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Path Integrals Applied to Stochastic Processes

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Abstract

This work aims to recreate the Feynman the path integral approach used in Quantum Field Theory in a reaction-diffusion process on a lattice. The evolution of the system is given by a master equation which describes the change in the probability of the lattice being in a certain configuration. Creation and annihilation operators are then introduced, allowing us to apply second quantisation to our theory. I treat the source term perturbatively, and compute correlation functions. The branching term is then introduced and perturbative expansion is again invoked to find the correlation functions. The results are then presented diagramatically using Feynman diagrams. Finally, I conclude the work with a set of propsed Feynman rules.

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Chapter 1

Introduction

1.1 History 101

1.1.1 Stochastic Processes

In the first 10 years of the 20th century, statistics and probability were barely regarded as a mathematical theories in their own right. On the probability theory side, all there was was the work of Laplace (which was not regarded as mathematically rigorous). Then it began developing as independent theories from multiple Russian mathematicians, such as Pafnuty Chebyshev, Andrey Markov, and Aleksandr Lyapunov, but was never a whole theory. In the 1920s, Richard von Mises, after a failed attempt to unify probability theory, claimed that it was not a mathematical theory. On the statistics side of things, the situation was the same. It was regarded as a field relating to collecting large amounts of data regarding demographic and economic facts rather than a mathematical theory.

The earliest trace of stochastic Poisson processes was in a paper from 1903 by F. Lundberg. He later developed a theory of risk which is now regarded as a particular kind of stochastic process with independent increments. In the 1920s, Jarl Waldemar Lindeberg worked on the central limit theorem by developing the Lindeberg conditions, which in the 10 years after were proved by William Feller to be necessary for the validity of the central limit theory in certain conditions. During around the same time, Norbert Wiener published a paper about the Wiener measure, which is the modern day Brownian motion process. In 1925, Paul Lévy published a probability theory book which satisfies the mathematical rigour required. During the entirety of the 1920s, the probability theory techniques were developed such that by the 1930s these techniques were well known among the community of mathematicians interested in this field. During the same period, the field of statistics was also rapidly developing. R.A. Fisher and others published multiple works on multidimensional distributions, correlation, and estimation. By the end of the 1920s, the field was drastically transformed by the well-known works of Jerzy Neyman and Egon Pearson.

In the first years of the 1930s, there was a breakthrough in the field of probability from the works of Paul Lévy, Aleksandr Khinchin, and especially Andrey Kolmogorov; his work gave an entire new approach to using measure theory. These laid the foundations of probability theory. In 1934, Will Feller published his work on the central limit theorem and Markov processes, building on the recently published work of Andrey Kolmogorov, resulting in the foundations of the Markov process. Also during that year, Aleksandr Khinchin introduced stationary stochastic processes and proved some of its fundamental propositions. Another notable contributor to probability theory was Francesco Paolo Cantelli, he had published multiple valuable papers in this area. During the Second World War, Harald Cramér began writing his book on the mathematical methods of statistics, in which he attempted to show how statistical methods could be founded on a mathematical probability theory. After the Second World War, the theory of stochastic processes was booming, with many applications of it in different fields coming up. In 1950, Ulf Grenander wrote his thesis on using statistical inferences in stochastic processes, a field in which he was regarded to be a pioneer [1].

1.1.2 Quantum Field Theory

From mid 1925 to early 1927, Werner Heisenberg, Erwin Schrödinger, and Max Born independently laid the foundations of non-relativistic quantum mechanics with the discovery of the Schrödinger equation and the uncertainty principle. This kick-started the formation of Quantum Field Theory; physicists including Werner Heisenberg, Pascual Jordan, Wolfgang Pauli, and Paul Dirac were putting their efforts in quantising the electromagnetic field and exploring relativistic quantum mechanics.

By by the early 1930s, physicists faced the ever-so-prominent divergence issues in QFT. However, their bewilderment was interrupted by the outbreak of the Second World War. Afterwards, in the late 1940s, younger physicists used their numerical and engineering skills that were cultivated during the war to find experimental evidence supporting QFT. Later on in 1947 and 1948, Julian Schwinger and Richard Feynman formulated multiple renormalisation techniques to solve the issue of divergences in quantum electrodynamics. In late 1949, Freeman Dyson showed the equivalence between both Feynman and Schwinger's approaches, and further proved that renormalisation works at arbitrary perturbative order in QED.

Although the field was quickly advancing, there were more issues that arose. For example, in 1957, experimental evidence showed that parity symmetry was violated in weak force interactions, which led to Murray Gell-Mann, Richard Feynman, and a few other theorists to publish parity violating models. However, these theories showcased unwanted behaviour at high energies, which led physicists such as Sheldon Glashow and Julian Schwinger to resort to C.N. Yang and Robert Mills' suggestion that nuclear forces were mediated by force-carrying particle that obey a gauge symmetry.

However, issues with the mass of these force-carrying particles arose. As a result, physicists spent the late 1950s to the mid-1960s studying spontaneous symmetry breaking, resulting in what we know now as the Higgs mechanism. Meanwhile, Feynman's path integral techniques, which he developed in his PhD thesis in the 1940s, attracted the attention of the theoretical physics community, particularly for its use in models with non-trivial gauge structure.

Additionally, physicists were unsure of how to approach strongly coupled particles, as the aforementioned perturbative model only applied to weak coupling. During the 1970s, Yoichiro Nambu, Murray Gell-Mann, and Harald Fritzsch introduced quantum chromodynamics, presenting a systematic way to deal with these strong couplings. Amidst all this, the previous work of Sheldon Glashow, Steven Weinberg, and Abdus Salam unifying electromagnetic and weak interactions began to attract attention. The electroweak theory along with QCD came together to form the Standard Model of particle physics, all thanks to QFT [2].

1.2 An Unconventional Marriage

Field theoric methods and the renormalisation group (RG) have provided a lot of insight in the realm of statistical physics. Scaling ideas led to a profound understanding of critical singularities near continuous phase transitions in thermal equilibrium. The dynamic scaling hypothesis, which generalises the scaling ansatz fort the static correlation function and introduces an additional dynamic critical exponent was successful in describing experiments of time-dependent properties near second order phase transitions in thermal equilibrium.

The RG method in critical phenomena has provided a solid conceptual foundation for phenomenological scaling theories. This was backed by exact solutions from some idealized models and computer simulations as well as experimental evidence. In addition to providing a framework for static and dynamical properties near a critical point, RG also enables us to describe the large-scale and low frequency response in stable thermodynamic phases, as well as phase transitions at zero temperatures driven by quantum fluctuations. Additionally, RG and scaling concepts look to be promising in describing phenomena for far from equilibrium systems [3].

1.3 Overview of this Work

In this work, we attempt to apply the concepts of field theory, namely path integrals, to particles hopping on a lattice. The transfer of our particles between lattice sites occurs as a random walk, therefore, in the first chapter we introduce the basics of Stochastic processes and discuss the special cases of the Markovian and Poisson processes that lie behind the idea of a random walk. In the second chapter we derive our master equation that describes the change in the probability of our lattice having the desired configuration. This will act as an analogue to our Hamiltonian, allowing us to write a Schrödinger-like equation and second quantising our theory in the same manner as in QFT. We then compute our path integral in chapter 4 allowing us to find a form for our operator expectation value. Finally, in chapter 5 we introduce the concept of branching and derive the Feynman rules for our system. Throughout this work, there is an emphasis on how our results compare to those of QFT. Multiple references were resorted to in the making of this thesis, specifically Prof. Gunnar Pruessner's notes titled None Equilibrium Statistical Mechanics [4], which was the main reference followed.

Chapter 2

Background

In this chapter, we will introduce some basic concepts to help us better understand the nature of the system we will be tackling in this work. Our study is on particles hopping in a lattice, and since the dynamics of our particles is stochastic, i.e. of random nature, we will begin this chapter by defining what a stochastic process is. Then we analyse what it means for a variable to be ”random”, and introduce the special Markov property, which showcases how the occurrence of a future state depends only on the present state and not on the past states [5][6]. Afterwards, we introduce an example of a Markovian process; random walks. They are a key concept for this work since our particles execute random walks are they transport throughout the lattice [7]. Finally, we look at the Poisson distribution,

2.1 Stochastic Processes

A stochastic process is the formal representation of real systems whole evolution in time or space can be assumed as random [5]. It is defined as a function Y of both time t and a stochastic variable X, which we define in the following subsection [8]

$$Y_X(t) = f(X, t) \tag{2.1}$$

2.1.1 Random Events

Consider a discrete set of random points labelled on a time axis, such as the counts in a Geiger counter. This random set of points is called a stochastic variable. Its sample space consists of states, each state containing a nonnegative integer s, and for each s a set of s real numbers t_i such that

$$\infty < t_1 < t_2 < ... < t_s < \infty \tag{2.2}$$

Note that this is analogous to the definition of a Fock space [6]. The probability distribution function in the domain (2.2) is given by $Q_s(t_1, t_2, ..., t_s)$. By imposing that a permutation of the times $(t_1, t_2, ..., t_s)$ corresponds to the same state, we are no longer constrained by (2.2) and t_i is allowed to range form $-\infty$ to ∞. As a consequence, we are able to write our normalisation condition as [6]

$$Q_0 + \sum_{s=1}^{\infty} \frac{1}{s!} \int_{-\infty}^{\infty} dt_1 dt_2 ... dt_s Q_s(t_1, t_2, ..., t_s) = 1 \tag{2.3}$$

The average is $\langle A \rangle$ is then given by [6]

$$\langle A \rangle = A_0 Q_0 + \sum_{s=1}^{\infty} \frac{1}{s!} \int_{-\infty}^{\infty} dt_1 dt_2 ... dt_s Q_s(t_1, t_2, ..., t_s) A_s(t_1, t_2, ..., t_s) \tag{2.4}$$

2.2 Markov Process

The Markov process is a stochastic process with the Markov property, which states that the probability density at time t_n depends only on the value at the most recent time y_{n-1} and does not depend on the values at earlier times [9].

$$P_{1|n-1}(y_n, t_n | y_1, t_1; ...; y_{n-1}, t_{n-1}) = P_{1|1}(y_n, t_n | y_{n-1}, t_{n-1}) \tag{2.5}$$

Where $P_{1|1}$ is called the transition probability. $P_j(y_j, t_j; y_i, t_i)$ is the joint probability density function which describes the probability of observing y_i at t_i and y_j at t_j. It is given by the following relation

$$[4]P_{1|1}(y_2, t_2|y_1, t_1) = \frac{P_2(y_2, t_2; y_1, t_1)}{P_1(y_1, t_1)} = \frac{P_{1|1}(y_1, t_1|y_2, t_2)P_1(y_2, t_2)}{P_1(y_1, t_1)} \tag{2.6}$$

An interesting property of Markov processes is that they are fully determined by the initial distribution and the transition probability [7]. For example

$$\begin{aligned} P_3(y_1, t_1; y_2, t_2; y_3, t_3) &= P_2(y_1, t_1; y_2, t_2)P_{1|2}(y_3, t_3|y_1, t_1; y_2, t_2) \\ &= P_1(y_1, t_1)P_{1|1}(y_2, t_2|y_1, t_1)P_{1|1}(y_3, t_3|y_2, t_2) \end{aligned} \tag{2.7}$$

Where we have used Bayes theorem. Integrating this result over y_2

$$\begin{aligned} &\int dy_2 P_2(y_1, t_1; y_2, t_2)P_{1|2}(y_3, t_3|y_1, t_1; y_2, t_2) \\ &= P_1(y_1, t_1)\int dy_2 P_{1|1}(y_2, t_2|y_1, t_1)P_{1|1}(y_3, t_3|y_2, t_2) \end{aligned} \tag{2.8}$$

Dividing by $P_1(y_1, t_1)$ and using the identity

$$P_1(y_2, t_2) = \int dy_1 P_{1|1}(y_2, t_2|y_1, t_1)P_1(y_1, t_1) \tag{2.9}$$

Equation (2.8) becomes

$$P_{1|1}(y_3, t_3|y_1, t_1) = \int dy_2 P_{1|1}(y_2, t_2|y_1, t_1)P_{1|1}(y_3, t_3|y_2, t_2) \tag{2.10}$$

Which is known as the Chapman-Kolomogrov equation [10][4].

2.3 Random Walks

Random walks are an important example of a Markovian process in which transitions can only happen to neighbouring states [11][12]. For discretised space and time, a random walker can move from one point to another in every discrete time step [13]. The probability density that a walker is located at point y after n steps is given by

$$P(y;n) = \int_{-\infty}^{\infty} f(y-y')P(y';n-1)dy' \tag{2.11}$$

This means the probability of finding a particle at position y after n steps is the probability of the particle arriving at y' in $n-1$ steps and making up the difference in displacements $y-y'$ in one additional step [14]. Lets define the Fourier transform and the inverse Fourier transform respectively as

$$\tilde{P}(k;n) = \int_{-\infty}^{\infty} P(y;n)e^{-iky}dy \tag{2.12}$$

$$P(y;n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{P}(k;n)e^{iky}dk \tag{2.13}$$

From (2.11), we can infer the following

$$\tilde{P}(k;n) = \tilde{f}(k)\tilde{P}(k;n-1) \tag{2.14}$$

so that

$$\tilde{P}(k;n) = \tilde{f}(k)\tilde{P}(k;n-1) = \tilde{f}^2(k)\tilde{P}(k;n-2) = ... = \tilde{f}^n(k) \tag{2.15}$$

The inverse Fourier transform of (2.15) then gives

$$P(y;n) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\tilde{f}(k)\right)^n e^{iky}dk \tag{2.16}$$

If the first two moments of $\tilde{f}(k)$ are finite as $n \to \infty$, $P(y;n)$ converges to a Gaussian

$$P(y;n) = \frac{1}{(2\pi Dn)^{\frac{1}{2}}} e^{-\frac{(y-vn)^2}{4Dn}} \tag{2.17}$$

Where $v \equiv \langle r \rangle$ and $D = \frac{\langle (r-\langle r \rangle)^2 \rangle}{2}$ and r is the step size [5].

2.4 The Poisson Distribution

The Poisson distribution is the special case that occurs when our aforementioned random dots are independent [4], meaning Q_s factorizes as such

$$Q_s(t_1, t_2, ..., t_s) = e^{-v} q(t_1) q(t_2) ... q_(t_s), Q_0 = e^{-v} \tag{2.18}$$

Where q is a nonnegative integrable function and the normalisation condition gives

$$v = \int_{-\infty}^{\infty} q(t) dt \tag{2.19}$$

The Poisson distribution, defined as the probability distribution of s independent random dots/events falling in a limited interval, is given by [6]

$$p_s = \frac{\langle s \rangle^s}{s!} e^{-\langle s \rangle} \tag{2.20}$$

A collection of Markovian random walks converges to a Poisson processes [15].

Chapter 3

Reaction-Diffusion Field Theory

3.1 General Overview

Our focus in this thesis will be on studying the reaction-diffusion process generated from particles hopping on a lattice. Reaction and diffusion is a non-equilibrum stochastic process, with certain quantities being invariant if the full dynamics of the system is considered [16]. Before proceeding to tackle this problem, let us set the scene. Our system is time-dependent, and at finite temperature. Our particles are classical ones, and they reside on a d-dimensional hypercubic lattice [17]. A particle in our lattice chooses with rate H a random site to migrate to (random walks). We will restrict our system to the diffusion of one species of particles, for simplicity. Later on we will look at branching, the statistical mechanics analogue to scattering.

3.1.1 On the Doi-Peliti Approach

In order to find a description of the evolution of our system, we will resort to the Doi-Peliti approach, also knows as the coherent state path integral [18]. The Doi-Peliti approach is done by constructing a master equation from a set of rules, which would be the rate of change of the probability of our lattice having a certain number of particles at a specific site, and later on using perturbation theory methods [18][19]. The master equation is going to act as the mirror of the Hamiltonian in Quantum Field Theory. We then move on to defining our creation and annihilation operators, allowing us to second quantise our master equation. What makes this approach unique is that it preserves the entity of our particles, which is something we lose when resorting to traditional "coarse-graining" techniques [20].

3.2 The Master Equation

Define the state of the system as the number of particles at each lattice site $\{n_{\boldsymbol{j}}\}$. This is to say that n particles are found at the sight $\boldsymbol{j}$. If we only consider diffusion, our particles are just allowed to jump to a random neighbouring sight [21]. We want to find the change in the probability of our system being in a specific state. Beginning with a simple example, lets focus on two neighbouring lattice sites. Assume we want to find the change in the probability of one particle residing in $\boldsymbol{j} = 1$ and zero particles in $\boldsymbol{j} = 2$. The influx towards achieving that configuration would be if our particle which is residing in $\boldsymbol{j} = 2$ (i.e. $\{n_{\boldsymbol{j}+\boldsymbol{e}}\}$ where $\boldsymbol{e}$ is our unit vector representing one lattice spacing) would jump with rate Q to $\boldsymbol{j} = 1$. The outflow however would be if the particle was in fact in the site $\boldsymbol{j} = 1$ (i.e, $\{n_{\boldsymbol{j}}\}$), but decided to jump with rate Q to the site $\boldsymbol{j} = 2$. The master equation is then [22]

$$\frac{d}{dt}P(\{n_1\};t) = QP(\{n_2\};t) - QP(\{n_1\};t) \tag{3.1}$$

Generalising this to our d-dimensional lattice, the total influx to the probability of our system being in the state $\{n_{\boldsymbol{j}}\}$ is

$$Q\sum_{\boldsymbol{e}}\sum_{i}(n_{\boldsymbol{i}+\boldsymbol{e}}+1)P(\{n_{\boldsymbol{i}}-1, n_{\boldsymbol{i}+\boldsymbol{e}}+1\};t) \tag{3.2}$$

Where q is the number of neighbours and we multiply by $(n_{\boldsymbol{i}+\boldsymbol{e}}+1)$ since each particle at the site $\boldsymbol{i}+\boldsymbol{e}$ could jump to the neighbouring site. The outflow is given by

$$Q\sum_{\boldsymbol{i}} n_{\boldsymbol{i}} P(\{n_{\boldsymbol{j}}\};t) \tag{3.3}$$

This gives us the following master equation [1]

$$\frac{d}{dt}P(\{n_{\boldsymbol{j}}\};t) = \frac{Q}{q}\sum_{\boldsymbol{e}}\sum_{\boldsymbol{i}}(n_{\boldsymbol{i}+\boldsymbol{e}}+1)P(\{n_{\boldsymbol{i}}-1, n_{\boldsymbol{i}+\boldsymbol{e}}+1\};t) - Q\sum_{\boldsymbol{i}} n_{\boldsymbol{i}} P(\{n_{\boldsymbol{j}}\};t) \tag{3.4}$$

In order to take into account the limited size of our lattice, we must impose a boundary condition $P\{n_{\boldsymbol{i}}\} = 0$ if $\{n_{\boldsymbol{i}}\}$ contains particles residing outside our lattice. Additionally, we must specify that we cannot have a negative number of particles in a lattice site, i.e. $P\{n_{\boldsymbol{i}}\} = 0$ if $n_{\boldsymbol{i}} < 0$.

Now we must consider particle extinction with rate ϵ. This would be if the number of particles in site $\boldsymbol{j}$ went down by 1, so our master equation for extinction is

$$\frac{d}{dt}P(\{n_{\boldsymbol{j}}\};t) = \epsilon\sum_{\boldsymbol{i}}\big((n_{\boldsymbol{i}}+1)P(\{n_{\boldsymbol{i}}+1\};t) - n_{\boldsymbol{i}}P(\{n_{\boldsymbol{j}}\};t)\big) \tag{3.5}$$

Finally, we consider a lattice system with particle creation (which we interpret as a source term); meaning a particle can be created with rate β at any lattice site. The master equation for particle creation is

$$\beta\sum_{\boldsymbol{i}}\big(P(\{n_{\boldsymbol{i}}-1\};t) - P(\{n_{\boldsymbol{j}}\};t)\big) \tag{3.6}$$

Combining these three factors together, which we can do since they are independent Poisson processes [4], we get a final master equation describing particles subject to diffusion, extinction, and creation on a lattice

[1] Be warned that the notation is slightly abusive

$$\begin{aligned}\frac{d}{dt}P(\{n_{\boldsymbol{j}}\};t) = \frac{Q}{q}\sum_{\boldsymbol{e}}\sum_{\boldsymbol{i}}\big((n_{\boldsymbol{i+e}}+1)P(\{n_{\boldsymbol{i}}-1,n_{\boldsymbol{i+e}}+1\};t) - qn_{\boldsymbol{i}}P(\{n_{\boldsymbol{j}}\};t)\big)\\ +\epsilon\sum_{\boldsymbol{i}}\big((n_{\boldsymbol{i}}+1)P(\{n_{\boldsymbol{i}}+1\};t) - n_{\boldsymbol{i}}P(\{n_{\boldsymbol{j}}\};t)\big)\\ +\beta\sum_{\boldsymbol{i}}\big(P(\{n_{\boldsymbol{i}}-1\};t) - P(\{n_{\boldsymbol{j}}\};t)\big)\end{aligned} \tag{3.7}$$

Great! Now we have an equation describing the change in the probability of our lattice configuration, analogous to the Hamiltonian (for the fellow Heisenbergs reading this) which describes the total energy of the system. Following in the footsteps of Quantum Field Theory, we move on to second quantising our master equation to find what we interpret to be our analogue to the Schrödinger equation.

3.3 Second Quantisation

3.3.1 States and Operators

To begin this process, we lay out all the artillery we will be needing inspired by those of Quantum Field Theory, beginning with the Fock space. The Fock space is the set of Hilbert spaces of a the vacuum state, a one particle state, a two particle state, and so on [23]. We define our basis states for the Fock space as follows

$$|\{n_{\boldsymbol{j}}\}\rangle = \prod_{\boldsymbol{j}} a^{\dagger n_{\boldsymbol{j}}}|0\rangle$$

Note that we are dropping the hats. This state describes the configuration of our lattice, where $a^\dagger$ is our creation operator. We can see from this definition that the vacuum state $|0\rangle$ is obtained by setting $n_{\boldsymbol{j}} = 0$, meaning that no particles occupy any lattice site.

The orthogonality condition is given by the following relation

$$\langle\{n_j\}|\{m_j\}\rangle = \prod_j \delta_{n_j,m_j} \tag{3.8}$$

We define the mixed state as follows [24]

$$|\psi(t)\rangle = \sum_{\{n_{\boldsymbol{j}}\}} P(\{n_{\boldsymbol{j}}\};t)|\{n_{\boldsymbol{j}}\}\rangle \tag{3.9}$$

The state of a particular lattice configuration acting on the mixed state will then give us the probability of our lattice being in that specific configuration

$$\langle\{n_{\boldsymbol{j}}\}|\psi(t)\rangle = P(\{n_{\boldsymbol{j}}\};t) \tag{3.10}$$

Going back to the creation operator we previously mentioned, we define it as the operator that creates n particles at position $\boldsymbol{j}$ in our lattice, and the annihilation operator as the operator that destroys said particle. They are given by the relations [20]

$$a^\dagger(\boldsymbol{j})|n_{\boldsymbol{j}}\rangle = |n_{\boldsymbol{j}}+1\rangle \tag{3.11}$$

$$a(\boldsymbol{j})|n_{\boldsymbol{j}}\rangle = n_{\boldsymbol{j}}|n_{\boldsymbol{j}}-1\rangle \tag{3.12}$$

These operators satisfy the following commutation relations

$$[a(\boldsymbol{j}), a^\dagger(\boldsymbol{k})] = \delta_{\boldsymbol{j},\boldsymbol{k}} \tag{3.13}$$

$$[a^\dagger(\boldsymbol{j}), a^\dagger(\boldsymbol{k})] = [a(\boldsymbol{j}), a(\boldsymbol{k})] = 0 \tag{3.14}$$

Using these we may define the particle number operator, whose eigenvalues give the number of particles at site $\boldsymbol{j}$ [25]

$$a^\dagger(\boldsymbol{j})a(\boldsymbol{j})|\{n_{\boldsymbol{j}}\}\rangle = n_{\boldsymbol{j}}|\{n_{\boldsymbol{j}}\}\rangle \tag{3.15}$$

We now move on to calculating expectation values. Beginning with the expected particle number in a specific lattice site $\boldsymbol{i}$, we should have

$$\langle n\rangle(\boldsymbol{i};t) = \sum_{\{n_{\boldsymbol{j}}\}} P(\{n_{\boldsymbol{j}}\};t)n_{\boldsymbol{i}} \tag{3.16}$$

Using (3.10), we can write (3.16) as

$$a^{\dagger}(\boldsymbol{i})a(\boldsymbol{i})|\psi(t)\rangle = \sum_{\{n_{\boldsymbol{j}}\}} P(\{n_{\boldsymbol{j}}\};t)a^{\dagger}(\boldsymbol{i})a(\boldsymbol{i})|\{n_{\boldsymbol{j}}\}\rangle = \sum_{\{n_{\boldsymbol{j}}\}} P(\{n_{\boldsymbol{j}}\};t)n_{\boldsymbol{i}}|\{n_{\boldsymbol{j}}\}\rangle \tag{3.17}$$

Which is close to what we would like to achieve. To proceed we need to introduce an operator that projects all $|\{n_{\boldsymbol{j}}\}\rangle$ to unity

$$\langle\Omega| = \sum_{n'_{\boldsymbol{j}}} \langle\{n'_{\boldsymbol{j}}\}| \tag{3.18}$$

Therefore

$$\begin{aligned}\langle\Omega|a^{\dagger}(\boldsymbol{i})a(\boldsymbol{i})|\psi(t)\rangle &= \sum_{\{n'_j\}}\sum_{\{n_j\}} P(\{n_{\boldsymbol{j}}\};t)n_{\boldsymbol{i}}\langle\{n'_j\}|\{n_j\}\rangle \\ &= \sum_{\{n_{\boldsymbol{j}}\}} P(\{n_{\boldsymbol{j}}\};t)n_{\boldsymbol{i}} = \langle n\rangle(\boldsymbol{i};t)\end{aligned} \tag{3.19}$$

A property of Ω is that it is invariant under $a^{\dagger}$ from the right, meaning it is an eigenvector of the creation operator [4]

$$\langle\Omega|a^{\dagger}(\boldsymbol{i}) = \sum_{n_{\boldsymbol{j}}=0}^{\infty} \langle\{n_{\boldsymbol{j}}\}|a^{\dagger}(\boldsymbol{i}) = \sum_{n_{\boldsymbol{j}}=1}^{\infty} \langle\{n_{\boldsymbol{j}}-1\}| + 0 = \langle\Omega| \tag{3.20}$$

using this fact, we can say

$$\langle n\rangle(\boldsymbol{i};t) = \langle\Omega|a(\boldsymbol{i})|\psi(t)\rangle \tag{3.21}$$

3.3.2 Second Quantisation of the Master Equation

At this point, we are able to write our master equation, which we will label as H, as a Hamiltonian in a Schrödinger-type equation [17],

$$\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \tag{3.22}$$

But first we must second quantise the master equation by promoting the states to operators, and by assuming normal ordering (which makes apparent the preservation of the particle entity by the DOI-Peliti approach [20]). Lets do this in detail. Beginning with the diffusion term,

$$\begin{aligned}\frac{d}{dt}|\psi(t)_Q\rangle &= H_Q|\psi(t)\rangle \\ &= \frac{Q}{q}\sum_{\boldsymbol{e}}\sum_{\boldsymbol{i}}\big((n_{\boldsymbol{i}+\boldsymbol{e}}+1)P(\{n_{\boldsymbol{i}}-1,n_{\boldsymbol{i}+\boldsymbol{e}}+1\};t) - qn_{\boldsymbol{i}}P(\{n_{\boldsymbol{j}}\};t)\big)|\{n_{\boldsymbol{j}}\}\rangle\end{aligned} \tag{3.23}$$

For the first term of (3.23), we can write our configuration $\{n_{\boldsymbol{j}}\}$ as $n_{\boldsymbol{j}}, n_{\boldsymbol{j}+\boldsymbol{e}}$ [4], and by using the definitions of the creation and annihilation operators we get

$$\begin{aligned}&\frac{Q}{q}\sum_{\boldsymbol{e}}\sum_{\boldsymbol{i}}\Big((n_{\boldsymbol{i}+\boldsymbol{e}}+1)P(\{n_{\boldsymbol{i}}-1,n_{\boldsymbol{i}+\boldsymbol{e}}+1\};t)|n_{\boldsymbol{j}},n_{\boldsymbol{j}+\boldsymbol{e}}\rangle\Big) \\ &\quad= \frac{Q}{q}\sum_{\boldsymbol{e}}\sum_{\boldsymbol{i}}\Big(a^\dagger(\boldsymbol{i})(n_{\boldsymbol{i}+\boldsymbol{e}}+1)P(\{n_{\boldsymbol{i}}-1,n_{\boldsymbol{i}+\boldsymbol{e}}+1\};t)|n_{\boldsymbol{j}}-1,n_{\boldsymbol{j}+\boldsymbol{e}}\rangle\Big) \\ &\quad= \frac{Q}{q}\sum_{\boldsymbol{e}}\sum_{\boldsymbol{i}}\Big(a^\dagger(\boldsymbol{i})a(\boldsymbol{i}+\boldsymbol{e})P(\{n_{\boldsymbol{i}}-1,n_{\boldsymbol{i}+\boldsymbol{e}}+1\};t)|n_{\boldsymbol{j}}-1,n_{\boldsymbol{j}+\boldsymbol{e}}+1\rangle\Big) \\ &\quad= \frac{Q}{q}a^\dagger(\boldsymbol{i})a(\boldsymbol{i}+\boldsymbol{e})|\psi(t)\rangle\end{aligned} \tag{3.24}$$

The second term of (3.23) is quite straightforward, by quickly glancing at (3.17) we see that it can be written as

$$\sum_{n_{\boldsymbol{j}}} n_{\boldsymbol{i}}P(\{n_{\boldsymbol{j}}\};t)|\{n_{\boldsymbol{j}}\}\rangle = a^\dagger(\boldsymbol{i})a(\boldsymbol{i})|\psi(t)\rangle \tag{3.25}$$

So the second quantised Schrödinger-like equation is then given by

$$\begin{aligned} H_Q|\psi(t)_Q\rangle &= Q\sum_{\boldsymbol{i}}\sum_{\boldsymbol{e}}\big(a^\dagger(\boldsymbol{i})a(\boldsymbol{i}+\boldsymbol{e}) - a^\dagger(\boldsymbol{i})a(\boldsymbol{i})\big)|\psi(t)\rangle \\ &= Q\sum_{\boldsymbol{i}}\sum_{\boldsymbol{e}}\big(a^\dagger(\boldsymbol{i})(a(\boldsymbol{i}+\boldsymbol{e}) - a(\boldsymbol{i}))\big)|\psi(t)\rangle \end{aligned} \tag{3.26}$$

In order to proceed we must use the following identity [4]

$$\begin{aligned} &\big(a^\dagger(\boldsymbol{i}) - a^\dagger(\boldsymbol{i}+\boldsymbol{e})\big)\big(a(\boldsymbol{i}+\boldsymbol{e}) - a(\boldsymbol{i})\big) \\ &\qquad = a^\dagger(\boldsymbol{i})\big(a(\boldsymbol{i}+\boldsymbol{e}) - a(\boldsymbol{i})\big) + a^\dagger(\boldsymbol{i}+\boldsymbol{e})\big(a(\boldsymbol{i}) - a(\boldsymbol{i}+\boldsymbol{e})\big) \end{aligned} \tag{3.27}$$

This first term is what we have in (3.26), which, as we have previously stated, describes the process of particles leaving the site $\boldsymbol{i}$ subtracted from particles hopping from the site $\boldsymbol{i}+\boldsymbol{e}$ to $\boldsymbol{i}$ in terms of the creation and annihilation operators. Comparing this to the second term, we see that it describes particles hopping away from $\boldsymbol{i}+\boldsymbol{e}$ and hopping onto $\boldsymbol{i}+\boldsymbol{e}$ from $\boldsymbol{i}$. So the identity is the addition of two terms describing the same process, where one is just shifted by $\boldsymbol{e}$, and after the shift it just reproduces the first term [4]. We can take advantage of this by dividing the unit vector in half, and substituting in (3.26),

$$H_Q|\psi(t)_Q\rangle = \frac{Q}{2}\sum_{\boldsymbol{i}}\sum_{\boldsymbol{e}}\big(a^\dagger(\boldsymbol{i}) - a^\dagger(\boldsymbol{i}+\boldsymbol{e})\big)\big(a(\boldsymbol{i}+\boldsymbol{e}) - a(\boldsymbol{i})\big)|\psi(t)\rangle \tag{3.28}$$

Moving on to the creation equation, we have

$$\frac{d}{dt}|\psi(t)_\beta\rangle = H_\beta|\psi(t)\rangle = \beta\sum_{\boldsymbol{i}}\sum_{\{n_{\boldsymbol{j}}\}}\big(P(\{n_{\boldsymbol{i}}-1\};t) - P(\{n_{\boldsymbol{j}}\};t)\big)|\{n_{\boldsymbol{j}}\}\rangle \tag{3.29}$$

With the first term being

$$\beta \sum_{\boldsymbol{i}} \sum_{\{n_j\}} P(\{n_{\boldsymbol{i}} - 1\}; t) |\{n_{\boldsymbol{j}}\}\rangle = \beta \sum_{\boldsymbol{i}} \sum_{\{n_j\}=0} a^\dagger(\boldsymbol{i}) P(\{n_{\boldsymbol{i}} - 1\}; t) |\{n_{\boldsymbol{j}} - 1\}\rangle$$
$$= 0 + \beta \sum_{\boldsymbol{i}} \sum_{\{n_j\}=1} a^\dagger(\boldsymbol{i}) P(\{n_{\boldsymbol{i}} - 1\}; t) |\{n_{\boldsymbol{j}} - 1\}\rangle \quad (3.30)$$

Where we have enforced our aforementioned boundary condition of not allowing a negative number of particles. The second term is simply the definition of our state $|\psi(t)\rangle$, so the creation equation is

$$\frac{d}{dt}|\psi(t)_\beta\rangle = \beta \sum_{\boldsymbol{i}} (a^\dagger(\boldsymbol{i}) - 1)|\psi(t)\rangle \quad (3.31)$$

Finally, we look at the extinction equation

$$\frac{d}{dt}|\psi(t)_\epsilon\rangle = \epsilon \sum_{\boldsymbol{i}} \sum_{\{n_j\}} \big((n_{\boldsymbol{i}} + 1) P(\{n_{\boldsymbol{i}} + 1\}; t) - n_{\boldsymbol{i}} P(\{n_{\boldsymbol{j}}\}; t)\big) |\{n_{\boldsymbol{j}}\}\rangle \quad (3.32)$$

The first term gives

$$\epsilon \sum_{\boldsymbol{i}} \sum_{\{n_j\}=0} \big((n_{\boldsymbol{i}} + 1) P(\{n_{\boldsymbol{j}} + 1\}; t)\big) |\{n_{\boldsymbol{j}}\}\rangle = \epsilon \sum_{\boldsymbol{i}} \sum_{\{n_j\}=0} a(\boldsymbol{i}) P(\{n_{\boldsymbol{j}} + 1\}; t) |\{n_{\boldsymbol{j}} + 1\}\rangle \quad (3.33)$$

Relabelling $n + 1 \to n$ then using the fact that $a|0\rangle$, (3.33) becomes

$$\epsilon \sum_{\boldsymbol{i}} \sum_{\{n_j\}=1} a(\boldsymbol{i}) P(\{n_{\boldsymbol{j}}\}; t) |\{n_{\boldsymbol{j}}\}\rangle = \epsilon \sum_{\boldsymbol{i}} \sum_{\{n_j\}=0} a(\boldsymbol{i}) P(\{n_{\boldsymbol{j}}\}; t) |\{n_{\boldsymbol{j}}\}\rangle \quad (3.34)$$

The second term is again, straightforward

$$\epsilon \sum_{\boldsymbol{i}} \sum_{\{n_j\}} n_{\boldsymbol{i}} P(\{n_{\boldsymbol{j}}\}; t) |\{n_{\boldsymbol{j}}\}\rangle = \epsilon \sum_{\boldsymbol{i}} a^\dagger(\boldsymbol{i}) a(\boldsymbol{i}) |\psi(t)\rangle \quad (3.35)$$

So our extinction equation is

$$H_\epsilon |\psi(t)_\epsilon\rangle = \epsilon \sum_{\boldsymbol{i}} \big(a(\boldsymbol{i}) - a^\dagger(\boldsymbol{i}) a(\boldsymbol{i})\big) |\psi(t)\rangle \quad (3.36)$$

We can now finally write out the full form of our Schrödinger-type equation after second quantising the master equation

$$\frac{d}{dt}|\psi(t)\rangle = \Big(\frac{Q}{2}\sum_{\boldsymbol{i}}\sum_{\boldsymbol{e}}\big(a^\dagger(\boldsymbol{i}) - a^\dagger(\boldsymbol{i}+\boldsymbol{e})\big)\big(a(\boldsymbol{i}+\boldsymbol{e}) - a(\boldsymbol{i})\big) + \beta\sum_{\boldsymbol{i}}(a^\dagger(\boldsymbol{i}) - 1) + \epsilon\sum_{\boldsymbol{i}}\big(a(\boldsymbol{i}) - a^\dagger(\boldsymbol{i})a(\boldsymbol{i})\big)\Big)|\psi(t)\rangle \quad (3.37)$$

Now that we have set the base of our theory, we can move on to the fun part: finding the path integral!

Chapter 4

Finding the Path Integral

4.1 On the Essence of Path Integrals

In quantum mechanics, the path integral discretises time into N small steps, and measures the superposition of every path the particle can take in the infinitesimal time interval, then goes on to find the whole quantum amplitude of every path the particle could take for every time interval [26]. To illustrate this, we look at the example of the double slit experiment: A particle is travelling from point S to point O, and there is between these points a screen with two slits, A_1 and A_2. The amplitude at point O is a superposition of the path to A_1 and the path to A_2. Now imagine adding more slits, the amplitude becomes the superposition of every path the particle can take. Now we complicate things a bit more; imagine adding an infinite number of screens, with an infinite number of slits. The quantum amplitude in this scenario is exactly what the path integral computes [27]. Now that we have understood the essence of the path integral, lets apply it to our lattice system.

4.2 The Operator Expectation Value

Lets begin with the formal solution to our Schrödinger equation (5.20) [28]

$$|\psi(t)\rangle = e^{Ht}|\psi(0)\rangle \tag{4.1}$$

We substitute this in our formula for the expected particle number (3.21)

$$\langle n\rangle(\boldsymbol{i};t) = \langle\Omega|a^{\dagger}(\boldsymbol{i})a(\boldsymbol{i})|\psi(t)\rangle = \langle\Omega|a^{\dagger}(\boldsymbol{i})a(\boldsymbol{i})e^{Ht}|\psi(0)\rangle \tag{4.2}$$

we can generalise this rule to find the expectation value of any operator

$$\langle O\rangle(t) = \langle\Omega|Oe^{Ht}|\psi(t)\rangle \tag{4.3}$$

Using this, and keeping in mind that the expectation value of unity should be 1 [28], we get

$$\langle 1\rangle(t) = \langle\Omega|\psi(t)\rangle = \langle\Omega|e^{Ht}|\psi(0)\rangle = 1 \tag{4.4}$$

Expanding (4.4) in small t

$$\langle\Omega|1 + Ht + \frac{(Ht)^2}{2!} + ...|\psi(0)\rangle = 1 \tag{4.5}$$

$$\langle\Omega|\psi(0)\rangle + \langle\Omega|Ht + \frac{(Ht)^2}{2!} + ...|\psi(0)\rangle = 1 \tag{4.6}$$

since the Ω projects everything to 1, we have

$$\langle\Omega|\psi(0)\rangle = 1 \tag{4.7}$$

(4.6) then becomes

$$\langle\Omega|Ht + \frac{(Ht)^2}{2!} + ...|\psi(0)\rangle = 0 \tag{4.8}$$

This allows us to say

$$\langle\Omega|H^n|\psi(0)\rangle = 0 \tag{4.9}$$

for all $n \geq 1$.

Since H represents a change in probability $\dot{P}(\{n_j\};t)$, the fact that it must equal 0 reflects conservation of probability. Now lets tackle the time evolution operator. As we have mentioned earlier, we begin by discretising time into N slices [22]. The time evolution operator is then given by

$$e^{-Ht} = \lim_{\Delta t\to 0}(1 - H\Delta t)^{\frac{t}{\Delta t}} = (1 - H(t_n - t_{n-1}))....(1 - H(t_1 - t_0)) \tag{4.10}$$

The resolution of unity for coherent states is given by [22]

$$\int \frac{d\phi(t)d\phi^*(t)}{\pi} e^{-\phi^*(t)\phi(t)} e^{\phi(t)a^\dagger}|0\rangle\langle 0|e^{\phi^*(t)a} \tag{4.11}$$

We are able to insert (4.11) in (4.10) between each time slice

$$e^{-Ht} = \left(\prod_{n=0}^{N}\int \frac{d\phi(t_n)d\phi^*(t_n)}{\pi} e^{-\phi^*(t_n)\phi(t_n)}\right)\Big(e^{\phi(t_N)a^\dagger}|0\rangle\langle 0|e^{\phi^*(t_N)a} \tag{4.12}$$

$$\big(1 - H(t_N - t_{N-1})\big)e^{\phi(t_{N-1})a^\dagger}|0\rangle....\langle 0|e^{\phi(t_1)a^\dagger}\big(1 - H(t_1 - t_0)\big)e^{\phi(t_0)a^\dagger}|0\rangle\langle 0|e^{\phi^*(t_0)a}\Big) \tag{4.13}$$

To simplify things, lets compute each term by itself. Beginning with the time slice from $N-1$ to N

$$\langle 0|e^{\phi^*(t_N)a}(1 - H\Delta t)e^{\phi(t_{N-1})a^\dagger}|0\rangle \tag{4.14}$$

$$= \langle 0|e^{\phi^*(t_N)a}e^{\phi(t_{N-1})a^\dagger}|0\rangle - \langle 0|e^{\phi^*(t_N)a}He^{\phi(t_{N-1})a^\dagger}|0\rangle \tag{4.15}$$

$$= \sum_{k,m=0}^{\infty}\frac{\phi^{*m}\phi^k}{k!}\langle m|k\rangle - \Delta t\langle 0|e^{\phi(t_N)a}He^{\phi(t_{N-1})a^\dagger}|0\rangle \tag{4.16}$$

Using

$$\langle m|k\rangle = \delta_{k,m}$$

The first term becomes

$$\sum_{k,m=0}^{\infty} \frac{\phi^{*m}\phi^{k}}{k!}\delta_{k,m} = \sum_{k=0}^{\infty} \frac{\left(\phi^{*}(t_{N-1})\phi(t_N)\right)^{k}}{k!} = e^{\phi(t_{N-1})\phi(t_N)} \tag{4.17}$$

Where we have Taylor expanded $e^{\phi a^\dagger}$ and $e^{\phi^* a}$ [29] and used the following property [4]

$$\langle 0|a^m = \langle m| \tag{4.18}$$

$$a^{\dagger k}|0\rangle = |k\rangle \tag{4.19}$$

To compute the second term, we make the assumption that the Hamiltonian is a function of creation and annihilation operators, i.e $H = a^{\dagger\gamma}a^{\sigma}$ where $\gamma, \sigma \geq 0$ [4]. This gives us

$$\Delta t\langle 0|e^{\phi^*(t_N)a}He^{\phi(t_{N-1})a^\dagger}|0\rangle \tag{4.20}$$

$$= \Delta t \sum_{k=\sigma,m=\gamma}^{\infty} \frac{\phi^*(t_{N-1})^m\phi(t_N)^k}{k!}\frac{k!}{(k-\sigma)!}\delta_{m-\gamma,k-\sigma} \tag{4.21}$$

$$= \Delta t \sum_{k,m=0}^{\infty} \frac{\phi^*(t_{N-1})^{m+\gamma}\phi(t_N)^{k+\sigma}}{k!}\frac{k!}{(k-\sigma)!}\delta_{m,k} \tag{4.22}$$

$$= \Delta t\phi^*(t_{N-1})^{\gamma}\phi(t_N)^{\sigma} \sum_{k,m=0}^{\infty} \frac{\phi^*(t_{N-1})^m\phi(t_N)^k}{k!}\delta_{m,k} \tag{4.23}$$

$$= \Delta t\phi^*(t_{N-1})^{\gamma}\phi(t_N)^{\sigma}e^{\phi^*(t_{N-1})\phi(t_N)} \tag{4.24}$$

Thus, (4.16) becomes

$$e^{\phi^*(t_{N-1})\phi(t_N)} - \Delta t\phi^*(t_{N-1})^{\gamma}\phi(t_N)^{\sigma}e^{\phi^*(t_{N-1})\phi(t_N)} \tag{4.25}$$

$$= e^{\phi^*(t_{N-1})\phi(t_N)}(1 - \Delta t\phi^*(t_{N-1})^{\gamma}\phi(t_N)^{\sigma}) \tag{4.26}$$

We now multiply (4.26) by the $e^{-\phi^*(t_{N-1})\phi(t_{N-1})}$ term to get

$$e^{\phi^*(t_{N-1})\phi(t_N)-\phi^*(t_{N-1})\phi(t_{N-1})}(1-\Delta t\phi^*(t_{N-1})^{\gamma}\phi(t_N)^{\sigma}) \tag{4.27}$$

Taking the continuum limit $\Delta t \to 0$

$$\prod_{n=0}^{N-1} e^{\phi^*(t_{n-1})\phi(t_n)-\phi^*(t_{n-1})\phi(t_{n-1})} \approx e^{-\int dt\phi\partial_t\phi^*} \tag{4.28}$$

The remaining terms in (4.13) are

$$e^{-\int dt\phi\partial_t\phi^*}\Big(\prod_{n=0}^{N}\int\frac{d\phi(t_n)d\phi^*(t_n)}{\pi}\Big)e^{-\phi^*(t_N)\phi(t_N)} \Big(\prod_{n=0}^{N-1}(1-\Delta t\phi^*(t_{n-1})^{\gamma}\phi(t_n)^{\sigma}\Big)e^{\phi(t_N)a^\dagger}|0\rangle\langle 0|e^{\phi^*(t_0)a} \tag{4.29}$$

Now that we have an expression for our time evolution operator, lets try and compute the expectation value of an operator

$$\langle O\rangle(t) = \langle\Omega|Oe^{-Ht}J|0\rangle \tag{4.30}$$

Where $J|0\rangle = |\psi(0)\rangle$. Lets assume that $J = a^{\dagger r}$ where r is a positive integer, which we can do since normal ordering would push all the annihilation operators to the right and $a|0\rangle = 0$ [4][30]. Substituting the result we got from (4.29) in the time evolution operator, we get this term from the right side of the equation

$$\langle 0|e^{\phi^*(t_0)a}J|0\rangle = \sum_{n=0}^{\infty}\langle n|\phi^*(t_0)^n a^{\dagger r}|0\rangle = \sum_{n=0}^{\infty}\langle n|\phi^*(t_0)^n|r\rangle = \phi^*(t_0)^r \tag{4.31}$$

Lets now divert our attention to the left side of (4.30), we assume that $O = a^j$ since normal ordering would force all the creation operators to be on the left and Ω is invariant under $a^\dagger$ [4][30], this becomes

$$\langle\Omega|Oe^{-\phi^*(t_N)\phi(t_N)}e^{\phi(t_N)a^\dagger}|0\rangle = e^{-\phi^*(t_N)\phi(t_N)}\langle\Omega|a^j e^{\phi(t_N)a^\dagger}|0\rangle \tag{4.32}$$

$$= e^{-\phi^*(t_N)\phi(t_N)}\sum_{n=0}^{\infty}\frac{\phi(t_N)^n}{n!}\langle\Omega|a^j|n\rangle \tag{4.33}$$

$$= e^{-\phi^*(t_N)\phi(t_N)}\sum_{n=l}^{\infty}\frac{\phi(t_N)^n}{n!}\frac{n!}{(n-l)!}\langle\Omega|n-l\rangle \tag{4.34}$$

$$= e^{-\phi^*(t_N)\phi(t_N)}\phi(t_N)^l\sum_{n=0}^{\infty}\frac{\phi(t_N)^n}{n!} = e^{-\phi^*(t_N)\phi(t_N)}\phi(t_N)^l e^{\phi(t_N)} \tag{4.35}$$

Thus, (4.30) becomes

$$\langle O\rangle(t) = \Big(\prod_{n=0}^{N}\int\frac{d\phi d\phi^*}{\pi}\Big)\Big(\prod_{n=0}^{N-1}(1-\Delta t\phi^*(t_{n-1})^\gamma\phi(t_n)^\sigma\Big)e^{-\int dt\phi\partial_t\phi^*} \tag{4.36}$$

$$e^{-\phi^*(t_N)\phi(t_N)}\phi(t_N)^l e^{\phi(t_N)}\phi^*(t_0)^r \tag{4.37}$$

Taking the continuum limit ($\Delta t \to 0$)

$$\langle O\rangle = \int D\phi\phi^l(t_N)e^{\phi(t_N)-\phi^*(t_N)\phi(t_N)+\int dt(\phi\partial_t\phi^*-\phi^*(t_{n-1})^\gamma\phi(t_n)^\sigma)}\phi^*(t_0)^r \tag{4.38}$$

For the sake of simple notation, we perform what is called a DOI shift [17]

$$\tilde{\phi}(t) = \phi^*(t) - 1 \tag{4.39}$$

Therefore, (4.30) becomes

$$\langle O\rangle = \int D\phi\phi^l(t_N)e^{\phi(t_N)-(\tilde{\phi}(t_N)+1)\phi(t_N)-\int dt(\phi\partial_t\tilde{\phi}+\phi^\sigma(\tilde{\phi}+1)^\gamma}\phi^*(t_0)^r \tag{4.40}$$

Integrating the $\int dt\phi\partial_t\tilde{\phi}$ term by parts

$$\int D\phi\phi^l(t_N)e^{\phi(t_N)-(\tilde{\phi}(t_N)+1)\phi(t_N)-\phi\tilde{\phi}|_{t_0}^{t_N}-\int dt(\tilde{\phi}\partial_t\phi+\phi^\sigma(\tilde{\phi}+1)^\gamma)}\phi^*(t_0)^r \tag{4.41}$$

$$= \int D\phi\phi^l(t_N)e^{\phi(t_N)-\tilde{\phi}(t_N)\phi(t_N)+\phi(t_N)-\phi(t_N)\tilde{\phi}(t_N)+\phi(t_0)\tilde{\phi}(t_0)-\int dt(\tilde{\phi}\partial_t\phi+\phi^\sigma(\tilde{\phi}+1)^\gamma}) \phi^*(t_0)^r \tag{4.42}$$

$$= \int D\phi\phi^l(t_N)e^{-\int dt(\tilde{\phi}\partial_t\phi+\phi^\sigma(\tilde{\phi}+1)^\gamma)}\phi^*(t_0)^r \tag{4.43}$$

where we have dropped the boundary term $\phi(t_0)\tilde{\phi}(t_0)$, the expectation value of our operator becomes [1]

$$\langle O\rangle = \int D\phi\phi^l(\tilde{\phi}(t_0)+1)^r e^{H_0} \tag{4.44}$$

where

$$H_0 = \int dt\big(\tilde{\phi}\partial_t\phi + \phi^\sigma(\tilde{\phi}+1)^\gamma\big) \tag{4.45}$$

We are now in a position to Fourier transform. Lets define the Fourier transforms as follows [4]

$$\phi(\boldsymbol{k},\omega) = \int dtd^dx\phi(\boldsymbol{x},t)e^{i\omega t-i\boldsymbol{k}\boldsymbol{x}} \tag{4.46}$$

$$\phi(\boldsymbol{x},t) = \int \frac{d\omega}{2\pi}\frac{d^dk}{2\pi}\phi(\boldsymbol{k},\omega)e^{-i\omega t+i\boldsymbol{k}\boldsymbol{x}} \tag{4.47}$$

The DOI-shifted term is a bit tricky, let us perform the Fourier transform of it explicitly

$$\begin{aligned}\tilde{\phi}(\omega) = \int dt\tilde{\phi}(t)e^{i\omega t} &= \int dt(\phi^*(t)-1)e^{i\omega t} \\ &= \int dt\phi^*(t)e^{i\omega t} - \int dte^{i\omega t} = \phi^*(-\omega) - 2\pi\delta(\omega)\end{aligned} \tag{4.48}$$

We now move on to Fourier transforming the action (4.45), beginning with the derivative term $\partial_t\phi$

$$\begin{aligned}\dot{\phi}(t) = \int_{-\infty}^{\infty} d\omega\dot{\phi}(\omega)e^{-i\omega t} &= \int d\omega\frac{d}{dt}\phi(\omega)e^{-i\omega t} \\ &= -\int d\omega\phi(\omega)\frac{d}{dt}e^{-i\omega t} = -\int d\omega\phi(\omega)(-i\omega)e^{-i\omega t}\end{aligned} \tag{4.49}$$

[1] Here we are dropping some notation for simplicity

In perturbation theory, we get the following term in our action [2]

$$\int_{-\infty}^{\infty} dt\big(\tilde{\phi}(t)\dot{\phi}(t) + \epsilon\tilde{\phi}(t)\phi(t)\big) \tag{4.50}$$

Fourier transforming it, we get

$$\begin{aligned}
&\int dt d\omega d\omega'\big(\tilde{\phi}(\omega')\dot{\phi}(\omega) + \epsilon\tilde{\phi}(\omega')\phi(\omega)\big) \\
&\qquad = -\int dt d\omega d\omega'\big(\tilde{\phi}(\omega')\phi(\omega)(-i\omega)e^{-i(\omega+\omega')t} + \epsilon\tilde{\phi}(\omega')\phi(\omega)e^{-i(\omega+\omega')t}\big) \\
&\quad = -\int dt d\omega d\omega'\tilde{\phi}(\omega')(\epsilon - i\omega)\phi(\omega)e^{-i(\omega+\omega')t} = -\int d\omega\tilde{\phi}(-\omega)(-i\omega+\epsilon)\phi(\omega) \\
&= -\int đ\omega\big(\phi^*(\omega) - \frac{\delta}{2\pi}(-\omega)\big)(-i\omega+\epsilon)\phi(\omega) = -\int đ\omega\phi^*(\omega)(-i\omega+\epsilon)\phi(\omega)
\end{aligned} \tag{4.51}$$

The Fourier transformed action with the extinction term is then

$$\tilde{H}_\epsilon = -\int d\omega'\big(\phi^*(\omega')(-i\omega'+\epsilon)\phi(\omega')\big) - \delta(\omega') \tag{4.52}$$

If it were not for the extra $\delta(\omega')$ term, our action would be local in ω. But to find the path integral, we require it to be in a Gaussian form i.e. $\frac{1}{\pi}\int dxdye^{-z*Az} = \frac{1}{A}$, not the form we currently have

$$\int \frac{d\phi^* d\phi}{2\pi i}e^{-\tilde{\phi}(-\omega)A\phi(\omega)} = \int \frac{d\phi^* d\phi}{2\pi i}e^{-(\phi^*(\omega)-\frac{\delta}{2\pi}(\omega))A\phi(\omega)} \tag{4.53}$$

We use the following identity [4]

$$\int \frac{dz^* dz}{2\pi i}e^{-(z+\zeta)^*(z+\xi)}(z+\zeta)^{*n}(z+\xi)^m = \delta_{n,m}n! \tag{4.54}$$

So (4.53) becomes

$$\int \frac{d\phi^* d\phi}{2\pi i}e^{-\phi^*(\omega)A\phi(\omega)}e^{\frac{\delta}{2\pi}(\omega)A\phi(\omega)} \tag{4.55}$$

$$= \int \frac{d\phi d\phi^*}{2\pi i} e^{-\phi^*(\omega)A\phi(\omega)} \tag{4.56}$$

From there we finally arrive at our Gaussian path integral

$$\int D\phi e^{\tilde{H}_\epsilon} = \int D\phi e^{-\int d\omega \phi^*(\omega)(-i\omega+\epsilon)\phi(\omega)} \tag{4.57}$$

$\tilde{H}_\epsilon$ now contains both the extinction term ϵ and the temporal dependence (from the time derivative). Now to generalize this to a lattice, we incorporate spacial dependence by summing over the lattice [4]

$$H_\epsilon = -\sum_{\boldsymbol{y}} \int đ\omega(\phi^*(\boldsymbol{y},\omega)(-i\omega+\epsilon)\phi(\boldsymbol{y},\omega)) \tag{4.58}$$

Rescaling becomes part of the normalisation, and the sum $\sum_{\boldsymbol{y}} a^d$ becomes $\int d^d y$, we then Fourier transform

$$H_\epsilon = -\int đ\omega \int dy \phi^*(\boldsymbol{y},\omega)(-i\omega+\epsilon)\phi(\boldsymbol{y},\omega) \tag{4.59}$$

$$= -\int dy đ\omega đ^d k đ^d k' \phi^*(k,\omega)(-i\omega+\epsilon)\phi(k',\omega)e^{i\boldsymbol{y}(k'-k)} \tag{4.60}$$

$$= -\int đ\omega đ^d k \phi^*(k,\omega)(-i\omega+\epsilon)\phi(k,\omega) \tag{4.61}$$

$$= -\int đ^d k đ\omega \tilde{\phi}(-k,-\omega)(-i\omega+\epsilon)\phi(k,\omega) \tag{4.62}$$

Which we can see is analogous to our Klein-Gordon action

$$-\frac{1}{2}\int d^{d-1}x dt \phi^*(\boldsymbol{x},t)(\Box+m^2)\phi(\boldsymbol{x},t) \tag{4.63}$$

Now its time to add the diffusion term (particle hopping from $\boldsymbol{y}$ to $\boldsymbol{y}+\boldsymbol{e}$). The action for diffusion is

$$H_D = -\frac{Q}{2q}\int dt\big(\tilde{\phi}(\boldsymbol{y}+\boldsymbol{e},t)-\tilde{\phi}(\boldsymbol{y},t)\big)\big(\phi(\boldsymbol{y}+\boldsymbol{e},t)-\phi(\boldsymbol{y},t)\big) \tag{4.64}$$

Using the approximation [4]

$$\begin{pmatrix} \phi(\boldsymbol{y}+\boldsymbol{e})-\phi(\boldsymbol{y},t) \\ \phi(\boldsymbol{x}+\boldsymbol{e})-\phi(\boldsymbol{x},t) \\ . \\ . \\ . \\ \phi(\boldsymbol{z}+\boldsymbol{e})-\phi(\boldsymbol{z},t) \end{pmatrix} = a\nabla\phi(\boldsymbol{y},t)+O(a^2) \tag{4.65}$$

The action (4.64) becomes

$$H_D = -\frac{Q}{2q}\int dt \sum_{\boldsymbol{y}} \Big(a^d \frac{2a^2\nabla\tilde{\phi}(\boldsymbol{y},t)\cdot\phi(\boldsymbol{y},t)}{a^d} + O(a^3)\Big) \tag{4.66}$$

Where

$$\boldsymbol{a} = |\boldsymbol{e_x}| = |\boldsymbol{e_y}| = ... = |\boldsymbol{e_z}|$$

$\boldsymbol{a}$ is the lattice spacing and e_i are the basis vectors.

In order to take the continuum limit ($a \to \infty$), we must take $\frac{Q}{2q}$ as a constant C, and $\sum_{\boldsymbol{y}} a^d \to \int d^d y$

$$H_D = -C\int dt d^d y \Big(\frac{\nabla\tilde{\phi}(\boldsymbol{y},t)\cdot\nabla\phi(\boldsymbol{y},t)}{a^d}\Big) \tag{4.67}$$

(The a^d gets absorbed into the Jacobian) [4]. Fourier transforming, we get

$$H_D = -C\int dt d^d y d^d k d^d k' d\omega d\omega' (i\boldsymbol{k}\cdot i\boldsymbol{k}')\tilde{\phi}(\boldsymbol{k}',\omega')\phi(\boldsymbol{k},\omega)e^{i(\boldsymbol{k}+\boldsymbol{k}')\boldsymbol{y}}e^{-i(\omega+\omega')t} \tag{4.68}$$

$$= -C\int d\omega d^d k \boldsymbol{k}^2 \tilde{\phi}(-\boldsymbol{k},-\omega)\phi(k,\omega) \tag{4.69}$$

Our action including extinction and diffusion is then

$$H_{D\epsilon} = -\int d^d k đ\omega \tilde{\phi}(-\boldsymbol{k},-\omega)(-i\omega + C\boldsymbol{k}^2+\epsilon)\phi(\boldsymbol{k},\omega) \tag{4.70}$$

So the expectation value of an observable is given by

$$\langle O\rangle(\boldsymbol{x},t;\boldsymbol{x}_0,t) = \langle \phi^l(\boldsymbol{x},t)\big(1+\tilde{\phi}(\boldsymbol{x}_0,t_0)\big)^{\gamma}\rangle = \int D\phi\phi^l(\boldsymbol{x},t)e^{H}\big(1+\tilde{\phi}(\boldsymbol{x}_0,t_0)\big)^{\gamma} \quad (4.71)$$

4.3 The Propagator

We are now in a position to write down our response propagator, which describes how our system responds after probing it at an initial time

$$\langle \phi(\boldsymbol{k},\omega)\tilde{\phi}(\boldsymbol{k}_0,\omega_0)\rangle \quad (4.72)$$

$$= \int D\phi\phi(\boldsymbol{k},\omega)\tilde{\phi}(\boldsymbol{k}_0,\omega_0)e^{-\int đ^d k đ\omega \tilde{\phi}(-\boldsymbol{k},-\omega)(-i\omega+C\boldsymbol{k}^2+\epsilon)\phi(\boldsymbol{k},\omega)} \quad (4.73)$$

$$= \frac{\frac{\delta}{2\pi}^d(\boldsymbol{k}+\boldsymbol{k}_0)\frac{\delta}{2\pi}(\omega+\omega_0)}{-i\omega+C\boldsymbol{k}^2+\epsilon} \quad (4.74)$$

For any correlator

$$\langle \phi(\boldsymbol{k}_1,\omega_1)...\phi(\boldsymbol{k}_n,\omega_n)\tilde{\phi}(\boldsymbol{k}'_1,\omega_1)...\tilde{\phi}(\boldsymbol{k}'_m,\omega'_m)\rangle \quad (4.75)$$

Wick's theorem states [26]

$$\langle \phi(\boldsymbol{k}_1,\omega_1)...\phi(\boldsymbol{k}_n,\omega_n)\tilde{\phi}(\boldsymbol{k}'_1,\omega_1)...\tilde{\phi}(\boldsymbol{k}'_m,\omega'_m)\rangle = \langle\phi_1\tilde{\phi}_1\rangle...\langle\phi_n\tilde{\phi}_n\rangle + \langle\phi_1\tilde{\phi}_2\rangle... + ... \quad (4.76)$$

In which (4.75) equals 0 for $n \neq m$ with $n,m \in \mathbb{N}$ We can write our bare propagator as

$$\langle \phi(\boldsymbol{k},\omega)\tilde{\phi}(\boldsymbol{k}_0,\omega_0)\rangle = \frac{\delta}{2\pi}^d(\boldsymbol{k}+\boldsymbol{k}_0)\frac{\delta}{2\pi}(\omega+\omega_0)G_0(\boldsymbol{k},\omega) \quad (4.77)$$

with $G(\boldsymbol{k},\omega) = \frac{1}{-i\omega+C\boldsymbol{k}^2+\epsilon}$. It will prove to be useful to inverse Fourier transform our bare propagator (4.77), beginning with the time component

$$\int đ\omega đ\omega_0 e^{-i\omega t}e^{-i\omega_0 t_0}\frac{\delta}{2\pi}^d(\boldsymbol{k}+\boldsymbol{k}_0)\frac{\delta}{2\pi}(\omega+\omega_0)G_0(\boldsymbol{k},\omega) \quad (4.78)$$

$$= \frac{\delta}{2\pi}^{d}(\boldsymbol{k}+\boldsymbol{k}_0)\int đ\omega e^{-i\omega(t+t_0)}G_0(\boldsymbol{k},\omega) \tag{4.79}$$

Inverse Fourier transforming G_0

$$G_0(\boldsymbol{k},t) = \int đ\omega G(\boldsymbol{k},\omega)e^{-i\omega t} \tag{4.80}$$

$$= \int đ\omega \frac{1}{-i\omega + C\boldsymbol{k}^2 + \epsilon}e^{-i\omega t} = \theta(t)e^{-t(C\boldsymbol{k}^2+\epsilon)} \tag{4.81}$$

Our Fourier transformed propagator is then

$$\int đ\omega đ\omega_0 e^{-i\omega t}e^{-i\omega_0 t_0}\langle\phi(\boldsymbol{k},\omega)\tilde{\phi}(\boldsymbol{k}_0,\omega_0)\rangle = \frac{\delta}{2\pi}^{d}(\boldsymbol{k}+\boldsymbol{k}_0)\int đ\omega e^{-i\omega(t+t_0)}G_0(\boldsymbol{k},\omega) \tag{4.82}$$

$$= \frac{\delta}{2\pi}^{d}(\boldsymbol{k}+\boldsymbol{k}_0)\int đ\omega e^{-i\omega(t+t_0)}\frac{1}{-i\omega + C\boldsymbol{k}^2+\epsilon} = \theta(t-t_0)e^{-(t-t_0)(C\boldsymbol{k}^2+\epsilon)} \tag{4.83}$$

Where $\theta(t)$ is our Heaviside function, which tells us we have no particle density to measure at negative time [2]. We can read from this result that the $\boldsymbol{k} = 0$ term remains only if there is no particle extinction.

Inverse Fourier transforming to get the spacial component, we get

$$\int đ^d k e^{-tC\boldsymbol{k}^2}e^{i\boldsymbol{k}\cdot\boldsymbol{x}} = \frac{1}{(4\pi Ct)^{\frac{d}{2}}}e^{\frac{-\boldsymbol{x}^2}{4Ct}} \tag{4.84}$$

Thus, the diffusion starting at x_0, t_0 with extinction rate ϵ is

$$\langle\phi(\boldsymbol{x},t)\tilde{\phi}(\boldsymbol{x}_0,t_0)\rangle = G_0(\boldsymbol{x}-\boldsymbol{x}_0,t-t_0) = \theta(t-t_0)\frac{e^{-(t-t_0)\epsilon}}{\big(4\pi C(t-t_0)\big)^{\frac{d}{2}}}e^{-\frac{(\boldsymbol{x}-\boldsymbol{x}_0)^2}{4C(t-t_0)}} \tag{4.85}$$

4.4 The Source Term

We now introduce our source term β and treat it perturbatively. Splitting the total action into two parts [2]

$$H = H_{D\epsilon} + H_\beta \tag{4.86}$$

$$e^H = e^{H_{D\epsilon}} e^{H_\beta} = e^{H_{D\epsilon}} \sum_{n=0}^{\infty} \frac{\beta^n}{n!} \tilde{\phi}^n(\mathbf{0}, 0) \tag{4.87}$$

where

$$H_\beta = \int đ^d k đ\omega \beta \tilde{\phi}(\boldsymbol{k}, \omega) \frac{\delta}{2\pi}^d (\boldsymbol{k}) \frac{\delta}{2\pi}(\omega) \tag{4.88}$$

$$H_{D\epsilon} = -\int đ^d k đ\omega \tilde{\phi}(-\boldsymbol{k}, -\omega)(-i\omega + C\boldsymbol{k}^2 + \epsilon)\phi(\boldsymbol{k}, \omega) \tag{4.89}$$

We deduced the form of H_β from (3.31) bt taking $a \to \phi$ and $a^\dagger \to \tilde{\phi}$. In order to calculate the expectation value, we must note the following property [6]

$$\langle O \rangle_{D\epsilon} = N_{D\epsilon} \int D\phi O e^{H_{D\epsilon}} \tag{4.90}$$

$$\langle O \rangle = N \int D\phi e^H \tag{4.91}$$

$$= \frac{N_{D\epsilon}}{N} \langle O e^{H_\beta} \rangle_{D\epsilon} \tag{4.92}$$

In which normalisation forces $N = N_{D\epsilon}$ or $\frac{N_{D\epsilon}}{N} = 1$ [4]. Using these relations, we can then conclude that the correlation propagator for our action including the source term becomes

$$\langle \phi_1 ... \phi_n \tilde{\phi}_1 ... \tilde{\phi}_n \rangle = \langle \phi(\boldsymbol{k}_1, \omega_1) ... \phi(\boldsymbol{k}_n, \omega_n) e^{H_\beta} \rangle_{normal} \tag{4.93}$$

$$= \langle \phi(\boldsymbol{k}_1, \omega_1)\phi(\boldsymbol{k}_2, \omega_2)...\phi(\boldsymbol{k}_n, \omega_n)\big(1 + \beta\tilde{\phi}(\mathbf{0}, 0) + \frac{\beta^2}{2!}\tilde{\phi}^2(\mathbf{0}, 0) + ... + \frac{\beta^m}{m!}\tilde{\phi}^m(\mathbf{0}, 0)\big)\rangle \tag{4.94}$$

$$= \frac{\beta^m}{m!} \langle \phi(\boldsymbol{k}_1, \omega_1)\phi(\boldsymbol{k}_2, \omega_2)...\phi(\boldsymbol{k}_n, \omega_n)\tilde{\phi}(\mathbf{0}, 0)\tilde{\phi}(\mathbf{0}, 0)...\tilde{\phi}(\mathbf{0}, 0)\rangle \tag{4.95}$$

$$= \beta^n \prod_{i=1}^{n} \frac{\frac{\delta}{2\pi}^d(\boldsymbol{k}_i)\frac{\delta}{2\pi}(\omega_i)}{-i\omega_i + C\boldsymbol{k}_i^2 + \epsilon} \tag{4.96}$$

$$= \left(\frac{\beta}{\epsilon}\right)^n \prod_{i=1}^{n} \frac{\delta}{2\pi}^d(\boldsymbol{k}_i)\frac{\delta}{2\pi}(\omega_i) \tag{4.97}$$

Note that the $n!$ in the denominator cancels with the $n!$ from Wick's theorem.

Now let's compute our response propagator

$$\langle \phi(\boldsymbol{k}, t)\tilde{\phi}(\boldsymbol{k}_0, \omega_0)\rangle = \langle \phi(\boldsymbol{k}, t)\tilde{\phi}(\boldsymbol{k}_0, \omega_0)e^{H_\beta}\rangle_{D\epsilon} \tag{4.98}$$

$$= \langle \phi(\boldsymbol{k}, t)\tilde{\phi}(\boldsymbol{k}_0, \omega_0)\rangle_{D\epsilon} \tag{4.99}$$

$$= \frac{\frac{\delta}{2\pi}^d(\boldsymbol{k} - \boldsymbol{k}_0)\frac{\delta}{2\pi}(\omega, \omega_0)}{-i\omega + C\boldsymbol{k}^2 + \epsilon} \tag{4.100}$$

Inverse Fourier transforming, we get

$$\theta(t - t_0)\frac{e^{-(t-t_0)\epsilon}}{\left(4\pi C(t - t_0)\right)^{\frac{d}{2}}}e^{-\frac{(\boldsymbol{x}-\boldsymbol{x}_0)^2}{4C(t-t_0)}} \tag{4.101}$$

An interesting case is when $n = 1$ in (4.97)

$$\langle \phi(\boldsymbol{k}, \omega)\rangle = \beta\langle \phi(\boldsymbol{k}, \omega)\tilde{\phi}(\boldsymbol{0}, 0)\rangle_{D\epsilon} = \frac{\beta}{\epsilon}\frac{\delta}{2\pi}^d(\boldsymbol{k})\frac{\delta}{2\pi}(\omega) \tag{4.102}$$

Performing the inverse Fourier transform give us

$$\langle \phi(\boldsymbol{x}, t)\rangle = \frac{\beta}{\epsilon} \tag{4.103}$$

This suggests that there is always a uniform background particle density present. However, this contradicts our result for (4.101), which suggests that the contribution of the field at an initial time eventually vanishes. Where is the inconsistency?

This issue is arising because we are calculating the wrong correlation function, what we want to calculate is $\langle \phi(\boldsymbol{x}, t)\phi^*(\boldsymbol{x}_0, t_0)\rangle$, and this is because our creation operator corresponds to ϕ^* rather than $\tilde{\phi}$ [4]. Our desired propagator is then

$$\langle \phi(\boldsymbol{x}, t)\big(1 + \tilde{\phi}(\boldsymbol{x}_0, t_0)\big)\rangle = \langle \phi(\boldsymbol{x}, t)\rangle + \langle \phi(\boldsymbol{x}, t)\tilde{\phi}(\boldsymbol{x}_0, t_0)\rangle \tag{4.104}$$

We inverse Fourier transform, so (4.104) becomes

$$\frac{\beta}{\epsilon} + \theta(t - t_0)\frac{e^{-(t-t_0)\epsilon}}{\big(4\pi C(t - t_0)\big)^{\frac{d}{2}}} e^{-\frac{(x-x_0)^2}{4Ct}} \tag{4.105}$$

which agrees of our prediction of always having a background particle density present.

Chapter 5

Branching

5.1 The Master Equation Revisited

For the final part of this work, we will discuss branching, our analogue for scattering in Quantum Field Theory. Branching is the process of a particle producing offspring and then dying off [15]. But how do we write its contribution to our master equation (3.7)? The influx towards our lattice having configuration $\{n_{\boldsymbol{j}}\}$ is achieved if a particle in site $\boldsymbol{j}$ has one particle too little $\{n_{\boldsymbol{j}} - 1\}$, and one of the particles turns into two particles with branching rate σ

$$\sigma \sum_{\boldsymbol{i}} (n_{\boldsymbol{i}} - 1) P(\{n_{\boldsymbol{i}} - 1\}; t) \tag{5.1}$$

On the other hand, the outflow from the probability of our desired lattice configuration occurs if the lattice was already in said configuration but one particle turned into two with branching rate σ, resulting in one particle too many

$$\sigma \sum_{\boldsymbol{i}} n_{\boldsymbol{i}} P(\{n_{\boldsymbol{j}}\}; t) \tag{5.2}$$

The branching master equation is then

$$H_\sigma = \sigma \sum_{\boldsymbol{i}} \big((n_{\boldsymbol{i}} - 1)P(\{n_{\boldsymbol{i}} - 1\}; t) - n_{\boldsymbol{i}} P(\{n_{\boldsymbol{j}}\}; t)\big) \tag{5.3}$$

So ultimately our master equation (3.7) after incorporating branching becomes

$$\begin{aligned} \frac{d}{dt}P(\{n_{\boldsymbol{j}}\}; t) = H \sum_{\boldsymbol{e}} \sum_{\boldsymbol{i}} \big((n_{\boldsymbol{i+e}} + 1)P(\{n_{\boldsymbol{i}} - 1, n_{\boldsymbol{i+e}} + 1\}; t) - n_{\boldsymbol{i}} P(\{n_{\boldsymbol{j}}\}; t)\big) \\ + \epsilon \sum_{\boldsymbol{i}} \big((n_{\boldsymbol{i}} + 1)P(\{n_{\boldsymbol{i}} + 1\}; t) - n_{\boldsymbol{i}} P(\{n_{\boldsymbol{j}}\}; t)\big) + \beta \sum_{\boldsymbol{i}} \big(P(\{n_{\boldsymbol{i}} - 1\}; t) - P(\{n_{\boldsymbol{j}}\})\big) \\ + \sigma \sum_{\boldsymbol{i}} \big((n_{\boldsymbol{i}} - 1)P(\{n_{\boldsymbol{i}} - 1\}; t) - n_{\boldsymbol{i}} P(\{n_{\boldsymbol{j}}\}; t)\big) \end{aligned} \tag{5.4}$$

5.2 Second Quantisation Revisited

Repeating the process we've previously done in section 3.3.1, we proceed to writing our branching master equation (5.3) in operator form to get our Schrödinger-like equation

$$\frac{d}{dt}|\psi(t)\rangle = H_\sigma |\psi(t)\rangle \tag{5.5}$$

$$= \sigma \sum_{\boldsymbol{i}} \sum_{\{n_j\}} \big((n_{\boldsymbol{i}} - 1)P(\{n_{\boldsymbol{i}} - 1\}; t)|\{n_{\boldsymbol{j}}\}\rangle - n_{\boldsymbol{i}} P(\{n_{\boldsymbol{j}}\}; t)|\{n_{\boldsymbol{j}}\}\rangle\big) \tag{5.6}$$

$$= \sigma \sum_{\boldsymbol{i}} \sum_{\{n_j\}} \big(a^\dagger(\boldsymbol{i})(n_{\boldsymbol{i}} - 1)P(\{n_{\boldsymbol{i}} - 1\})|\{n_{\boldsymbol{j}} - 1\}\rangle - a^\dagger(\boldsymbol{i})a(\boldsymbol{i})P(\{n_{\boldsymbol{j}}\})|\{n_{\boldsymbol{j}}\}\rangle\big) \tag{5.7}$$

$$= \sigma \sum_{\boldsymbol{i}} \sum_{\{n_j\}} \big(a^{\dagger^2}(\boldsymbol{i})a(\boldsymbol{i})P(\{n_{\boldsymbol{i}} - 1\})|\{n_{\boldsymbol{j}} - 1\}\rangle - a^\dagger(\boldsymbol{i})a(\boldsymbol{i})P(\{n_{\boldsymbol{j}}\})|\{n_{\boldsymbol{j}}\}\rangle\big) \tag{5.8}$$

$$= \sigma \sum_{\boldsymbol{i}} \big(a^{\dagger^2}(\boldsymbol{i})a(\boldsymbol{i}) - a^\dagger(\boldsymbol{i})a(\boldsymbol{i})\big)|\psi(t)\rangle \tag{5.9}$$

For notational convenience, we write our branching operator as follows [4]

$$H_\sigma = \sigma \sum_{\boldsymbol{i}} \big(a^{\dagger^2}(\boldsymbol{i})a(\boldsymbol{i}) - a^\dagger(\boldsymbol{i})a(\boldsymbol{i})\big) = \sigma \sum_{\boldsymbol{i}} \big(\tilde{a}(\boldsymbol{i})a(\boldsymbol{i}) + \tilde{a}^2(\boldsymbol{i})a(\boldsymbol{i})\big) \tag{5.10}$$

where

$$\tilde{a}(\boldsymbol{i}) = a^{\dagger}(\boldsymbol{i}) - 1 \tag{5.11}$$

Since the first term of (5.10) is bilinear, it is essentially a mass shift due to the change in the present number of particles. The second term is what accounts for the branching. Translating the operators into fields, the total action becomes

$$H = \int đ^d k đ\omega \tilde{\phi}(-\boldsymbol{k}, -\omega)(-i\omega' + C\boldsymbol{k}^2 + \epsilon - \sigma)\phi(\boldsymbol{k}, \omega) + \int đ^d k đ\omega \beta \tilde{\phi}(\boldsymbol{k}, \omega) \frac{\delta}{2\pi}^d(\boldsymbol{k}) \frac{\delta}{2\pi}(\omega) \tag{5.12}$$

$$+\sigma \int d^d k_1 d^d k_2 d\omega_1 d\omega_2 \tilde{\phi}(\boldsymbol{k}_1, \omega_1)\tilde{\phi}(\boldsymbol{k}_2, \omega_2)\phi(-(\boldsymbol{k}_1 + \boldsymbol{k}_2), -(\omega_1 + \omega_2)) \tag{5.13}$$

We note here that we must enforce $\epsilon - \sigma \geq 0$ to keep the integral from diverging. Our new bare propagator is given by

$$\langle \phi(\boldsymbol{k}, \omega)\tilde{\phi}(\boldsymbol{k}_0, \omega_0)\rangle = \frac{\frac{\delta}{2\pi}^d(\boldsymbol{k} + \boldsymbol{k}_0)\frac{\delta}{2\pi}(\omega + \omega_0)}{-i\omega + C\boldsymbol{k}^2 + \epsilon - \sigma} \tag{5.14}$$

Since we are treating branching in a perturbative manner, we split the action [2]

$$H = H_{D\epsilon} + H_{\beta} + H_{\sigma} \tag{5.15}$$

$$e^H = e^{H_{D\epsilon}} e^{H_\beta} e^{H_\sigma} \tag{5.16}$$

$$= e^{H_{D\epsilon}} \sum_{n=0}^{\infty} \frac{\beta^n}{n!}\tilde{\phi}^n(\boldsymbol{0}, 0) \sum_{m=0}^{\infty} \frac{\sigma^m}{m!} \left(\tilde{\phi}(\boldsymbol{k}_1', \omega_1')\tilde{\phi}(\boldsymbol{k}_2', \omega_2')\phi(-(\boldsymbol{k}_1' + \boldsymbol{k_2}'), -(\omega_1' + \omega_2'))\right)^m \tag{5.17}$$

Thus, the expectation value would be

$$\langle O \rangle = \frac{N_{D\epsilon}}{N} \langle O e^{H_\beta} e^{H_\sigma} \rangle_{D\epsilon} \tag{5.18}$$

Where the relation $\frac{N_{D\epsilon}}{N} = 1$ still holds.

5.3 An Example

Now lets try and see branching in action. As an example, we will compute the first few terms of the correlator

$$\langle \phi(\boldsymbol{k}_2,\omega_2)\phi(\boldsymbol{k}_1,\omega_1)\tilde{\phi}(\boldsymbol{k}_0,\omega_0)\rangle \tag{5.19}$$

Beginning with the case of $n=1$ and $m=0$ in (5.17)

$$\langle \phi(\boldsymbol{k}_2,\omega_2)\phi(\boldsymbol{k}_1,\omega_1)\tilde{\phi}(\boldsymbol{k}_0,\omega_0)\beta\tilde{\phi}(\boldsymbol{0},0)\rangle_{D\epsilon} \tag{5.20}$$

$$= \beta\langle \phi_2\tilde{\phi}_0\rangle_{D\epsilon}\langle \phi_1\tilde{\phi}(\boldsymbol{0},0)\rangle_{D\epsilon} + \beta\langle \phi_1\tilde{\phi}_0\rangle_{D\epsilon}\langle \phi_1\tilde{\phi}(\boldsymbol{0},0)\rangle_{D\epsilon} \tag{5.21}$$

$$= \frac{\frac{\delta}{2\pi}^d(\boldsymbol{k}_2+\boldsymbol{k}_0)\frac{\delta}{2\pi}(\omega_2+\omega_0)}{-i\omega_2+C\boldsymbol{k}_2^2+\epsilon-\sigma}\beta\frac{\frac{\delta}{2\pi}^d(\boldsymbol{k}_1)\frac{\delta}{2\pi}(\omega_1)}{-i\omega_1+C\boldsymbol{k}_1^2+\epsilon-\sigma} \tag{5.22}$$

$$+\frac{\frac{\delta}{2\pi}^d(\boldsymbol{k}_1+\boldsymbol{k}_0)\frac{\delta}{2\pi}(\omega_1+\omega_0)}{-i\omega_1+C\boldsymbol{k}_1^2+\epsilon-\sigma}\beta\frac{\frac{\delta}{2\pi}^d(\boldsymbol{k}_2)\frac{\delta}{2\pi}(\omega_2)}{-i\omega_2+C\boldsymbol{k}_2^2+\epsilon-\sigma} \tag{5.23}$$

The first term (5.22) gives rise to the disconnected Feynman diagram [1]

$$\tilde{\phi}_0 \longrightarrow \phi_2 \quad \tilde{\phi}_{\boldsymbol{0},0} \longrightarrow \phi_1$$

While the second term (5.23) gives

$$\tilde{\phi}_0 \longrightarrow \phi_1 \quad \tilde{\phi}_{\boldsymbol{0},0} \longrightarrow \phi_2$$

Inverse Fourier transforming (5.22), we get

$$\int d\omega_0 d\omega_1 d\omega_2 e^{-i\omega_0 t_0}e^{-i\omega_1 t_1}e^{-i\omega_2 t_2}\frac{\delta}{2\pi}^d(\boldsymbol{k}_2+\boldsymbol{k}_0)\frac{\delta}{2\pi}(\omega_2+\omega_0)\frac{\delta}{2\pi}^d(\boldsymbol{k}_1)\frac{\delta}{2\pi}(\omega_1)G(\boldsymbol{k}_2,\omega_2)G(\boldsymbol{k}_1,\omega_1) \tag{5.24}$$

$$= \frac{\delta}{2\pi}^d(\boldsymbol{k}_2+\boldsymbol{k}_0)\frac{\delta}{2\pi}^d(\boldsymbol{k}_1)\int d\omega_0 d\omega_2 e^{-i\omega_0 t_0}e^{-i\omega_2 t_2}\frac{\delta}{2\pi}(\omega_2+\omega_0)G(\boldsymbol{k}_2,\omega_2)G(\boldsymbol{k}_1,\omega_1) \tag{5.25}$$

$$= \frac{\delta}{2\pi}^d(\boldsymbol{k}_2+\boldsymbol{k}_0)\frac{\delta}{2\pi}^d(\boldsymbol{k}_1)G(\boldsymbol{k}_1,0)\int d\omega_2 e^{i\omega_2 t_0}e^{-i\omega_2 t_2}G(\boldsymbol{k}_2,\omega_2) \tag{5.26}$$

[1] The flow of time is from left to right

$$= \frac{\delta}{2\pi}^{d}(\boldsymbol{k}_2+\boldsymbol{k}_0)\frac{\delta}{2\pi}^{d}(\boldsymbol{k}_1)G(\boldsymbol{k}_1,0)\int d\omega_2 e^{-i\omega_2(t_2-t_0)}G(\boldsymbol{k}_2,\omega_2) \tag{5.27}$$

$$= \frac{\delta}{2\pi}^{d}(\boldsymbol{k}_2+\boldsymbol{k}_0)\frac{\delta}{2\pi}^{d}(\boldsymbol{k}_1)G(\boldsymbol{k}_1,0)\theta(t_2-t_0)e^{-(t_2-t_0)(C\boldsymbol{k}_2^2+\epsilon)} \tag{5.28}$$

$$= \int d^d\boldsymbol{k}_0 d^d\boldsymbol{k}_1 d^d\boldsymbol{k}_2 e^{i\boldsymbol{k}_0\boldsymbol{x}_0}e^{i\boldsymbol{k}_1\boldsymbol{x}_1}e^{i\boldsymbol{k}_2\boldsymbol{x}_2}\frac{\delta}{2\pi}^{d}(\boldsymbol{k}_2+\boldsymbol{k}_0)\frac{\delta}{2\pi}^{d}(\boldsymbol{k}_1)G(\boldsymbol{k}_1,0)\theta(t_2-t_0)e^{-(t_2-t_0)(C\boldsymbol{k}_2^2+\epsilon)} \tag{5.29}$$

$$= \frac{\beta}{\epsilon-\sigma}\int d^d\boldsymbol{k}_0 d^d\boldsymbol{k}_2 e^{i\boldsymbol{k}_0\boldsymbol{x}_0}e^{i\boldsymbol{k}_2\boldsymbol{x}_2}\frac{\delta}{2\pi}^{d}(\boldsymbol{k}_2+\boldsymbol{k}_0)\theta(t_2-t_0)e^{(t_2-t_0)(C\boldsymbol{k}_2^2+\epsilon)} \tag{5.30}$$

$$= \frac{\beta}{\epsilon-\sigma}\int d^d\boldsymbol{k}_2 e^{i\boldsymbol{k}_2(\boldsymbol{x}_2-\boldsymbol{x}_0)}\theta(t_2-t_0)e^{(t_2-t_0)(C\boldsymbol{k}_2^2+\epsilon)} \tag{5.31}$$

$$\frac{\beta}{\epsilon-\sigma}\theta(t_2-t_0)\frac{e^{-(t_2-t_0)\epsilon}}{(4\pi D(t_2-t_0))^{\frac{d}{2}}}e^{-\frac{(\boldsymbol{x}_2-\boldsymbol{x}_0)^2}{4C(t_2-t_0)}} \tag{5.32}$$

$$= \frac{\beta}{\epsilon-\sigma}G_0(\boldsymbol{x}_2-\boldsymbol{x}_0,t_2-t_0) \tag{5.33}$$

Inverse Fourier transforming (5.23) in the same manner, (5.20) becomes

$$\frac{\beta}{\epsilon-\sigma}G_0(\boldsymbol{x}_2-\boldsymbol{x}_0,t_2-t_0;\epsilon-\sigma)+\frac{\beta}{\epsilon-\sigma}G_0(\boldsymbol{x}_1-\boldsymbol{x}_0,t_1-t_0;\epsilon-\sigma) \tag{5.34}$$

Now considering the case of $n=0$ and $m=1$ in (5.17)

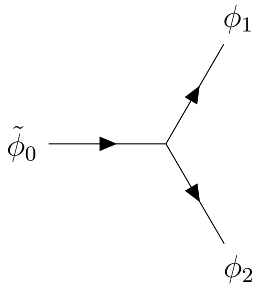
$$\langle\phi(\boldsymbol{k}_1,\omega_1)\phi(\boldsymbol{k}_2,\omega_2)\tilde{\phi}(\boldsymbol{k}_0,\omega_0)\tilde{\phi}(\boldsymbol{k}_1',\omega_1')\tilde{\phi}(\boldsymbol{k}_2',\omega_2)\phi\big(-(\boldsymbol{k}_1'+\boldsymbol{k}_2'),-(\omega_1'+\omega_2')\big)\rangle \tag{5.35}$$

$$= \sigma\langle\phi_1\tilde{\phi}_1\rangle_{D\epsilon}\langle\phi_2\tilde{\phi}_2\rangle_{D\epsilon}\langle\phi_0\tilde{\phi}_0\rangle_{D\epsilon} \tag{5.36}$$

$$= \sigma\frac{\frac{\delta}{2\pi}^{d}(\boldsymbol{k}_1+\boldsymbol{k}_1')\frac{\delta}{2\pi}^{d}(\boldsymbol{k}_2+\boldsymbol{k}_2')\frac{\delta}{2\pi}^{d}(\boldsymbol{k}_0-\boldsymbol{k}_1'-\boldsymbol{k}_2')\frac{\delta}{2\pi}(\omega_1+\omega_1')\frac{\delta}{2\pi}(\omega_2+\omega_2')\frac{\delta}{2\pi}(\omega_0-\omega_1'-\omega_2')}{(-i\omega_1+C\boldsymbol{k}_1+\epsilon-\sigma)(-i\omega_2+C\boldsymbol{k}_2+\epsilon-\sigma)(-i\omega_0+C\boldsymbol{k}_0+\epsilon-\sigma)} \tag{5.37}$$

$$= \sigma\frac{\frac{\delta}{2\pi}^{d}(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_0)\frac{\delta}{2\pi}(\omega_1+\omega_2+\omega_0)}{(-i\omega_1+C\boldsymbol{k}_1+\epsilon-\sigma)(-i\omega_2+C\boldsymbol{k}_2+\epsilon-\sigma)(-i\omega_0+C\boldsymbol{k}_0+\epsilon-\sigma)} \tag{5.38}$$

This produces the following Feynman diagram



Inverse Fourier transforming (5.38), we get

$$\sigma \int dt d^d x G_0(\boldsymbol{x}_1 - \boldsymbol{x}, t_1 - t) G_0(\boldsymbol{x}_2 - \boldsymbol{x}_0, t_2 - t_0) G_0(\boldsymbol{x} - \boldsymbol{x}_0, t - t_0) \tag{5.39}$$

Therefore

$$\begin{aligned} \langle \phi_2 \phi_1 \tilde{\phi}_0 \rangle = & \frac{\beta}{\epsilon - \sigma} \big(G_0(\boldsymbol{x}_1 - \boldsymbol{x}_0, t_1 - t_0) + G_0(\boldsymbol{x}_2 - \boldsymbol{x}_0, t_2 - t_0) \big) \\ & + 2\sigma \int dt d^d x G_0(\boldsymbol{x}_1 - \boldsymbol{x}, t_1 - t) G_0(\boldsymbol{x}_2 - \boldsymbol{x}_0, t_2 - t_0) G_0(\boldsymbol{x} - \boldsymbol{x}_0, t - t_0) \end{aligned} \tag{5.40}$$

Where the factor of 2 comes from swapping ϕ_2 and ϕ_1. However, the same problem we had in section 4.4 arises; it is apparent that (5.40) doesn't take into consideration the effects of the background particle density. That is because we have committed the same mistake of calculating the wrong observable. What we want instead is

$$\langle \phi_2^* \phi_2 \phi_1^* \phi_1 \phi_0^* \rangle \tag{5.41}$$

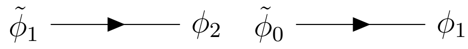
This correlator is describing the effect of planting a particle at an initial time and position and measuring the resulting particle densities ($a^\dagger a$) in terms of correlations at later times and positions [4]. Expanding (5.41) we get

$$\begin{aligned} & \langle (\tilde{\phi}_2 + 1)\phi_2(\tilde{\phi}_1 + 1)\phi_1(\tilde{\phi}_0 + 1) \rangle \\ & = \langle \tilde{\phi}_2 \phi_2 \tilde{\phi}_1 \phi_1 \tilde{\phi}_0 \rangle + \langle \tilde{\phi}_2 \phi_2 \tilde{\phi}_1 \phi_1 \rangle + \langle \tilde{\phi}_2 \phi_2 \phi_1 \tilde{\phi}_0 \rangle + \langle \tilde{\phi}_2 \phi_2 \phi_1 \rangle \\ & + \langle \phi_2 \tilde{\phi}_1 \phi_1 \tilde{\phi}_0 \rangle + \langle \phi_2 \tilde{\phi}_1 \phi_1 \rangle + \langle \phi_2 \phi_1 \tilde{\phi}_0 \rangle + \langle \phi_2 \phi_1 \rangle \end{aligned} \tag{5.42}$$

The first four terms of (5.42) vanish since there is no $t \geq t_2$. The contribution from the fifth term is

$$\langle \phi_2 \tilde{\phi}_1 \phi_1 \tilde{\phi}_0 \rangle = \langle \phi_2 \tilde{\phi}_1 \rangle_{D\epsilon} \langle \phi_1 \tilde{\phi}_0 \rangle_{D\epsilon} \tag{5.43}$$

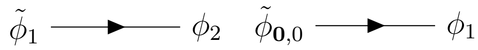
$$= G_0(\boldsymbol{x}_2 - \boldsymbol{x}_1, t_2 - t_1) G_0(\boldsymbol{x}_1 - \boldsymbol{x}_0, t_1 - t_0) \tag{5.44}$$



The sixth term is similar to what we previously found in (5.40), the only difference is that the $\phi_1 \tilde{\phi}_1$ term vanishes. The contribution is given by

$$\langle \phi_2 \tilde{\phi}_1 \phi_1 \rangle = \langle \phi_2 \tilde{\phi}_1 \rangle_{D\epsilon} \beta \langle \phi_1 \tilde{\phi}(\mathbf{0}, 0) \rangle_{D\epsilon} + \langle \phi_1 \tilde{\phi}_1 \rangle_{D\epsilon} \beta \langle \phi_2 \tilde{\phi}(\mathbf{0}, 0) \rangle_{D\epsilon} \tag{5.45}$$

$$= \frac{\beta}{\epsilon - \sigma} G_0(\boldsymbol{x}_2 - \boldsymbol{x}_1, t_2 - t_1) \tag{5.46}$$

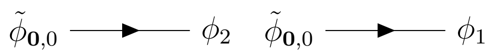


The seventh term is exactly the same as what we have already found in (5.40).

The eighth term gives a contribution from both the source and the branching terms. The contribution solely from the source term arises from $n = 2, m = 0$ in (5.17)

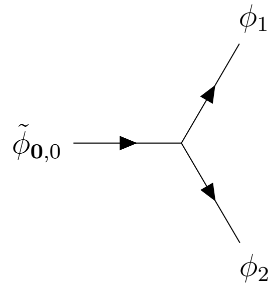
$$\left(\frac{\beta}{\epsilon - \sigma}\right)^2 \langle \phi_2 \phi_1 \tilde{\phi}(\mathbf{0}, 0) \tilde{\phi}(\mathbf{0}, 0) \rangle = \left(\frac{\beta}{\epsilon - \sigma}\right)^2 \left(\langle \phi_2 \tilde{\phi}(\mathbf{0}, 0) \rangle_{D\epsilon} + \langle \phi_1 \tilde{\phi}(\mathbf{0}, 0) \rangle_{D\epsilon} \right) \tag{5.47}$$

$$= \left(\frac{\beta}{\epsilon - \sigma}\right)^2 \tag{5.48}$$



There is another contribution from the eighth term given from $n = 1, m = 1$ in (5.17)

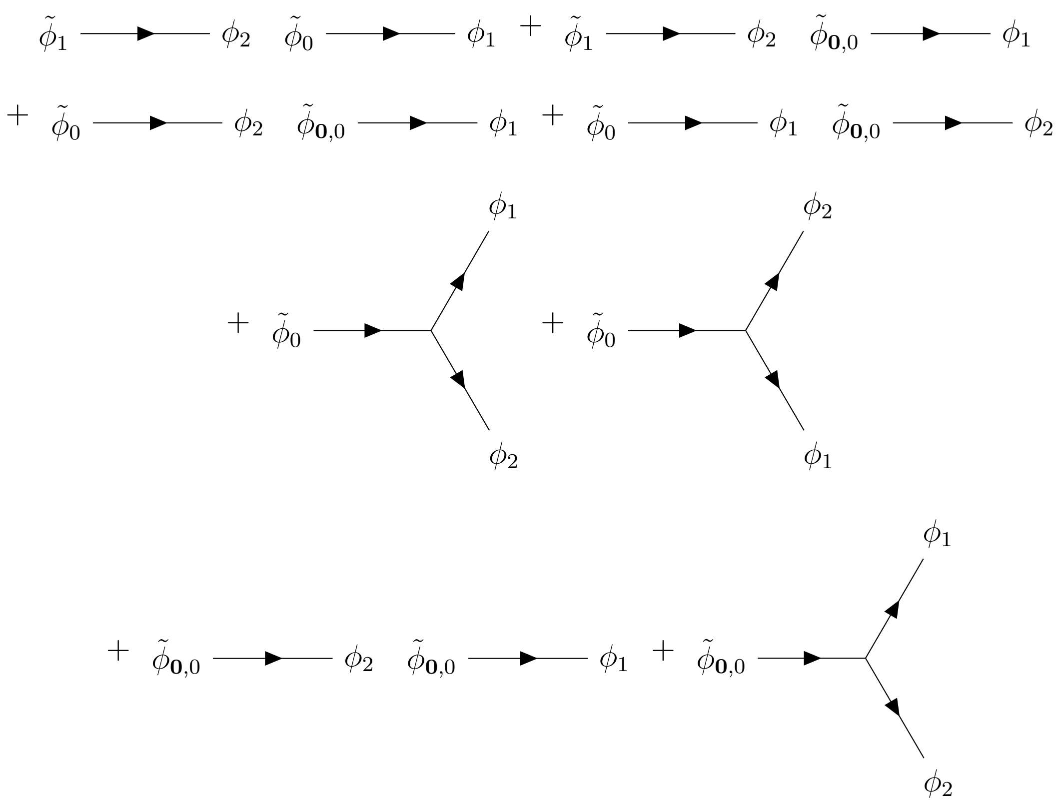
$$\frac{\beta\sigma}{\epsilon - \sigma} \int dt d^d x G_0(\boldsymbol{x}_1 - \boldsymbol{x}, t_1 - t; \epsilon - \sigma) G_0(\boldsymbol{x}_2 - \boldsymbol{x}, t_2 - t; \epsilon - \sigma) \tag{5.49}$$



The total contribution is then

$$
\begin{aligned}
&\langle \phi_2^* \phi_2 \phi_1^* \phi_1 \phi_0^* \rangle \\
&= G_0(\boldsymbol{x}_2 - \boldsymbol{x}_1, t_2 - t_1) G_0(\boldsymbol{x}_1 - \boldsymbol{x}_0, t_1 - t_0) + \frac{\beta}{\epsilon - \sigma} G_0(\boldsymbol{x}_2 - \boldsymbol{x}_1, t_2 - t_1) + \frac{\beta}{\epsilon - \sigma} G_0(\boldsymbol{x}_1 - \boldsymbol{x}_0, t_1 - t_0) \\
&+ \frac{\beta}{\epsilon - \sigma} G_0(\boldsymbol{x}_2 - \boldsymbol{x}_0, t_2 - t_0) + 2\sigma \int dt d^d x G_0(\boldsymbol{x}_1 - \boldsymbol{x}, t_1 - t) G_0(\boldsymbol{x}_2 - \boldsymbol{x}_0, t_2 - t_0) G_0(\boldsymbol{x} - \boldsymbol{x}_0, t - t_0) \\
&+ \left(\frac{\beta}{\epsilon - \sigma} \right)^2 + \frac{\beta \sigma}{\epsilon - \sigma} \int dt d^d x G_0(\boldsymbol{x}_1 - \boldsymbol{x}, t_1 - t; \epsilon - \sigma) G_0(\boldsymbol{x}_2 - \boldsymbol{x}, t_2 - t; \epsilon - \sigma) \quad (5.50)
\end{aligned}
$$

Which is in terms of Feynman diagrams is



5.4 Feynman Rules

We have now finally arrived at a position to derive Feynman rules based on our results from the previous section. Note that this work does not discuss loops, so the listed Feynman rules apply for tree level diagrams only.

- A line from $(\boldsymbol{x}_i, t_i)$ directly to $(\boldsymbol{x}_j, t_j)$ gives a factor;

$$G_0(\boldsymbol{x}_j - \boldsymbol{x}_i, t_j - t_i; \epsilon - \sigma)$$

- A line from a source term directly to $(\boldsymbol{x}_i, t_i)$ gives a factor;

$$\frac{\beta}{\epsilon - \sigma}$$

- An interaction vertex at $(\boldsymbol{x}_i, t_i)$ is integrated over and the vertex carries a coupling factor;

$$\sigma \int d^d x_i dt_i$$

- A line from an interaction vertex at $(\boldsymbol{x}_i, t_i)$ to $(\boldsymbol{x}_j, t_j)$ gives a factor;

$$G_0(\boldsymbol{x}_j - \boldsymbol{x}_i, t_j - t_i; \epsilon - \sigma)$$

- A line from a source term to an interaction vertex at $(\boldsymbol{x}_i, t_i)$ gives a factor;

$$\frac{\beta}{\epsilon - \sigma}$$

Chapter 6

Conclusion

6.1 Summary

To conclude this work, we will briefly re-discuss what was presented in this thesis. We began by introducing the concept of a stochastic process, which is essentially a function whose argument is a random variable [6]. We discussed the special case of the Markov process which enforces ”memory loss” on our stochastic process [5]. A special case of this is the random walk, which is the underlying concept behind the development of our lattice system. We then introduce the Poisson processes, and it becomes apparent later on that the collection of Markovian random walks converges to a Poisson processes [15].

Moving on to constructing out field theory, we briefly defined the Doi-Peliti approach, which is the method we use throughout this thesis, and discussed its advantages over other methods. We then derived our master equation, which describes the change in the probability of our lattice having a certain configuration, and found a Schrodinger-type equation to describe the dynamics of our system. Defining operators and states, we were able to second quantise our system, thus forming a statistical field theory.

We then began the process of finding the path integral in order to enable us to calculate the expectation values of our operators, permitting us to understand all the possible ways our system can evolve. Equipped with the method of discretising time and using resolution of unity of coherent states, we were able to achieve our objective. We later invoked perturbative methods in order to deal with our source term, thus reaching the final form of our propagator.

Finally, we looked at branching, which is the process of one particle producing two new particles and dying off. We also utilised our perturbation theory toolbox here to include the branching (or interaction) term, thus finding our new correlator. We then looked at an example and computed the correlator of two fields and one conjugate field . This provided us with insight on the nature of particle diffusion throughout a lattice and showed us that it exponentially decays with a background particle density always being present. We then illustrated our interactions with Feynman diagrams. Finally, we summarised our results in a set of Feynman rules.

6.2 Similar Approaches

The methods of field theory gave a lot of insight into the behaviour of critical phenomena [31]. However, there are other methods that proved to accurate in this field. For example, exact solutions for simple one dimensional models are an excellent tool for approximate analysis [3]. On the other hand, we have the Langevin equation, which is a stochastic partial differential equation that describes the time evolution of the observable rather than the time evolution of the probability distribution function. It is regarded by many as the most convenient way to describe critical phenomena, especially for Brownian motion [32][33][34]. One could also use the Fokker-Plank equation, which is an equation of motion of the probability density [4].

In addition to the DOI-Peliti approach, there are other field theoric methods used for non-equilibrium critical many particle systems. For example, the lesser known Martin–Siggia–Rose–Janssen–De Dominicis Field theory that is derived from Dean's equation, however it the particle entity in this formalism must be enforced, as opposed to the DOI-Peliti formalism in which the particle entity is built into it [35].

6.3 Further Applications

Field renormalisation has been used by John Cardy to study reaction diffusion processes for systems with two particle species; namely $A + B \to 0$ and $A + A \to C$ [17]. Taüber uses the renormalisation group (RG) to analyse continuous transitions from active to inactive absorbing states [36]. The differential renormalisatoin group (DRG) was also reviewed for ctirical phenomena [37]. Field theoric methods have been used to study entropy production in active matter systems [38]. Random walks on lattices have been used to model bacterial dynamics [39]. Additionally, the path integral approach has been applied to non-equilibrium Bose-Einstein condensates, non equilibrium quantum processes in the early universe, and non-equilibrium relativistic heavy ion collisions and disoriented chiral condensates [40].

Bibliography

[1] Harald Cramér. Half a century with probability theory: some personal recollections. The annals of probability, 4(4):509–546, 1976. pages 2

[2] Sidney Coleman. Quantum field theory: lectures of Sidney Coleman. 2019. pages 4, 31, 35, 36, 42

[3] Uwe C Täuber. Critical dynamics: a field theory approach to equilibrium and non-equilibrium scaling behavior. Cambridge University Press, 2014. pages 4, 51

[4] Gunnar Pruessner. Non-equilibrium statistical mechanics, 2015. pages 5, 9, 11, 15, 18, 19, 20, 27, 28, 29, 30, 31, 32, 33, 36, 38, 41, 45, 51

[5] Maria Francesca Carfora. Stochastic processes. In Shoba Ranganathan, Michael Gribskov, Kenta Nakai, and Christian Schönbach, editors, Encyclopedia of Bioinformatics and Computational Biology, pages 747–752. Academic Press, Oxford, 2019. pages 7, 11, 50

[6] Nicolaas Godfried Van Kampen. Stochastic processes in physics and chemistry, volume 1. Elsevier, 1992. pages 7, 8, 11, 36, 50

[7] Oliver C Ibe. Elements of random walk and diffusion processes. John Wiley & Sons, 2013. pages 7, 9

[8] Grigorios A Pavliotis. Stochastic processes and applications. Springer, 2016. pages 7

[9] Elliott W Montroll and George H Weiss. Random walks on lattices. ii. Journal of Mathematical Physics, 6(2):167–181, 1965. pages 8

[10] H Haken and G Mayer-Kress. Chapman-kolmogorov equation and path integrals for discrete chaos in presence of noise. Zeitschrift für Physik B Condensed Matter, 43(2):185–187, 1981. pages 9

[11] George H Weiss and Robert J Rubin. Random walks: theory and selected applications. Adv. Chem. Phys, 52:363–505, 1983. pages 10

[12] Samuel Karlin and James McGregor. Random walks. Illinois Journal of Mathematics, 3(1):66–81, 1959. pages 10

[13] Naoki Masuda, Mason A Porter, and Renaud Lambiotte. Random walks and diffusion on networks. Physics reports, 716:1–58, 2017. pages 10

[14] Joseph Klafter and Igor M Sokolov. First steps in random walks: from tools to applications. OUP Oxford, 2011. pages 10

[15] Vitaly Bergelson, Peter March, and Joseph Rosenblatt. Convergence in ergodic theory and probability, volume 5. Walter de Gruyter, 2011. pages 11, 40, 50

[16] John L Cardy and Uwe C Täuber. Field theory of branching and annihilating random walks. Journal of statistical physics, 90:1–56, 1998. pages 13

[17] John Cardy. Reaction-diffusion processes. A+ A, 100:26, 2006. pages 13, 19, 29, 52

[18] Luca Peliti. Path integral approach to birth-death processes on a lattice. Journal de Physique, 46(9):1469–1483, 1985. pages 13

[19] Alex Kamenev. Keldysh and doi-peliti techniques for out-of-equilibrium systems. Strongly Correlated Fermions and Bosons in Low-Dimensional Disordered Systems, pages 313–340, 2002. pages 13

[20] Marius Bothe, Luca Cocconi, Zigan Zhen, and Gunnar Pruessner. Particle entity in the doi–peliti and response field formalisms. Journal of Physics A: Mathematical and Theoretical, 56(17):175002, 2023. pages 14, 17, 19

[21] NH Christ, R Friedberg, and T_D Lee. Random lattice field theory: General formulation. Nuclear Physics B, 202(1):89–125, 1982. pages 14

[22] John Cardy. Field theory and nonequilibrium statistical mechanics. Lectures Presented at the Troisieme Cycle de la Suisse Romande, 1999. pages 14, 26

[23] David Skinner. Quantum field theory ii. Lecture notes, Part III of the Mathematical Tripos, University of Cambridge, 2018. pages 16

[24] John Cardy, Gregory Falkovich, and Krzysztof Gawędzki. Non-equilibrium statistical mechanics and turbulence. Number 355. Cambridge University Press, 2008. pages 17

[25] Masao Doi. Second quantization representation for classical many-particle system. Journal of Physics A: Mathematical and General, 9(9):1465, 1976. pages 17

[26] Mark Srednicki. Quantum field theory. Cambridge University Press, 2007. pages 24, 34

[27] Soren J. Helhoski. Canonical quantizatinon and the path integral formulation: A brief comparison, 2020. pages 24

[28] Lewis H Ryder. Quantum field theory. Cambridge university press, 1996. pages 25

[29] Daniel Zwillinger and Alan Jeffrey. Table of integrals, series, and products. Elsevier, 2007. pages 27

[30] Claude Itzykson and Jean-Bernard Zuber. Quantum field theory. Courier Corporation, 2012. pages 28, 29

[31] R Folk and Günter Moser. Critical dynamics: a field-theoretical approach. Journal of Physics A: Mathematical and General, 39(24):R207, 2006. pages 51

[32] William Coffey and Yu P Kalmykov. The Langevin equation: with applications to stochastic problems in physics, chemistry and electrical engineering, volume 27. World Scientific, 2012. pages 51

[33] Roberto Livi and Paolo Politi. Nonequilibrium statistical physics: a modern perspective. Cambridge University Press, 2017. pages 51

[34] Rafael Benguria and Mark Kac. Quantum langevin equation. Physical review letters, 46(1):1, 1981. pages 51

[35] Marius Bothe and Gunnar Pruessner. Doi-peliti field theory of free active ornstein-uhlenbeck particles. Physical Review E, 103(6):062105, 2021. pages 52

[36] Uwe C Täuber. Field-theory approaches to nonequilibrium dynamics. In Ageing and the Glass Transition, pages 295–348. Springer, 2007. pages 52

[37] Thomas S Chang, Dimitri Dimitrievich Vvedensky, and Jeff F Nicoll. Differential renormalization-group generators for static and dynamic critical phenomena. Physics Reports, 217(6):279–360, 1992. pages 52

[38] Cesare Nardini, Étienne Fodor, Elsen Tjhung, Frédéric Van Wijland, Julien Tailleur, and Michael E Cates. Entropy production in field theories without time-reversal symmetry: quantifying the non-equilibrium character of active matter. Physical Review X, 7(2):021007, 2017. pages 52

[39] AB Slowman, MR Evans, and RA Blythe. Jamming and attraction of interacting run-and-tumble random walkers. Physical review letters, 116(21):218101, 2016. pages 52

[40] Esteban A Calzetta and Bei-Lok B Hu. Nonequilibrium quantum field theory. Cambridge University Press, 2009. pages 52