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Department of Physics

M.Sc. Quantum Fields and Fundamental Forces

Leggett-Garg Inequalities: A Test of Realism

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Abstract

This master's thesis delves into the study of Leggett-Garg inequalities and exploring its associated assumptions of macroscopic realism and non-invasive measurability that come along with it. The study aims to provide a comprehensive analysis of the macroscopic quantum behaviour and the role of quantum correlations in various physical systems. Quantum mechanics has long challenged our classical understanding of reality, and the Leggett-Garg inequalities offer a unique perspective to explore the boundary between classical and quantum behaviours.

The thesis involves a systematic exploration of Leggett-Garg inequalities in diverse scenarios, ranging from individual quantum systems to macroscopic ensembles. The study explores the violation of these inequalities as a signature of quantum coherence and correlations, shedding light on the intricate interplay between quantum states and their measurements. Both analytical and computational techniques are employed to contextualize the significance of Leggett-Garg inequalities in the broader landscape of quantum foundations.

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Chapter 1

Introduction

From times immemorial, people have always been curious towards the quest for 'truth' and over the time the word truth has also interchangeably been used with 'reality'. Physics itself has its origins in the Greek word 'phusis' which means Nature. Even today the question still remains unanswered as to what can be classified as real. This also acts as a part of the fields dealing with consciousness. However within the context of Physics, with the advent of quantum mechanics this question picked up even more complexities. Given the lack of an observer in the formalism of quantum mechanics it no longer leaves room for a deterministic answer. An encounter with quantum theory had opened various doors of counter-intuitive phenomenon. Note that counter-intuitive is substantial to the transition from a classical world-view to a quantum-mechanical one. Moreover, the existence of physical phenomenon that cannot yet be explained through the formalism of quantum mechanics namely Gravity has only pushed the yearning to testify a true description of reality.

Therefore, it has since then become a quest to come up with certain set of restrictions or conditions that helps us test the notion of realism with regards to our understanding of nature involving physical phenomena. In chapter 2 we start to discuss one of the first important contributions from Bell [1]. We see how local-realism and causality form the pillars of Bells' arguments. We describe the key elements involved with such arguments such as that of probability distributions for the simplest setup i.e. EPRB. We then move to discuss an important result by Fine [2, 3] that adds to the general picture. Finally we discuss the Bell inequalities and their vital implications with regards to the local hidden variable theories.

In chapter 3, we discuss the Leggett-Garg formalism. Although Leggett and Garg did not intend to propose the idea [4] as a basis of formalism yet considering today's context and some of the most recent developments it is treated as yet another framework. We see how the LG picture differentiates itself from that of Bells' inequalities and tries to testify for a converse argument namely Macroscopic Realism. We then walk through the derivation of the Leggett-Garg inequalities. Reviewing the applicability of Fine's theorem in this framework, we move on to discuss the relevance of this framework and try to refine our notion of Macrorealism and try to gauge the scope of it's macroscopicity.

Moving to chapter 4 we discuss extensively how one applies the Leggett-Garg framework within the formalism of quantum mechanics. We look that the maximum possible violation that one can get with regards to the inequalities. Moving along we emphasize on one of the important postulates of this formalism which is the Non-Invasive Measurability. Describing various experimental violations for the inequalities we move ahead to describe a rather special regime [5] where the Leggett-Garg framework reveals some discrepancies. This is realized by considering examples of discrete systems. Towards the end we also view a stronger set of conditions which when combined with the original picture attests to a much stronger notion of macrorealism.

In chapter 5, we then move on describe some of the very recent developments of the Leggett-Garg inequalities in the context of continuous variable systems. We present the formalism for the famous Quantum Harmonic Oscillator [6] and try to build an understanding in terms of the violations that arise from the superpositions of the ground- and first excited- states. In a similar context we further discuss more with respect to the coherent states [7]. Consequently we shift our discussion to present the wide range of applicability of the Leggett-Garg formalism such as to test the quantumness of gravity [8, 9]. Lastly, we briefly present the Tsirelson inequalities that constitute yet another framework which again seems to build on the very lacking of the Leggett-Garg picture, namely upon the Non-Invasive Measurability criterion.

Finally, we present a summary of our discussion and discuss some plausible lines of future exploration.

Chapter 2

Notions of Realism

We start with a general description of Quantum Mechanical realism in section 2.1. We proceed to present the Bell inequalities by shedding light on its formalism in section2.2. Discussing Fine's theorem and the Tsirelson bound in section 2.3 we then present the key inferences in section 2.4.

2.1 Local Realism

Based on the EPR paper [10], Bell [1] was the first one to actually put to test the notion of realism in the context of quantum mechanics.

The motivation for Bell's inequalities comes from an era where we had various models (interpretations) of quantum mechanics coming up. As for in all the models, it only started to become more and more apparent that quantum mechanics is inherently a probabilistic theory as opposed to our pre-conceived notions of classical mechanics. These pre-conceived notions can be stated as '*causality*' and '*locality*'. Causality refers to the notion that the past affects the future and not vice-versa. In the context of quantum mechanics where we deal with operators, generally represented as matrices, the previously made measurements i.e. the expectation value of the quantum mechanical operator affects any of its subsequent set of measurements. On the other hand locality refers to the notion of the interplay of causality on the observables only as a virtue of objects' immediate surroundings. This is the fact that doesn't allow the infamous 'spooky action at a distance' Thus Bell challenged this view by putting forward a proposition by envisaging a setup of systems being separated spatially. We would like to emphasize on the *spacelike separation* of the systems in the context of Bell's proposition as this gives us a key insight

into the motivations for other forms if inequalities.
This came to be the first formal instance of the pre-existence of the results of the measurements we choose to perform on the quantum mechanical system at hand. This has been stated in the context of the macroscopic world in various forms. One famous example can be stated by taking into consideration a pair of socks (literally anything imbibing chirality). Now supposing one sets the stage by separating the left and the right handed pieces and a measurement is made of one of the black boxes (nothing to do with colour, black is just used to specify the lack of knowledge of the result of the measurement before making it), instantly one would be able to determine the result of the other without actually making a measurement on the other system. This is however not to be confused with the speed at which information could travel. This spooky action is what we refer to as the objects (in the macroscopic) and states (in the quantum mechanical context) being present as an entangled state. Although the nature of entanglement remains a mystery yet there is an interesting deep-dive in an attempt to describe the 'speed' of entanglement [11] for the interested reader. We now move on to describe in the originally proposed context of quantum mechanics as proposed by Bell, precisely in the case of spin-1/2 systems. Note that the subsequent section forms the basis of Leggett-Garg inequalities.

2.2 Bell's Formalism

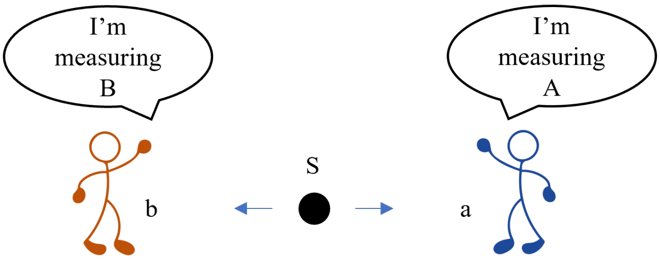

Fig. 2.1: An oversimplified demonstration of the EPRB experimental setup with the source S creating an entangled pair of particles

The Bell's setup equivalently also known as the "EPR-Bohm-Bell" setup is prescribed involves two observers equipped with making the choice of the observable they wish to measure on their system. The system is generally assumed to be prepared by a common source to employ the very fact that the results of the experiment already exist before making the measurement. Mathematically we can then assign a new variable λ to the set of all possible values of the system unless the measurement has been made. Consequently, in a quantum mechanical formalism there exists a probability distribution $\rho = \rho(\lambda)$ such that it obeys

$$\int_{\Lambda} d\lambda \rho(\lambda) = 1 \tag{2.1}$$

i.e. a normalised probability distribution where Λ denotes the full range of values that λ can take. Let suppose the first observer chooses to measure an observable A and the other chooses to measure B and they get the results as a and b respectively. Then in case we assume the measurements to be completely independent of each other, the joint probability of the setup is given by

$$p(ab|AB) = p(a|A)\ p(b|B) \tag{2.2}$$

where, $p(a|A)$ denotes the probability of getting the outcome 'a' for the observable 'A'. However, Bell showed that even if we stick to the *hidden variable formalism* of quantum mechanics and make use of the normalised probability distribution as a function of the hidden variable, the equality turns out to be a fallacy. Thus taking into account the hidden variable formalism we may compute the expectation value of the resulting experiment as

$$\int_{\Lambda} d\lambda\ \rho(\lambda)\ p(ab|AB) \tag{2.3}$$

$$\int_{\Lambda} d\lambda\ \rho(\lambda)\ p(a|A, \lambda)\ p(b|B, \lambda) \tag{2.4}$$

keeping note of the fact that we have now included the dependence of the hidden variable (λ in our case) explicitly on the choice of the observable. Therefore, given that the locality assumption is true we would expect the expectation value for the setup described above, to be governed by (2.4). Now consider the case for a dichotomic variable i.e. restricting the sample space for the observed values of 'a' and 'b' to ± 1 and for further simplicity we also restrict the measurement choices for the observers A and B to two values. The expectation value is then given as,

$$\langle AB \rangle = \sum_{a,b} abp(ab|AB) \tag{2.5}$$

$$= p(+1, +1|AB) - p(-1, +1|AB) - p(+1, -1|AB) + p(-1, -1|AB) \tag{2.6}$$

Thus, imbibing the locality assumption, the following expression

$$S = \langle A_1 B_1 \rangle + \langle A_1 B_2 \rangle + \langle A_2 B_1 \rangle - \langle A_2 B_2 \rangle \tag{2.7}$$

yields the following inequality,

$$S \leq 2 \tag{2.8}$$

This particular inequality forms a special case of Bell inequalities and was first demonstrated by [12]. Utilizing the above inequality we may easily resort the quantum mechanical context as illustrated by Bell in the case for spin-1/2 systems. Now since the source is assumed to initially prepare a pair of entangled particles, we can measure the spin component of the particles along a specific direction like so, $\vec{\sigma} \cdot \vec{A}$ where $\vec{\sigma}$ denotes the Pauli vector and $\vec{A}$ denotes the unit vector along the direction we wish to measure the spin of the particle. Inferring from the principle of entanglement (complete anti-correlation between the measured values of the spin in this context) we now expect the quantum mechanical expectation value i.e. $\langle \vec{\sigma} \cdot \vec{A}\, \vec{\sigma} \cdot \vec{B} \rangle = -\vec{a} \cdot \vec{b}$. Following the illustration by [13], explicitly assigning the choices of the measurement $(\vec{A}, \vec{B})$ as,

- $\vec{A} \in \{\ A_1 \equiv \hat{x},\ A_2 \equiv \hat{y}\}$
- $\vec{B} \in \{\ B_1 \equiv \frac{1}{\sqrt{2}}(\hat{x} + \hat{y}),\ B_2 \equiv \frac{1}{\sqrt{2}}(-\hat{x} + \hat{y})\}$

thus,

$$\langle A_1 B_1 \rangle = \langle A_1 B_2 \rangle = \langle A_2 B_1 \rangle = 1/\sqrt{2} \tag{2.9}$$

$$\langle A_2 B_2 \rangle = -1/\sqrt{2} \tag{2.10}$$

plugging the above in (2.7) we see that $S = 2\sqrt{2} \geq 2$ thus implying the violation of the locality assumption. Therefore in a nutshell Bell inequalities plays a pivotal role in ruling out any possible hidden elements of reality in quantum mechanics.

2.3 Fine's Theorem and Tsirelson Bound

Arthur Fine showed the equivalence between the existence of an underlying joint probability distribution for the case of Bell/CHSH inequalities [2] to match a specific set of marginals. He showed how the Bell/CHSH constitute the necessary and sufficient conditions for local realism. The explicit proof has been constructed via [3]. An argument is presented separately for the cases of a *deterministic hidden variable model* and that of a *stochastic* one. The difference just involves associating a probability distribution to the possible values that can be obtained within the experiment in the stochastic case as compared to dealing with specific states for the system under consideration. Recall that Eq.(**??**) corresponds to the stochastic case. Fine argued that in the case for a deterministic type of hidden model, it is necessary and sufficient for triple probabilities distributions of the form $P(A, B, B')$, $P(A', B, B')$ to exist whose marginals yield the same value for the distribution associated with the non-commuting pair of observables B, B' i.e. $P(BB')$. This is illustrated by an explicit construction of the distribution $P(A, A', B, B')$ by setting

$$P(A, A', B, B') = \frac{P(A, B, B')\ P(A', B, B')}{P(BB')} \tag{2.11}$$

following which a Bell-type inequality is derived and vice-versa. Subsequently, he then moves on to show that the existence of a deterministic model guarantees that an associated stochastic model is factorizable. Note that factorizability encapsulates (2.2). Thus in conclusion, the violation of Bell/CHSH inequalities implies a direct violation of the local hidden variable models in general.

Having been convinced that Bell/CHSH inequalities indeed form the set of necessary and sufficient conditions for local-realism, one of the natural endeavours is to be able to compute the extent of violation of such inequalities. Thus in the context of quantum mechanics, [14] showed that the CHSH inequality (2.8) is bounded above by a value of $2\sqrt{2}$. Later on [15] illustrated a higher bound by incorporating the no-signaling principle [1] As a further refinement, [16] realized a slightly lesser value for this bound which has also been attested repeatedly by various experimental trials [17–20]. We now demonstrate briefly the derivation for the Tsirelson bound in the quantum mechanical context. We proceed following the footsteps of Tsirelson [14] wherein recasting the expressions involving squares of the observables (which in the quantum mechanical context are seen as hermitian

[1] we would come back to the no-signaling principle in detail in the later chapters.

operators) then the inequality $S = A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2$ can

$$\begin{aligned}\frac{1}{\sqrt{2}}\left(A_1^2 + A_2^2 + B_1^2 + B_2^2\right) - \frac{\sqrt{2}-1}{8}&\Big((\sqrt{2}+1)(A_1 - B_1) + A_2 - B_2\Big)^2\\ - \frac{\sqrt{2}-1}{8}&\Big((\sqrt{2}+1)(A_1 - B_2) - A_2 - B_1\Big)^2\\ - \frac{\sqrt{2}-1}{8}&\Big((\sqrt{2}+1)(A_2 - B_1) + A_1 + B_2\Big)^2 -\\ - \frac{\sqrt{2}-1}{8}&\Big((\sqrt{2}+1)(A_2 + B_2) - A_1 - B_1\Big)^2\end{aligned} \tag{2.12}$$

Recall we had set the stage for Bell/CHSH type inequalities by allowing the observables to only take values between ± 1. Thus yielding the following bound in the context of quantum mechanics,

$$S \leq \frac{1}{\sqrt{2}}\left(A_1^2 + A_2^2 + B_1^2 + B_2^2\right) \leq 2\sqrt{2} \cdot I \tag{2.13}$$

where I denotes the identity matrix. It is thus illustrated in literature that such spatially separated quantum mechanical systems always impose the above bound on the corresponding correlators.

2.4 Why should we care about Bell inequalities?

To this end, we understand how Bell gave a profound description of testing the local hidden variables interpretation of quantum mechanics. Thus the violation of Bell inequalities clearly ruled out the hidden variables argument. These hidden elements of reality are what we refer to as local-realism. One of the first inferences from the Bell's description is that for an intrinsically probabilistic theory such as that of quantum mechanics, realism would manifest itself in form of a joint probability distribution that should be able to describe the outcome of a particular set of measurements before actually carrying out the experiment. Furthermore, the actual rigour of preparing a Bell's setup and testing the described inequalities brings with it various loopholes. The first one is the Detection loophole. This broadly refers to the incompatibility of the measuring apparatus. To be precise, even in the cases where we consider a dichotomic entity such as the spin state of the particles there might arise situations where the measuring device doesn't detect anything thus classifying such a trial as null. Thus in a case where the detector efficiency is significantly low, one can reinstate the hidden-variable formalism. In fact [21, 22] explore this possibility by explicit construction utilizing the inefficiency of the measurement devices. The second vital lacking in Bell's formalism is what is often coined as the locality loophole. This refers to the possibility of any communication between the two spacelike separated sites, which if violated again renders the whole system fairly useless. One then needs to ensure that the measurement sites are sufficiently separated in space such that the measurement time is shorter than the time taken by any signal to reach from one site to another. Note that this also formed one of the most vital assumptions in Bell's formalism which can precisely be stated as,

$$p(a|A)_{B=b} = p(a|A)_{B=b'} \tag{2.14}$$

$$p(b|B)_{A=a} = p(b|B)_{A=a'} \tag{2.15}$$

The locality loophole has been also dealt with great rigour employing various mechanisms such as the use of time-varying analyzers while conducting experiments with an entangled pair of photons. The idea being, to be able to manipulate the settings of the measurement during the time of flight of the particle. Various experiments such as [23–27] also lay the foundations of settling the debate of quantum mechanics being an inherently probabilistic. As a matter or fact these contributions are exactly the ones for which the *Nobel Prize in Physics has been awarded in 2022*.

We can summarize our discussion so far as being very much motivated to consistently stitch our notion of reality which explicitly avoids any spooky action at a distance together with the seemingly counter-intuitive theory of quantum mechanics. We described, what is often coined as local realism in conventional literature. To this end, we would like to take exactly an opposite route and now assuming that we are fully convinced of the dynamics of the microscopic world one naturally would wish to examine the classical world with the lens of quantum mechanics. The next of the chapter now takes a deep dive into what is known as the "Macroscopic realism".

Chapter 3

Leggett-Garg Inequalities

We start by giving a description of Macroscopic Realism in section 3.1. We derive the Legget-Garg Inequalities in section 3.2. Extending Fine's theorem in section 3.3 we finally look at the main conclusion from this formalism in section 3.4.

3.1 Macroscopic Realism

This concept is introduced as a formal description of the classical world. We do carry a notion of the macroscopic world in terms of it yielding definite values for a particular experiment. However, we still lack a formal way of distinguishing between the large scale phenomenon from that of the smaller ones. Leggett and Garg [4, 28] thereby attempt to provide a formal categorization of the classical world based on our intuitive understanding of nature. In the light of the same, Macroscopic Realism or Macrorealism (MR) refers to a set of properties that any system that is described via classical laws will satisfy. Let us motivate this with the help of a small thought experiment. Let us consider the case of a coin. It's consideration immediately implies the existence of more than one macroscopically distinct states i.e. Heads or Tails. We also always observe it to be in either of the states at a given time. Moving on we can argue that our observations with regards to the coin does not act as an invasive procedure to measure its state since the coin doesn't behave in a funny manner by changing its state once it has been measured. Also, any of our future measurements remain completely unaffected by the preceding ones and vice-versa. Therefore, we can formalize this description of classicality that also is in sync with our usual intuitions of the macroscopic world as a logical conjunction of Macrorealism per se, Non-Invasive Measurability and Induction where

- Macrorealism per se (MRps): A classical system that exhibits more than two macroscopically distinct states always exists in one of such states.
- NIM: The technique to determine the state of the system without, in principle, perturbing the dynamics of the system.
- Induction (Ind): The future measurements do not affect the previously made measurements.

we can see this as a codification of the behaviour of classical objects i.e.

$$MR = MRps \wedge NIM \wedge Ind \tag{3.1}$$

Thus we find the need to devise a qualitative test for such a notion of classicality which brings us to the proposal put forth by Leggett and Garg namely the Leggett-Garg Inequalities (LGI).
LGI(s) act as a probe to test this notion of classicality. Note that these differ from the Bell's formalism by means of their experimental setup. In Bell's case the spatial separation between the systems being observed in a way restricts our ways of dealing with the problem in a pragmatic way. These inequalities therefore improves upon this criterion by introducing measurements being made at different time-stamps. This can also be taken to be a time-like separation between the sub-systems. For this reason LGI(s) are often referred to as the temporal version of Bell/CHSH inequalities. This is realized in the following way. Considering a dichotomic variable $Q \in \{\pm 1\}$ we can compute a similar quantity S as in (2.7). Although, the measurements would now correspond to the measured values of Q over two distinct time stamps. Denoting the measurements at time stamp t_i the corresponding value is then $Q(t_i)$. The expectation value between the time stamps t_i and t_j is then expressed as $\langle Q(t_i)Q(t_j)\rangle$. For the case involving four time-stamps we get the following quantity

$$S = \langle Q(t_1)Q(t_2)\rangle + \langle Q(t_2)Q(t_3)\rangle + \langle Q(t_3)Q(t_4)\rangle - \langle Q(t_1)Q(t_4)\rangle \tag{3.2}$$

However there is a subtlety when dealing with NIM in the context of quantum mechanics. The problem arises due to the collapse of the wavefunction upon making a measurement on the system which clearly questions the ways one would expect to compute the expectation values that appear in the expression for S. Let us now look at a scenario that provides an insight to carry out such a measurement.

Suppose we begin with an ensemble of students who wish to reach Imperial College South Kensington campus via walking or taking a bus from Clapham Junction (we implicitly assume that this particular ensemble won't be using any other means of transport). Now given that a student can opt either Bus-49 (B49) or Bus-345(B345) we now wish to determine the route taken by the student without any means of interaction with them. Let suppose there are chocolates at the reception for the students who arrived via bus. Thus, in such a scenario, even though we won't have interacted with any of the students (in the quantum mechanical context we would not have perturbed the walk-able route) we would be sure from the number of chocolates left of the number of students who preferred walking upto the campus. The aforementioned scenario can then be classified as an ideal negative result experiment. This notion of ideal negative-result experiments allows us to infer information without direct interaction. Note that we could repeat a similar experiment on the subsequent days by only interacting with a subset of the system thus allowing for us to compute the results for at least one of the paths. A similar strategy can then be applied to determine the values for the expectation values in the quantum mechanical context for the dichotomic operators. Thus, for the two distinct values assumed by the operator an ideal-negative result for one implies the existence for the other and vice-versa.
We now turn towards an explicit derivation for LGI(s) and look at its associated technicalities.

3.2 Deriving the Leggett-Garg Inequalities

Let us first look at an important feature that is central within the LG framework. The idea that whether a system exhibits MR relies on the question of being able to find an underlying joint probability distribution involving various parameters of the system. It is therefore natural to infer that such an exploration is highly dependent on the experimental data.
In principle one could deal with various types of data-sets however for most of the explorations done in the context of LGI(s) people majorly build upon *standard data-sets*. These denote the single-time averages $\langle Q(t_i)\rangle = \langle Q_i\rangle$ adjoined with $n-$cycle of second order correlators denoted by $C_{ij} = \langle Q_i Q_j\rangle$. As an example, the case involving measurements at four-times constitute the correlators given by C_{12}, C_{23}, C_{34} and C_{14}. Note that one may specify the values for the missing correlators C_{13} and C_{24} but doing so falls out of the class

of standard data-sets. In the Fig.(3.1) we provide an illustration for the case of $n-$times. Although major discussions with respect to LGI(s) are done via standard data-sets yet

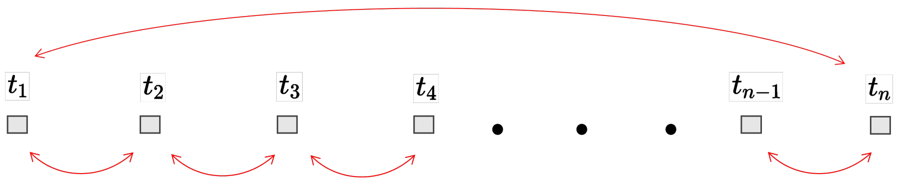

Fig. 3.1: The n-cycle corresponding to the standard data set, for the case of $n-$time measurements.

there are other types of data-sets as well upon which extensive explorations have been carried out. We will mention these 'other' data-sets in the later part of this section. It is also worth mentioning that we proceed with an explicit derivation for LGI(s) in a general (classical) context for the purpose of this chapter which is to say that the dichotomic variables are not yet treated as operators as in the quantum mechanical context.

Let us now describe LGI(s). Note that MR indicates towards the existence of an underlying joint probability distribution. This leads us to express the underlying probabilities as a function of the single-time averages and second order correlators of the dichotomic variables. Considering a dichotomic variable Q that is measured at two time-stamps t_1, t_2 yielding the values s_1, s_2 respectively, allows us to express the expected joint probability of outcomes $p(s_1, s_2)$ as

$$p(s_1, s_2) = 1 + s_1\langle Q_1\rangle + s_2\langle Q_2\rangle + s_1 s_2\langle C_{12}\rangle \geq 0 \tag{3.3}$$

We emphasize on the word 'expected' to highlight the fact that we are only making use of MRps to state the above expression. However following from our previous arguments in order to completely testify MR we also need to invoke the NIM postulate. This then allows for us to non-invasively measure the single- and two- time averages which renders (3.3) as a two-time LGI which is also often denoted as LG2. Notice that the non-invasive measurements for the involved parameters refer to the process of determining them in separate experimental trials. Similarly, let us look at the case for three-times t_1, t_2 and t_3. Clearly, this now includes the correlators specified from the standard data-sets which are generally expressed as

$$L : 1 + s_1 s_2 C_{12} + s_2 s_3 C_{23} + s_1 s_3 C_{13} \geq 0 \tag{3.4}$$

expanding on the different possible values for $s_i = \pm 1$ we get the following set of inequalities

$$1 + C_{12} + C_{23} + C_{13} \geq 0 \tag{3.5}$$

$$1 - C_{12} - C_{23} + C_{13} \geq 0 \tag{3.6}$$

$$1 + C_{12} - C_{23} - C_{13} \geq 0 \tag{3.7}$$

$$1 - C_{12} + C_{23} - C_{13} \geq 0 \tag{3.8}$$

Following a similar notation we generally express the three-time LGI(s) by LG3(s). Let us now look at two plausible ways to derive LG3(s). The first method comes from considering the following expression

$$(s_1 Q_1 + s_2 Q_2 + s_3 Q_3)^2 \geq 1 \tag{3.9}$$

At first glance it might seem that we make use of the positive definiteness of the squared expression however such is not the case. The bound is simply obtained as a result of the possible values the variables could take namely ± 1. We outline an explicit proof for the same in Appendix A. There is also an alternate and more elaborate way of deriving the LG3(s). The second method starts with an assumption of MRps only. Note that, invoking MRps only tells us that there must exist an underlying probability distribution for the possible outcomes without any the measurements being actually performed on the system. In this case we could express the correlator arising just from MRps as C_{ij}^{MRps} which can then be represented as an explicit sum over all the possible value for the three time i.e.

$$\begin{aligned} C_{12}^{MRps} =& p^{MRps}(-,-,-) + p^{MRps}(-,-,+) - p^{MRps}(-,+,-) - p^{MRps}(-,+,+) \\ & - p^{MRps}(+,-,-) - p^{MRps}(+,-,+) + p^{MRps}(+,+,-) + p^{MRps}(+,+,+) \end{aligned} \tag{3.10}$$

similarly we may also express the other two correlators. Now following the completeness relation of probabilities i.e.

$$\sum_{s_1, s_2, s_3} p^{MRps}(s_1, s_2, s_3) = 1 \tag{3.11}$$

thus just from imposing the MRps criterion we find the following inequality

$$\begin{aligned} L^{MRps} &= 1 + C_{12}^{MRps} + C_{23}^{MRps} + C_{13}^{MRps} \\ &= 4[p^{MRps}(+,+,+) + p^{MRps}(-,-,-)] \geq 0 \end{aligned} \tag{3.12}$$

However in order for us to derive the LG3 we must now invoke the NIM criterion which then allows for the underlying probability distribution actually obtained from the measurements to match with that obtained from just MRps i.e. $p^{MRps}(s_1, s_2, s_3) = p(s_1, s_2, s_3)$ and thus yielding the corresponding LG3(s). Note that the negativity of (3.4) implies a violation of the associated LGI(s).

Furthermore we may extend the form of such inequalities involving higher order correlators such as $D_{123} = \langle Q_1 Q_2 Q_3 \rangle$ (third order correlator) or $E_{1234} = \langle Q_1 Q_2 Q_3 Q_4 \rangle$ (fourth order). Thus, we can define the order of an LGI as a characteristic of the correlator involved in the moment expansion. Subsequently, we can express the underlying joint probability distribution for a set of n such dichotomic variables $Q_1, Q_2, Q_3, \ldots, Q_n$ with their respective outcomes being $s_1, s_2, s_3, \ldots, s_n$ as

$$p(s_1, s_2, \ldots, s_n) = \frac{1}{2^n} \left\langle \prod_{i=1}^{n} (1 + s_i Q_i) \right\rangle \geq 0 \tag{3.13}$$

Consequently, a third-order LGI is expressed as

$$p(s_i, s_j, s_k) = \frac{1}{8} \left(1 + \sum_i s_i \langle Q_i \rangle + \sum_{i<j} s_i s_j C_{ij} + s_i s_j s_k D_{ijk} \right) \geq 0 \tag{3.14}$$

Now that we understand the construction of the joint probabilities and thus the LGI(s) we are most easily drawn towards an attempt to list the necessary and sufficient conditions for MR given a particular system and data-sets. This brings us to the next section where we discuss an extension of Fine's theorem in the temporal context.

3.3 Fine's Theorem in the LG Context

Given the experimental setup of EPRB there wouldn't be an issue of matching the underlying joint distribution to those of the associated marginals as the black boxes are spatially separated as well as the measurements if made inherently rules out any communication within the involved systems. However, we run into a problem as soon as we wish to ex-

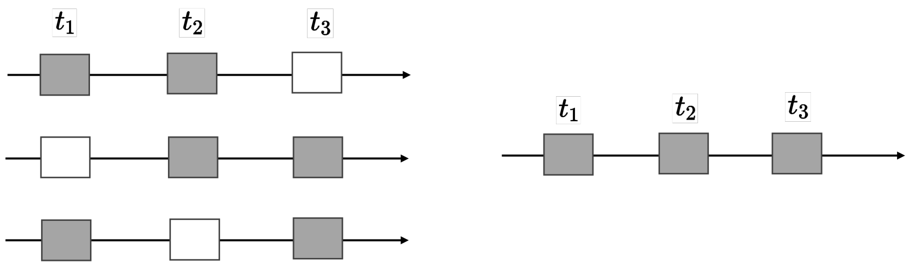

Fig. 3.2: Piece-wise measurements on the left. Sequential measurements on the right.

tend a similar argument for the case of LGI(s). This is because even in the cases of ideal negative measurements inherently what an experimentalist seeks out, are the temporal correlators. As a requirement for Fine's theorem to hold the involved two time marginals should always take up a non-negative value which in the cases for LGI(s) would not always be guaranteed. Therefore, nothing stops the underlying joint distribution to not yield the marginals upfront. Moreover, in the LG case the measurements can be classified being conducted in a piecewise manner or in a sequential manner as depicted in the figure. Therefore, as an immediate consequence we see that the LG3(s) as described by (3.5) - (3.8) only form a necessary set of conditions for MR. In order to extend Fine's theorem to the LG case we would then require additional constraints that ensure that the involved marginals as well remain positive. Thus for LG3(s) the full set of necessary and sufficient conditions constitute the above four along with the 12 LG2(s) i.e. four set of inequalities for each second order correlator C_{12}, C_{23} and C_{13}.

Now the form for the second order correlator we would similarly have,

$$C_{ij} = \sum_{s_1 s_2 s_3} s_i s_j p\left(s_1, s_2, s_3\right) = \langle Q_i Q_j \rangle \tag{3.15}$$

$$D = \sum_{s_1 s_2 s_3} s_1 s_2 s_3 p\left(s_1, s_2, s_3\right) = \langle Q_1 Q_2 Q_3 \rangle \tag{3.16}$$

Now under the standard data set assumption, we would have B_i(s) and C_{ij}(s) fixed. Thus the only way we can ensure a non-negative value for $P(s_1, s_2, s_3)$ is when

$$\mathcal{P}\left(s_1, s_2, s_3\right) \equiv 1 + \sum_i B_i s_i + \sum_{i<j} C_{ij} s_i s_j \geq -D s_1 s_2 s_3 \tag{3.17}$$

Thus yielding four maximum values for D_{ijk} for the case when $s_1 s_2 s_3 = -1$ and similarly four minimum possible values for the case $s_1 s_2 s_3 = 1$ i.e.

$$\mathcal{P}\left(s_1, s_2, s_3\right) \geq D_{ijk} \geq -\mathcal{P}\left(s_1, s_2, s_3\right) \tag{3.18}$$

A subtle part that remains to complete our argument follows from the definition of the third order correlator itself. Again, following the similar line of argument as put forth by [29], it follows from $P(s_i, s_j) \geq 0$ that

$$P(s_1, s_2) + P(s_2, s_3) + P(s_1, s_3) \geq 0 \tag{3.19}$$

substituting in form of the corresponding correlators we get,

$$\begin{aligned} (1 + s_1 B_1 + s_2 B_2 + s_1 s_2 C_{12}) + (1 + s_2 B_2 + s_3 B_3 + s_2 s_3 C_{23}) \\ + (1 + s_1 B_1 + s_1 B_3 + s_1 s_3 C_{13}) \geq 0 \end{aligned} \tag{3.20}$$

$$\Rightarrow 3 + 2 \sum_i B_i s_i + \sum_{i<j} C_{ij} s_i s_j \geq 0 \tag{3.21}$$

Furthermore, the above set of necessary conditions for LG3(s) can be expressed as

$$1 + \sum_{i<j} C_{ij} s_i s_j \geq 0 \tag{3.22}$$

summing up (3.21) and (3.22) we obtain

$$2 + \sum_i B_i s_i + \sum_{i<j} C_{ij} s_i s_j \geq 0 \tag{3.23}$$

$$\Rightarrow \mathcal{P}(s_1, s_2, s_3) \geq -1 \tag{3.24}$$

The above discussion thus concludes a successful extension of Fine's theorem for LG3(s) thus giving rise to the set of 16 necessary and sufficient conditions for MR.

More generally speaking, we can now extend Fine's theorem to LG4 case. Once again based on our choice of measurements i.e. working in sync with the standard data set in the LG4 case, the correlators C_{13} and C_{24} act as free parameters. In order to derive the necessary and sufficient conditions in this case one now utilizes Fine's ansatz for the four spin case. This is where one resorts to a clever choice to recast (2.11). One of the motivations would be reduce the matching problem to obtain the probability distributions corresponding to the free parameters. Subsequently we may choose to proceed with any one of the following choices

$$P(s_1, s_2, s_3, s_4) = \frac{P(s_1, s_2, s_4)P(s_2, s_3, s_4)}{P(s_2, s_4)} \tag{3.25}$$

$$P(s_1, s_2, s_3, s_4) = \frac{P(s_1, s_2, s_3)P(s_1, s_3, s_4)}{P(s_1, s_3)} \tag{3.26}$$

Picking up (3.25) we see that $P(s_2, s_3, s_4)$ will be non-negative as long as the three 2-time marginals $P(s_2, s_3)$, $P(s_3, s_4)$ and $P(s_2, s_4)$ are non-negative as well as the associated set of four LG3(s) hold. In a concise manner this set of LG3(s) can be expressed as

$$-1 + |C_{23} + C_{34}| \leq C_{24} \leq 1 - |C_{23} - C_{34}| \tag{3.27}$$

thus setting the bounds for C_{24}. A similar argument for $P(s_1, s_2, s_4)$ yields the another bound i.e.

$$-1 + |C_{12} + C_{14}| \leq C_{24} \leq 1 - |C_{12} - C_{14}| \tag{3.28}$$

Further as $P(s_2, s_4)$ not only is a part of the denominator in (3.25) but also as a marginal for both the three-time probabilities, thus requires an additional constraint for it to be non-negative. This can be represented by the corresponding single-order correlators as

$$\left.\begin{aligned} 1 + B_2 + B_4 + C_{24} \geq 0 \\ 1 + B_2 - B_4 - C_{24} \geq 0 \\ 1 - B_2 + B_4 - C_{24} \geq 0 \\ 1 - B_2 - B_4 + C_{24} \geq 0 \end{aligned}\right\} \Rightarrow -1 + |B_2 + B_4| \leq C_{24} \leq 1 + |B_2 - B_4| \tag{3.29}$$

Thus, together with (3.27) and (3.28) gives the range for the value of C_{24} such that the fixed correlators are so bounded,

$$|C_{12} \mp C_{14}| + |C_{23} \pm C_{34}| \leq 2 \tag{3.30}$$

yielding the set of 8 inequalities. The final piece of the puzzle would be to then check for the compatibility of the bounds with those governed by (3.29). Here we can make use of the non-negativity of the fixed correlators thus enabling us to construct the following inequalities

$$P(s_2, s_3) + P(-s_3, s_4) \geq 0 \tag{3.31}$$
$$P(-s_1, s_2) + P(s_1, s_4) \geq 0 \tag{3.32}$$

where an explicit expansion in in terms of the correlators give

$$2 + s_2B_2 + s_4B_4 \geq -s_2s_3C_{23} + s_3s_4C_{34} \tag{3.33}$$
$$2 + s_2B_2 + s_4B_4 \geq -s_1s_4C_{14} + s_1s_2C_{12} \tag{3.34}$$

Thus making this set of 8 LGI(s) the necessary and sufficient conditions for four times. One can subsequently proceed with a generalization of Fine's ansatz upto n-time LGI(s). Where essentially for higher 'n' we could break the problem down to ensure the non-negativity of the subsequent $(n-1)-$, $(n-2)-$, ... , 2-time probabilities. [29] put forth a set of 2^{n-1} set of LGI(s) as the necessary conditions. They then turn towards proving that these indeed form the set of sufficient conditions for n-times. Note that all of our discussion has been under the assumption of working with the standard dataset. However one is not obliged to proceed with the standard dataset and thus may include various variations of the datasets. For instance, [30] illustrates the case considering all possible two-time correlators.

3.4 Violations of LGI(s): Conclusion?

In our discussion so far we have presented the basic framework for the LGI(s). Note that although we refrained to work within the quantum mechanical context yet in the hindsight the measurements intuitively refer to have been performed mostly on the microscopic systems that enables us to gather a deeper understanding of quantum mechanics. To this end we turn towards the analysis by [31] in the context of MR. Notice that unlike the Bell/CHSH inequalities that ruled out the hidden variable argument we lack a conclusive understanding when it comes to LG tests. Even at the time Maroney and Timpson presented their analysis various experimental groups had reported violations of the LGI inequalities for instance [32–34] and many more yet it isn't precisely clear as to what do the violations indicate. An argument is highlighted taking specific examples that forms the basis to challenge the conventional view of MR as was originally proposed by Leggettand

Garg. For instance, the De-Broglie-Bohm interpretation of quantum mechanics [35–38] allows for the existence of a superposition of states however as it falls in the category of deterministic hidden variable models thus each measurement yields a definite value of the state. In the context of this formalism these definite values would be determined by the Born's approximation rule. Thus one is able to uniquely determine the results of the measurement given that the initial phase-space configuration, the initial wavefunction for the whole system and the interacting Hamiltonian are specified. Now since the theory coincides with the dynamics of quantum mechanics hence we would indeed get a violation of the associated LGI(s) even though it still classifies as being a MR theory in itself.

Thus in case of de-Broglie-Bohm interpretation we inherently get a violation as it empirically resonates with other interpretations of quantum mechanics which imply a violation of MR but since it satisfies MRps it still accounts for a perfectly valid macrorealistic theory, hence the contradiction. One other instance comes from a brief account of the Everett interpretation [39–41]. This formalism pushes the boundaries further and questions the essence of MR itself. There still exists a wave-function associated to the whole system however it refrains from imposing a super-selection rule to pick out a definite state. It rather encourages the various possibilities of the different states being as *real* as any other state. This leads one to make an explicit distinction when discussing of macroscopic realism as opposed to the discussion with regards to the realism about the macroscopic.

The reason for the lack of any sort of conclusion in some part also lies with the NIM assumption. It is clear from the nature of quantum mechanics that in principle it is impossible to conduct an experiment without involving a wave-function collapse. It is for this reason a distinction is proposed by categorising the the process of making a measurement. They classify these as being *Operational non-disturbance* (OPND) and *Ontic non-invasiveness* (ONI). OPND can be thought of as being more of a statistical notion rather than it being strictly non-invasive. A kind of measurement that only fiddles with the microstates of the system but has no effect on the overall macroscopic variable. Under such an assumption we would consequently require that the transition probabilities among different microstates remain unaffected by the act of measurement. Whereas in the case for ONI one needs to ensure that the measurement process does not change the microstates of the system i.e. for a given hidden variable associated with the system (equivalently the predefined distribution of weights) remains unaltered. In a summary one clearly sees that ONI guarantees OPND but not the converse.

This leads to an even broader classification of the MR within the context of LGI(s).

Maroney and Timpson therefore propose the following classification of MR,

- Operational Eigenstate Mixture MR
- Operational Eigenstate Support MR
- Supra Eigenstate Support MR

In an essence MR should then be thought of as the existence of a set of value-definite ontonic states corresponding to a macroscopically observable quantity Q. Thus the violation of LGI(s) only rules out a subclass of MR theories namely the operational eigenstate mixture MR which only requires one to prepare a particular statistical mixture of states via the ontonic variable beforehand and thus leaving the measurement process to unveil a proper distribution of the probabilities. As for the scope of this article we refrain from diving into the other sub-classes of MR theories having extracted the notion of MR originally in-tune with that put forth by Leggett and Garg.

Chapter 4

Leggett-Garg Tests in Quantum Mechanics

We start with the derivation of the correlators in the quantum context in section 4.1. Discussing maximum violations for the inequalities in section 4.2 we move ahead to describe the experimental limitations in section 4.3. We then discuss some actual experiments that have reported violations of LG tests and thus Mr in section 4.4. We then look at higher order violations of LGI(s) in section 4.5. Finally we look at a stronger classification of MR in section 4.6.

4.1 Quantum Correlators: Derivation

In order for us to start dealing with specific quantum systems we first look at the formalism of correlators. It turns out that the correlators take up different forms in the spatial and temporal contexts. Once again starting the EPRB setup we try to look at the derivation. Let us recall the EPRB setup we introduced in the beginning. Since we are only interested in the underlying joint probability for measurements that correspond to different time stamps, we briefly go over the spatial correlators for a pair of spins and illustrate how the formalism is extended for the case of temporal correlators as shown by [42]. A majority of the work within an operator formalism is dealt in terms of the projection operators associated to the dichotomic operators[1]. Thus, for a two-level system such as a pair of spins if we consider the observables (operators in the quantum context) Q_1 and Q_2 with

[1]Notational point: we avoid the use of '^' to denote operators for simplicity. Nevertheless it is important to keep this distinction in mind as opposed to the classical scenario.

their corresponding outputs as $s_1, s_2 \in \{+1, -1\}$ then the projective operators are given as

$$\mathcal{P}_{s_1}(Q_1) = \frac{1}{2}(1 + s_1 Q_1) \tag{4.1}$$

$$\mathcal{P}_{s_2}(Q_2) = \frac{1}{2}(1 + s_2 Q_2) \tag{4.2}$$

Subsequently, for a particular quantum state $\psi|$ we may express the underlying joint probability as

$$P(s_1, s_2) = \frac{1}{8}\langle\psi|(1 + s_1 Q_1)\cdot(1 + s_2 Q_2)\cdot(1 + s_1 Q_1)|\psi\rangle \tag{4.3}$$

Following which the spatial correlators are expressed as the trace of the corresponding density matrix ρ

$$C_{Q_1 Q_2} = tr[\rho(Q_1 \otimes Q_2)] \tag{4.4}$$

At this point, invoking Fritz theorem that guarantees the existence of two-times, temporal correlators if and only if there exists an analogous version of theirs in the spatial context to get

$$C_{Q_1 Q_2} = tr\left(\rho \cdot \frac{1}{2}\{Q_1, Q_2\}\right) \tag{4.5}$$

. Therefore, we can extend the formalism to the LG framework where the measurement corresponding to the i^{th} time-stamp for a dichotomic operator $Q(t_i) = Q_i$ with their corresponding outputs being represented with s_i to derive the form for higher order correlators. Here we illustrate the case for third order and fourth order correlator. For the third order correlator we get

$$D_{ijk} = \sum_{s_i, s_j, s_k} s_i s_j s_k P(s_i, s_j, s_k) \tag{4.6}$$

$$= \sum_{s_i, s_j, s_k} s_i s_j s_k tr[\mathcal{P}_{s_k}(t_k)\mathcal{P}_{s_j}(t_j)\mathcal{P}_{s_i}(t_i)\rho\mathcal{P}_{s_i}(t_i)\mathcal{P}_{s_j}(t_j)] \tag{4.7}$$

$$= \frac{1}{4} tr\left[Q_k Q_j Q_i \rho + Q_k Q_j \rho Q_i + Q_k Q_i \rho Q_j + Q_k \rho Q_i Q_j\right] \tag{4.8}$$

$$= \frac{1}{4}\langle\{Q_i, \{Q_j, Q_k\}\}\rangle \tag{4.9}$$

where going from (4.7) to (4.8) we made used the following results

$$\sum_{s_i} s_i \mathcal{P}_{s_i}(t_i)\, \rho \mathcal{P}_{s_i}(t_i) = \frac{1}{2}\left(Q_i \rho + \rho Q_i\right), \tag{4.10}$$

$$\sum_{s_i} s_i \mathcal{P}_{s_i}(t_i)\, \rho = Q_i \rho \tag{4.11}$$

Similarly we see that the fourth order correlator takes the following form,

$$E_{ijkl} = \frac{1}{8}\langle\{Q_i, \{Q_j, \{Q_k, Q_l\}\}\}\rangle \tag{4.12}$$

4.2 Luders Bound

Now that we have derived the form for the correlators in the quantum mechanical context, we wish to look at the maximum possible violation that one could get in such a framework. This maximum violation is what is referred to as the Luders' bound. We now turn towards the deriving the same. Recall (3.9) which we used to derive the LG3(s). However in the quantum mechanical context since we deal with expectation values with the variables being promoted to operators which yields

$$\left\langle (s_1\hat{Q}_1 + s_2\hat{Q}_2 + s_3\hat{Q}_3)^2 \right\rangle \geq 0 \tag{4.13}$$

which immediately yields the following inequality,

$$1 + s_1 s_2 C_{12} + s_2 s_3 C_{23} - s_1 s_3 C_{13} \geq -\frac{1}{2} \tag{4.14}$$

Thus the maximum possible violation for a quantum mechanical system for LG3(s) is $-\frac{1}{2}$. This is what we refer to as the Luders bound. We can also view this from a different perspective by considering the case for LG2(s). Note that we may express the LG2(s) (3.3) as

$$\frac{1}{2}\langle (1 + s_1\hat{Q}_1 + s_2\hat{Q}_2)^2 - 1\rangle \tag{4.15}$$

which clearly reduces to the following eigen-value equation when the maximum violation is achieved,

$$(s_1\hat{Q}_1 + s_2\hat{Q}_2)|\psi\rangle = -|\psi\rangle \tag{4.16}$$

The violation of LGI(s) upto the Luders bound have also been confirmed as a result various experiments. Furthermore, it is observed that, this value for the violation only appears as a characteristic of two-level systems. For many-valued systems that the maximum value may take up different values. [43] illustrates this case for multi-level systems characterized by the spins of the particles, keeping von-Neumann method of making measurements at its base.
The Luders bound can be seen as a result of the possibility of two-time correlators taking an infinite number of values between 1 and -1 as opposed to the classical case. We would discuss the form for the correlators explicitly in the later part of this chapter. For now it suffices to note that for a two-level systems such as that of the spin-1/2 particles for a specific choice for the Hamiltonian is given in terms of $\cos(\omega(t_i - t_j))$ where $t_{i/j}$ denotes the time-stamps between which the measurements are made. Subsequently, for a clever choice for the parameters one gets the maximum value of the corresponding expression as $-\frac{1}{2}$. A recent development by [44] also explores these maximum violations as being characterized by the dynamics of the system, which is usually taken to be Markovian. They further proceed to show that even in the case of non-Markovian dynamics it is impossible to reach the maximum violation as a result of the corresponding Bloch vector shrinking to less than half of its value. Similarly, there have been other proposals to justify these violations such as the one proposed by [45]. There is also an extensive exploration of the Luders bound for systems involving PT symmetric evolution as one of its dynamical features where the maximal violation has been achieved via manipulations of the postulates of the LG framework [45].

4.3 Encountering Loopholes

Now that we understand two main features of the puzzle that is testing the notion of MR given a particular choice of data sets where we would mostly resort to standard data-sets and secondly being able to compute the correlators in the quantum mechanical context. Even though in principle we may be very well equipped to lay the foundations to test MR yet experimentally various issues sweep in whilst ensuring the credibility of the lab conditions. These issues are mostly referred to as the 'loopholes'. Most of the loopholes arise as a by product of making sure that a particular experimental setup abides by the NIM conditions. However given the formalism of quantum mechanics this leaves the experimentalist with the only choice of arguing upto a certain degree of precision adhering to the credibility of the LGI violations being obtained. As a part of this section we briefly go over some of the major loopholes that are often discussed in this regard.

The first one was argued by [46] which they coined as the *clumsiness loophole*. The key idea was to suggest that the violations of the LGI(s) can not only be achieved as a characteristic of the non-classicality (or quantumness) but also due to noise (inherent disturbances) within the experimental setup. Thus for a stubborn macrorealist the LGI violations implies in favour of either of the two scenarios, the system is (i) non-macrorealistic or it is (ii) macrorealistic but vulnerable to certain measurement techniques. To resolve the clumsiness loophole, they suggest an 'adroit' measurement. Consider the case for three-time measurements, where the first, second and third measurement Q_1, Q_2 and Q_3 respectively. If either of the three measurements does not affect the correlators of the remaining two then such a measurement would be considered as adroit upto a precision of $\epsilon \geq 0$. It is then argued with the help of a certain experimental setup how a series of adroit measurements yield a violation of the corresponding LGI. Consequently it contribute to making the second implication a bit more precise. Furthermore this helps to replace the NIM assumption by implying that a violation may indicate towards a macrorealistic model but with the property that two adroit measurements coalesce somehow to perturb the system rather vigorously. They illustrate the above in case of qubit system constituting the observables σ_z and $\sigma_\theta = \cos(\theta)\sigma_z + \sin(\theta)\sigma_x$ with the adroit measurements begin depicted as a series of measurements, starting from an initial state ρ. They further move on to generalize this setup to a series of $n-$adroit measurements to illustrate the sensitivity of the range for which a violation is obtained as a function of n.

Another loophole emerges as a result of the *inadequacy of the particle detectors*. Under the postulates for the LG framework, one further needs to ensure whether the sample distributions for different particle pairs obtained as a result of various experimental runs are in a complete statistical agreement with those predicted by the entire distribution of all possible pairs. In principle this would be the case if the detectors being utilized operate with a hundred percent efficiency. However, this is not the case when actually dealing with such experimental setups. This is a loophole that equivalently exists for Bell/CHSH type inequalities. In the light of the same, there have been various approaches to account for this inefficiency of the detectors however we shall briefly give an account of the major two. [47] argues that the minimum required efficiency of the detectors reduces to 83% as opposed to previously demonstrated efficiencies. The proceed by assigning the null measurement as well, a probability and thus describes the new set of correlators as a function of the efficiency η of the detectors and the original correlators i.e. preceding the null-assignment. Following on the lines of [48] and [49] they move on to derive the necessary and sufficient conditions corresponding with the post-assignment data set. Note that certain symmetries associated with the probabilities are imposed to derive the new set of inequalities accounting for the detectors' inefficiency. In contrast [50] illustrates the same result without requiring to assign probabilities to null-detection measurements. He argues via a 'change of ensemble' mechanism that in the case when repeating the experiment over the same ensemble yields a different pair of particles as opposed to the case when the Bell/CHSH inequalities are satisfied this would not have been the case. Thus the Bell/CHSH scenario under the assumption of an ensemble change still uphold the notions of local realism, perfect anti-correlation and the dichotomy of the of the observed values. This enables them to classify the detector inefficiency as a subclass of measurement inefficiency as well as further introducing the coincidence inefficiency. A distinction is made on the basis of the choice of the bound.

The last of the major loopholes that we would like to mention is the *initial state preparation loophole*. This would simply correspond to having background interference as a characteristic of the 'source' in the experiment. For instance, when the experiment involves photons the measurement results could yield a discrepancy in case of a background source of light. Having discussed some of the key issues involved with actual measurement setups when testing for the notion of realism we now move on to describe a few examples of setups where various experimental groups have observed violations of LGI(s).

4.4 Experimental Violations of LGI(s)

Various experimental violations have been reported for LGI(s) involving super-conducting qubits [51], nuclear spins [52, 53], quantum dot qubits [54], Nitrogen atom vacancies [55] as well as other setups for photons [56–60].

To this end, let us turn towards describing some of the violations obtained for LGI(s). The motivation for almost all experiments is to put the framework to test on the systems which are known to behave quantum mechanically. Thus for systems as simple as involving electrons and photons indeed takes a step forward to discern MR as an underlying notion of reality. [61] expect a violation of LGI(s) in the phenomenon of quantum transport. They point out some of the emerging difficulties of the setup such as time-resolved measurements i.e. making sure that the time intervals between two successive measurements is less than the decoherence time of the system. In light of this and the clumsiness loophole they resort to the electronic Mach-Zehnder Interferometer (MZI). They proceed in the following two ways with MZI. The first one being, transporting the electrons in two spatially distinct ways thus making it one of the most natural ways to incorporate NIM. Then for the position label of the detector α and for the cases of detection $(n = 1)$ or failure $(n = 0)$, the probability obtained from the detectors is denoted by $P^D_{\alpha\pm}(n)$. Subsequently one computes the correlators by placing in the detectors at different positions in the setup as shown in Fig.4.1,

$$C_{31} = P^D_{3+}(1) - P^D_{3-}(1) \tag{4.17}$$

$$C_{21} = P^D_{2+}(1) - P^D_{2-}(1) \tag{4.18}$$

$$C_{32} = -\sum_{q,q'=\pm} qq' P^D_{3q,2q'}(1,\cdot) \tag{4.19}$$

where $P^D_{3q,2q'}(1,\cdot)$ denotes the probability obtained from placing the detectors at positions $2(+or-)$ and $3(+or-)$ for the case when the electron is detected at the later point. Following which the constructed LGI leads yields a violations. Note that the authors also report achieving the Luders bound as well by proceeding with calculations in terms of the current operators.

The second one involving Quantum Dots (QD) that are sensitive to the charge of the electron and thus act as quantum point contacts. In the second technique however, the electrons are never diverted out of MZI and the detectors only influence the system as a dephasing effect. They then compute the probabilities associated with the illustrated

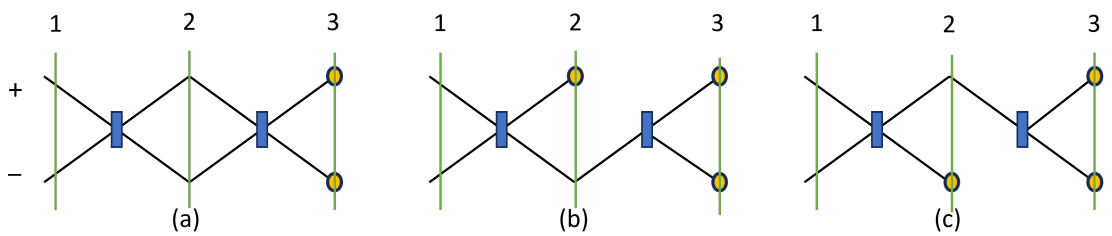

Fig. 4.1: (a) Denoting the basic MZI setup with the detectors only placed at the instant t_3 (b) Placing an additional detector on the $+$ arm (c) Placing an additional detector on the $-$ arm

sequential measurements and subsequently obtain the correlators C_{13} from (a) and C_{12}, C_{23} by the combining the probabilities obtained from (b) and (c). Note however, they assume the detectors to be hundred percent efficient which automatically leaves room for the detector inefficiency.

As another example, consider the violations obtained by [62]. They utilise a microwave pulse as an analogy of a coin toss to yield a randomly assigned state for a Cesium atom. The two position labelled states are distinguished by nature of its polarization. The experiment is carried out at time-stamps with a suitable choice of labelling $t_1 = 0, t_2 = 1 and t_3 = 4$. This is followed by a further choice for the dichotomic variable Q separately for three times. For $t_1 = 0$ the variable is assigned as $Q(t_1) = 1$. For $Q(t_2)$ the state of the atom is measured with with regards to the underneath labelling.

$$(\uparrow, x = -1); \quad (\downarrow, x = +1) \tag{4.20}$$

Finally for the $Q(t_3)$ a measurement is made with regards to the atom's position such that it returns the value -1 when $x \leq 0$ and $+1$ when $x > 0$. Notice that this forms an example for preparation of the state for which the authors ignore the time-evolution which only invokes NIM whilst measuring $Q(t_2)$. Thus in such an assignment the correlators are obtained as

$$C_{12} = \langle Q(t_1)Q(t_2)\rangle = 1 \tag{4.21}$$
$$C_{13} = \langle Q(t_1)Q(t_3)\rangle = \langle Q(t_3)\rangle \tag{4.22}$$
$$C_{23} = \sum_{x=\pm 1} P\left(t_2; x\right)\left\langle Q\left(t_3\right)\right\rangle_x \tag{4.23}$$

where $P(t_2; x)$ denotes the probability of finding the atom in x at t_2 and $\langle Q(t_3)\rangle$ denotes the expectation value obtained with regards to a negative result experiment at t_2. Thus substituting the obtained values in the LGI, a violation of the inequalities is reported.

They proceed further to utilize these violations as a measure of the degree of quantumness of the system. The implication being a contradiction between the classically expected trajectories to those actually being observed. Even though issues such as the possibility of unexpected excitation of the position states and measurement inefficiency are justified, the setup remains vulnerable towards the clumsiness loophole, which they humbly attest.

However, recent experiments conducted by [63] stands out as one of the best argued violations of the inequalities. Taking into account the clumsiness, measurement inefficiency loopholes alongside keeping track of multi-particle emission, preparation state and coincidence inadequacies they were able to test for LG violations dealing with one photon at a time. Consequently given our discussion so far, one might classify this as a profound test of MR as well. New developments with regards to testing the non-classicality of certain phenomenon such as the Casimir effect are also picking up interest leading to insights on the problem of quantum gravity [64].

4.5 MR Violations: Higher order LGI(s)

Let us explore a different regime characterized by LGI(s). From our discussion until now we realize that the violation of LGI(s) indicate a violation of MR. However, we also find violations of MR when a certain class of LGI(s) are satisfied [5]. These violations then arise as a result of transitioning from lower- to higher- order LGI(s).
Data-sets take the centre stage as part of this exploration. Treating the standard data-sets as the starting point we aim is to find suitable parameter choices that lead to violations for higher order inequalities. Keeping this thought in mind we extend our understanding as illustrated by Fig.4.2. This is implemented by employing the course graining procedure.

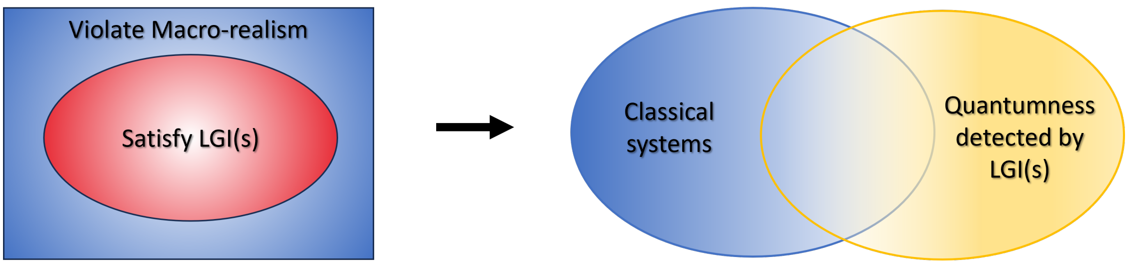

Fig. 4.2: Illustrating the transition from our original notion with regards to the violations of LGI(s).

The key idea is to argue that the standard data set satisfies the set of lower-times LGI(s) but yield violations of certain higher order LGI(s). We emphasize again the choice of data-sets play a vital role whilst exploring this regime. It might as well be the case that a given data set is in sync with the classical notion for the system where as a different choice of the involved parameters might then highlight non-classicality that is inherent to the system. Let us begin by considering a spin-1/2 system with the dichotomic operator $Q = \vec{c} \cdot \vec{\sigma}$ where $\vec{c}$ is a unit vector and $\vec{\sigma}$ is the Pauli vector. Further starting from the initial state (in form of the density matrix formalism) as

$$\rho = \frac{1}{2}(1 + \vec{v} \cdot \vec{\sigma}) \tag{4.24}$$

such that $\vec{v} \cdot \vec{v} \leq 1$ for pure states. Furthermore to include the notion of making a measurement for a particular time-stamp we follow the same notation for $\vec{c}$ as

$$Q_i = e^{iHt}\, Q e^{-iHt} \equiv \vec{c}_i \cdot \vec{\sigma} \tag{4.25}$$

As a first step to explore higher order LGI(s) we note that (4.5), (4.6) and (4.12) yield the form for the second, third, fourth order correlators. Then for the corresponding initial state described by the density matrix we get

$$\langle Q_i \rangle = \vec{c}_i \cdot \vec{v} \tag{4.26}$$

$$C_{ij} = \vec{c}_i \cdot \vec{c}_j \tag{4.27}$$

$$D_{ijk} = (\vec{c}_j \cdot \vec{c}_k)\,(\vec{c}_i \cdot \vec{v}) \tag{4.28}$$

$$E_{ijkl} = (\vec{c}_i \cdot \vec{c}_j)\,(\vec{c}_k \cdot \vec{c}_l) \tag{4.29}$$

To derive the above expressions we make repetitive use of the properties of the Pauli matrices i.e. $(\vec{a} \cdot \vec{\sigma})(\vec{b} \cdot \vec{\sigma}) = (\vec{a} \cdot \vec{b})1 + i(\vec{a} \times \vec{b}) \cdot \vec{\sigma}$. Now consider the case for LG2(s) for three times which is

$$1 + s_i\vec{c}_i \cdot \vec{v} + s_j\vec{c}_j \cdot \vec{v} + s_i s_j \vec{c}_i \cdot \vec{c}_j \geq 0 \tag{4.30}$$

$$(\vec{v} + s_i\vec{c}_i + s_j\vec{c}_j)^2 \geq 1 \tag{4.31}$$

similarly for three-times the LG3(s) take the following form

$$1 + s_1 s_2 \vec{c}_1 \cdot \vec{c}_2 + s_2 s_3 \vec{c}_2 \cdot \vec{c}_3 + s_1 s_3 \vec{c}_1 \cdot \vec{c}_3 \geq 0 \tag{4.32}$$

$$(s_1\vec{c}_1 + s_2\vec{c}_2 + s_3\vec{c}_3)^2 \geq 1 \tag{4.33}$$

Furthermore, the third-order LGI(s) give the following

$$\begin{split}\frac{1}{8}(1 + s_1\vec{c}_1 \cdot \vec{v} + s_2\vec{c}_2 \cdot \vec{v} + s_3\vec{c}_3 \cdot \vec{v} + s_1 s_2 \vec{c}_1 \cdot \vec{c}_2 + s_1 s_3 \vec{c}_1 \cdot \vec{c}_3 + s_2 s_3 \vec{c}_2 \cdot \vec{c}_3 \\ + s_1 s_2 s_3\,(\vec{c}_1 \cdot \vec{v})\,(\vec{c}_2 \cdot \vec{c}_3)) \geq 0\end{split} \tag{4.34}$$

Now since we are interested to see if we can fit the given data set abiding by a classical distribution or not let us look at the following choices for the parameters involved in the above equations. Note that there could be various choices that exhibit the non-classical behaviour, therefore here we make a different choice (following the same procedure as shown in the paper) for the involved parameters and subsequently obtain the same result. We make the following set of choices,

$$\vec{c}_1 = -\vec{c}_3 \tag{4.35}$$

$$\vec{v} = -\vec{c}_2 \tag{4.36}$$

which reduce two of the four distinct LG3(s) to $\vec{c}_2 \cdot \vec{c}_2 \geq 1$ and the other two to $1 \pm \vec{c}_2 \cdot \vec{c}_3 \geq 1$. Moreover, we also see that the LG2(s) for the pair (t_1, t_3) are trivially satisfied. Therefore, this particular choice of the parameters satisfy the corresponding LG2(s) and LG3(s). However, a further extension to the third-order LGI for the case $s_1 = s_2 = s_3 = 1$ yields

$$-1 + (\vec{c}_3 \cdot \vec{v})^2 \geq 0 \tag{4.37}$$

but since we start with the restriction on $\vec{v}$, this in turn gives a violation of the third-order LGI. Further along, the third-order moment expansion as $p(s_1, s_2, s_3)$

$$p(s_1, s_2, s_3) = \frac{1}{8}[1 - s_2 - s_1 s_3 + [s_3(1 - s_2) - s_1(1 - s_2)](c_3 \cdot v) + s_1 s_2 s_3 D_{123}] \tag{4.38}$$

At this point we break the argument in two parts. Consider the first case where we utilize the freedom over D_{ijk} that we gain from the lack of it being specified in the standard data set. In this case we already see that the LG2(s) and LG3(s) are satisfied and the only way $p(s_1, s_2, s_3)$ takes a non-negative value is when $D_{ijk} = 1$. Thus we see that $p(s_1, s_2, s_3) = 0$ for $s_2 = \pm 1$ and $s_1 = s_3$, whereas for the case when $s_2 = -1$ and $s_3 = -s_1$, we get

$$1 + s_3\,(c_3 \cdot v) \geq 0 \tag{4.39}$$

The second case in which we presume that the third-order correlator is fixed (i.e. already specified within the provided data set) then $D_{ijk} = (c_3 \cdot v)^2$. Now, the correlator can also assume values other than 1 which immediately falls in the regime where LG2(s) and LG3(s) are specified but the third order probability isn't. Thus implying that a finely and appropriately grained system might lead to violations of higher order LGI(s) whilst satisfying some of the lower ones.

Similar violations are achievable for a much relevant Hamiltonian, such as $H = \omega\sigma_x/2$ and choosing the dichotomic operator to be along the conventional z-direction i.e. σ_z. Then depending on the spin system under consideration, one makes an appropriate choice (keeping track of hermiticity and dichotomy of eigenvalues) for the dichotomic operator as discussed in section 3.1 previously. Generalized expressions for the second, third and fourth order correlators derived in [5] are

$$\langle Q_i \rangle = v_z \cos(\omega t_i) + v_y \sin(\omega t_i) \tag{4.40}$$
$$C_{ij} = \cos[\omega(t_j - t_i)] \tag{4.41}$$
$$D_{ijk} = \cos[\omega(t_k - t_j)][v_z \cos(\omega t_i) + v_y \sin(\omega t_i)] \tag{4.42}$$
$$E_{ijkl} = \cos[\omega(t_j - t_i)] \cos[\omega(t_l - t_k)] \tag{4.43}$$

wherein for the following choice for the parameters

$$\langle Q_1 \rangle = \langle Q_2 \rangle = 0 \tag{4.44}$$
$$\langle Q_3 \rangle = \frac{1}{2} \sin(\omega t_3) \tag{4.45}$$
$$C_{12} = -1 \tag{4.46}$$
$$C_{13} = C_{23} = \cos(\omega t_3) \tag{4.47}$$
$$D_{123} = 0 \tag{4.48}$$

again it is found that the corresponding LG2(s) and LG3(s) are satisfied as opposed to the third-order LGI. Violations for certain two level and three level systems have also been realized in previous experiments [65, 66].

The reason for the interest in such class of theories comes from the motivation to list the necessary and sufficient conditions for MR. Had it been the case that there did not exist a data set for which we obtain violations for certain microscopic systems apart from the set of inequalities that an extension of Fine's theorem suggests, there would not have been the question of such exploration. In such a case we would automatically rest our faith in the diagram on the left. However, it is revealed that such is not the case. [5] extend their discussion to another data set namely the one that involves specifying all the single-time averages and also specifying all the possible second-order correlators upto the number of time-stamps the measurement is inclusive of. This is expressed as

$$n + 2\sum_{i>j}^{n} s_i s_j C_{ij} \geq \begin{cases} 1 & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases} \tag{4.49}$$

The above type of data sets were first proposed by [30] a generalized form of the inequalities is realized in [29]. For the specific case of $n = 5$ the above inequality is coined as the pentagon inequalities (PI). Therefore, a subsequent generalization of the same procedure allows us to tease out such a regime for higher order LGI(s) as well. One may also extend a similar discussion to systems exhibiting more than two states (or values) and therefore generalize the case for dichotomic variables. Such systems are often called many-valued systems involving many-valued variables.

Let us now look at the conditions for the case involving many-valued variables. Note that the notion of a system assuming more than one value comes from the possibilities exhibited as a result of the measurements being carried out. For instance a simple spin-1/2 system is a two valued system. Since the formalism for dichotomic variables is well studies in conventional literature, it only makes sense to construct the many valued case utilizing those. One of the proposed extensions come from [67] where a string of N projection operators

$$Q = \sum_{n=1}^{N} \epsilon(n)\mathcal{P}_n \tag{4.50}$$

is generalized to construct a many valued variable where $\sum_{n=1}^{N} \mathcal{P}_n = 1$ and $\epsilon(n)$ takes the values ± 1. Even though one is free to choose an arbitrary assignment of ϵ yet it suffices to have at least one of the assignment as $+1$ with the rest of all being -1. One might consider further generalizations by carrying out an explicit analysis for the three- or four-time cases however, as a starting point it is best to build the case for two-times for varying N. In such a scenario we wish to determine the necessary and sufficient set of inequalities for MR. Thus the general candidate for the associated probability is

$$p(n_1, n_2) = \langle \mathcal{P}_2(n_2)\mathcal{P}_1(n_1) \rangle \tag{4.51}$$

whose non-negativity implies

$$1 + \langle Q_1(n_1) \rangle + \langle Q_2(n_2) \rangle + \langle Q_1(n_1)Q_2(n_2) \rangle \geq 0 \tag{4.52}$$

which turns out to be analogous to LG2(s) under the consideration of the relation between the two operators. This also tells us that for the two-time case it is only necessary to perform the measurements on $N - 1$ of the dichotomic variables. Therefore, for $N = 2$ case we find that it suffices for the two dichotomic variables to satisfy the relation

$Q(2) = -Q(1)$ which is result of a rather general expression $\sum_{n=1}^{N} Q(n) = (2-N)I$, where I denotes the identity matrix. For the case $N = 3$, consider the dichotomic operators Q, R and S with the corresponding projective operators being $\mathcal{P}_Q, \mathcal{P}_R$ and $\mathcal{P}_S$. Following the identity $\mathcal{P}_Q + \mathcal{P}_R + \mathcal{P}_S = 1$ satisfied by the projection operators we get

$$\frac{1}{2}(1+Q) + \frac{1}{2}(1+R) + \frac{1}{2}(1+S) = 1 \tag{4.53}$$

$$1 + Q + R + S = 0 \tag{4.54}$$

which yield a full set of nine inequalities out of which the underneath five are derived as a result of the constraint relation between the dichotomic variables.

$$\langle Q_1Q_2\rangle + \langle Q_1R_2\rangle + \langle R_1Q_2\rangle + \langle R_1R_2\rangle \geq 0 \tag{4.55}$$

$$\langle Q_1\rangle + \langle R_1\rangle + \langle Q_1Q_2\rangle + \langle R_1Q_2\rangle \leq 0 \tag{4.56}$$

$$\langle Q_1\rangle + \langle R_1\rangle + \langle Q_1R_2\rangle + \langle R_1R_2\rangle \leq 0 \tag{4.57}$$

$$\langle Q_2\rangle + \langle R_2\rangle + \langle Q_1Q_2\rangle + \langle Q_1R_2\rangle \leq 0 \tag{4.58}$$

$$\langle Q_2\rangle + \langle R_2\rangle + \langle R_1Q_2\rangle + \langle R_1R_2\rangle \leq 0 \tag{4.59}$$

Thus these form the necessary and sufficient conditions for MR for $N = 3$ as illustrated in the paper. One can keep introducing arbitrary number of variables as well as derive the set of inequalities for arbitrary times of measurement however we will just proceed to illustrate the case for the two-time inequalities for $N = 4$ case as most of the literature does not find interest in the set of conditions for more than four or five level experiments. Note also that increasing the number of variables in an experimental setup only contributes to an increase in the total number of measurements and therefore it only makes the setup clumsy. For sake of completeness, let us look at the $N = 4$ case. Consider the dichotomic operators Q, R, S, T with $\mathcal{P}_Q, \mathcal{P}_R, \mathcal{P}_S, \mathcal{P}_T$ as the projection operators respectively. Then from the property of projection operators we get

$$Q + R + S + T = -2 \tag{4.60}$$

We also note that at first glance for the two-time case we would have a full set of 16 inequalities which can be represented as the combinations of measurements made at t_1 and t_2 as depicted in Fig.4.3 However, in this case it suffices to make measurements for three of the variables such that the inequalities involving the remaining variable can be recast to

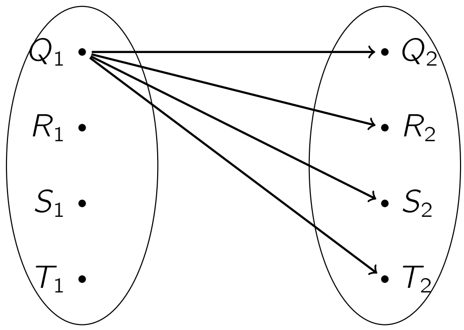

Fig. 4.3: This only represents four of the sixteen possibilities, the other twelve come from R_1, S_1, T_1 as well.

yield the bounds which allow for the scope of actually performing the measurements and testing the set of inequalities. This immediately results in the following nine inequalities involving Q, R and S

$$1 + \langle X_1 \rangle + \langle Y_2 \rangle + \langle X_1 Y_2 \rangle \geq 0 \tag{4.61}$$

where $X, Y \in \{Q, R, S\}$ along with a set of the following seven inequalities

$$\langle X_1 \rangle + \langle Y_1 \rangle + \langle X_1 Z_2 \rangle + \langle Y_1 Z_2 \rangle \leq 0 \tag{4.62}$$

$$\langle X_2 \rangle + \langle Y_2 \rangle + \langle Z_1 X_2 \rangle + \langle Z_1 Y_2 \rangle \leq 0 \tag{4.63}$$

$$\sum_{X,Y,X \neq Y} \langle X_1 Y_2 \rangle \leq 2 \tag{4.64}$$

where $X, Y, Z \in \{Q, R, S\}$ and $X \neq Y \neq Z$. Thus we have derived the necessary and sufficient conditions for MR involving four variables for two-times. Having worked out the above conditions, we now proceed to highlight the robustness of the test of MR the above formalism yields.

4.6 No-Signaling In Time

It turns out that the above probe only attests towards a weak notion of MR. Note that in all of our discussion until now we either try to match the underlying probability distribution or implicitly assume the tautology of the involved marginals and subsequently proceed to build the structure for higher-order LGI(s) or for the cases involving more than two projective measurements. Even though such a notion lies in a perfect sync with a macrorealisitic point of view however when dealing with actual measurements within the framework of quantum mechanics such might not be the case. Therefore, this boils down to the question of whether the equality of the marginals hold both-ways? The red arrow

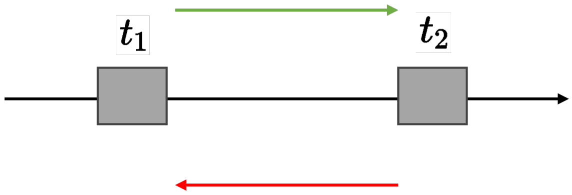

Fig. 4.4: The diagram illustrates the forward (green) and backward (red) signals corresponding to the measurements in two-time case

in Fig.4.4 denotes the possibility of a backward signal in which case the succeeding measurement affects the preceding measurement which in turn produces a violation of NIM. This forms the basis for the argument of No-Signaling in Time (NSIT). Note that the green arrow isn't part of the argument as it is trivially satisfied by *Induction*. In Quantum Mechanics the probabilities obtained as result of a specific measurement might not match with the underlying marginals thus NSIT conditions are then illustrated in the following way. Consider, we perform a projective projective measurement starting from an initial state ρ, then the probability at single time t is

$$p(s) = tr(\mathcal{P}_s(t)\rho) \tag{4.65}$$

where $\mathcal{P}_s(t) = e^{iHt}\mathcal{P}_s e^{-iHt}$, then as we saw earlier while deriving the form for correlators in the quantum mechanical context, the probability for a sequential projective measurement for t_1, t_2 is

$$p\left(s_1, s_2\right) = tr[\mathcal{P}_{s_2}\left(t_2\right)\mathcal{P}_{s_1}\left(t_1\right)\rho\mathcal{P}_{s_1}\left(t_1\right)] \tag{4.66}$$

It is clear from this expression that although summing over the final measurement s_2 does exactly yield the marginal associated with the single measurement at t_1 however the summing over s_1 i.e. the initial measurement gives

$$\sum_{s_1} p(s_1, s_2) = tr[\mathcal{P}_{s_2}(t_2)\rho_M(t_1)] \tag{4.67}$$

where ρ_M denotes the density operator of the outcome received after the measurement at t_1 which is not necessarily equal to the associated marginal for the time stamp t_2. In literature, one distinguishes notation of the measured probabilities by introducing subscripts such as $p_{12}(s_1, s_2)$ as opposed to the usual probabilities $p(s_1, s_2)$. The NSIT condition for two times then takes the following form,

$$\sum_{s_1} p_{12}(s_1, s_2) = p_2(s_2) \tag{4.68}$$

Here, $p_{12}(s_1, s_2)$ would then represent the probability obtained under a measurement at both times t_1 and t_2 and $p_2(s_2)$ denotes the probability of measurement at time t_2 only, with no measurements being performed at the preceding time stamps. The NSIT argument was first proposed by [68]. Presented in a much refined manner [69] illustrated how the NSIT conditions form a stronger set of conditions for MRps, and when combined with the LGI(s) for MR. This study introduced a quasi-probability approach to gain better clarity over NSIT and it's implications for MR. This approach utilises the redefinition of quasi-probabilities associated with the marginals that we get by summing over the individual measurements. Subsequently one makes use of the decoherence functional

$$D(s_1, s_2|s_1', s_2) = tr[\mathcal{P}_{s_2}(t_2)\mathcal{P}_{s_1}(t_1)\rho\mathcal{P}_{s_1'}(t_1)] \tag{4.69}$$

to track the interference terms that might arise in contrast to the expected like so

$$q(s_1, s_2) = p(s_1, s_2) + 2Re[D(s_1, s_2| - s_1, s_2)] \tag{4.70}$$

thus the NSIT condition comes as a result when the decoherence functional vanishes. Note that, one can also report an LG violation that comes as a result of the violation of the NSIT conditions rather than it emerging from the non-classicality of the considered system. Thus a rigorous treatment in terms of quasi-probabilities tells us that the NSIT conditions combined with the LGI(s) yield s stronger test for the MR. This analysis also helps us classify the originally proposed LGI(s) as being a weaker test for MR. The reason we delve into a sneak-peak of the NSIT conditions is motivated by the fact if we wish to extend the analysis to the case involving many-valued variables. Thus in the case for N variables (4.68) can be generalized to

$$\sum_{n_1=1}^{N} p_{12}(n_1, n_2) = p_2(n_2) \tag{4.71}$$

where we still deal with $N-1$ independent conditions. However as we previously noted, considering more than two variables also gives us the choice of determining any one of the variables first for which the NSIT condition is recasted as

$$\sum_{s_1} p_{12}^{Q}(s_1, n_2) = p_2(n_2) \tag{4.72}$$

therefore one can equivalently define the decoherence functional as in (4.69) incorporating the distinction between the choice of the variable whose measurement is being carried out by defining the following

$$D(n_1, n_2|n_1', n_2) = tr[\mathcal{P}_{n_2}(t_2)\mathcal{P}_{n_1}(t_1)\rho\mathcal{P}_{n_1'}(t_1)] \tag{4.73}$$

$$D(s_1, n_2|s_1', n_2) = tr[\mathcal{P}_{n_2}(t_2)\mathfrak{P}_{s_1}(t_1)\rho\mathfrak{P}_{s_1'}(t_1)] \tag{4.74}$$

where the projective operators $\mathfrak{P}_s$ corresponds to the dichotomic variable Q upon which the measurement has already been made. This can then be represented as $\mathfrak{P}_s = \sum_n c_{sn}\mathcal{P}_n$. Note that a double subscript is used within the coefficients where s denotes the possible outcome of the measurement and n denotes the dichotomic variable that is being taken into consideration. Consequently the coefficients either pick up a null value or returns the value 1. Thus the two decoherence functionals can now be related by summing over different histories of the measurement i.e. n_1, n_1' yielding

$$D\left(s_1, n_2 \mid s_1', n_2\right) = \sum_{n_1, n_1'} c_{s_1 n_1} c_{s_1' n_1'} D\left(n_1, n_2 \mid n_1', n_2\right) \tag{4.75}$$

This in turn helps us to compute $p_2(n_2)$ from both of the above expressions by summing over n_1, n_1' for (4.73) and over s_1, s_1' for (4.74) thus implying that in order for the NSIT conditions to hold we would require all the possible interference terms to vanish. Extending a similar analysis to the two-time case for $N = 3$ case we would now have to introduce further terms corresponding to the interference from each pair of dichotomic variables. Following the same procedure, we find six interference terms contributing towards the violation of NSIT conditions where each pairwise interference term yields two independent values for a specified history. However, the choice of first measurement among the three variables also pick up interference terms involving the decoherence functionals of the type (4.74). Note that at this point, the construction of the dichotomic variables also matters thus helping us explore the subtlety between the number of interference terms involved to the total number of independent choices for the dichotomic variables. Under a specific choice for the variables such as,

$$Q = \mathcal{P}_A - \overline{\mathcal{P}_A}; \quad R = \mathcal{P}_B - \overline{\mathcal{P}_B}; \quad S = \mathcal{P}_C - \overline{\mathcal{P}_C} \tag{4.76}$$

and combining the expected probability distribution from the generalization of the NSIT conditions with the three choices as previously stated, we get a full set of eight constraints (two for each variable) for NSIT to hold. In the above expression $\overline{\mathcal{P}_i} = 1 - \mathcal{P}_i$. However, since we only require to constrain the six interference terms, this allows us to constrain only six of the eight conditions to ensure that NSIT is satisfied. Thus leading to the case where some (not all) combinations of the interference terms do vanish but at the same time the underlying two-time LGI(s) are violated. Subsequent conditions for NSIT can be derived for arbitrary-times however in such a case, one is required to judiciously choose a specific form for the dichotomic variables such that the discrepancy between the total number of independent choices for the dichotomic variables and those arising from the involved interference terms can be teased out adequately. Thus, one can construct various plausible conditions corresponding to strong, intermediate and weak violations of MR.

Chapter 5

A step closer to macroscopicity

We look at the case for Quantum Harmonic Oscillator in section 5.1. Reviewing the case for Gaussian states of the harmonic oscillator in section 5.2. We then extend the LG formalism to a gravitational context in section 5.3. Finally we present an alternate formalism in light of recent developments in section 5.4

So far, we have discussed how the LG framework yields a set of inequalities to test the notion of MR inherent within the system into consideration. However, it is remarkable how many experimental groups have focused only on particular types of systems i.e. discrete systems such as those involving spins of particles, photons, electrons, neutrino oscillations, characteristics of certain atoms like Nitrogen vacancies and many more. Indeed they have been able to report suitable violations of the LGI(s). Only recently, there seems to be an increasing interest within the community to utilize the LG framework to set the stage [70] in order to detect traces of non-classicality within certain macroscopic systems such as the ones making use of the interference within C_{60} and C_{70} atoms [71,72] and phenomenons involving relatively large organic molecules [73]. Some of the recent experiments have also proposed to put the LGI(s) to test in the cases where it is possible to trap fairly massive particles via harmonic potentials [74]. This can be noticed as an advancement towards the notion of MR where in we witness subsequent satisfaction and (or) violations of LGI(s), arising from the continuous-valued variables embedded within such systems. Note that these systems have already found solace within the general framework of quantum mechanics, therefore quite amusingly it can also be seen as posing yet another layer of analysis of the well-understood dynamics of such systems but only to deepen our understanding of the relatively recent LG framework.

5.1 Quantum Harmonic Oscillator

We now take a deep dive into the particular case of the quantum harmonic oscillator (QHO). The reason such a system can be categorized within a continuous-valued variable is due to the freedom within the parameter space of such systems. An added motivation to deal with systems involving continuous variables comes from the fact that QHO forms one of the most well-understood quantum mechanical systems not only in terms of its dynamics but also in terms of vital role that forms the basis of the study of areas such as that of quantum field theory, string theory, super-symmetry etc. To this end, we now resort to [6] that illustrates a detailed analysis for this case.

We first note that a major part of the analysis within the QHO setting is done within the position space formalism. Although one can ideally work out analogous computations within the momentum representations however it would then constitute the difficulty of extracting physical implications obtained from satisfaction or violation of the LGI(s). Thus it remains convenient to work within the position representation itself. The very first step in our puzzle would be to suitably choose a specific form for the dichotomic variable Q. Such a proposal comes from [74] by defining $Q = sgn(x)$, where x denotes the position variable of the QHO. Note however this only only forms as one of the ways to probe such continuous valued-systems. There have been other approaches as well, which involve either introducing suitable couplings with other quantum systems [75, 76] or by an explicit construction of Gaussian states [77] and thus proceeding with the analysis. In contrast the procedure illustrated by Bose et. al enables one to carry out an independent analysis of the system itself taking into consideration the involved correlations. As one of the most natural choice for the states of QHO we restrict the discussion in terms of the energy eigenstates and their superpositions. Thus if H denotes[1] the Hamiltonian of the system then $|n\rangle$ represents the energy eigenstate with the corresponding eigenvalues E_n. Secondly, we now choose a specific representation for the associated projection operator as

$$\mathcal{P}_s = \frac{1}{2}(1 + sQ) = \theta(sx) \tag{5.1}$$

where $\theta(x)$ is the Heaviside step function i.e. $\theta(x) := \{1|x \geq 0; \quad 0|x < 0\}$. Owing to the time-independence within the observables and their corresponding projection operators it

[1] All the quantities that act as operators within the framework of quantum mechanics implicitly bear the $\hat{()}$ symbol which we choose to omit for notational convenience. We would make the distinction clear wherever deemed necessary.

turns out best to work within the Heisenberg picture. Then, following the usual notation of $s_i \equiv$ measurement outcomes associated with the corresponding projections we can express the two-time probability obtained from projective sequential measurements as

$$p\left(s_1, s_2\right) = \mathrm{Re}\,\mathrm{tr}\left[\mathcal{P}_{s_2}\left(t_2\right)\mathcal{P}_{s_1}\left(t_1\right)\rho\mathcal{P}_{s_1}\left(t_1\right)\right] \tag{5.2}$$

whose moment expansion would differ from that of the moment expansion of the two-time quasi-probability by a term involving the decoherence functional. Note also that the quasi-probability for this setup can be expressed as

$$q\left(s_1, s_2\right) = \mathrm{Re}\,\mathrm{tr}[\mathcal{P}_{s_2}\left(t_2\right)\mathcal{P}_{s_1}\left(t_1\right)\rho] \tag{5.3}$$

$$q\left(s_1, s_2\right) = \mathrm{Re}\,\mathrm{tr}\left[e^{iHt_2/\hbar}\theta\left(s_1 x\right)e^{-iH(t_2-t_1)/\hbar}\theta\left(s_2 x\right)e^{-iHt_1/\hbar}\rho\right] \tag{5.4}$$

since the evolution of the projective operators would involve the time differences between the initial and evolved state hence for further convenience we would indicate the $t_2 - t_1 = \tau$. From here, one proceeds to make use of the corresponding energy eigenstates to obtain an explicit form for the quasi-probability as shown in the Appendix B.
Subsequently, we may proceed to compute the corresponding single-time averages and the respective correlators. Notice that the single-time averages that appear in the moment expansion of the quasi-probability would vanish[2], like so

$$\langle Q\rangle = \langle n|\,\mathrm{sgn}(x)|n\rangle = \langle n| - n\rangle = 0 \tag{5.5}$$

thus yielding the quasi-probability as

$$q(s_1, s_2) = \frac{1}{4}\left(1 + s_1\langle Q_1\rangle + s_2\langle Q_2\rangle + s_1 s_2 C_{12}\right) \tag{5.6}$$

$$q(s_1, s_2) = \frac{1}{4}\left(1 + s_1 s_2 C_{12}\right) \tag{5.7}$$

From this point onwards we employ yet another notation so as to distinguish the correlators on the basis of the energy eigenstate with regards to which they may be computed. Thus for an energy eigenstate $|n\rangle$ the corresponding correlator is $C_{ij}^{|n\rangle}$.
Let us briefly recall that for QHO the Hamiltonian of the system is generally given as

$$H = \frac{p_o^2}{2m} + \frac{1}{2}m\omega^2 x_o^2 \tag{5.8}$$

[2]The $sgn()$ function essentially flips the parity.

where it seems much more convenient to deal with dimensionless versions of the physical position (x_o) and momentum (p_o) operators so that we won't have to care about dimensional matching in the end for the corresponding parameters. Thus we would work with the following quantities

$$x_o = x\sqrt{\frac{\hbar}{m\omega}} \quad ; p_o = p\sqrt{\frac{m\omega}{\hbar}}. \tag{5.9}$$

The energy eigenvalues corresponding to the eigenstate $|n\rangle$ of QHO are simply given by

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega = \epsilon_n \hbar\omega \tag{5.10}$$

Then the analytical solution for the wavefunctions satisfying the Schrodinger's equation is expressed in terms of the Hermite polynomials $H_n(x)$

$$\psi_n(x) \equiv \langle n \mid x\rangle = \frac{1}{\sqrt{2^n n!}}\pi^{-1/4}\exp\left(-\frac{1}{2}x^2\right)H_n(x) \tag{5.11}$$

We illustrate the procedure to compute the correlators and the single-time averages in the Appendix []. This involves two means to get the form for the correlators.

The key point of difference between the approaches discussed in the appendix can be summarized as follows. The first approach only allows one to compute an approximate form for the correlators within an estimation of the truncation error (usually denoted by $\Delta_n(m)$). Whereas utilizing the second approach, although involves lengthy and non-trivial calculations yet it yields an exact form for the correlator. For instance, computation for the specific case of $q(+,+)$ would involve the freedom to choose the values for the initial time stamp t_1 and number of terms upto which we wish to seek a truncation of the infinite sum. The most convenient choice goes as setting $t_1 = 0$ for the initial state. Subsequently, making use of an explicit calculation of the corresponding values for the partial overlap of the eigenstates J_{kl} we find the approximated expressions for the correlator of the first excited state $C_{12}^{|1\rangle}$ upto first three ($\Delta_1(2)$)- and first four ($\Delta_1(4)$)- non-zero terms is

$$C_{12}^{|1\rangle} = \frac{3}{\pi}\cos(\omega\tau) \approx \cos(\omega\tau); \quad \Delta_1(2) = 0.011 \tag{5.12}$$

$$C_{12}^{|1\rangle} \approx \frac{36}{37}\cos\omega\tau + \frac{1}{37}\cos 3\omega\tau; \quad \Delta_1(4) = 0.005 \tag{5.13}$$

Note that for the ground state one would require a mot more terms so as to achieve convergence and thus an argument is illustrated only for the first excited state using the first approach. In contrast, since it is feasible to compute an exact form for the correlators using the second approach. Thus yielding the following expressions for the ground- and first excited- states

$$C_{ij}^{|0\rangle} = \frac{2}{\pi} \operatorname{Re} \arctan \left(\frac{1}{f(\tau)} \right) \tag{5.14}$$

$$C_{ij}^{|1\rangle} = \frac{2}{\pi} \operatorname{Re} \left[\arctan \left(\frac{1}{f(\tau)} \right) + f(\tau) \right] \tag{5.15}$$

where $f(\tau) = -ie^{-i(\omega\tau/2)}\sqrt{2i \sin \omega\tau}$. Having obtained the expressions for the correlators we can now plug these in the expression for the quasi-probability to obtain for a specific state of the QHO. We illustrate the corresponding violations in Fig.5.1 Now, considering

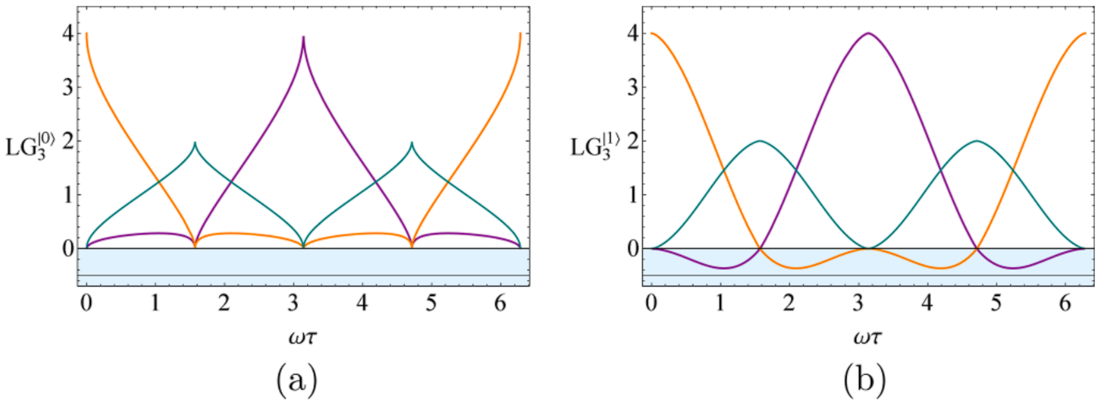

Fig. 5.1: (a) Plots for LG3(s) for the ground state of QHO, where we get only satisfactions (b) Plots for LG3(s) for the first excited state yielding violations for inequalities represented in orange and purple. This figure is taken from [6]

a general superposition of the ground- and the first excited- states of the form

$$|\psi\rangle = a|0\rangle + b|1\rangle \tag{5.16}$$

we find the following expression for the quasi-probability Eq.(5.1.10)

$$q(s_1, s_2) = 1 + s_1 \left(2\sqrt{\frac{2}{\pi}} \operatorname{Re} a^* b e^{i\omega t_1} \right) + s_2 \left(2\sqrt{\frac{2}{\pi}} \operatorname{Re} a^* b e^{i\omega t_2} \right) + s_1 s_2 \left(|a|^2 C_{ij}^{|0\rangle} + |b|^2 C_{ij}^{|1\rangle} \right) \tag{5.17}$$

upto a factor of 1/4. Thereafter separate analysis for the cases of pure and mixed states is carried out. For pure states we find that the LG2(s) are trivially satisfied arising as a consequence of Eq.(5.1.8). However, LG3(s) that is the set of inequalities of the form

$$L_1^{|n\rangle} = 1 + C_{12}^{|n\rangle} + C_{13}^{|n\rangle} + C_{23}^{|n\rangle} \tag{5.18}$$

for the ground state is always satisfied as opposed to the violations obtained for the case of the first excited state. In the case for superposition states, one relies on a judicious parametrization of the LG3(s). This is achieved most easily by expressing the LG3(s) superpositions where the coefficients are positive definite and add up to unity. Then a superposed LG3 can be expressed as

$$L_i(\theta) = \cos^2\frac{\theta}{2}L_i^{|0\rangle} + \sin^2\frac{\theta}{2}L_i^{|1\rangle} \tag{5.19}$$

Considering this, it is found that one seemingly enters a regime where the LG2(s) are violated and the corresponding LG3(s) are either satisfied or violated Fig.5.2. One may

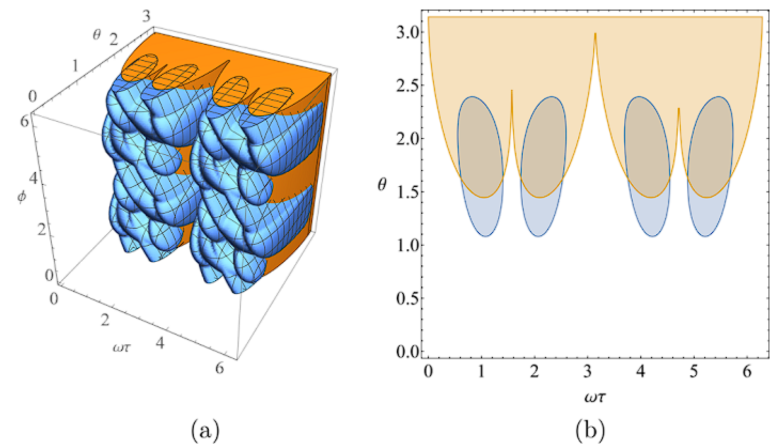

Fig. 5.2: (a) Represents the behaviour of LG2 and LG3 for all possible superpositions of the ground- and first excited- state. Here the orange non-meshed region denotes the violations of at least one of the LG3(s) and similarly the blue region for LG2(s) (b) Since the authors illustrate the coarse graining details, this represents a slice of the corresponding parameter space. This figure is taken from [6].

extend a similar analysis to higher order LGI(s) for higher times are explore a much wider class satisfactions and violations of certain LGI(s). From our discussion about the QHO some of the natural branches of exploration arise for the underneath cases,

- An analysis for the case in which a different choice for the dichotomic operator is taken into consideration. The authors have explore this possibility by two means. For the first case they compute the LGI(s) for a refined version of the previously considered dichotomic operator, where refinement refers to the course-graining done on the position space. In such a case an LG2 violation is reported under a specific setting for the involved parameters. As a second means, a different operator is taken into consideration such as the case for the parity operator.

- We note that, the analysis presented in this section lies completely within the QHO like potentials. However, one can utilize a similar framework for more such bound potential functions. One such potential using which the corresponding LG violations were explored is the Morse potential. In this case, similar observations are made regarding the LG violations. Consequently, it is found that the LG3(s) and LG4(s) are significantly violated for the first excited state.

In the next section we explore the LG framework for coherent states for the QHO explicitly.

5.2 Gaussian states

Let us first quickly review the Coherent states in the QHO. Recall that coherent states are the eigenstates of the lowering ladder operator of the QHO usually represented as $|\alpha\rangle$. These take up the following form in the energy eigenbasis

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{5.20}$$

We also recall that the coherent states can be generated from the displacements of the ground state of the QHO. This is done by defining the Displacement operator as

$$D(\alpha) = e^{(\alpha a^{\dagger} - \alpha^* a)} \tag{5.21}$$

and subsequent;y acting over the ground state to generate a coherent state i.e. $|\alpha\rangle = D(\alpha)|0\rangle$. Moreover, one can see that the coherent states are those which most closely resemble the dynamics of a classical harmonic oscillator. This behaviour comes as a consequence of the unitarity of the displacement operator and can be seen from the expression of the wavefunction of a coherent state

$$\psi_\alpha(x) = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{\infty} \frac{\left(\alpha a^\dagger\right)^n}{n!} \frac{1}{\sqrt{x_0\sqrt{\pi}}} \exp\left(-\frac{x^2}{2x_0^2}\right) \tag{5.22}$$

where x_0 demotes the standard deviation of the state. Clearly, the probability density for such a wavefunction is a Gaussian, which forms the central idea of this section. For sake of completeness we also mention briefly the time evolved coherent state that is

$$\psi_\alpha(x,t) = \psi_{\alpha(\mathrm{t})}(x) e^{-\frac{i\omega t}{2}} \tag{5.23}$$

where following the ease of notation from the previous section we again choose to work with the dimensionless variables for position and momentum and express the time-evolved wavefunction as

$$\psi_\alpha(x,t) = \frac{1}{\pi^{\frac{1}{4}}} \exp\left[-\frac{1}{2}(x-x_t)^2 + i\frac{p_t}{\hbar}x + i\gamma(t)\right] \tag{5.24}$$

where $\gamma(t)$ denotes the evolved phase for the coherent state. Lastly, the classical trajectories for such coherent states can be expressed in terms of the expectation values of the dimensionless variables x and p which are related as

$$x_i = \langle x(t_i)\rangle = \sqrt{2}\,\mathrm{Re}\,\alpha(t); \quad p_i = \langle p(t_i)\rangle = \sqrt{2}\,\mathrm{Im}\,\alpha(t) \tag{5.25}$$

Note that the overall procedure to analyse the satisfaction and (or) violations of the LGI(s) rests upon the computation of the involved single-time averages and the second-order correlators.

We now proceed to demonstrate the analysis carried out in [7] in the context of LGI(s). Note that, for sake of simplicity we still make of the dichotomic operator as defined in the previous section. Then for the coherent states, the quasi-probability takes a similar form as

$$q(s_1,s_2) = \mathrm{Re}\left\langle \alpha \left| e^{i\frac{it_2}{\hbar}} \theta(s_2 x)\, e^{-\frac{iH\tau}{\hbar}} \theta(s_1 x)\, e^{-\frac{iHt_1}{\hbar}} \right| \alpha \right\rangle \tag{5.26}$$

which for a general Gaussian measurement denoted by $m(x)$ and for the specific case of

$q(+,+)$ yields

$$q(+,+) = \mathrm{Re}\left\langle 0\left|D^{\dagger}(\alpha)e^{\frac{iHt_2}{\hbar}}m(\hat{x})e^{-\frac{iH\tau}{\hbar}}m(\hat{x})e^{-\frac{iHt_1}{\hbar}}D(\alpha)\right|0\right\rangle \tag{5.27}$$

$$= \mathrm{Re}\, e^{\frac{i\omega\tau}{2}}\left\langle 0\left|m\left(\hat{x}+x_2\right)e^{-iH\tau}m\left(\hat{x}+x_1\right)\right|0\right\rangle \tag{5.28}$$

therefore by suitable introduction of the partial overlaps of the energy eigenstates as illustrated in the previous section we find

$$q(+,+) = \mathrm{Re}\sum_{n=0}^{\infty} e^{-in\omega\tau} J_{0n}\left(x_1,\infty\right) J_{0n}\left(x_2,\infty\right) \tag{5.29}$$

where by computing the partial overlaps yields

$$\begin{aligned} q\left(s_1,s_2\right) &= 1 + s_1\,\mathrm{erf}\left(x_1\right) + s_2\,\mathrm{erf}\left(x_2\right) \\ &\quad + s_1 s_2\left[\mathrm{erf}\left(x_1\right)\mathrm{erf}\left(x_2\right) + 4\sum_{n=1}^{\infty}\cos(n\omega\tau)J_{0n}\left(x_1,\infty\right)J_{0n}\left(x_2,\infty\right)\right] \end{aligned} \tag{5.30}$$

We can now compare the expression for the expected form of the quasi-probability to the one obtained above to yield the form of the correlator as

$$C_{12} = \mathrm{erf}\left(x_1\right)\mathrm{erf}\left(x_2\right) + 4\sum_{n=1}^{\infty}\cos(n\omega\tau)J_{0n}\left(x_1,\infty\right)J_{0n}\left(x_2,\infty\right) \tag{5.31}$$

where again the explicit forms for the partial overlaps are specified in the paper. The authors consequently report violations of the corresponding LG2(s), LG3(s) and LG4(s). We recall from section4.5 that a major part of detecting violations or satisfaction of the LGI(s) depends on a choice for the parameters involved in the system. Thereby, it turns out that for the analysis involving Gaussian states, this freedom arises via the equal time interval τ, the standard deviation in the position basis x_o and the corresponding deviation in the momentum space p_o. One can utilize the tools of numerical analysis to detect violations of the inequalities involved. An interesting feature however is noticed with regards to the maximum possible violations of the LGI(s). The authors illustrate via extensive numerical search how the maximum violations reach 22% for LG2(s), 28% for LG3(s) and 26% for LG4(s).

One can thereafter extend a similar analysis to analogous quantities in the much wider class of quantum mechanical phenomenon. Note that, by analogous quantities we refer to the formalism of other such quantum mechanical phenomenon involving Gaussian states. An analysis with regards to the quantum mechanical currents is discussed in [7]. We now proceed to demonstrate the case involving the computation of Bohmian trajectories [37, 78]. The corresponding analysis can be dealt with in two ways. One of the ways deals with the time evolution of an initial plane wave in one dimension with restriction imposed over the position arguments (precisely $x < 0$). This approach utilizes Moshinsky's function [79, 80] work thereby enabling interpretations of temporal analogue of diffraction which he coined as 'diffraction in time'. The way to proceed with this method comes

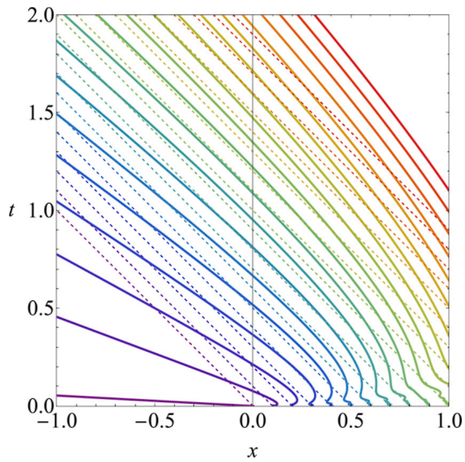

Fig. 5.3: An illustration of Bohmian trajectories under the consideration of Moshinsky function. Here the dotted lines denote the classically expected behaviour of the quantum-mechanical currents. This figure is taken from [7]

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from the numerical analysis carried out to find the solutions to the guidance equation for Bohm trajectories Fig.5.3. Subsequently one computes the quasi-probability associated with the involved quantum mechanical currents to analyze the behaviour of LGI(s). Their analysis then hints towards a deviation as illustrated in Fig.5.4 from the expected classical behaviour. Note that for the restricted Bohm trajectory which is considered by the authors, a classical behaviour would essentially refer to the scenario in which the particle undergoes a Zeno effect. However it is noted that the forms of the quantum mechanical currents contribute towards the emergence of a non-classical behaviour or an anti-Zeno effect and subsequently violate the LG2(s).

Another way to analyze the behaviour of LGI(s) in this context is using the Wigner-Leggett-Garg Inequalities (WGLI). As a short detour we would first demonstrate the key ideas which constitute the WGLI(s). In our discussions from the previous chapter, we

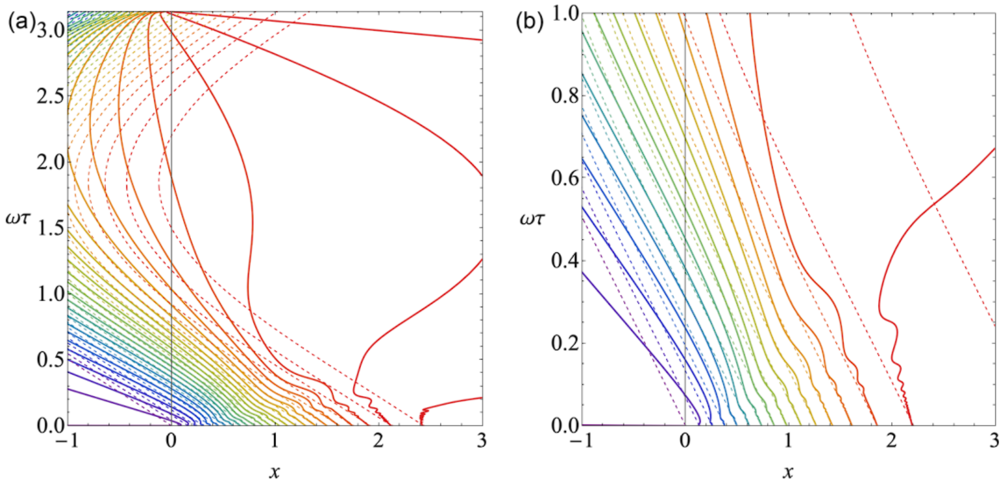

Fig. 5.4: (a) These represent the Bohm trajectories for a particular choice from the phase-space with regards to the behaviour of quantum mechanical currents associated with the particle to be found in the region $x > 0$. (b) This a zoomed in version of (a) resembling to the classically expected case as before. This figure is taken from [7]

saw how the NSIT conditions form a stronger set of conditions for MR as opposed to the originally proposed LGI(s). WGLI(s) in the similar spirit lies midway between the NSIT and the originally proposed LGI(s) in terms of their strongness to attest to MR. These are mainly built from Wigner's inequality [81] for the EPRB-Bohm setup for spatially distinct spin-1/2 particles. Keeping this at the base [82] derive a set of analogous inequalities which we will go over now. The key difference between the originally LGI(s) and WGLI(s) is that the analogous Wigner-type inequalities involve sequential measurements in a single experimental trial. Thereby making use of appropriate marginalization of the underlying joint probability distribution we then match the two-time probabilities as previously seen in chapter-4. The for the dichotomic variable Q, Wigner considers the following combinations obtained from the two-time marginals as

$$P\left(Q_2+, Q_3-\right) - P\left(Q_1+, Q_2+\right) - P\left(Q_1-, Q_3-\right) \leq 0 \tag{5.32}$$

subsequently other forms for the WGLI can be mentioned making use of various combinations of the observable joint probabilities. A generalized version for the case of n-time WGLI(s) can then be expressed as

$$P\left(Q_1+, Q_n-\right) - \sum_{i=1}^{n-1} P\left(Q_i+, Q_{i+1}-\right) \leq 0. \tag{5.33}$$

which can be derived from the relation between the marginals of the overall underlying probability distributions. [82] then proceed to compute the WGLI(s) and the original LGI(s). On comparison for a two-state oscillatory system, they find that the violations in the quantum mechanical context for the original LGI(s) remain independent of the initial state of the system was opposed to the violations obtained for the case of WGLI(s). We once again emphasize on the role of data sets while performing such computations. The calculations with regards to the WGLI(s) constitutes the standard data sets. Furthermore when carrying out an analysis considering the NSIT conditions for the system leads them to conclude that indeed WGLI(s) encapsulate a weaker notion of MR than the NSIT conditions but a stronger notion relative to the original LGI(s).

Returning to our discussion regarding the Gaussian states, we see that within the context of WGLI(s) the two time quasi-probabilities can be expressed in terms of the projection operator and their respective measurements as,

$$q^W(s_1, s_2) = 1 - \langle P_{-s_1}(t_1)\rangle - \langle P_{-s_2}(t_2)\rangle + p_{12}(-s_1, -s_2) \tag{5.34}$$

where the corresponding parameters are determined in a single experimental trial. Thus a comparison of the violations between the two formalism for LGI(s) yields a larger violation for WLGI(s) relative to those obtained from the original set of LG2(s). As a side remark, this once again attests to the classification of WGLI(s) with regards to it's sensitivity to detect non-classical behaviour. [7] proceeds further to employ various techniques such as,

- the introduction of coherent state projectors instead of the usual projectors associated with the dichotomic operator for the system,

- Carrying out explicit mathematical computations for the quasi-probability by splitting the integral in terms of the probability currents corresponding to either sides of the origin

All of these techniques shall thus be seen as further refinements to analyze the maximum violations of LGI(s) as functions of the parameters in the phase-space. Other examples of cases involving Gaussian states like that of the *squeezed states* of the QHO and the *thermal states* has also been illustrated to analyze their sensitivity to LG violations.

5.3 Testing quantumness in gravity

Now that we have seen the basic building blocks of LGI(s), it is interesting to note that this framework of testing the notion of MR has recently been picking up quite a lot of interest within the community. Similar to the case of Bell/CHSH type inequalities where various experimental groups work towards the refinement within the considered experimental setup and try to look for traces of local-causality (or local realism), the Leggett-Garg framework is recently been applied to various other macroscopic systems where the violations of the corresponding LGI(s) hints towards the non-classicality inherent to the system in some form. Recent developments have been made where the LG framework has been attempted to put to test on actual classically understood systems. Note that even during our discussions for the QHO, although dealt with a macroscopic system but one which has quite extensively been studied by the quantum theory.

This leads to an interesting question of whether we can devise tests for quantumness within gravity assuming that a theory of quantum gravity exists. We emphasize that we are not dealing with the arguments of whether gravity can be quantized or not, rather we build upon some of the recent proposals for the case and thereby explore the LG picture in that context. Through this section we aim to demonstrate the scope of the LG framework. For the following discussion we would mostly resort to the two proposals explored by [8] and [9].

Let us first begin with a demonstration put forth by [9]. The key idea to this procedure lies within the construction of a suitable Hamiltonian for a specific system that also involves gravitational interactions. Such constructions mostly lie in the category of Hybrid systems. For this exploration the authors consider the Newton-Schrodinger approach where the gravitational potential is given by the expectation values of matter distributions of the associated states. One such hybrid Hamiltonian which is considered here is

$$H = \Omega\sigma^z + \omega a^\dagger a + H_{grav} \tag{5.35}$$

where $\omega a^\dagger a$ denotes a free Hamiltonian, H_{grav} term encapsulates the gravitational interactions within the system and $\Omega\sigma^z$ stands responsible of the phenomenon that is a consequence of the Larmor precession in the two states. Note that, to test non-classicality within a classical setting of theories (such as gravity in our case) one implicitly starts with an assumption that let gravity abide by the formalism of quantum mechanics. Therefore if one is consequently able to detect violations of the analogous LGI(s), this would hint

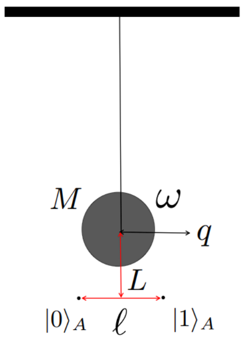

Fig. 5.5: The figure is taken from [9]. The system constitutes of an oscillator and a particle that is a superposition of the two spatially different states $|0\rangle_A$ and $|1\rangle_A$. Here M denotes the mass and ω the angular frequency. L denotes the length between the text particle and the oscillator and l denotes the distance between the two spatially localizations of the particle.

towards a non-classical behaviour for the theory. On the other hand the satisfaction of the LGI(s) would favour the well-understood macrorealistic behaviour of the theory. Therefore, keeping this in mind we remark that σ^z in the hybrid Hamiltonian then represents the two spatially localized states of the oscillatory particle. We can then utilize the hybrid Hamiltonian to yield the form for the analogous unitary evolution operator that can be associated with the states of the system. The explicit form such a unitary operator is then expressed in terms of a constant (which they label as g) and time-dependent complex variables $\alpha(t)$,$\beta(t)$. Thereby yielding

$$U(t) = e^{-i\left(\Omega\sigma^z + \omega a^\dagger a\right)t} e^{g\sigma^z\left(\alpha(t)a - \alpha^*(t)a^\dagger\right)} \tag{5.36}$$

where

$$\alpha(t) = \frac{e^{-i\omega t} - 1}{\omega}, \quad \beta(t) = \frac{1}{\omega}\left(t - \frac{\sin\omega t}{\omega}\right) \tag{5.37}$$

Now, consider an initial state of the hybrid system such as

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle_A + |1\rangle_A\right) \otimes |0\rangle \tag{5.38}$$

where $|0\rangle$ denotes the ground state of the QHO and whose corresponding time evolved state is given as

$$|\psi(t)\rangle = U(t)|\psi_0\rangle \tag{5.39}$$
$$= \frac{1}{\sqrt{2}}\left(e^{-i\Omega t}|0\rangle_A|g\alpha(t)\rangle_C + e^{i\Omega t}|1\rangle_A|-g\alpha(t)\rangle_C\right) \tag{5.40}$$

where the index C denotes the coherent states, that is to say that we express the time evolved states in terms of the coherent states of the QHO. Thereby dealing within the operator formalism which essentially helps us deal with the density matrix $\rho = \psi_0\rangle\langle\psi_0|$ and a particular choice for the basis of the measurement i.e.

$$\boldsymbol{n} = (\cos\varphi, \sin\varphi, 0) \tag{5.41}$$

which at specific times t_i is used to give the expression for the corresponding single time averages and the correlators (similar to what we saw earlier in the spin-1/2 case) as

$$\left\langle \hat{Q}_1 \right\rangle = \mathrm{tr}\left[\boldsymbol{n}\cdot\boldsymbol{\sigma}\left(t_1\right)\rho_0\right] = \cos\left(2\Omega t_1 - \varphi\right) e^{-8\lambda^2 \sin^2 \frac{\omega t_1}{2}} \tag{5.42}$$
$$\left\langle \hat{Q}_2 \right\rangle = \mathrm{tr}\left[\boldsymbol{n}\cdot\boldsymbol{\sigma}\left(t_2\right)\rho_0\right] = \cos\left(2\Omega t_2 - \varphi\right) e^{-8\lambda^2 \sin^2 \frac{\omega t_2}{2}} \tag{5.43}$$

$$\begin{aligned} C\left(t_2, t_1\right) &= \frac{1}{2}\mathrm{Tr}\left[\left\{\boldsymbol{n}\cdot\boldsymbol{\sigma}\left(t_1\right), \boldsymbol{n}\cdot\boldsymbol{\sigma}\left(t_2\right)\right\}\rho_0\right] \\ &= \cos\Theta\left(t_2, t_1\right)\cos\left(2\Omega\left(t_2 - t_1\right)\right) e^{-8\lambda^2 \sin^2 \frac{\omega\left(t_2 - t_1\right)}{2}} \end{aligned} \tag{5.44}$$

where

$$\Theta\left(t_2, t_1\right) = 16\lambda^2 \sin\frac{\omega\left(t_2 - t_1\right)}{2}\sin\frac{\omega t_2}{2}\sin\frac{\omega t_1}{2} \tag{5.45}$$

with $\lambda = g/\omega$ thus yielding the form for the quasi-probability for such a hybrid system as

$$\begin{aligned} q_{s_1 s_2}\left(t_1, t_2\right) = \frac{1}{4}\Big(&1 + s_1\cos\left(2\Omega t_1 - \varphi\right) e^{-8\lambda^2\sin^2\frac{\omega t_1}{2}} + s_2\cos\left(2\Omega t_2 - \varphi\right) e^{-8\lambda^2\sin^2\frac{\omega t_2}{2}} \\ &+ s_1 s_2\cos\Theta\left(t_2, t_1\right)\cos\left(2\Omega\left(t_2 - t_1\right)\right) e^{-8\lambda^2\sin^2\frac{\omega\left(t_2 - t_1\right)}{2}}\Big) \end{aligned} \tag{5.46}$$

Having obtained an expression for the involved quasi-probability we can now split the discussion in two parts to analyse the violations or satisfaction of the corresponding LGI(s). The first case ($t_1 = 0$ and $\Omega \neq 0$) it is observed that the gravitational interaction suppresses the violations obtained from LGI(s). This can be seen by substituting the value $\lambda = 0$ within the exponential factor that acts as the source for gravitational interactions.

Note that, setting $\lambda = 0$ presumably removes any contributions coming as a result of the gravitational interactions and therefore the hybrid system can be taken to act as only a microscopic system which is why we assume that the LGI(s) without the influence of gravity is trivially violated. Therefore, we see that the quasi-probability in such a case yields the following inequality

$$1 + s_1 \cos\varphi + s_2 \cos(2\Omega t_2 - \varphi) + s_1 s_2 \cos(2\Omega t_2) < 0 \tag{5.47}$$

which is always satisfied unless $\phi = 0$. However, clearly the large exponential factors arising from gravitational interactions suppresses the expected violations. Physically this situation is then understood in terms of the argument that gravitational interaction plays an active role to entangle the oscillator and the particle. Furthermore, the second case ($t_1 \neq 0$ and $\Omega = 0$) yields the following expression for quasi-probability

$$q_{s_1 s_2}(t_1, t_2) = \frac{1}{4}\left(1 + s_1 e^{-8\lambda^2 \sin^2 \frac{\omega t_1}{2}} + s_2 e^{-8\lambda^2 \sin^2 \frac{\omega t_2}{2}} + s_1 s_2 \cos\Theta(t_2, t_1)\, e^{-8\lambda^2 \sin^2 \frac{\omega(t_2 - t_1)}{2}}\right) \tag{5.48}$$

In this case, we see that setting $\lambda = 0$ always satisfies the quasi-probability via

$$q_{s_1 s_2}(t_1, t_2) = \frac{1}{4}(1 + s_1 + s_2 + s_1 s_2) \geq 0 \tag{5.49}$$

whereas in the case $\lambda \leq 1$ the expression for quasi-probability reduces to

$$\begin{aligned} q_{s_1 s_2}(t_1, t_2) \simeq & \frac{1}{4}(1 + s_1 + s_2 + s_1 s_2) \\ & - 2\lambda^2 \left(s_1 \sin^2 \frac{\omega t_1}{2} + s_2 \sin^2 \frac{\omega t_2}{2} + s_1 s_2 \sin^2 \frac{\omega(t_2 - t_1)}{2}\right) \end{aligned} \tag{5.50}$$

which for a suitable choice for the parameters $s_i(s)$ may take up negative values and therefore violate the inequalities. However, this only forms as one of the ways to take a step ahead in the direction to test the quantumness of gravity. A more recent exploration helps to provide an insight into the analogous condition of NIM to that in the gravitational context. Let us quickly recall that according to NIM, a classical system should remain unaffected by the act of an ideal measurement. Which as we saw earlier is strongly put to test via the NSIT conditions. The analogous version in the context of classical systems is known as the Non-Disturbance Condition (NDC). Therefore an equivalent argument can be constructed where an NDC condition is derived for a carefully set-up experiment and subsequently tested for.

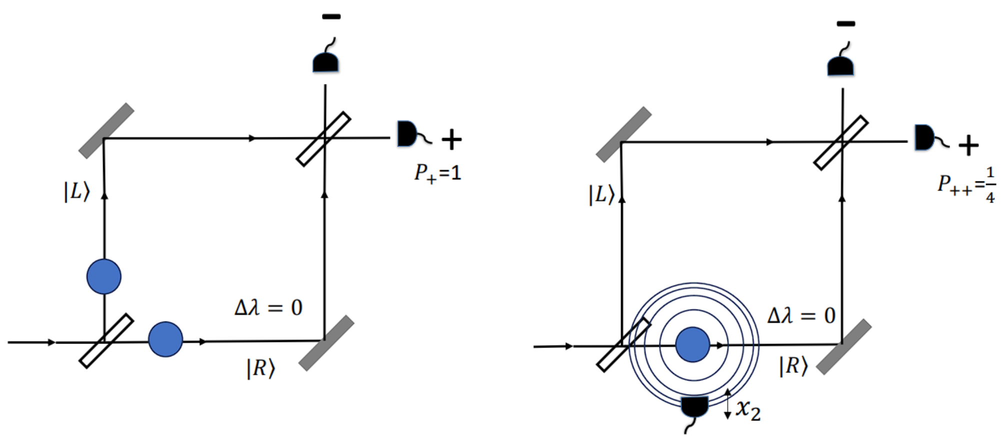

Fig. 5.6: The MZI setup to test the quantumness of gravity. Left hand side denotes the case involving two detectors only at the ends of the arms of MZI. The figure on the right denotes the placement of an extra detector in one of the arms. This figure is taken from [8].

This is described using a MZI-type setup (in Fig.5.6) as discussed in [8]. Then a *source mass* is made to undergo interferometry with equal amplitudes within the two arms of the MZI setup. The states corresponding to the two arms are conveniently labelled as the left $|L\rangle$ and the right $|R\rangle$ states. Then under the following set of assumptions,

- We suppose that the mass is described via quantum theory yet it is large enough to produce a gravitational field that can be detected by a suitable detector.
- The phase difference between the arms of MZI is taken to vanish.
- The involved masses never interact with each other which is to say that we ignore any form of interference arising from the interaction of two gravitational fields.

we expect the probability of detecting a positive (negative) output at the two detection points. Now if in a second trial one places one of the detectors midway of the interferometry this would essentially act as a 'probe mass' within the setup and measures the output of the detector placed at the end, it is then possible to express an NDC condition in terms of the difference between the values of the probabilities obtained. Then NDC states that

$$P_{+(-)}(\text{no midwaymeasurements}) - P_{+(-)}(\text{post midway measurements}) = 0 \tag{5.51}$$

If we now extend the assumption of our setup being inherently quantum mechanical, then one would express the above mentioned NDC condition in terms of the probabilities from the initial setup to that of the sum of probabilities between the intermediate and final detector in the second setup. In such a case we can recast the NDC as

$$P_{+} - (P_{+,+} + P_{-,+}) = \frac{1}{2} \tag{5.52}$$

where $P_{+,+}$ denotes the probability of obtaining a $+$ value at the intermediate detector and $+$ value at the final detector and likewise for $P_{-,+}$. Note that, treating the whole setup as being quantum mechanical collapses the state of the once a measurement is made (the detector measures the gravitational potential of the masses in this context). Thus following the simple rules of probabilities, we get $P_{+,+} = P_{-,+} = 1/4$ and thus we expect the total probability of obtaining either of the states as $1/2$ as on the right hand side of Eq.(). Which clearly gives a violation to the classical case. This in-turn would indicate that gravity is subjected to non-classicality in some form. The authors then present an argument, restricting again to the Newton-Schrodinger approach to justify for the vanishing phase difference between the arms of interferometry. Such recent developments indicate towards a large scale applicability of the LG framework and it's associated concepts.

5.4 Beyond LGs: Tsirelson Inequalities

To this end we turn towards an interesting aspect with regards to testing the notion of realism within various systems. We looked at the Leggett-Garg formalism through the previous chapters and their subsequent sections. We also note that the LG framework constitutes the postulate of NIM that itself leaves little room for the experimentalists to verify the results of such inequalities. Thus, even though we are able to draw meaningful implications with respect to the non-classicality that might be inherent within various systems yet it comes at its own cost. To this end we present an alternative framework to test non-classicality within a system.

In this section therefore, we discuss a quite recently noticed way to look for traces of non-classicality. The way this framework differs from the LG one can be understood in terms of the basic postulates itself. Although we still work with the MRps assumption however it is noticed that even without requiring the NIM condition one may produce a discrepancy between the classically predicted and quantum mechanically predicted characteristic of a particular system. The main idea constitutes the dynamics of the system or precession of the states of the system. In this way one could completely avoid making an actual measurement on the system. This framework although was first proposed by Tsirelson [83] however, it is only recently that people are trying to explore as well as extend it's applicability.

Since these act as an alternative framework it is natural to once again apply this to discrete (e.g. spin) systems [84] and systems involving continuous variables [85]. Following the footsteps of [85] we present the Tsirelson inequalities for two- and three- level systems in terms of conditional probabilities. The authors also illustrate a conditional probability matrix (that constitutes all possible pairs or triplets) by constructing appropriate polytopes. Considering the case for a two-level system, the conditional probability $p(s_i|t_i)$ then represents the probability of obtaining the value s_i conditioned on the time-stamp t_i. Employing the constraints on the joint probability distribution $p(11) = 0$, $p(12) = p(21) = p(22) = 1$ yields the corresponding polytope to be a triangle Fig.5.7. Therefore, yielding the an in-

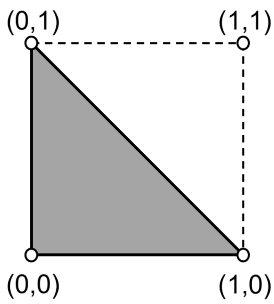

Fig. 5.7: Polytope corresponding to a two-level system.

equality

$$p(1|1) + p(1|2) \leq 0 \tag{5.53}$$

This is the Tsirelson inequality for such a two level system. Similarly the Tsirelson inequality for a three-level system under the constraint $p(111) = 0$ is gives,

$$p(1|1) + p(1|2) + p(1|3) \leq 2 \tag{5.54}$$

However the same inequality under the constraints $p(111) = 0$ and $p(222) = 0$ gives

$$1 \leq p(1|1) + p(1|2) + p(1|3) \leq 2 \tag{5.55}$$

With regards to the QHO, these inequalities arise from the specification of a particular precession of states. The key point comes from breaking the time period into equal intervals. Such a characterisation of the dynamics of the system is most easily done via the Wigner function. It can be seen as a quantity that comes along when working within the Wigner-Weyl picture of quantum mechanics. Remember that the Wigner-Weyl transform, maps the hermitian operators to real phase-space functions. Subsequently, for a wavefunction $|\psi\rangle$ that lies in the Hilbert space $\mathcal{H}$ of QHO the Wigner function is given as

$$W_\psi(q,p) = \frac{1}{\pi\hbar}\int_{\mathbb{R}} \bar{\psi}(q+x)\psi(q-x)e^{i\frac{2px}{\hbar}}\,\mathrm{d}x \tag{5.56}$$

where q, p denote the position and momentum coordinates in the phase-space. Then for a given Winger function W_ψ that replaces the wavefunctions in the Winger-Weyl formalism of quantum mechanics, the probability distribution as a characteristic of the position is expressed as

$$P(q) = \int_{\mathbb{R}} W_\psi(q,p)dp \tag{5.57}$$

Now, using the fact that the time evolution of Wigner functions is described by the Hamilton equations leads us to conclude that precession of states for the classical and quantum harmonic oscillator remains the same. As a result, for the case of QHO, non-classicality only emerges due to the negativity of the Wigner function.

Following from Tsirelson's work it is concluded that for a division of the time-period of the QHO into for example three equal time intervals the precession of the states as a characteristic of its position yields that at least one of the states should pick-up a negative value of the phase. [85] demonstrates this concept in terms of the conditional probabilities and it's full set which they call as the conditional probability matrix. They then define the the Tsirelson inequality as a linear inequality that is satisfied by all conditional probability matrices for which a constrained (with regards to the dynamics) classical model exists but at the same time is also violated due to the existence some other such matrix. Finally they justify the wide range of applicability of such a framework, one that not only tests for the non-classicality within the system but can also be extended to real life situations such as that of the magic tricks involving deception.

Chapter 6

Summary and Conclusion

Through this thesis we looked at mainly two notions of realism that have been explored quite extensively in literature. We first understand how Bell's inequalities stand on the assumptions of local realism and causality and understand that its violations rule out the hidden-variable formalism of quantum mechanics. We understand how the existence of an underlying joint probability distribution containing the information for the result of the experiments before performing actual measurements remains central to the idea of realism. This brings along an important aspect of Fritz theorem that enables us to understand that for local-realism Bell/CHSH type inequalities constitute the necessary and sufficient conditions. This is done considering the factorizability of the joint probability distributions in terms of the independently measured outcomes. We further explore the LGI(s) that essentially deals with testing for the notion of Macrorealism. Building from the basics involving sequential projective measurements associated for dichotomic variables in the general context we look at explicit examples of LG violations in various quantum mechanical systems.

We see how the NIM conditions acts as a major roadblock to experimentally testify this framework. Thus, only allowing to argue with respect to NIM upto a certain precision from an experimentalists point of view. In order to better our argument from the perspective of a hardcore macrorealist we look at an extension of Fine's theorem in the LG context. Playing around with standard data-sets for discrete quantum systems we look at scenarios leading to satisfaction of lower order LGI(s) however yielding violations for higher-orders. Thereby emphasizing the need to verify the full set of inequalities when carrying out actual experiments. We also look at the NSIT conditions and WLGI(s) which constitute a much stronger notion of macrorealism. Finally we look at examples of the applying the Leggett-Garg framework to continuous-variable systems such as that of Quantum Harmonic Oscillator and its variations with regards to the initial states. Treating coherent states of the oscillator as a starting point, we conclude that indeed the WLGI(s) imply a stronger violation of MR relative to the original LGI(s).

Having discussed the inequalities in the context of discrete and continuous variable systems we explore some of the recent developments with regards to applying this framework to test the quantumness of gravity (assuming that a theory of quantum-gravity exists). So far most part of this exploration is done within the Newton-Schrodinger formalism of quantum gravity. This further allows plenty room for someone to explore the possibility of applying the Leggett-Garg machinery within the usual context of gravitational theories such as general relativity. Towards the end, we briefly review another proposed framework that could be considered as a slight variation of the Leggett-Garg picture. These are essentially called the Tsirelson inequalities. A possible extension of the work could also encompass applying Tsirelson inequalities in various other discrete or continuous systems.

In conclusion, the quest for a complete classification of realism still remains an open question however we may keep refining the scientific machinery to tackle with the associate problems via more convincing arguments. What we have been able to infer from our discussions only forms a small subset of systems namely those involving dichotomic variables and related extensions via standard data-sets. Even though we discuss the case for many-valued variables quite extensively, one may take a deep dive to explore various other ways of constructing such variables and thereof carry out a similar analysis.

In a nutshell, exploring variations of the LG framework such as the one discussed briefly in the last section, as well as extend its applicability would eventually enable us to better our understanding of the distinction between classical and quantum. This in turn seems to be a fascinating path to tread for future research works.

Appendix A

Explicit proof for LG3(s)

Consider the expression

$$E : (s_1Q_1 + s_2Q_2 + s_3Q_3)^2 \tag{A.1}$$

we first convince ourselves of RHS as shown in the table we may then expand LHS as

s_1	s_2	s_3	Q_1	Q_2	Q_3	E
1	1	1	1	1	1	1
1	1	-1	1	1	-1	1
1	-1	1	1	-1	1	3
1	-1	-1	1	-1	-1	3
-1	1	1	-1	1	1	3
-1	1	-1	-1	1	-1	3
-1	-1	1	-1	-1	1	3
-1	-1	-1	-1	-1	-1	1

Table A.1: A table explicitly showing the bound for the considered expression.

$$3 + 2(s_1s_2Q_1Q_2 + s_2s_3Q_2Q_3 + s_1s_3Q_1Q_3) \geq 1 \tag{A.2}$$

$$1 + 2(1 + s_1s_2C_{12} + s_2s_3C_{23} + s_1s_3C_{13}) \geq 1 \tag{A.3}$$

$$1 + s_1s_2C_{12} + s_2s_3C_{23} + s_1s_3C_{13} \geq 0 \tag{A.4}$$

which exactly resembles the form LG3(s). Note that going from (A.2) to (A.3) requires us to average out on both sides of the inequality.

Appendix B

Expression of quasi-probability for QHO

Picking up from Eq.(5.4) we may explicitly write the quasi-probability as

$$q(s_1, s_2) = \mathrm{Re} \sum_{m=0,n=0}^{\infty} \langle \psi \mid m \rangle \left\langle m \left| \mathcal{P}_{s_2}(t_2)\, \mathcal{P}_{s_1}(t_1) \right| n \right\rangle \langle n \mid \psi \rangle \tag{B.1}$$

for the case $s_1 = +, s_2 = +$ we get

$$q(+,+) = \mathrm{Re} \sum_{m=0,n=0}^{\infty} \langle \psi \mid m \rangle \langle n \mid \psi \rangle q_{mn} \tag{B.2}$$

where,

$$q_{mn} = e^{i(E_m/\hbar)t_2 - i(E_n/\hbar)t_1} \int_0^{\infty} \int_0^{\infty} dx dy \langle m \mid x \rangle \left\langle x \left| e^{-iH\tau/\hbar} \right| y \right\rangle \langle y \mid n \rangle \tag{B.3}$$

Notice that due to the infinite ladder or eigenstates for the QHO, all of the above expressions involve an infinite sum. These computations are carried out via two ways that are described in full detail in [6].

The first could be to make use of the energy eigenbasis (which can be utilized via resolution of unity) consequently leading to a truncation of the infinite sum. In such a case Eq.(5.1.7) reduces to

$$q_{mn} = e^{i(E_m/\hbar)t_2 - i(E_n/\hbar)t_1} \sum_{k=0}^{\infty} e^{-i(E_k/\hbar)(\tau)} \int_0^{\infty} \int_0^{\infty} dx dy \langle m|x\rangle \langle x|k\rangle \langle k|y\rangle \langle y|n\rangle \tag{B.4}$$

$$q_{mn} = e^{i(E_m/\hbar)t_2 - i(E_n/\hbar)t_1} \sum_{k=0}^{\infty} e^{-i(E_k/\hbar)(\tau)} J_{mk} J_{nk} \tag{B.5}$$

where the above expression is expressed in terms of the partial overlap of the eigenstates i.e.

$$J_{kl} = \int_0^\infty dx \langle k|x\rangle\langle x|l\rangle \tag{B.6}$$

thus yielding the following general expression for a particular energy eigenstate

$$q_n(+,+) = \mathrm{Re}\, e^{i(E_n/\hbar)\tau} \sum_{k=0}^{\infty} e^{-i(E_k/\hbar)\tau} J_{nk} J_{nk} \tag{B.7}$$

A second way to deal with this would be to plug in the explicit form of the wavefunctions that come out as the solutions for QHO. Therefore in this case, we find that Eq.(5.1.7) takes the following form in terms of the Hermite polynomials upto an overall normalization factor $\mathcal{N}_{nm}$ i.e.

$$\begin{aligned} q_{mn} = \mathcal{N}_{mn}(\tau) e^{i(E_m/\hbar)t_2 - i(E_n/\hbar)t_1} \int_0^\infty \int_0^\infty & dr ds H_m(r) H_n(s) e^{-r^2/2} e^{-s^2/2} \\ & \times \exp\left(i \frac{1}{2\tan(\omega\tau)} \left(r^2 + s^2\right) - i \frac{1}{\sin(\omega\tau)} rs \right) \end{aligned} \tag{B.8}$$

Note that here the overall normalization factor is not a constant on the other hand depends of the time interval of the evolution of the state that is to say $\mathcal{N}_{nm} = \mathcal{N}_{nm}(\tau)$. The explicit form of the normalization factor is also stated in the paper, however since we are focusing on the implications of the LG framework we would leave it up to the reader to refer to.

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