac{1}{2\pi}A_1 \\ X_4^\mu &= -\frac{1}{16\pi^2}\varepsilon^{\mu\alpha\beta\gamma}\mathrm{tr}\Big(A_\alpha F_{\beta\gamma} - \frac{2}{3}A_\alpha A_\beta A_\gamma\Big) & X_3 &= -\frac{1}{16\pi^2}\varepsilon^{ijk}\mathrm{tr}\Big(A_i F_{jk} - \frac{2}{3}A_i A_j A_k\Big), \end{aligned} \tag{B.3}$$

where $i, j, k = \mu$. These objects are evidently integrable in 1- and 3-dimensional space, respectively, and their independence of the metric means that they may still be considered topological entities.

B.2.1 Interpretation of Topological Quantities

What the integrals of the Chern-Simons terms reveal about the topological content of their spaces is dependent upon what they represent. The integral of the Abelian form of the 3-dimensional term,

$$H(A) := \int X_3 \mathrm{d}^3x = \int \varepsilon^{ijk}A_i\partial_j A_k \mathrm{d}^3x, \tag{B.4}$$

is interpreted as the *magnetic helicity*, $H(A)$, (which measures the extent to which the field lines are linked/wrapped around one another) when A_i is the electromagnetic vector potential and $\varepsilon^{ijk}\partial_j A_k = B^i$ is the magnetic field [85, 86]. If one also considers a smooth (C^∞), arc-length-parametrized curve, γ, in three dimensions, then one can define the *writhing number*,

$\mathrm{Wr}(\gamma)$, which measures the extent to which the curve, γ, is linked/wrapped around itself and is expressed as [86]

$$\mathrm{Wr}(\gamma) = \frac{1}{4\pi}\int_{\gamma\times\gamma}\left(\frac{dx}{dt}\times\frac{dy}{ds}\right)\frac{x-y}{|x-y|^3}\mathrm{d}s\mathrm{d}t. \tag{B.5}$$

It does not seem particularly surprising that the writhing number is related to the helicity, and this relationship can be expressed succinctly as $H(A) = \mathrm{Flux}(A)^2\mathrm{Wr}(\gamma)$ [86, 87].

An alternative context for the calculation of topological content arises when we identify $A_i \equiv v_i \equiv$ (the velocity of a fluid), then $\varepsilon^{ijk}\partial_j A_k \equiv w^i \equiv$ (the vorticity of the fluid), whose integral (*i.e.* integrating the appropriate n-dimensional Chern-Simons current/term in (B.3)) is the *kinetic vorticity* of the fluid [85].

Finally, it is worth considering the gravitational perspective. In this case, we obtain a 3-dimensional Chern-Simons term from a 4-dimensional *Hirzebruch-Pontryagin* density [61]

$$\partial_\mu X^\mu := \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}R_{\mu\nu\rho\sigma}R_{\alpha\beta}{}^{\rho\sigma}. \tag{B.6}$$

The Chern-Simons term appearing in (3.35) comes (as was done previously) by performing the 3-dimensional integral over the unused indices.

The links between topology and this work (and, in fact, physics in general) is a continuously developing field, and all of the links cannot possibly be discussed here. We point the reader to the citations herein for a brief overview of links to this work, but also recommend [70] as a useful guide to explore the relationships between topology and physics.

C Useful Code - xAct in Action

In this section of the appendices, we include some of the code used in obtaining the results of this work. As with all 'good' science, results should be reproducible by a colleague utilising the same equipment/tools. The installation instructions *etc,* for the xAct and diffgeo packages are not included here and are left to the reader, but some instructions can be found at [34]. All code extracts reproduced in this section (for this work) were written by the author, however, a particularly useful introduction can be found at [88].

In §2.2, we calculated the Ricci tensor for a generic Kerr-Schild metric. This can easily be done using xAct [34], and the code employed is presented below. The next calculation included is the calculation of the gravitational equations of motion. Of course, the latter can be done by hand despite being tedious, but it is useful to show that the same result can be obtained using xAct.

In §3.3, it is noted that the "diffgeo" package [77] may be used – although it is emphasized that this package is *not* a part of the xAct package. However, it was found to be quite practical in verifying calculations done by hand, and Mathematica users unfamiliar with xAct and xCoba will likely find it more intuitive to learn (despite being vastly more limited than xAct in its capabilities).

Kerr-Schild Ricci Tensor

We attempt to reproduce known results for the form of the Ricci Tensor in Kerr-Schild Coordinates

Import Packages

In[1]:=
```
<< xAct`xTensor`
<< xAct`xPert`
```

```
------------------------------------------------------------
Package xAct`xPerm`  version 1.2.3, {2015, 8, 23}
CopyRight (C) 2003-2020, Jose M. Martin-Garcia, under the General Public License.
Connecting to external mac executable...
Connection established.
------------------------------------------------------------
Package xAct`xTensor`  version 1.2.0, {2021, 10, 17}
CopyRight (C) 2002-2021, Jose M. Martin-Garcia, under the General Public License.
------------------------------------------------------------
These packages come with ABSOLUTELY NO WARRANTY; for details type
  Disclaimer[]. This is free software, and you are welcome to redistribute
  it under certain conditions. See the General Public License for details.
------------------------------------------------------------
------------------------------------------------------------
Package xAct`xPert`  version 1.0.6, {2018, 2, 28}
CopyRight (C) 2005-2020, David Brizuela, Jose M. Martin-Garcia
  and Guillermo A. Mena Marugan, under the General Public License.
------------------------------------------------------------
These packages come with ABSOLUTELY NO WARRANTY; for details type
  Disclaimer[]. This is free software, and you are welcome to redistribute
  it under certain conditions. See the General Public License for details.
------------------------------------------------------------
** Variable $PrePrint assigned value ScreenDollarIndices
** Variable $CovDFormat changed from Prefix to Postfix
** Option AllowUpperDerivatives of ContractMetric changed from False to True
** Option MetricOn of MakeRule changed from None to All
** Option ContractMetrics of MakeRule changed from False to True
```

Define Manifold, Metric, Metric Perturbation

In[3]:=
```
DefManifold[M3, 3, {α, β, γ, ρ, σ, τ, μ, ν, λ}]
DefMetric[-1, metric[-α, -β],  CD,  PrintAs → "g" ]

(*Colour power indices*)
Unprotect[IndexForm];
IndexForm[LI[x_]] := ColorString[ToString[x], RGBColor[0, 0, 1]];
Protect[IndexForm];

DefMetricPerturbation[metric, metpert, ϵ];
PrintAs[metpert] ^= "h";
```

```
** DefManifold: Defining manifold M3.
** DefVBundle: Defining vbundle TangentM3.
** DefTensor: Defining symmetric metric tensor metric[-α, -β].
** DefTensor: Defining antisymmetric tensor epsilonmetric[-α, -β, -γ].
** DefCovD: Defining covariant derivative CD[-α].
** DefTensor: Defining vanishing torsion tensor TorsionCD[α, -β, -γ].
** DefTensor: Defining symmetric Christoffel tensor ChristoffelCD[α, -β, -γ].
** DefTensor: Defining Riemann tensor RiemannCD[-α, -β, -γ, -λ].
** DefTensor: Defining symmetric Ricci tensor RicciCD[-α, -β].
** DefCovD:  Contractions of Riemann automatically replaced by Ricci.
** DefTensor: Defining Ricci scalar RicciScalarCD[].
** DefCovD:  Contractions of Ricci automatically replaced by RicciScalar.
** DefTensor: Defining symmetric Einstein tensor EinsteinCD[-α, -β].
** DefTensor: Defining vanishing Weyl tensor WeylCD[-α, -β, -γ, -λ].
** DefTensor: Defining symmetric TFRicci tensor TFRicciCD[-α, -β].
** DefTensor: Defining Kretschmann scalar KretschmannCD[].
** DefCovD:  Computing RiemannToWeylRules for dim 3
** DefCovD:  Computing RicciToTFRicci for dim 3
** DefCovD:  Computing RicciToEinsteinRules for dim 3
** DefTensor: Defining weight +2 density Detmetric[]. Determinant.
** DefParameter: Defining parameter ϵ.
** DefTensor: Defining tensor metpert[LI[order], -α, -β].
```

Find Perturbed Ricci Tensor

In[10]:=
```
PerturbedRicci =
  Perturbed[RicciCD[α, -β], 1] // ExpandPerturbation // ToCanonical //
   ContractMetric
```

Out[10]=

$$R[\nabla]^{\alpha}{}_{\beta} - \epsilon\, h^{1\alpha\gamma}\; R[\nabla]_{\beta\gamma} - \frac{1}{2}\epsilon\, h^{1\gamma}{}_{\gamma;\beta}{}^{;\alpha} + \frac{1}{2}\epsilon\, h^{1}{}_{\beta}{}^{\gamma;\alpha}{}_{;\gamma} + \frac{1}{2}\epsilon\, h^{1\alpha\gamma}{}_{;\beta;\gamma} - \frac{1}{2}\epsilon\, h^{1}{}_{\beta}{}^{\alpha;\gamma}{}_{;\gamma}$$

Apply Simplifying Rules

In[11]:=
```
(*set ricci tensor to zero*)
ricciflattensor = RicciCD[x_, y_] :→ 0
(*killing form of metpert*)
tracelesspert = metpert[LI[1], x_, -x_] :→ 0
```

Out[11]= $R[\nabla]^{xy} :\to 0$

Out[12]= $h^{1x}{}_{x} :\to 0$

In[13]:=
```
PerturbedRicci /. ricciflattensor
```

Out[13]= $-\frac{1}{2}\epsilon\, h^{1\gamma}{}_{\gamma;\beta}{}^{;\alpha} + \frac{1}{2}\epsilon\, h^{1}{}_{\beta}{}^{\gamma;\alpha}{}_{;\gamma} + \frac{1}{2}\epsilon\, h^{1\alpha\gamma}{}_{;\beta;\gamma} - \frac{1}{2}\epsilon\, h^{1}{}_{\beta}{}^{\alpha;\gamma}{}_{;\gamma}$

In[14]:=
```
% /. tracelesspert
```

Out[14]= $\frac{1}{2}\epsilon\, h^{1}{}_{\beta}{}^{\gamma;\alpha}{}_{;\gamma} + \frac{1}{2}\epsilon\, h^{1\alpha\gamma}{}_{;\beta;\gamma} - \frac{1}{2}\epsilon\, h^{1}{}_{\beta}{}^{\alpha;\gamma}{}_{;\gamma}$

In[15]:=
```
ChangeCovD[%, CD, PD] /. (ChristoffelCD[x__] :→ 0) // ContractMetric
```

Out[15]= $\frac{1}{2}\epsilon\, h^{1}{}_{\beta}{}^{\gamma,\alpha}{}_{,\gamma} + \frac{1}{2}\epsilon\, h^{1\alpha\gamma}{}_{,\beta,\gamma} - \frac{1}{2}\epsilon\, h^{1}{}_{\beta}{}^{\alpha,\gamma}{}_{,\gamma}$

TMG Equations of Motion

We attempt to use xAct to find the EoMs for TMG

Import Packages

In[∘]:=
```
<< xAct`xTensor`
<< xAct`xPert`
$PrePrint = ScreenDollarIndices;
```

```
------------------------------------------------------------

Package xAct`xPerm`  version 1.2.3, {2015, 8, 23}

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Connecting to external mac executable...

Connection established.

------------------------------------------------------------

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------------------------------------------------------------

------------------------------------------------------------

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  and Guillermo A. Mena Marugan, under the General Public License.

------------------------------------------------------------

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  Disclaimer[]. This is free software, and you are welcome to redistribute
  it under certain conditions. See the General Public License for details.

------------------------------------------------------------

** Variable $PrePrint assigned value ScreenDollarIndices

** Variable $CovDFormat changed from Prefix to Postfix

** Option AllowUpperDerivatives of ContractMetric changed from False to True

** Option MetricOn of MakeRule changed from None to All

** Option ContractMetrics of MakeRule changed from False to True
```

Define Manifold, Metric, Perturbations, Constants

In[∘]:=
```
DefManifold[M3, 3, {α, β, γ, ρ, σ, τ, μ, ν, λ}]
DefMetric[-1, metric[-α, -β], CD, {";", "∇"}, PrintAs → "g"]

(*Colour 'power' indices*)
Unprotect[IndexForm];
IndexForm[LI[x_]] := ColorString[ToString[x], RGBColor[1, 0, 0]];
Protect[IndexForm];

DefMetricPerturbation[metric, metpert, ξ]
PrintAs[metpert] ^= "δg"

DefConstantSymbol[κ]
DefConstantSymbol[m]
```

```
** DefManifold: Defining manifold M3.
** DefVBundle: Defining vbundle TangentM3.
** DefTensor: Defining symmetric metric tensor metric[-α, -β].
** DefTensor: Defining antisymmetric tensor epsilonmetric[-α, -β, -γ].
** DefCovD: Defining covariant derivative CD[-α].
** DefTensor: Defining vanishing torsion tensor TorsionCD[α, -β, -γ].
** DefTensor: Defining symmetric Christoffel tensor ChristoffelCD[α, -β, -γ].
** DefTensor: Defining Riemann tensor RiemannCD[-α, -β, -γ, -λ].
** DefTensor: Defining symmetric Ricci tensor RicciCD[-α, -β].
** DefCovD:  Contractions of Riemann automatically replaced by Ricci.
** DefTensor: Defining Ricci scalar RicciScalarCD[].
** DefCovD:  Contractions of Ricci automatically replaced by RicciScalar.
** DefTensor: Defining symmetric Einstein tensor EinsteinCD[-α, -β].
** DefTensor: Defining vanishing Weyl tensor WeylCD[-α, -β, -γ, -λ].
** DefTensor: Defining symmetric TFRicci tensor TFRicciCD[-α, -β].
** DefTensor: Defining Kretschmann scalar KretschmannCD[].
** DefCovD:  Computing RiemannToWeylRules for dim 3
** DefCovD:  Computing RicciToTFRicci for dim 3
** DefCovD:  Computing RicciToEinsteinRules for dim 3
** DefTensor: Defining weight +2 density Detmetric[]. Determinant.
** DefParameter: Defining parameter ξ.
** DefTensor: Defining tensor metpert[LI[order], -α, -β].
```

Out[∘]= δg

```
** DefConstantSymbol: Defining constant symbol κ.
** DefConstantSymbol: Defining constant symbol m.
```

```
In[•]:= Lex = 1/κ^2 Sqrt[-Detmetric[]] (-RicciScalarCD[])
Lexpert = Perturbation[Lex]
Print["Expanding Perturbation"]
%% // ExpandPerturbation
Print["Contracting Metric"]
%% // ContractMetric
Print["Canonicalising"]
%% // ToCanonical
Lexpert = %;

RHSex =
 2 (-VarD[metpert[LI[1], α, β], CD][Lexpert] / Sqrt[-Detmetric[]] /. delta[-LI[1],
          LI[1]] → 1 // SeparateMetric[metric] //
      RicciToEinstein) // Expand // ContractMetric // ToCanonical
Print["EOM is:"]
0 == RHSex // FullSimplify
```

Out[•]= $-\dfrac{\sqrt{-\tilde{\tilde{g}}}\ R[\nabla]}{\kappa^2}$

Out[•]= $-\dfrac{\sqrt{-\tilde{\tilde{g}}}\ \Delta[R[\nabla]] - \dfrac{\Delta[\tilde{\tilde{g}}]\ R[\nabla]}{2\sqrt{-\tilde{\tilde{g}}}}}{\kappa^2}$

Expanding Perturbation

Out[•]= $-\dfrac{1}{\kappa^2}\left(-\dfrac{\tilde{\tilde{g}}\ \delta g^{1}{}_{\lambda}{}^{\lambda}\ R[\nabla]}{2\sqrt{-\tilde{\tilde{g}}}} + \sqrt{-\tilde{\tilde{g}}}\left(-\delta g^{1\alpha\beta}\ R[\nabla]_{\alpha\beta} + g^{\alpha\beta}\left(\dfrac{1}{2}\left(-\delta g^{1\gamma}{}_{\gamma;\beta;\alpha} - \delta g^{1\gamma}{}_{\beta;\gamma;\alpha} + \delta g^{1}{}_{\beta\gamma}{}^{;\gamma}{}_{;\alpha}\right) + \dfrac{1}{2}\left(\delta g^{1\gamma}{}_{\beta;\alpha;\gamma} + \delta g^{1\gamma}{}_{\alpha;\beta;\gamma} - \delta g^{1}{}_{\beta\alpha}{}^{;\gamma}{}_{;\gamma}\right)\right)\right)\right)$

Contracting Metric

Out[•]= $\dfrac{\sqrt{-\tilde{\tilde{g}}}\ \delta g^{1\alpha\beta}\ R[\nabla]_{\alpha\beta}}{\kappa^2} - \dfrac{\sqrt{-\tilde{\tilde{g}}}\ \delta g^{1}{}_{\alpha}{}^{\alpha}\ R[\nabla]}{2\kappa^2} + \dfrac{\sqrt{-\tilde{\tilde{g}}}\ \delta g^{1\beta}{}_{\beta}{}^{;\alpha}{}_{;\alpha}}{2\kappa^2} + \dfrac{\sqrt{-\tilde{\tilde{g}}}\ \delta g^{1\beta\alpha}{}_{;\beta;\alpha}}{2\kappa^2} - \dfrac{\sqrt{-\tilde{\tilde{g}}}\ \delta g^{1\alpha}{}_{\beta}{}^{;\beta}{}_{;\alpha}}{2\kappa^2} - \dfrac{\sqrt{-\tilde{\tilde{g}}}\ \delta g^{1\beta\alpha}{}_{;\alpha;\beta}}{\kappa^2} + \dfrac{\sqrt{-\tilde{\tilde{g}}}\ \delta g^{1\alpha}{}_{\alpha}{}^{;\beta}{}_{;\beta}}{2\kappa^2}$

Canonicalising

Out[•]= $\dfrac{\sqrt{-\tilde{\tilde{g}}}\ \delta g^{1\alpha\beta}\ R[\nabla]_{\alpha\beta}}{\kappa^2} - \dfrac{\sqrt{-\tilde{\tilde{g}}}\ \delta g^{1\alpha}{}_{\alpha}\ R[\nabla]}{2\kappa^2} - \dfrac{\sqrt{-\tilde{\tilde{g}}}\ \delta g^{1\alpha\beta}{}_{;\alpha;\beta}}{\kappa^2} + \dfrac{\sqrt{-\tilde{\tilde{g}}}\ \delta g^{1\alpha}{}_{\alpha}{}^{;\beta}{}_{;\beta}}{\kappa^2}$

Out[∘]= $-\frac{2\,G[\nabla]_{\alpha\beta}}{\kappa^2}$

EOM is:

Out[∘]= $\frac{G[\nabla]_{\alpha\beta}}{\kappa} == 0$

Define Lagrangian

In[∘]:= $\mathcal{L} = \frac{1}{\kappa^2}$ Sqrt[-Detmetric[]]

(-RicciScalarCD[] - $\frac{1}{2\,m}$ epsilonmetric[μ, ν, ρ] (ChristoffelCD[α, -μ, -σ] ×
PD[-ν][ChristoffelCD[σ, -α, -ρ]] + $\frac{2}{3}$ ChristoffelCD[α, -μ, -σ] ×
ChristoffelCD[σ, -ν, -β] × ChristoffelCD[β, -ρ, -α]))

Out[∘]= $$\frac{\sqrt{-\tilde{\tilde{g}}}\left(-R[\nabla] - \frac{\epsilon g^{\mu\nu\rho}\left(\frac{2}{3}\Gamma[\nabla]^{\alpha}{}_{\mu\sigma}\,\Gamma[\nabla]^{\beta}{}_{\rho\alpha}\,\Gamma[\nabla]^{\sigma}{}_{\nu\beta}+\Gamma[\nabla]^{\alpha}{}_{\mu\sigma}\,\Gamma[\nabla]^{\sigma}{}_{\alpha\rho,\nu}\right)}{2\,m}\right)}{\kappa^2}$$

Perform variation as is done in example - **NB: Take note of ToCanonical method used here**

In[∘]:= Lpert = ToCanonical[#, UseMetricOnVBundle → None] &@
ContractMetric@ExpandPerturbation@Perturbation@$\mathcal{L}$

Out[∘]= $$\frac{\sqrt{-\tilde{\tilde{g}}}\;\delta g^{1\alpha\beta}\;R[\nabla]_{\alpha\beta}}{\kappa^2} - \frac{\sqrt{-\tilde{\tilde{g}}}\;\delta g^{1\alpha}{}_{\alpha}\;R[\nabla]}{2\,\kappa^2} - \frac{\Gamma[\nabla]^{\alpha}{}_{\beta\gamma}\;\Gamma[\nabla]^{\beta}{}_{\lambda\mu}\;\sqrt{-\tilde{\tilde{g}}}\;\epsilon g^{\gamma\mu\nu}\;\delta g^{1\lambda}{}_{\nu;\alpha}}{2\,m\,\kappa^2} -$$
$$\frac{\sqrt{-\tilde{\tilde{g}}}\;\delta g^{1\alpha}{}_{\beta}{}^{;\beta}{}_{;\alpha}}{2\,\kappa^2} - \frac{\sqrt{-\tilde{\tilde{g}}}\;\delta g^{1\alpha\beta}{}_{;\alpha;\beta}}{2\,\kappa^2} + \frac{\sqrt{-\tilde{\tilde{g}}}\;\delta g^{1\alpha}{}_{\alpha}{}^{;\beta}{}_{;\beta}}{\kappa^2} +$$
$$\frac{\Gamma[\nabla]^{\alpha}{}_{\beta\gamma}\;\Gamma[\nabla]^{\beta}{}_{\lambda\mu}\;\sqrt{-\tilde{\tilde{g}}}\;\epsilon g^{\gamma\mu\nu}\;\delta g^{1}{}_{\alpha\nu}{}^{;\lambda}}{2\,m\,\kappa^2} - \frac{\Gamma[\nabla]^{\alpha}{}_{\beta\gamma}\;\Gamma[\nabla]^{\beta}{}_{\lambda\mu}\;\sqrt{-\tilde{\tilde{g}}}\;\epsilon g^{\gamma\mu\nu}\;\delta g^{1}{}_{\alpha}{}^{\lambda}{}_{;\nu}}{2\,m\,\kappa^2} -$$
$$\frac{\sqrt{-\tilde{\tilde{g}}}\;\epsilon g^{\beta\lambda\mu}\;\delta g^{1}{}_{\alpha\beta}{}^{;\gamma}\;\Gamma[\nabla]^{\alpha}{}_{\gamma\lambda,\mu}}{4\,m\,\kappa^2} + \frac{\sqrt{-\tilde{\tilde{g}}}\;\epsilon g^{\gamma\lambda\mu}\;\delta g^{1\alpha}{}_{\beta;\gamma}\;\Gamma[\nabla]^{\beta}{}_{\alpha\lambda,\mu}}{4\,m\,\kappa^2} +$$
$$\frac{\sqrt{-\tilde{\tilde{g}}}\;\epsilon g^{\beta\lambda\mu}\;\delta g^{1\alpha}{}_{\beta;\gamma}\;\Gamma[\nabla]^{\gamma}{}_{\alpha\lambda,\mu}}{4\,m\,\kappa^2} + \frac{\Gamma[\nabla]^{\alpha}{}_{\beta\gamma}\;\sqrt{-\tilde{\tilde{g}}}\;\epsilon g^{\gamma\lambda\mu}\;\delta g^{1\beta}{}_{\lambda;\alpha,\mu}}{4\,m\,\kappa^2} -$$
$$\frac{\Gamma[\nabla]^{\alpha}{}_{\beta\gamma}\;\sqrt{-\tilde{\tilde{g}}}\;\epsilon g^{\gamma\lambda\mu}\;\delta g^{1}{}_{\alpha\lambda}{}^{;\beta}{}_{,\mu}}{4\,m\,\kappa^2} + \frac{\Gamma[\nabla]^{\alpha}{}_{\beta\gamma}\;\sqrt{-\tilde{\tilde{g}}}\;\epsilon g^{\gamma\lambda\mu}\;\delta g^{1}{}_{\alpha}{}^{\beta}{}_{;\lambda,\mu}}{4\,m\,\kappa^2}$$

Now take variational derivatives

In[∘]:=
```
RHS =
  (-2 VarD[metpert[LI[1], μ,  ν],  CD][Lpert] / Sqrt[-Detmetric[]] /. delta[-LI[1],
          LI[1]] → 1 // SeparateMetric[metric] // RicciToEinstein) // Expand //
    ContractMetric // ToCanonical[#, UseMetricOnVBundle → None] & // FullSimplify
```

Out[∘]=

$$\frac{1}{4\,m\,\kappa^2}\Big(-8\,m\,G[\nabla]_{\mu\nu} + \epsilon \mathbf{g}_{\nu\beta\gamma}\left(\Gamma[\nabla]^{\alpha}{}_{\mu}{}^{\beta;\gamma}{}_{;\alpha} - \Gamma[\nabla]_{\mu}{}^{\alpha\beta;\gamma}{}_{;\alpha}\right) +$$
$$\epsilon \mathbf{g}_{\mu\beta\gamma}\left(\Gamma[\nabla]^{\alpha}{}_{\nu}{}^{\beta;\gamma}{}_{;\alpha} - \Gamma[\nabla]_{\nu}{}^{\alpha\beta;\gamma}{}_{;\alpha}\right) + \epsilon \mathbf{g}_{\nu}{}^{\beta\gamma}\,\Gamma[\nabla]^{\alpha}{}_{\mu\beta,\gamma;\alpha} + \epsilon \mathbf{g}_{\mu}{}^{\beta\gamma}\,\Gamma[\nabla]^{\alpha}{}_{\nu\beta,\gamma;\alpha} +$$
$$\epsilon \mathbf{g}_{\alpha\beta}{}^{\gamma}\left(\Gamma[\nabla]_{\mu\nu}{}^{\alpha;\beta}{}_{;\gamma} + \Gamma[\nabla]_{\nu\mu}{}^{\alpha;\beta}{}_{;\gamma}\right) - \epsilon \mathbf{g}_{\nu}{}^{\lambda\rho}\,\mathbf{g}^{\beta\gamma}\,\mathbf{g}_{\mu\alpha}\,\Gamma[\nabla]^{\alpha}{}_{\beta\lambda,\rho;\gamma} +$$
$$\mathbf{g}_{\nu\alpha}\left(-\epsilon \mathbf{g}_{\mu}{}^{\lambda\rho}\,\mathbf{g}^{\beta\gamma}\,\Gamma[\nabla]^{\alpha}{}_{\beta\lambda,\rho;\gamma} + \epsilon \mathbf{g}^{\beta\gamma\lambda}\,\Gamma[\nabla]^{\alpha}{}_{\mu\beta,\gamma;\lambda}\right) + \epsilon \mathbf{g}^{\beta\gamma\lambda}\,\mathbf{g}_{\mu\alpha}\,\Gamma[\nabla]^{\alpha}{}_{\nu\beta,\gamma;\lambda}\Big)$$

Quick DiffGeo Usage

Before importing one must define the coordinates being used and the form of the metric. Here we use the 3D Kerr-Schild metric.

In[1]:=
```
coord = {u, v, x};
metric = {{κ ϕ[u, x], -1, 0}, {-1, 0, 0}, {0, 0, 1}};
$Assumptions = And[u ∈ Reals, v ∈ Reals, x ∈ Reals, κ ∈ Reals, m > 0];
metricsign = -1;
```

Import diffgeo

In[5]:=
```
<< diffgeo`
```

Immediately have access to all sorts of quantities (see https://people.brandeis.edu/~headrick/Mathematica/diffgeoManual.nb for more)

In[6]:=
```
display[Christoffel]
```

Out[6]=

{v, u, x} {v, x, u} {x, u, u}	$-\frac{1}{2}\kappa\,\phi^{(0,1)}[u, x]$
{v, u, u}	$-\frac{1}{2}\kappa\,\phi^{(1,0)}[u, x]$

In[7]:=
```
display[Riemann]
```

Out[7]=

{u, x, u, x} {u, x, x, v}	$-\frac{1}{2}\kappa\,\phi^{(0,2)}[u, x]$
{x, u, u, x} {x, u, x, v}	$\frac{1}{2}\kappa\,\phi^{(0,2)}[u, x]$

In[8]:=
```
display[Cotton]
```

Out[8]=

{u, u}	$-\frac{1}{2}\kappa\,\phi^{(0,3)}[u, x]$

Can now easily verify *e.g.* that the choice of k-vector is covariantly conserved. Also verify k·V = 1.

In[9]:=
```
k = zeroTensor[1];
k[[v]] = 1;
k
V = zeroTensor[1];
V[[u]] = -1;
(*lower index on k*)
kl = lower[k]
covariant[kl]
kl.V
```

Out[11]= {0, 1, 0}

Out[14]= {-1, 0, 0}

Out[15]= {{0, 0, 0}, {0, 0, 0}, {0, 0, 0}}

Out[16]= 1

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