Effective Theories on or near Branes

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Abstract

This dissertation discusses lower-dimensional effective gravitational theories with non-compact transverse space. We start with the introduction of some classic supergravity theories and general brane solutions on a flat background. Following that, we discuss the Kaluza-Klein dimensional reduction with compact transverse space. We then consider two types of effective gravity constructions, type I and type III. In type I construction, we discuss its characteristics and provide some examples including embedding general relativity and supergravity on the worldvolume. In type III construction, we focus on the effective theory based on the uplifted Salam-Sezgin vacuum solution in 10 dimensions. We investigate the effective gravity on aspects like Newton constant, Newtonian potential, and gravity spectrum. Finally, we do a similar investigation of the Randall-Sundrum model.

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Contents

Chapter 1

Introduction

Supergravity is a gravitational theory compatible with supersymmetry. It was first proposed [1, 2] to address the difficulties in the quantization of gravity. Even though supergravity was found not sufficient to tame the notorious ultraviolent divergences arising from the perturbation theory, its various physical contents and beautiful mathematical structure win the favor of physicists. The abundant field content of supergravity, ranging from spin-0 to spin-2, set up a big stage. On the other hand, the stringent constraints of supersymmetry restrict the combination of fields, making supergravity practical. Furthermore, it was realized that string theory and M-theory might instead be the promising foundation for quantum gravity while supergravity theories are their low-energy effective field theories.

In supergravity, there are extended dynamic objects called branes. Specifically, p-branes in D-dimensional spacetime are objects with $(p+1)$- dimensional worldvolume (including p spatial directions and 1 temporal direction) whose transverse space is $(D-p-1)$-dimensional. For example, strings are 1-branes and particles are 0-branes. Brane solutions are non-linear supergravity solutions like Schwarzschild black holes in general relativity. By studying brane solutions, we can understand supergravity better and hence string theory and M-theory.

The higher-dimensional nature of string theory and M-theory seem nonsensical because all the experimental observations convince us that our world is (3+1)- dimensional. Standard model matter cannot propagate a long distance in extra dimensions without conflict with observations. Besides, Newton's $1/r^2$ law and general relativity suggest a 4-dimensional spacetime. The first problem can be avoided if the standard model is confined to a $(3 + 1)$-dimensional subspace in the higher dimensions [3–5]. However, this solution does not work for gravity as gravity is the dynamics of spacetime. Hence, the idea of gravity localization on a 4-dimensional Minkowski-signature subspace appears. It has been of interest to cosmology [6].

A standard method to generate a 4-dimensional effective theory is Kaluza-Klein dimensional reduction with *compact* transverse space [7]. Compactness guarantees discrete Fourier modes, on which transverse-space functions expand. Then the transverse-space part of higher-dimensional equations of motion is replaced with eigenvalues of Fourier modes, serving as mass terms for the residual space. Due to the infinite number of Fourier modes, the existence of extra compact dimensions will generate an infinite tower of massive particles on the residual space. Their mass is inversely proportional to the length scale of the compact direction, which is assumed to be Planck length. Hence, these massive modes are too heavy to generate in colliders. To describe 4-dimensional gravity with massless graviton, the transverse-space problem must admit a zero eigenvalue. A mathematically attractive case is when the interactions of the massless modes decouple from the massive sector, allowing us to consistently truncate out the massive modes. However, consistent truncations are not available in many physically interesting constructions.[1] Thus, to construct a low-energy effective gravity, the existence of a *mass gap* is more essential.

Another way to obtain a lower-dimensional effective theory is dimensional reductions on *non-compact* transverse space. It was first proposed in ref. [9] to construct non-compact gauge symmetries through dimensional reduction, as the higher-dimensional diffeomorphism on the transverse space transforms into the gauge symmetries on the reduced subspace. Based on [10], ref.[11] provided a taxonomy of braneworld gravities derived from dimension reduction with non-compact transverse space. Instead of the types of gravitational sources[2], the categorization depends on the boundary condition of the transverse problem. Type I constructions equip a sort of Dirichlet boundary condition, which fixes the way perturbations depend on transverse space. It involves a consistent truncation of the higher-dimensional supergravity to a lower-dimensional theory on the worldvolume and gives rise to fully non-linear lower-dimensional theories. The gravitational theories on the worldvolume can be Einstein's theory [12, 13] or even supergravities [14, 15]. On the other hand, Type II and Type

[1] One notable instance is reductions on compact Calabi-Yau spaces without Killing symmetries, in which we integrate out the massive modes producing higher-derivative terms suppressed by compactification-space volume.[8]

[2] NS5-brane generates lower-dimensional gravity near the brane in [10].

III localizations equip a Dirichlet condition on the asymptotically away requiring fields to vanish and Neuman and Robin conditions, respectively, near the source. With different boundary conditions, they admit different sets of eigenfunctions. Since there is no zero eigenfunction in Type II localization, we will focus on Type III constructions. They are not based on a consistent truncation and give only a perturbative realization of lower-dimensional gravitational theories.

We begin Chapter 2 by introducing some classic supergravity theories. Following this, we discuss brane solutions of single-charge action, which is the building block of general brane solutions, based on the ref. [16]. To tell a full story of brane solutions, we will briefly discuss super p-brane actions, which play the roles of sources in brane solutions. We end this chapter with Kaluza-Klein dimensional reduction, a classic method to generate lower-dimensional physics.

In the remaining chapters, we will start to discuss constructing effective gravity with non-compact transverse space, which is our main interest. In Chapter 3, we will illustrate the Type I constructions in ref. [11]. We begin this chapter by introducing key characteristics of type I effective theories. Then we start with the simplest case with general relativity on branes. In this case, we replace the worldvolume with Ricci-flat metric, e.g. Schwarzschild metric, and achieve the black strings model constructed in ref. [13]. Moreover, we discuss embedding supergravity on branes following ref. [14, 15]. Different from the general relativity on branes, the theories admit the existence of gauge fields, *i.e.* charges of branes, on worldvolume, and hence more possible sources on the lower-dimensional space. For example, Reissner–Nordström black holes and a class of stationary black holes constructed with the non-linear sigma model.

In Chapter 4, we will discuss the Type III construction. We focus on the effective gravity based on the Salam-Sezgin-CGP background [10]. The transverse space geometry determines a special transverse problem, which can be transformed into a Schrodinger equation with a Posher-Teller type of potential. After carefully choosing the Robin boundary condition, the transverse spectrum comprises a zero eigenvalue bound state and continuum scattering states separated by a mass gap. The mass gap allows a lower-dimensional effective theory with only small massive-mode-induced correction. Then, we discuss the Randall-Sundram model ref. [17], in which no mass gap exists. Because of the transverse space geometry, it can still produce a lower-dimensional effective gravity in a different way.

Chapter 2

Supergravity and Branes

Let's briefly recall supersymmetry before discussing supergravity, which merges supersymmetry into general relativity. As an exception to the no-go theorem of Coleman and Mandula, supersymmetry unifies the spacetime symmetries of the Poincare group with internal symmetries by introducing the anticommutation relations and allowing the transformation between the bosonic and fermionic generators [18]. The simplest 4D $\mathcal{N} = 1$ supersymmetry extends the Poincare algebra with

$$\{Q_\alpha, Q_\beta\} = 2(C\Gamma^\mu)_{\alpha\beta} P_\mu, \quad [Q_\alpha, P_\mu] = 0, \quad [Q_\alpha, M^{\mu\nu}] = \frac{1}{2}(\Gamma^{\mu\nu})_\alpha{}^\beta Q_\beta, \tag{2.1}$$

where Q_α are supercharges, the generator of supersymmetry transformation. It is available to generalize the algebra with the generators of internal symmetry called R-symmetry and central charges that commute with all the generators. Since supersymmetry is the symmetry between bosonic and fermionic fields, the number of bosonic and fermionic degrees of freedom should be equal. A practical way to construct a supersymmetry theory is using supermultiplets which are composed of equal bosonic and fermionic degrees of freedom.

In supergravity, since gravity theory admits diffeomorphism generated by P_μ, the supersymmetry should be local. Besides, supersymmetry extends the massless graviton in general relativity to the gravity multiple including graviton, gauge fields, scalars, gravitini and spinors. Supergravity admits non-perturbative solutions known as brane solutions, analog to black hole solutions in general relativity. Branes are extended objects playing the roles of sources for both gauge field and gravity. The dimension of branes can range from 0, particles, to $D-2$, domain walls, in a D-dimensional spacetime. The super p-brane action is obtained by generalizing the Nambu-Goto action with supersymmetry. We can deepen our understanding of supergravity by exploring brane solutions.

In this chapter, we will present some classic supergravity theories. Then we derive single-charge brane solutions in arbitrary dimensions, which is the foundation of more general brane solutions. To provide a full story of brane solutions, we briefly discuss the super p-brane action to make a complete story. Finally, we discuss the Kaluza-Klein dimensional reduction, which allows us to connect theories and solutions in different dimensions.

2.1 Supergravity

Let's begin with the $D = 11$ supergravity which is the lower-dimensional effective theory of the M-theory. For a supergravity theory, it must contain gravity supermultiple composed of one graviton g_{MN} and one gravitino Ψ_M. In 11 dimensions, they have 44 and 128 (on-shell) degrees of freedom respectively. As the supersymmetry requires equality between the degrees of freedom of bosonic and fermionic sectors, we need a 3-form gauge field A_{MNP} with 84 (on-shell) degree of freedom to compensate for the difference between graviton and gravitino.

Before writing down the action, we can investigate the supersymmetry algebra first. In 11 dimensions, each Majorana spinor has 32 degrees of freedom. There is another fact that, for any dimension, 32 is the maximum number of supercharges we can have without getting into fields with spin higher than 2. Hence, we can only have one set of supercharges, *i.e.* $\mathcal{N} = 1$, in 11 dimensions. The simplest supersymmetry algebra is

$$\{Q_\alpha, Q_\beta\} = (C\Gamma^A)_{\alpha\beta} P_A, \quad \alpha, \beta = 1, \cdots, 32. \quad A = 0, \cdots, 10, \tag{2.2}$$

which can be generalized with central charges. Considering the symmetry of between α, β, the central charges can be rank $n = 1, 2, 5, 6, 9, 10$ due to the symmetry property of $C\Gamma^{(n)}$ in 11-dimensional. The rank n Gamma matrices are defined by antisymmetrizing n Gamma matrices with weight 1. Further, in $(2m + 1)$-dimension, rank n Gamma matrices are dual to the rank-$(2m + 1 - n)$ with the epsilon symbol [19]. Thus, the generalized supersymmetry algebra

$$\{Q, Q\} = C\left(\Gamma^A P_A + \Gamma^{AB} U_{AB} + \Gamma^{ABCDE} V_{ABCDE}\right), \tag{2.3}$$

where we drop the spinor indices for simplicity. And we can check that the left-hand-side has $32 \times 33/2 = 528$ degrees of freedom, while the right-hand-side has $11 + 11!/(10! \cdot 2!) + 11!/(5! \cdot 6!) = 528$ degrees of freedom. This means that the supersymmetry algebra is maximally extended. Actually, U_{AB} and V_{ABCDE} are related to the electric and magnetic charges coupled to M2- and M5-branes in 11D supergravity [16].

We can construct the action of 11D $\mathcal{N} = 1$ supergravity through the Noether method, starting with the kinetic terms of these field contents with rigid supersymmetry. Then, we gauge the supersymmetry by adding more interaction terms and correcting the local supersymmetry transformation step by step [19]. Here, we write the final result

$$\begin{aligned} S_{11} = \frac{1}{2\kappa^2} \int d^{11}xe \Bigg[& e_M^{\underline{E}} e^{\underline{F}N} R_{MN\underline{EF}}(\omega) - \bar{\psi}_M \Gamma^{MNP} D_N \left(\frac{1}{2}(\omega + \hat{\omega}) \right) \psi_P - \frac{1}{48} F_{[4]}^2 \\ & - \frac{1}{192} \bar{\psi}_E \left(\Gamma^{ABCDEF} + 12 \Gamma^{AB} g^{CE} g^{DF} \right) \psi_F \left(F_{ABCD} + \hat{F}_{ABCD} \right) \\ & - \frac{1}{(144)^2} \epsilon^{A'B'C'D'ABCDEFG} F_{A'B'C'D'} F_{ABCD} A_{EFG} \Bigg], \end{aligned} \tag{2.4}$$

where $F_{[4]} = dA_{[3]}$ is field strength of the 3-form field and $\Gamma^{M \cdots N}$ are rank-n Gamma matrices in 11 dimensions with weight-1. The "hatted" connection and field strength are the supercovariant counterparts given by

$$\begin{aligned} \omega_{M\underline{EF}} &= \omega_{M\underline{EF}}(e) + K_{M\underline{EF}}, \\ \hat{\omega}_{M\underline{EF}} &= \omega_{M\underline{EF}}(e) - \frac{1}{4} \left(\bar{\psi}_M \Gamma_{\underline{F}} \psi_{\underline{E}} - \bar{\psi}_{\underline{E}} \Gamma_M \psi_{\underline{F}} + \bar{\psi}_{\underline{F}} \Gamma_{\underline{E}} \psi_M \right), \\ K_{M\underline{EF}} &= -\frac{1}{4} \left(\bar{\psi}_M \Gamma_{\underline{F}} \psi_{\underline{E}} - \bar{\psi}_{\underline{E}} \Gamma_M \psi_{\underline{F}} + \bar{\psi}_{\underline{F}} \Gamma_{\underline{E}} \psi_M \right) + \frac{1}{8} \bar{\psi}_N \Gamma^{NP}{}_{M\underline{EF}} \psi_P, \\ \hat{F}_{MNPQ} &= 4 \partial_{[M} A_{NPQ]} + \frac{3}{2} \bar{\psi}_{[M} \Gamma_{NP} \psi_{Q]}. \end{aligned} \tag{2.5}$$

In the expression, we use capital letters $\{A, B, \cdots, M, N, \cdots\}$ to indicate the spacetime indices and the letters with underline to indicate the tangent space. The tetra inside the bracket $\omega_{M\underline{EF}}(e)$ means that the spin connection is the unique one determined by the metric compatibility and torsion-free conditions,

$$\omega_M^{\underline{EF}}(e) = 2e^{N[\underline{E}} \partial_{[M} e_{N]}^{\underline{F}]} - e^{N[\underline{E}} e^{\underline{F}]P} e_{M\underline{G}} \partial_N e_P^{\underline{G}}. \tag{2.6}$$

The covariant derivative D_M with respect to the spin connection ω is defined as

$$D_M(\omega)\psi_N = \partial_M \psi_N + \frac{1}{4} \omega_{M\underline{EF}} \Gamma^{\underline{EF}} \psi_N. \tag{2.7}$$

The action is invariant under the local supersymmetry transformations

$$\begin{aligned}\delta e_{M}^{\underline{E}} &= \frac{1}{2}\bar{\epsilon}\Gamma^{\underline{E}}\psi_{M},\\ \delta\psi_{M} &= D_{M}\left(\hat{\omega}\right)\epsilon+\frac{1}{288}\left(\Gamma^{ABCD}{}_{M}-8\Gamma^{BCD}\delta_{M}^{A}\right)\hat{F}_{ABCD}\epsilon\equiv\tilde{D}_{M}\epsilon,\\ \delta A_{MNP} &= -\frac{3}{2}\bar{\epsilon}\Gamma_{[MN}\psi_{P]},\end{aligned}\tag{2.8}$$

where the supersymmetry transformation parameter $\epsilon=\epsilon\left(x\right)$ depends on the space-time position.

When studying classical brane solutions, physicists will concentrate on the bosonic sector and set zero for the fermionic sector consistently. The reduced action written in differential form is

$$S_{11}=\frac{1}{2\kappa_{11}^{2}}\int R*\mathbf{1}-\frac{1}{2}F_{[4]}\wedge*F_{[4]}-\frac{1}{6}F_{[4]}\wedge F_{[4]}\wedge A_{[3]},\tag{2.9}$$

with equations of motion and Bianchi identity

$$\begin{aligned}R_{MN} &= \frac{1}{12}\left(F_{MN}^{2}-\frac{1}{12}g_{MN}F^{2}\right),\\ d*F_{[4]} &= -\frac{1}{2}F_{[4]}\wedge F_{[4]},\\ dF_{[4]} &= 0.\end{aligned}\tag{2.10}$$

Correspondingly, to preserve supersymmetry transformation in Eq.(2.8), we need

$$\delta\psi_{M}=D_{M}\left(\omega\right)\epsilon+\frac{1}{288}\left(\Gamma^{ABCD}{}_{M}-8\Gamma^{BCD}\delta_{M}^{A}\right)F_{ABCD}\epsilon=0.\tag{2.11}$$

The spinor satisfying the condition is called *Killing spinor*. The number of killing spinors in a solution is exactly the amount of preserved supersymmetry. For killing spinors $\epsilon^{i},\epsilon^{j}$, the vector $K^{ijM}\equiv\bar{\epsilon}^{i}\Gamma^{M}\epsilon^{j}$ are Killing vectors. And the diagonal Killing vectors K^{ii} are either time-like or null [20, 21].

We can obtain the type IIA supergravity from $D=11$ supergravity by dimensional reduction on S^{1}. The bosonic sector of the type IIA supergravity is composed of metric $g_{\mu\nu}$, a dilaton ϕ, anti-symmetric 2-from $B_{\mu\nu}$, 1-form R-R vector $A_{[1]}$, and 3-form R-R field $A_{[3]}$. The action of the bosonic sector is

$$\begin{aligned}S_{IIA}=&\frac{1}{2\kappa_{10}^{2}}\int R*\mathbf{1}-\frac{1}{2}d\phi\wedge*d\phi-\frac{1}{2}e^{\frac{3}{2}\phi}F_{[2]}\wedge*F_{[2]}\\ &-\frac{1}{2}e^{\frac{1}{2}\phi}F_{[4]}\wedge*F_{[4]}-\frac{1}{2}e^{\phi}F_{[3]}\wedge*F_{[3]}+\frac{1}{2}dA_{[3]}\wedge dA_{[3]}\wedge A_{[3]},\end{aligned}\tag{2.12}$$

where the field strength are defined as $F_{[2]} = dA_{[1]}$, $F_{[3]} = dA_{[2]}$ and $F_{[4]} = dA_{[3]} - dA_{[2]} \wedge A_{[1]}$. We can obtain the supersymmetry conditions for type IIA supergravity with the fermions truncated by substituting the gravitino decomposition into Eq.(2.11). The 11-dimensional gravitino ψ_M decomposes into two gravitini Ψ_μ and two dilatini λ defined as

$$\lambda = e^{-\phi/6}\psi_{11}, \quad \Psi_\mu = e^{-\phi/6}\Big(\psi_\mu + \frac{1}{2}\Gamma_\mu\Gamma_{11}\psi_{11}\Big). \tag{2.13}$$

The two gravitini and dilatini are each having opposite chiralities. Then, we can derive the Killing spinor equations as

$$\begin{aligned} \delta\Psi_\mu &= \Big(\nabla_\mu - \frac{1}{4}F_{\mu\nu\rho}\Gamma^{\nu\rho 11} - \frac{1}{8}e^\phi F_{\alpha\beta}\Gamma_\mu{}^{\alpha\beta 11} + \frac{1}{8}e^\phi F_{\alpha\beta\rho\sigma}\Gamma^{\alpha\beta\rho\sigma}\Gamma_\mu\Big)\tilde{\epsilon} = 0 \\ \delta\lambda &= \Big(-\frac{1}{3}\partial_\mu\phi\Gamma^{\mu 11} + \frac{1}{6}F_{\mu\nu\rho}\Gamma^{\mu\nu\rho} - \frac{1}{4}e^\phi F_{\mu\nu}\Gamma^{\mu\nu} + \frac{1}{12}e^\phi F_{\alpha\beta\rho\sigma}\Gamma^{\alpha\beta\rho\sigma 11}\Big)\tilde{\epsilon} = 0. \end{aligned} \tag{2.14}$$

where we have redefined the Killing spinos as $\tilde{\epsilon} = e^{\phi/6}\epsilon$. We will discuss the dimensional reduction soon in sec.2.4.

2.2 Single-charge Action and Brane Solutions

Even though consider only the bosonic sector of a supergravity theory, we must confront a Lagrangian with complicated field contents composed of the graviton, antisymmetric-tensor field strengths, and various scalars. For example, the 4D $\mathcal{N} = 8$ supergravity has 1 graviton, 28 gauge fields and 70 scalars. To obtain a more tractable system, we shall make a *consistent truncation* of the action down to, in most cases, a simple system in D dimensions comprising the metric g_{MN}, a scalar field ϕ and a single $(n-1)$-form gauge potential $A_{[n-1]}$ with field strength $F_{[n]}$. The truncated *single-charge action* is

$$I = \int R * \mathbf{1} - \frac{1}{2}d\phi \wedge *d\phi - \frac{1}{2}e^{a\phi}F_{[n]} \wedge *F_{[n]}. \tag{2.15}$$

The parameter a controls the interaction of the scalar field ϕ with the field strength $F_{[n]}$. Varying the action, we obtain the equations of motion

$$\begin{aligned} R_{MN} &= \frac{1}{2}\partial_M\phi\partial^M\phi + \frac{1}{2(n-1)!}e^{a\phi}\left(F_{M\ldots}F_N^{\ \ldots} - \frac{n-1}{n(D-2)}F^2 g_{MN}\right), \\ 0 &= \nabla_{M_1}\left(e^{a\phi}F^{M_1\cdots M_n}\right), \\ \Box\phi &= \frac{a}{2n!}e^{a\phi}F^2. \end{aligned} \tag{2.16}$$

A consistent truncation of field variables is a restriction on the variables that commutes with the variation of the action to produce the field equations. Equivalently, a restriction that solutions to the equations for the restricted variables are also solutions to the equations for the unrestricted variables. Since setting zero for the fermionic sector is also a consistent truncation, the solutions we get from single-charge action are still solutions for the supergravity theory itself.

The single-charge action admits p-brane solutions written as $M_D = M_d \times B_{D-d}$ with $d = p+1$. Just like the strategy of solving the black hole solutions in general relativity, we need some ansatz and try our luck. We shall be looking for solutions preserving certain unbroken supersymmetries, which require unbroken translational symmetries according to Eq.(2.2). Similar to the spherical symmetry imposed in deriving Schwarzschild metric, we consider p-brane solutions with $(\text{Poincaré})_d \times \text{SO}\,(D-d)$ symmetry and write the manifold as $M_D = \mathbb{R}^{1,d-1} \times \mathbb{R}^{D-d}$. The metric can be written in the form

$$ds^2 = e^{2A(r)} dx^\mu dx^\nu \eta_{\mu\nu} + e^{2B(r)} dy^m dy^n \delta_{mn}, \tag{2.17}$$

where $r = \sqrt{y^m y^m}$, $x^M = (x^\mu, y^m)$ with x^μ $(\mu = 0, 1, \cdots, p = d-1)$ being the coordinates adapted to the $(\text{Poincaré})_d$ isometries on the worldvolume and y^m $(m = d, \cdots, D-1)$ being the coordinates transverse to the worldvolume. The corresponding ansatz for the scalar field $\phi(x)$ is simply

$$\phi = \phi(r) \tag{2.18}$$

For the $(n-1)$-form gauge potential $A_{[n-1]}$, we have three possibilities for the ansatz, electric, magnetic, and dyonic.

Before providing the specific expression for the gauge potential in different ansatz. We can first solve the Ricci tensor given the metric in Eq.(2.17). Introduce vielbeins $\left\{\hat{e}^{\underline{E}}\right\} = \{\hat{e}^{\underline{\mu}}, \hat{e}^{\underline{m}}\}$ of the metric

$$\begin{aligned} \hat{e}^{\underline{\mu}} &= e^{A(r)} dx^\mu, \quad a = 0, 1, \cdots, p = d-1, \\ \hat{e}^{\underline{m}} &= e^{B(r)} dy^m, \quad m = d, \cdots, D-1 \end{aligned} \tag{2.19}$$

so that $ds^2 = \eta_{\underline{EF}} \hat{e}^{\underline{E}} \otimes \hat{e}^{\underline{F}}$. Here, we donate p-brane tangent coordinates with $\hat{e}^{\underline{\mu}}$ and transverse tangent coordinates with $\hat{e}^{\underline{m}}$.

The metric compatibility requires that the connection 1-forms are antisymmetric $\omega_{\underline{EF}} = -\omega_{\underline{FE}}$ where we raise or lower the indexes by $\eta^{\underline{EF}}$ or $\eta_{\underline{EF}}$ with $\eta_{\underline{mn}} = \delta_{\underline{mn}}$. The torsion-free condition written in Cartan's structure equation is

$$T^{\underline{E}} = d\hat{e}^{\underline{E}} + \omega^{\underline{E}}{}_{\underline{F}} \wedge \hat{e}^{\underline{F}} = 0 \tag{2.20}$$

Because of the antisymmetry, the connection 1-form should take the form

$$\omega^{\underline{\mu\nu}} = \left(\mathcal{A}^{\nu}\hat{e}^{\underline{\mu}} - \mathcal{A}^{\mu}\hat{e}^{\underline{\nu}}\right), \ \omega^{\underline{mn}} = \left(\mathcal{B}^{n}\hat{e}^{\underline{m}} - \mathcal{B}^{m}\hat{e}^{\underline{n}}\right). \tag{2.21}$$

After calculation, we can get connection 1-forms

$$\begin{aligned}
\omega^{\underline{\mu\nu}} &= 0, \\
\omega^{\underline{\mu m}} &= \partial_m A e^{A(r)-B(r)} dx^{\mu} = \partial_m A e^{-B(r)}\hat{e}^{\underline{\mu}}, \\
\omega^{\underline{nm}} &= \partial_m B dy^n - \partial_n B dy^m = e^{-B(r)}\left(\partial_m B\hat{e}^{\underline{n}} - \partial_n B\hat{e}^{\underline{m}}\right).
\end{aligned} \tag{2.22}$$

It is easy to check the antisymmetry.

To calculate the curvature 2-form, we use the second Cartan's structure equation

$$R^{\underline{EF}} = d\omega^{\underline{EF}} + \eta_{\underline{IJ}}\omega^{\underline{EI}} \wedge \omega^{\underline{JF}}, \tag{2.23}$$

where we have raised the index with $\eta^{\underline{EF}}$. The calculation is straightforward, and we summarize the result here

$$\begin{aligned}
R^{\underline{\mu\nu}} &= -\partial_m A\partial_m A e^{-2B(r)}\hat{e}^{\underline{\mu}} \wedge \hat{e}^{\underline{\nu}}, \\
R^{\underline{\mu m}} &= \left[\partial_n\partial_m A + \partial_m A\partial_n A - \partial_m A\partial_n B - \partial_n A\partial_m B\right] e^{-2B(r)}\hat{e}^{\underline{n}} \wedge \hat{e}^{\underline{\mu}} \\
&\quad - \partial_n A\partial_n B e^{-2B(r)}\hat{e}^{\underline{\mu}} \wedge \hat{e}^{\underline{m}}, \\
R^{\underline{nm}} &= e^{-2B(r)}\big(\left(\partial_k\partial_m B - \partial_k B\partial_m B\right)\hat{e}^{\underline{k}} \wedge \hat{e}^{\underline{n}} \\
&\qquad - \left(\partial_k\partial_n B - \partial_n B\partial_k B\right) d\hat{e}^{\underline{k}} \wedge \hat{e}^{\underline{m}} - \partial_k B\partial_k B\hat{e}^{\underline{n}} \wedge \hat{e}^{\underline{m}}\big).
\end{aligned} \tag{2.24}$$

We can use the definition of the curvature 2-form $R^{\underline{EF}} = \frac{1}{2}R^{\underline{EF}}{}_{\underline{IJ}}\hat{e}^{\underline{I}} \wedge \hat{e}^{\underline{J}}$, to find the Riemann curvature tensor. After transforming back into the original coordinate with the component of tetra, $\hat{e}^{\underline{\mu}}{}_{\nu} = \delta^{\mu}{}_{\nu}e^{A(r)}$, $\hat{e}^{\underline{m}}{}_{n} = \delta^{m}{}_{n}e^{B(r)}$, we achieve the non-vanished components of the Ricci tensor

$$\begin{aligned}
R_{\mu\nu} &= -\eta_{\mu\nu}e^{2(A-B)}\left(A'' + dA'^2 + \tilde{d}A'B' + \frac{\tilde{d}+1}{r}A'\right), \\
R_{mn} &= -\delta_{mn}\left(B'' + dA'B' + \tilde{d}B'^2 + \frac{2\tilde{d}+1}{r}B' + \frac{d}{r}A'\right) \\
&\quad - \frac{y^m y^n}{r^2}\left(\tilde{d}B'' + dA'' - 2dA'B' + dA'^2 - \tilde{d}B'^2 - \frac{\tilde{d}}{r}B' - \frac{d}{r}A'\right).
\end{aligned} \tag{2.25}$$

using $\partial_m A = A' r^{-1} y^m$, $\quad \partial_n\partial_m A = A'' r^{-2} y^m y^n - A' r^{-3} y^m y^n + A' r^{-1}\delta_{mn}$, and similar for $B\left(r\right)$ and $\tilde{d} = D - d - 2$.

We now discuss different gauge-field ansatz and their corresponding solutions.

2.2.1 Electric p-brane solutions

In *elementary ansatz* (or electric ansatz), we take the gauge field to be

$$A_{[n-1]} = e^{C(r)}\mathrm{vol}(M_{n-1}), \tag{2.26}$$

with non-vanishing components

$$A_{\mu_1\cdots\mu_{n-1}} = \varepsilon_{\mu_1\cdots\mu_{n-1}}e^{C(r)} \tag{2.27}$$

as $M_{n-1} = \mathbb{R}^{1,n-2}$ in our ansatz. In this case, the gauge potential needs to couple to a $(n-2)$-brane. Hence, the elementary ansatz will provide us a flat $(n-2)$-brane solution with $d_{el} = n-1$.

Substituting Eq.(2.18), (2.25), and (2.27) into the equations of motion Eq.(2.16), we have the

$$\begin{aligned}
A'' + d_{el}A'^2 + \tilde{d}_{el}A'B' + \frac{\tilde{d}_{el}+1}{r}A' &= \frac{\tilde{d}_{el}}{2(D-2)}S^2,\\
B'' + d_{el}A'B' + \tilde{d}_{el}B'^2 + \frac{2\tilde{d}_{el}+1}{r}B' + \frac{d}{r}A' &= -\frac{d_{el}}{2(D-2)}S^2,\\
\tilde{d}_{el}B'' + d_{el}A'' - 2d_{el}A'B' + d_{el}A'^2 - \tilde{d}_{el}B'^2 &\\
-\frac{\tilde{d}_{el}}{r}B' - \frac{d_{el}}{r}A' + \frac{1}{2}\phi'^2 &= \frac{1}{2}S^2,\\
\phi'' + d_{el}A'\phi' + \tilde{d}_{el}B'\phi' + \frac{\tilde{d}_{el}+1}{r}\phi' &= -\frac{1}{2}aS^2,
\end{aligned} \tag{2.28}$$

with

$$S = \exp\left(\frac{1}{2}a\phi - dA + C\right)C'. \tag{2.29}$$

The differential equations are still daunting. Fortunately, since we want a solution preserving a portion of supersymmetry, we can impose further restrictions

$$d_{el}A + \tilde{d}_{el}B = 0, \quad A = -\frac{\tilde{d}_{el}}{a(D-2)}\phi, \tag{2.30}$$

and introduce a new notation [1]

$$a^2 = \Delta - \frac{2d\tilde{d}}{D-2}. \tag{2.31}$$

Then, Eq.(2.28) are reduced to

$$\nabla^2\phi + \frac{\Delta}{2a}\phi'^2 = 0, \quad S^2 = \frac{\Delta\phi'^2}{a^2} \tag{2.32}$$

[1] We drop the subscripts here because the definition of the new notation Δ is suited for all ansatz.

or simply

$$\nabla^2 e^{\frac{\Delta}{2a}\phi} = 0 \tag{2.33}$$

The Laplacian operator is defined on the transverse space. Acting on angular independence variables, the Laplacian operator has the form $\nabla^2\phi(r) = \phi'' + (\tilde{d}_{el}+1)r^{-1}\phi'$. Defining the harmonic function on the transverse space

$$e^{\frac{\Delta}{2a}\phi} \equiv H(y), \tag{2.34}$$

we get the solution

$$\begin{aligned} ds_D^2 &= H^{-\frac{4\tilde{d}_{el}}{\Delta(D-2)}}\eta_{\mu\nu}dx^\mu dx^\nu + H^{\frac{4d_{el}}{\Delta(D-2)}}dy^m dy^m, \\ A_{[n-1]} &= \frac{2}{\sqrt{\Delta}}H^{-1}\text{vol}(M_{n-1}), \quad e^\phi = H^{\frac{2a}{\Delta}}, \quad \nabla^2 H = 0, \end{aligned} \tag{2.35}$$

with the worldvolume manifold $M_{n-1} = \mathbb{R}^{1,n-2}$. The simplest non-trivial solution for the harmonic function is

$$H(y) = 1 + \frac{k}{r^{\tilde{d}_{el}}}, \quad k > 0, \tag{2.36}$$

describing a brane sitting at $r = 0$ electrically coupled to the gauge-field $A_{[n-1]}$. Here, k is determined by the tension of the brane, *i.e.* source, which is taken to be positive to prevent the naked singularities. We can compare our results with the charged black hole in general relativity.

The harmonic function $H(y)$ admits multi-center solutions

$$H(y) = 1 + \sum_i \frac{k}{(y-y_i)^{\tilde{d}_{el}}}, \tag{2.37}$$

corresponding to the solutions with parallel and similarly oriented p-branes. Physically, it is the result of cancellation between attractive gravitational and scalar-field forces against repulsive antisymmetric-tensor forces. In charged black hole solutions, we have a similar result for the extremal case with $M = Q$. Indeed, the solutions in Eq.(2.35) satisfy the Bogomol'ny bounds $\mathcal{E} = U$ preserving half supersymmetries called *BPS-brane solutions* [16]. The energy density $\mathcal{E}$ can be calculated with the ADM formula. The electric charge density is defined by

$$U = \int_{\partial B} e^{a\phi} * F_{[n]}, \tag{2.38}$$

as the equation of motion for the antisymmetric-field strength $d(e^{a\phi} * F_{[n]}) = 0$.

Finally, we provide the M2-brane in 11D supergravity Eq.(2.9) as an instance for the electric brane solution. With the electric ansatz Eq.(2.27), the FFA term vanishes identically. There is no scalar field, we have thus $a = 0$, $\Delta = 4$ and the solution for M2-brane

$$\begin{aligned} ds_{11}^2 &= (1+\frac{k}{r^6})^{-2/3}\eta_{\mu\nu}dx^\mu dx^\nu + (1+\frac{k}{r^6})^{1/3}dy^m dy^m, \\ A_{\mu\nu\lambda} &= \varepsilon_{\mu\nu\lambda}(1+\frac{k}{r^6})^{-1}, \quad \phi = 0, \end{aligned} \tag{2.39}$$

in isotropic coordinates. The solution has a similar causal structure as the extremal black hole solution. For more details, see the ref. [16].

2.2.2 Magnetic p-brane solutions

The other choice is the *solitonic ansatz* (or magnetic ansatz). In the solitonic solution, it is the dual gauge potential $K_{[D-n-1]}$ of $A_{[n-1]}$ coupling to a $(D-n-2)$-brane. Thus, the solitonic ansatz will provide us a $(D-n-2)$-brane solution with $d_{so} \equiv \tilde{d}_{el} = D-n-1$. The dual field strength $G_{[D-n]} = dK_{[D-n-1]}$ related to the original one by

$$F_{[n]} = e^{-a\phi} * G_{[D-n]}, \tag{2.40}$$

and the Lagrangian written in new variables is

$$I = \int R*\mathbf{1} - \frac{1}{2}d\phi\wedge *d\phi - \frac{1}{2}e^{-a\phi}G_{[D-n]}\wedge *G_{[D-n]}. \tag{2.41}$$

Note that the sign of the coupling constant on the exponential flips. Similar to the Eq.(2.26), we choose the gauge field to be

$$K_{[D-n-1]} = e^{C(y)}\mathrm{vol}(M_{D-n-1}). \tag{2.42}$$

We can directly write down the solution compared with the electric brane solutions Eq.(2.35),

$$\begin{aligned} ds_D^2 &= H^{-\frac{4d_{el}}{\Delta(D-2)}}\eta_{\mu\nu}dx^\mu dx^\nu + H^{\frac{4\tilde{d}_{el}}{\Delta(D-2)}}dy^m dy^m, \\ K_{[D-n-1]} &= \frac{2}{\sqrt{\Delta}}H^{-1}\mathrm{vol}(M_{D-n-1}), \quad e^\phi = H^{\frac{-2a}{\Delta}}, \quad \nabla^2 H = 0. \end{aligned} \tag{2.43}$$

Again, $H(y)$ is a harmonic function on the transverse space. In the original variables, this is

$$F_{m_1\cdots m_n} = \frac{2}{\sqrt{\Delta}}\varepsilon^i{}_{m_1\cdots m_n}\partial_i H. \tag{2.44}$$

The solutions are also BPS solutions preserving half supersymmetries because they saturate the mass-charge inequality $\mathcal{E} = V$, with the magnetic charge density defined by

$$V = \int_{\partial B} e^{a\phi} F_{[n]}. \tag{2.45}$$

given the Bianchi identity $dF_{[n]} = 0$.

The corresponding magnetic ansatz example in 11D supergravity is the M5-brane solution

$$\begin{aligned} ds_{11}^2 &= (1+\frac{k}{r^3})^{-1/3}\eta_{\mu\nu}dx^\mu dx^\nu + (1+\frac{k}{r^3})^{2/3}dy^m dy^m, \\ F_{m_1\cdots m_4} &= 3k\varepsilon_{m_1\cdots m_4 p}\frac{y^p}{r^5}, \quad \phi = 0. \end{aligned} \tag{2.46}$$

Identically, the FFA term vanishes with the solitonic ansatz and $a = 0,\ \Delta = 4$.

2.2.3 Dyonic p-brane solutions

Dyonic branes are charged under a self-dual or anti-self-dual field strength and only occur in even dimensions with $p + 2 = D/2$ for odd p. In this case, the scalar field decouples as it is sourced by $F_{[p+2]} \wedge *F_{[p+2]} = \pm F_{[p+2]} \wedge F_{[p+2]} = 0$, Eq.(2.16). Hence, we can set $a = 0$. The resulting solution is

$$\begin{aligned} & ds_D^2 = H^{-\frac{2}{\Delta}}\eta_{\mu\nu}dx^\mu dx^\nu + H^{\frac{2}{\Delta}}dy^m dy^m, \quad \nabla^2 H = 0, \\ & F^{\pm}_{[p+2]} = L_{[p+2]} \pm *L_{[p+2]}, \quad L_{[p+2]} = \mp\frac{\sqrt{2(D-2)}}{d}H^{-2}dH \wedge \mathrm{vol}(M_{p+1}), \end{aligned} \tag{2.47}$$

where the $\pm$ indicates whether the field strength is self-dual or anti-self-dual. Again, $H(y)$ is a harmonic function on the transverse space. The conserved charge associated with the dyonic brane is both electric and magnetic.

There is no dyonic brane in 11D supergravity. One classic dyonic brane solution is the D3-brane in type IIB supergravity coupling to the gauge field with a self-dual 5-form field strength in the R-R sector [20].

2.2.4 Preserved Supersymmetry

Let's discuss these solutions from the perspective of supersymmetry. As the discussions depend on the specific solutions, we only discuss the M2-brane solution, Eq.(2.39), other cases are similar.

In M2-brane solution, we would like to preserve SO(1, 2) × SO(8) covariance in consistent with the ISO(1, 2) × SO(8) invarant background. Hence, we need to adopt a suitable basis for the Gamma matrices. An appropriate one can be

$$\Gamma_A = \left(\gamma_\mu \otimes \tilde{\Sigma}_9, \mathbf{1} \otimes \Sigma_m\right), \tag{2.48}$$

where γ_μ are Gamma matrices with Spin group $\mathrm{SO}(1,2)$ in 3 dimensions, Σ_m are Gamma matrices with $\mathrm{SO}(8)$ in 8 dimensions and $\tilde{\Sigma}_9 = \Sigma_3\Sigma_4 \cdots \Sigma_{10}$. Then a general spinor field in the M2-brane background can be written as

$$\epsilon(x,y) = \epsilon_2 \otimes \eta(r), \tag{2.49}$$

where ϵ_2 is a *constant* $\mathrm{SO}(1,2)$ spinor and $\eta(r)$ is an $\mathrm{SO}(8)$ spinor. Substituting this expression and Eq.(2.22) into the Killing spinor equation, Eq.(2.11), we can solve them with

$$\begin{aligned} &dA + \tilde{d}B = 0, \quad C'e^C = 3A'e^{3A}; \\ &\eta(r) = H^{-1/6}(y)\eta_0, \quad (\mathbf{1} + \tilde{\Sigma}_9)\eta_0 = 0, \end{aligned} \tag{2.50}$$

in which η_0 is a constant $\mathrm{SO}(8)$ spinor.

Here, the first-line equations are the supersymmetry conditions. The first one is exactly the restriction we imposed when solving the p-brane solution, Eq.(2.30), while the second one can be derived from the restriction Eq.(2.30) with Eq.(2.29) and Eq.(2.32). It means that the restrictions we imposed to solve the p-brane solutions are in fact required for preserving supersymmetries. On the other hand, the second-line equations specify the form of Killing spinors. The first one says the M2-brane solution breaks an infinite number of *local* supersymmetries to a finite number of *rigid* supersymmetries in the special form $\epsilon(x,y) = H^{-1/6}(y)\epsilon_2 \otimes \eta_0$. The second equation indicates that the surviving rigid supersymmetries are chiral in the transverse part, and hence, only $2 \cdot 8 = 16$, *i.e.* half of the 32, supersymmetries survive.

The Killing spinor equations are highly restricted and, in many cases, they determine the p-brane solution. When there is a time-like Killing spinor, whose corresponding Killing vector is time-like, the geometry solving the Killing spinor equations will solve all the equations of motions spontaneously provided that the antisymmetric-field strength $F_{[n]}$ satisfies the Bianchi identity $dF_{[n]} = 0$ and the equation of motion $d * F_{[n]} + \frac{1}{2}F_{[n]} \wedge F_{[n]} = 0$. When all the Killing spinors are null, the additional condition for the geometry is satisfying just one component of the Einstein equations [21].

We can also count the preserved supersymmetry through the supersymmetry algebra. For the M2-brane oriented in the $\{012\}$ directions, which satisfies $\mathcal{E} = U = U_{12}$, the 11D supersymmetry algebra, Eq.(2.3) becomes

$$\frac{1}{\mathrm{vol}(M_2)}\{Q_\alpha, Q_\beta\} = -(C\Gamma^0)_{\alpha\beta}\mathcal{E} + (C\Gamma^{12})_{\alpha\beta}U_{12} = 2\mathcal{E}P_{012}, \quad P_{012} = \frac{1}{2}(\mathbf{1} + \Gamma^{012}). \tag{2.51}$$

using $C = \Gamma_0$ and $\Gamma_0^2 = 1$. The P_{012} is a projection operator satisfying $P_{012}^2 = P_{012}$ with eigenvalues $+1$ and 0. Since the supercharges Q_α are the generators of the supersymmetry transformations, we can get the preserved supersymmetries by counting the number of zero eigenvalues of $\{Q_\alpha, Q_\beta\}$ which is simply $\mathrm{tr}(\mathbf{1} - P_{012})$. As $\mathrm{tr}(\Gamma^{012}) = 0$, there are only 16 supersymmetries preserved for the M2-brane solutions.

The above discussion can be easily generalized to other brane solutions.

2.3 Super p-brane Action

In brane solutions like Eq.(2.35-2.36), we can set the dilaton ϕ to vanish consistently when $a = 0$. In such cases, $r = 0$ is not a physical singularity, as we can check with the Riemann tensor in the orthonormal frame. We can conduct an analytical continuation. For example, in the M2-brane solution Eq.(2.39), we can choose $r = (\tilde{r}^6 - k)^{\frac{1}{6}}$. Here, $r = 0$ is a degenerate horizon, while $\tilde{r} = 0$ is a curvature singularity. The $\tilde{r} = 0$ is actually an extended subspace $(x^\mu, y^m = 0)$ suggesting an M2-brane sitting there. For the cases $a \neq 0$, $r = 0$ is singular which can be seen from the scalar equation. Thus, we consider the p-brane is sitting at $r = 0$ when $a \neq 0$. To describe the sources, *i.e.* p-branes, we need to consider the p-brane action.

Let's start from the bosonic part p-brane action in the Polyakov form,

$$I = \frac{1}{2}\int d^{p+1}\xi\sqrt{-\gamma}\left[\gamma^{ij}\partial_i x^m \partial_j x^n g_{mn} - (p-1)\right]. \tag{2.52}$$

Here, γ_{ij} is the auxiliary worldvolume metric with $i = 0, 1, \cdots, p$ depending on the worldvolume coordinate ξ^i, and $m = 0, 1\cdots, D-1$ is the spacetime indices used only in this section, which is different from our previous convention.

We need to generalize the target space x^m to superspace $Z^M = (x^m, \theta^\alpha)$ to get an action consistent with supersymmetry. For simplicity, we introduce vielbeins $E^A(Z) = dZ^M E_M^A(Z)$ with the world indices M and tangent space indices A ranging from both bosonic values $m, a = 0, 1, \cdots, D-1$ and fermionic values α depending on the spinor dimensions. Thus, we can describe the supergravity background with vielbeins $E_M^A(Z)$

and a superspace $(p+1)$-form $B = \frac{1}{(p+1)!}E^{A_1}\cdots E^{A_{p+1}}B_{A_{p+1}\cdots A_1}$[2].

With $Z^M(\xi)$ serving the map from the worldvolume to the target space, we have $E^A(Z) = dZ^M E^A_M(Z) = d\xi^i \partial_i Z^M(\xi) E^A_M(Z(\xi)) \equiv d\xi^i E^A_i(\xi)$ on the super p-brane worldvolume. Then, generalizing the metric g_{mn} with vielbeins and adding super $(p+1)$-form, we get the super p-brane action in Green-Schwarz form

$$I = \int d^{p+1}\xi \left\{ \frac{1}{2}\sqrt{-\gamma}\left[\gamma^{ij}E^a_i E^b_j \eta_{ab} - (p-1)\right] \right. \\ \left. + \frac{1}{(p+1)!}\varepsilon_{i_1\cdots i_{p+1}} E^{A_1}_{i_1}\cdots E^{A_{p+1}}_{i_{p+1}} B_{A_{p+1}\cdots A_1} \right\}. \tag{2.53}$$

In such a target space supersymmetry invariant form, we encounter a new question of how to balance the bosonic and fermionic sectors on the worldvolume. For example, on the supermembrane in 11 dimensions, the bosonic coordinates x^m take 11 values while Majorana spinors have 32 degrees of freedom. Due to the reparameterizations on the worldvolume of supermembrane, there are $11 - 3 = 8$ on-shell bosonic degrees of freedom left. Further, considering the bosonic equations of motion are second-order differential equations while the fermionic equations are first-order, we expected that there would be 16 worldvolume fermionic degrees of freedom instead of 32. Thus, we expect that there exists a fermionic gauge symmetry on the super p-brane action Eq.(2.53).

This fermionic gauge symmetry is called κ *symmetry* [22, 23], which will kill half of the fermionic degrees of freedom. The fermionic gauge transformation is generated by a spinor labeled by κ^α,

$$\delta z^a = 0, \quad \delta z^\alpha = \frac{1}{2}(1+\Gamma)^\alpha_\beta \kappa^\beta(\xi), \tag{2.54}$$

with

$$\Gamma = \frac{(-1)^{\frac{(p+1)(p-2)}{4}}}{(p+1)!\sqrt{\gamma}}\varepsilon^{i_1\cdots i_{p+1}} E^{a_1}_{i_1}\cdots E^{a_{p+1}}_{i_{p+1}}(\Gamma_{a_1\cdots a_{p+1}}), \quad \gamma_{ij} = E^a_i E^b_j \eta_{ab}. \tag{2.55}$$

It is easy to check that $\Gamma^2 = 1$ and, hence, $P = \frac{1}{2}(1+\Gamma)$ is a projection operator with $\mathrm{tr}P = \frac{1}{2}\mathrm{tr}1$. Thus, we can set half z^α to zero and hence balance the bosonic and fermionic sectors in the supermembrane case.

[2] Note the order of indices.

2.4 Kaluza-Klein Dimensional Reduction

Supergravity and superstring theories are essentially higher-dimensional theories. To generate a 4-dimensional effective theory, we need a technique to reduce dimensions. On the other hand, Supergravity theories can be obtained via dimensional reduction from higher-dimensional supergravity like 11D supergravity. In this section, we will introduce the classical dimensional reduction technique, Kaluza-Klein dimensional reduction. For simplicity, we only consider the bosonic sector here.

We consider the case when the spacetime can be written as a product manifold $M_D = M_d \times B_{D-d}$. The dimension reduction on a compact manifold B_{D-d} is well-understood for physicists, while the reduction on non-compact B_{D-d} was first discussed in [9]. For our main interests, we will consider only non-compact B_{D-d} when discussing how to get effective theories on or near the lower-dimensional branes M_d in the rest chapters. In this section, however, we discuss the dimensional reduction in a compact transverse space, while the non-compact cases are direct generalizations.

When the transverse space B_{D-d} is compact, a general variable in the higher-dimension M_D can be written as

$$\Phi(x, y) = \sum_n \phi_n(x)\zeta_n(y), \tag{2.56}$$

where x^μ and y^m are coordinate on M_d and B_{D-d} respectively and $\zeta_n(y)$ are a set of Fourier modes on B_{D-d} labeled by eigenvalues n. Here, we suppress all the possible indices of spacetime or internal symmetries. Then, higher-dimensional equations of motion for $\Phi(x, y)$ will generate an infinite tower of lower-dimensional equations of motion for $\phi_n(x)$. For example, considering a massless scalar field $\Phi(x, y)$ on $\mathbb{R}^{1,3} \times S^1$ with the length of S^1 being l, we have

$$\begin{aligned} 0 &= \Box_5 \Phi(x, y) \\ &= \Box_5 \left[\sum_{n=-\infty}^{\infty} a_n \phi_n(x) e^{in\pi y/l} \right] \\ &= \sum_{n=-\infty}^{\infty} a_n e^{in\pi y/l} \times \left[\Box_4 - \left(\frac{n\pi}{l} \right)^2 \right] \phi_n(x). \end{aligned} \tag{2.57}$$

The linear independence of the Fourier modes implies the equations of motion of the set of modes with different masses

$$\left[\Box_4 - \left(\frac{n\pi}{l} \right)^2 \right] \phi_n(x) = 0, \quad n \in \mathbb{Z}. \tag{2.58}$$

Note that the mass $n\pi/l$ is inversely proportional to the length scale l of the compact space. If we take $l \sim l_{\mathrm{pl}}$, the massive modes will be too heavy to be generated in the recent energy scale in colliders.

The essential step after writing in lower-dimension is a consistent truncation of the field variables to make the theory more tractable. We always leave only the modes independent of the reduced coordinate y^m in Kaluza-Klein reduction, such as the constant mode in the previous example.[3] The consistent truncation premises that the reduced lower-dimensional solutions are also solutions for the original theory.

2.4.1 Reduction from $D+1$ to D dimensions

Let's first consider the dimensional reduction from $D+1$ to D dimensions. The metric in $M_D = M_{D-1} \times B_1$ can be written as

$$d\hat{s}^2 = e^{2\alpha\varphi} ds^2 + e^{2\beta\varphi}\left(dz + \mathcal{A}_M dx^M\right)^2 , \tag{2.59}$$

where ds^2 is the metric on M_{D-1}. Introduce vielbeins on M_D

$$\hat{\theta}^A = \begin{cases} e^{\alpha\varphi}\theta^a, & A = a = 0, \cdots, D-1, \\ e^{\beta\varphi}\left(dz + \mathcal{A}_M dx^M\right), & A = D, \end{cases} \tag{2.60}$$

with θ^a being the vielbeins on metric ds^2. We use hatted and unhatted variables to distinguish the variables in $D+1$ and D dimensions.

Using the first Cartan's structure equation and metric compatibility condition, we solve the connection 1-form

$$\begin{aligned} \hat{\omega}^D_{\ \ a} &= e^{-\alpha\varphi}\left(\beta\partial_a\varphi\hat{\theta}^D + e^{\beta\varphi}\frac{1}{2}\mathcal{F}_{ab}\theta^b\right), \\ \hat{\omega}^a_{\ \ b} &= \omega^a_{\ \ b} + \alpha\partial_b\varphi\theta^a - \alpha\partial^a\varphi\eta_{db}\theta^d - e^{(\beta-2\alpha)\varphi}\frac{1}{2}\mathcal{F}^a_{\ \ b}\hat{\theta}^D, \end{aligned} \tag{2.61}$$

with $\mathcal{F}_{ab} = \partial_a\mathcal{A}_b - \partial_b\mathcal{A}_a$. After a long and tedious but straightforward calculation, we can express the Ricci scalar with lower-dimensional variables. Combining with

[3] The constant mode and consistent truncation do not always exit on the compact transverse space. A practical method to find a consistent truncation is utilizing the group theory and keeping only the singlets. Because the multiplication of singlets is always singlet, the interaction between singlets will not generate other representations of the group.

$\sqrt{\hat{g}} = e^{(D\alpha-\beta)\varphi}\sqrt{g}$, we have

$$\begin{aligned}\sqrt{\hat{g}}\hat{R}(\hat{g}) = e^{((D-2)\alpha+\beta)\varphi}\Big[&R(g) - e^{2(\beta-\alpha)\varphi}\frac{1}{4}\mathcal{F}_{ab}\mathcal{F}^{ab} - 2\left((D-1)\,\alpha+\beta\right)D_c\left(\partial^c\varphi\right)\\ &- \left[(D-1)\,(D-2)\,\alpha^2 + 2\beta((D-2)\,\alpha+\beta)\right]\partial_c\varphi\partial^c\varphi\Big]\,.\end{aligned} \tag{2.62}$$

Different combinations of α and β are related by Weyl transformations. We can choose $(D-2)\alpha+\beta=0$ to make reductions of the Einstein-frame form gravitational action in $D+1$ dimensions to Einstein-frame form gravitational actions in D dimensions. Further, we can choose $\alpha^2 = [2(D-1)(D-2)]^{-1}$ to make a canonical kinetic term for φ. In these coefficient choices, we have

$$\sqrt{\hat{g}}\hat{R}(\hat{g}) = \sqrt{-g}\left[R(g) - \frac{1}{2}\partial_c\varphi\partial^c\varphi - e^{-2(D-1)\alpha\varphi}\frac{1}{4}\mathcal{F}_{ab}\mathcal{F}^{ab}\right]. \tag{2.63}$$

Next, we need to establish the reduction ansatz for a $(D+1)$-dimensional anti-symmetric tensor gauge field $\hat{F}_{[n]} = d\hat{A}_{[n-1]}$. We have the decomposition

$$\hat{A}_{[n-1]}(x,z) = B_{[n-1]}(x) + B_{[n-2]}(x)\wedge dz, \tag{2.64}$$

using x to indicate the coordinates in the residual D dimensions. That is, $A_{[n-1]}$ is decomposed into two parts with 0 or 1 index take the value z. Due to a certain Chern-Simons structure appearing upon dimensional reduction, instead of defining $G'_{[n]} = dB_{[n-1]}$, we define the reduced field strengths to be

$$G_{[n]} = dB_{[n-1]} - dB_{[n-2]}\wedge\mathcal{A}_{[1]}, \quad G_{[n-1]} = dB_{[n-2]}. \tag{2.65}$$

In such an ansatz, the dimensional reduction of the gauge field kinetic term is given by

$$\begin{aligned}\frac{1}{2}\hat{F}_{[n]}\wedge\hat{*}\hat{F}_{[n]} =& e^{[-2(n-1)\alpha+(D-2)\alpha+\beta]\varphi}\frac{1}{2}G_{[n]}\wedge *G_{[n]}\\ &+ e^{[2(D-n)\alpha-((D-2)\alpha+\beta)]\varphi}\frac{1}{2}G_{[n-1]}\wedge *G_{[n-1]}.\end{aligned} \tag{2.66}$$

Here, $\hat{*}$ is defined on $D+1$ dimensions while $*$ is defined on D dimensions. And finally, for a scalar field, we define

$$\hat{\phi}(x,z) = \phi(x). \tag{2.67}$$

We can use the result to check the dimensional reduction from 11D supergravity Eq.(2.9) to type IIA supergravity directly Eq.(2.12). The 11-dimensional metric is decomposed into 10-dimensional metric, the dilaton and the R-R vector $A_{[1]}$ shown in the first line of Eq.(2.12). The 11-dimensional 3-from $A_{[3]}$ becomes the R-R 3-form field $A_{[3]}$ and the NS-NS 2-form $B_{[2]}$ in the second line. The $dA \wedge dA \wedge A$ origins from the FFA term.

Let's consider the dimension reduction of a single-charge action Eq.(2.15) to the Einstein frame with a canonical kinetic term for the Kaluza-Klein scalar. Substituting Eq.(2.63), Eq.(2.66) and Eq.(2.67) into

$$\hat{I} = \int d^{D+1}x\sqrt{-\hat{g}}\left[R\left(\hat{g}\right) - \frac{1}{2}\hat{\nabla}_M\phi\hat{\nabla}^M\phi - \frac{1}{2n!}e^{\hat{a}\phi}\hat{F}_{[n]}^2\right] \tag{2.68}$$

we have

$$\begin{aligned} I = \int d^Dx\sqrt{-g}\Big\{&R - \frac{1}{2}\nabla_m\varphi\nabla^m\varphi - \frac{1}{2}\nabla_m\phi\nabla^m\phi - e^{-2(D-1)\alpha\varphi}\frac{1}{4}\mathcal{F}_{cd}\mathcal{F}^{cd} \\ &-\frac{1}{2n!}e^{-2(n-1)\alpha\varphi+\hat{a}\phi}G_{[n]}^2 - \frac{1}{2\left(n-1\right)!}e^{2(D-n)\alpha\varphi+\hat{a}\phi}G_{[n-1]}^2\Big\}. \end{aligned} \tag{2.69}$$

Note that each of the antisymmetric-tensor field strengths couples the scalar fields with the factor $e^{a_r\tilde{\phi}_r}$ where $\tilde{\phi}_r$ is the SO(m)-rotated combination of m scalar fields. Here $m = 2$. The Δ, defined in Eq.(2.31), of each a_r satisfying

$$a_r^2 = \Delta - \frac{2d_r\tilde{d}_r}{D-2} = \Delta - \frac{2(r-1)(D-r-1)}{D-2} \tag{2.70}$$

is the same as the Δ of the parent coupling parameter $\hat{a}$ satisfying

$$\hat{a}_n^2 = \Delta - \frac{2d_n\tilde{d}_n}{(D+1)-2} = \Delta - \frac{2(n-1)(D-n)}{D-1}. \tag{2.71}$$

We can do a consistent truncation further and leave only one field strength and the corresponding scalar field to get a lower-dimensional single-charge action [16].

Dimensional reduction can not only be applied to supergravity theories but also to brane solutions. Note that in the p-brane solutions we get like Eq.(2.35), variables are independent of the worldvolume, which provides a naturally dimensional reduction on the worldvolume direction. Reductions on worldvolume directions are called *diagonal* dimensional reduction. On the other hand, the existence of multi-center solutions of the harmonic function Eq.(2.37) allows us to stack an infinite number of parallel branes with the same orientation along a transverse direction. Without loss of generality, we choose the stacked direction to be y^{D-1} and relabel it with z. In this case, the harmonic function becomes

$$H(y) = 1 + \int_{-\infty}^{+\infty}\frac{kdz}{\left(\hat{r}^2+z^2\right)^{\tilde{d}/2}} = 1 + \frac{\tilde{k}}{\tilde{r}^{\tilde{d}-1}}, \tag{2.72}$$

with

$$\hat{r}^2 = \sum_{m=d}^{D-2} y^m y^m, \quad \hat{k} = \frac{\sqrt{\pi} k \Gamma(\tilde{d} - \frac{1}{2})}{2\Gamma(\tilde{d})}. \tag{2.73}$$

The result solution is independent of the stacked direction, and hence we can dimensional reduction on it. Reduction on the transverse space called *vertical* dimensional reduction. These dimensional reductions on the brane solutions are commute and reversible.

The dimensional reduction can be performed repeatedly for n times which is equivalent to a dimensional reduction on $B_n = T^n$. Dimensional reduction on different transverse spaces B_{D-d} will result in different d-dimensional supergravity theories. One of the differences is the symmetries in the lower-dimensional supergravity theories. The general covariant symmetries of the original theory will result in global symmetries and gauge symmetries, which depend on the transverse space, in addition to the diffeomorphism in the reduced theory. When dimensional reduction on a complicated transverse space, *e.g.* S^n, instead of expressing the action with reduced variables we do in this subsection, we will first express the higher-dimensional equations of motion with reduced variables and then find an appropriate action for them. [24] provides a good review of Kaluza-Klein dimensional reduction.

Chapter 3

Type I - Effective Theories on Branes

Supergravity and superstring are essentially higher-dimensional. However, all the observations up to now suggest that we live in 4-dimensional spacetime. Contrast to the issue of the non-gravitational matters in the standard model can be solved by confining them on a 3-brane, gravity cannot be confined on a lower-dimensional subspace as it is the dynamic of spacetime itself. Getting effective gravitational theories on branes through the Kaluza-Klein dimensional reduction on compact transverse space has been well-studied by physicists. One of the most popular models assumes that there are six extra compact and extremely small dimensions space described by Calabi-Yau manifolds [25, 26]. In this dissertation, we focus on constructing effective theories with non-compact transverse spaces.

Unfortunately, it is not that straightforward to get a well-behaved effective gravity with a non-compact transverse space. Before starting with any specific models, we can first contemplate the characteristics of effective gravitational theories we want. Firstly, the gravity spectrum must include a *massless* mode. Similar to the Kaluza-Klein dimensional reduction, the existence of transverse space will generate an infinite tower of gravitons with different masses. To have a massless graviton, the transverse space problem must admit a zero eigenvalue. Then, we need a finite *Newton constant* G_d. If we simply consider direct product space $M_D = M_d \times B_{D-d}$, their Newton constants are related by $G_d = G_D/\text{Vol}_B$. The infinite volume of the non-compact transverse space B_{D-d} then suppresses G_d to zero. Besides, we need to check the Newtonian limit of gravity and we want to get a $1/r$ *gravitational potential* at the first order, which is the characteristic of gravity in three spatial dimensions. In the relativity limit, we need to check the *gravitational self-coupling* and the *coupling constant* between gravity and matter fields. Because of the self-coupling, we also need to promise the insignificance of *energy loss* induced by the coupling of the zero mode and to massive modes which do not ultimately couple back to matter on the brane.

Even if we have a theory satisfying the above requirement, we must guarantee that the lower-dimensional behavior of the effective theory is *stable*. A trivial example of non-compact reductions is starting from a higher-dimensional Minkowski space and requiring translational invariance in all dimensions higher than 4. There is a continuous massive sector spectrum right down to zero in this example and hence any small excitation of the massive modes causes the theory to behave like a higher dimensional theory. This construction is obviously unstable.

Ref.[11] provides a taxonomy for the effective gravitational theories on or near lower-dimensional branes. In this chapter, we discuss the type I construction of effective theory. We start it with a summary of the characteristics of type I constructions and then discuss it with specific models.

3.1 Characteristics of Type I Effective Theories

Type I constructions admit consistent truncation of the higher-dimensional theories and give rise to fully non-perturbative gravity on the worldvolume. One can consider type I effective theories as embedding lower-dimensional gravity into higher-dimensional space. Indeed, we set up the type I gravity by replacing the worldvolume variables of a brane solution with four-dimensional gravitational solutions. In addition to the simplest Ricci-flat Einstein gravity, we can substitute the worldvolume metric with supergravity. As a type I gravity is constructed by embedding, they satisfy almost all the criteria for a lower-dimensional gravity, except for a well-defined Newton constant because of the trombone symmetries.

The metrics of a type I gravity can be written in the form as

$$ds_D^2(x,y) = e^{2\alpha(y)} g_{\mu\nu}(x) dx^\mu dx^\nu + ds^2_{(D-4)}(y). \tag{3.1}$$

In a perturbative way, the gravitational effect is described by the perturbations $h_{\mu\nu}(x)$ only along the *worldvolume* dressed by a specific factor depending on the transverse space,

$$ds^2_D(x,y) = e^{2\alpha(y)}\left(\eta_{\mu\nu} + h_{\mu\nu}(x)\right) dx^\mu dx^\nu + ds^2_{(D-4)}(y). \tag{3.2}$$

It promises the lower-dimensional nature of the effective theory. As there is no dependence on the worldvolume part, it naturally admits a consistent truncation for transverse space.

We can investigate the type I construction from the perspective of a transverse space problem. Let's consider there is only one non-compact direction ρ in the transverse space. Then, a perturbation with both worldvolume and transverse space spherical symmetric can be written as

$$g_{\mu\nu}(x,y) = \eta_{\mu\nu}(x) + h_{\mu\nu}(r,\rho). \tag{3.3}$$

The higher-dimensional Einstein equations at first order will result in a Green's function problem

$$\Delta h_{\mu\nu}(x,\rho) = \left(\partial_r^2 + \frac{2}{r}\partial_r + g^2\Delta_\rho\right) h_{\mu\nu}(x,\rho) = 0, \tag{3.4}$$

where $\partial_r^2 + \frac{2}{r}\partial_r$ and Δ_ρ is the harmonic operator in the three-dimensional and transverse space respectively, and g is a coupling constant. Accordingly, we posit an expansion

$$h_{\mu\nu}(x,\rho) = \sum_i h^{(\lambda_i)}_{\mu\nu}(x)\xi_{(\lambda_i)}(\rho) + \int_{\Lambda_{\text{edge}}}^{\infty} h^{(\lambda)}_{\mu\nu}(x)\xi_{(\lambda)}(\rho), \tag{3.5}$$

in which the ξ_{λ_i} are discrete states and the ξ_λ are scattering states with eigenvalues λ starting from the edge of the continuous spectrum Λ_{edge}. The expansion reduces the problem further into the eigenvalue problem in the transverse space

$$\Delta_\rho \xi_{(\lambda)} + m^2_\lambda \xi_{(\lambda)} = 0. \tag{3.6}$$

Since we require a massless graviton, there must exist a zero eigenvalue $m_0^2 = 0$. As a homogeneous differential equation, the eigenfunction with zero eigenvalue can be solved by

$$\xi_{(0)}(\rho) = c_1 + c_2\zeta(\rho) \tag{3.7}$$

with arbitrary constants c_1 and c_2.

We need $c_2 = 0$ for type I construction to restrict the perturbations along the worldvolume as Eq.(3.2). The combinations of c_1 and c_2 can be selected with proper choices of boundary conditions. Hence, the type I construction, whose perturbations $h_{\mu\nu}(x)$ have fixed transverse space dependence, is based on a *Dirichlet* boundary condition on the transverse space problem.

3.2 Einstein Gravity on Branes

3.2.1 The Black String Solutions

It was first found in ref.[12] that we can replace the worldvolume of magnetic braneworlds like Eq.(2.43) with Ricci-flat metric. Based on the Randall-Sundrum model[1] Eq.(4.69), ref.[13] provided a model describing Schwarzschild black holes setting on domain walls in a AdS_5 background. As the domain wall solution is a magnetic solution, the construction is consistent with the conclusion in ref.[12]. The metric takes the form

$$ds^2 = \frac{l^2}{z^2}\left(-\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + dz^2\right) \quad (3.8)$$

in the Poincare patch coordinate, where l is the AdS radius. The induced metric on a domain wall at $z = z_0$ can be transformed into the standard Schwarzschild metric by rescaling the coordinate t and r.

It is important to note that the Schwarzschild spacetime is not localized in a specific transverse position and the singularity at $r = 0$ is not generically a point in the AdS_5 background. Thus, the singularities $r = 0$ spread out in the transverse space and stake up like a string. That is the reason why the model is called the black-string solution. The black-string solution provides a type I construction with general relativity *on* 3-branes.

To incorporate the Schwarzschild metric, the AdS_5 structure becomes non-trivial. The extensional nature of the black string has a significant impact on the AdS_5 spacetime geometry resulting in singular AdS horizon $z = \infty$.

We closely follow the discussion in ref.[13], in which they first considered the square of the Ricci tensor

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{1}{l^4}\left(40 + \frac{48M^2z^4}{r^6}\right). \quad (3.9)$$

It diverges at the singularity at each slice of 3-brane $r = 0$ as expected, while it is still singular at the AdS horizon $z = \infty$ with finite r. We can investigate the singular horizon by considering null or timelike geodesics

$$z = -\frac{z_1 l}{\lambda}, \qquad z = -\frac{z_1}{\sin(\lambda/l)}, \quad (3.10)$$

[1] We will provide more details for it in section 4.2.

that will approach the horizon $z = \infty$. Here, z_1 is a constant and $z \to \infty$ as $\lambda \to 0^-$. For simplicity, we define new coordinates $\tilde{r} = z_1 r/l, \tilde{t} = z_1 t/l$ and new parameter $\nu = -z_1^2/\lambda$ for null geodesics and $\nu = -(z_1^2/l)\cot(\lambda/l)$ for timelike geodesics. Then, the radial motions on the 3-brane of both null and timelike geodesics are effective 4-dimensional timelike orbits of Schwarzschild black holes

$$\left(\frac{d\tilde{r}}{d\nu}\right)^2 + \left(1 + \frac{\tilde{L}^2}{\tilde{r}^2}\right)\left(1 - \frac{2\tilde{M}}{\tilde{r}}\right) = \tilde{E}. \tag{3.11}$$

with the effective energy $\tilde{E} = z_1 E/l$, effective angular momentum $\tilde{L} = z_1^2 L/l^2$, and effective mass $\tilde{M} = z_1 M/l$. That is, the motions in the transverse space give rise to an effective mass in four dimensions. Once again, we see the effective gravity on 3-branes embedded in a higher-dimensional AdS_5 spacetime. In 5 dimensions, the 4-dimensional bound geodesic will stay at finite r and spiral along with the black string until ultimately hitting the singularity at the AdS horizon at $z = \infty$.

The unbound geodesics that reach $r = \infty$, however, have a well-defined square of the Ricci tensor because of their late time behavior $r \sim \frac{z_1 l}{l}\sqrt{\tilde{E}^2 - 1}$. To check the singular structure at the $z \to \infty, r \to \infty$, we can check the Riemann tensor in an orthonormal frame, which is set up by the tangent vector u^M and a unit normal n^M on the geodesic. [13] showed that the component

$$R_{(u)(n)(u)(n)} = R_{MNPQ} u^M n^N u^P n^Q = \frac{1}{l^2}\left(1 - \frac{2Mz^4}{z_1^2 r^3}\right) \tag{3.12}$$

is divergent as $r \to \frac{z_1 l}{l}\sqrt{\tilde{E}^2 - 1}$ along the unbound geodesics.

What's more, the Schwarzschild black in Randall-Sundrum background is not an extremal solution, $M \neq Q$, and will suffer a Gregory-Laflamme type of instability [27] near the AdS horizon. The black string will pinch off near the AdS horizon and give rise to a stable black cigar extend in the AdS space. That is, the black-string solution will not be the ultimate stable state.

3.2.2 Doubly-Ricci-flat Branes and Black Spoke Solutions

It is *possible* to replace both the worldvolume and transverse space with arbitrary Ricci-flat metrics for supersymmetric brane solutions. We call such a solution doubly-Ricci-flat brane whose metric is given by

$$ds^2 = H^\alpha(y)\bar{g}_{\mu\nu}(x)dx^\mu dx^\nu + H^\beta(y)\tilde{g}_{ij}(y)dy^i dy^j. \tag{3.13}$$

Here, α, β are two appropriate constants, $H(y)$ is a harmonic function on the transverse space, $\bar{g}_{\mu\nu}$ and $\tilde{g}_{ij}$ are Ricci-flat metrics on worldvolume and transverse space respectively. The harmonic function $H(y)$ is trivial in a compact transverse space and has singularities in a non-compact one.

Depending on the transverse space dimension, the transverse Euclidean space can be replaced by Calabi-Yau, hyper-Kähler, $G_{(2)}$, or Spin(7) manifold. The Killing spinor must be a singlet of the special holonomy group [20]. On the other hand, replacing the Minkowski worldvolume metric with a general Ricci-flat metric allows us to construct a worldvolume Einstein's gravity. The black-spoke solutions that we will discuss are the examples with Einstein's gravity, described by Schwarzschild metric, on the worldvolume.

It was proven that the metric in Eq.(3.13) along with appropriate scalars and gauge fields are supersymmetric provided that $\bar{g}_{\mu\nu}$ and $\tilde{g}_{ij}$ admit covariantly constant spinors with appropriate projection conditions. We can build supersymmetric brane solutions similar to those in section 2.2. For example, the electric p-brane solution in Eq.(2.35) becomes [11]

$$\begin{aligned} ds_D^2 &= H^{-\frac{4\tilde{d}_{el}}{\Delta(D-2)}} ds^2(M_{p+1}) + H^{\frac{4d_{el}}{\Delta(D-2)}} ds^2(B_{D-p-1}), \\ A_{[p+1]} &= \frac{2}{\sqrt{\Delta}} H^{-1} \mathrm{vol}(M_{p+1}), \quad e^{\phi} = H^{\frac{2a}{\Delta}}, \quad \tilde{\nabla}^2 H = 0. \end{aligned} \tag{3.14}$$

The resulting brane solutions have the same structure as those with flat worldvolume embedded in a flat background. Similar situations happen to the magnetic and dyonic brane solutions.

The doubly-Ricci-flat solutions Eq.(3.14) admit three global symmetries called trombone symmetries. The first one is

$$g_{MN} \mapsto k_H^2 g_{MN}, \quad \phi \mapsto \phi, \quad F_{[p+2]} \mapsto k_H^{p+1} F_{[p+2]}, \tag{3.15}$$

and the other two are individual rescaling of the worldvolume and transverse metric

$$\bar{g}_{\mu\nu} \mapsto k_W^2 \bar{g}_{\mu\nu}, \quad \tilde{g}_{ij} \mapsto k_T^2 \tilde{g}_{ij}. \tag{3.16}$$

It is the existence of trombone symmetries prevents a well-defined 4-dimensional Newton constant as we can scale the Newton constant to any value by a trombone transformation.

Not restricted to embedding a 4-dimensional Schwarzschild metric into the Randall-Sundrum type AdS_5 background, we can simply substitute the worldvolume of the doubly-Ricci-flat spacetime in Eq.(3.13) with a n-dimensional Schwarzschild metric

$$ds_n^2 = -\left(\frac{1-\frac{M}{r^{n-3}}}{1+\frac{M}{r^{n-3}}}\right)^2 dt^2 + \left(1+\frac{M}{r^{n-3}}\right)^{\frac{4}{n-3}} \left(dr^2 + r^2 ds^2(S^{n-2})\right). \tag{3.17}$$

Because there could be not only one transverse dimension, the singularities spread out radially in the transverse space like a spoke rather than a string. Ref. [11] named it the black-spoke solution.

We can write the black spoke metric in a perturbation picture around a doubly-Ricci-flat brane with a $\mathbb{R}^{1,n-1}$ worldvolume,

$$ds^2 = H^\alpha \left(\eta_{\mu\nu}dx^\mu dx^\nu + M h_{\mu\nu}dx^\mu dx^\nu\right) + H^\beta \tilde{g}_{ij}dy^i dy^j + \mathcal{O}(M^2), \tag{3.18}$$

with

$$h_{00} = \frac{4}{r^{n-3}}, \quad h_{mn} = \frac{4}{(n-3)r^{n-3}}\delta_{mn}. \tag{3.19}$$

It is easy to check that the perturbation is not traceless and obeys the de Donder gauge

$$\partial^\mu h_{\mu\nu} - \frac{1}{2}\partial_v h^\mu_\mu = 0, \tag{3.20}$$

sourced by the stress tensor

$$T_{MN} = M\delta_{M0}\delta_{N0}f(y)\frac{\delta(r)}{r^{n-2}}. \tag{3.21}$$

$f(y)$ depends on the transverse space geometry. There is only a δ-function in the worldvolume, *i.e.* the source spread out in the whole transverse space. We can see the important characteristic of type I gravity that the perturbations only along the worldvolume, while the transverse space dependence is fixed by the factor $H^\alpha(y)$.

3.3 Supergravities on Branes

By counting the preserved supersymmetries on the AdS_5 spacetime and the 3-branes, ref.[14] recognized that, in the Rundall-Sundrum model, the worldvolume gravity can be viewed as a 4D $\mathcal{N}=1$ supergravity, whose bosonic sector comprises only the metric, while the whole spacetime theory is a 5D $\mathcal{N}=2$ gauged supergravity. Two supergravity theories are related by a consistent dimensional reduction. It can be viewed as the worldvolume supergravity is embedded into the spacetime supergravity. Analog to it, they found a consistent dimensional reduction from the 5D $\mathcal{N}=4$ gauged supergravity to the 4D $\mathcal{N} = 2$ Einstein-Maxwell supergravity, whose bosonic sector comprises the metric and the Maxwell gauge field.

Because of the gauge field, the 3-brane allows for the Reissner–Nordström black hole, which in the 5-dimensional spacetime can be viewed as a string. The solution describes a string ending on a D3-brane in type IIB supergravity after being lifted to 10 dimensions with S^5. That is, in this manner, we can construct brane-on-brane solutions with lower-dimensional supergravity embedded into higher-dimensional ones.

Realizing $\mathrm{AdS}_5 \times S^5$ is the near-horizon geometry of a D3-brane in Type IIB supergravity, ref.[15] promoted the embedding of supergravity in ref.[14] to embedding the 4D $\mathcal{N} = 4$ supergravity in Type IIB supergravity with the D3-brane severing as the skeleton. In such case, the worldvolume 4D $\mathcal{N} = 4$ supergravity has the same amount of supersymmetries as the D3-brane solutions in Type IIB supergravity. By finding null geodesics on the scalar coset space, they found a certain class of stationary black hole solution. And they also constructed 6D $\mathcal{N} = (4,0)$ worldvolume supergravity embedded in 11D supergravity with the M5-brane severing as the skeleton.

3.3.1 4D $\mathcal{N} = 2$ supergravity from 5D $\mathcal{N} = 4$ supergravity

The bosonic sector of 5-dimensional $\mathcal{N} = 4$ supergravity is composed by the metric, a scalar ϕ, SU(2) Yang-Mills potentials $A^i_{[1]}$, a U(1) gauge potential $B_{[1]}$, and two 2-form potentials $A^\alpha_{[2]}$ which transforms as a charged doublet under the U(1) symmetry. The Lagrangian is written as [28] [2]

$$\begin{aligned}\mathcal{L}_{5DN4} =& R\tilde{*}\mathbf{1} - \frac{1}{2}d\phi \wedge \tilde{*}d\phi - \frac{1}{2}X^4 G_{[2]} \wedge \tilde{*}G_{[2]} - \frac{1}{2}X^{-2}\left(F^i_{[2]} \wedge \tilde{*}F^i_{[2]} + A^\alpha_{[2]} \wedge \tilde{*}A^\alpha_{[2]}\right) \\ &+ \frac{1}{2g}\varepsilon_{\alpha\beta}A^\alpha_{[2]} \wedge dA^\beta_{[2]} - \frac{1}{2}A^\alpha_{[2]} \wedge A^\alpha_{[2]} \wedge B_{[1]} - \frac{1}{2}F^i_{[2]} \wedge F^i_{[2]} \wedge B_{[1]} \\ &+ 4g^2\left(X^2 + 2X^{-1}\right)\tilde{*}\mathbf{1},\end{aligned} \tag{3.22}$$

where $X = e^{-\frac{1}{\sqrt{6}}\phi}$, $F^i_{[2]} = dA^i_{[1]} + \frac{1}{\sqrt{2}}g\varepsilon^{ijk}A^j_{[1]} \wedge A^k_{[1]}$ and $G_{[2]} = dB_{[1]}$, and $\tilde{*}$ donates the 5-dimensional Hodge dual. For simplicity, we can define $A_{[2]} \equiv A^1_{[2]} + iA^2_{[2]}$. Varying

[2] Our convention for the Hodge dual is different from that in [14, 28].

the Lagrangian, we get the equations of motion

$$
\begin{aligned}
d(X^{-1}\tilde{*}dX) =& \frac{1}{3}X^4 G_{[2]} \wedge \tilde{*}G_{[2]} - \frac{1}{6}X^{-2}\left(F^i_{[2]} \wedge \tilde{*}F^i_{[2]} + \bar{A}_{[2]} \wedge \tilde{*}A_{[2]}\right) \\
& - \frac{4}{3}g^2\left(X^2 - X^{-1}\right)\tilde{*}\mathbf{1}, \\
d(X^4\tilde{*}G_{[2]}) =& -\frac{1}{2}F^i_{[2]} \wedge F^i_{[2]} - \frac{1}{2}\bar{A}_{[2]} \wedge A_{[2]}, \\
d(X^{-2}\tilde{*}F^i_{[2]}) =& \sqrt{2}g\varepsilon^{ijk}X^{-2}F^j_{[2]} \wedge \tilde{*}A^k_{[1]} - F^i_{[2]} \wedge G_{[2]}, \\
X^2\tilde{*}F_{[3]} =& -igA_{[2]}, \\
R_{MN} =& 3X^{-2}\partial_M X \partial_N X - \frac{4}{3}g^2\left(X^2 + 2X^{-1}\right)g_{MN} \\
& + \frac{1}{2}X^4(G_M{}^P G_{NP} - \frac{1}{6}g_{MN}G^2_{[2]}) + \frac{1}{2}X^{-2}(F^i_M{}^P F^i_{NP} - \frac{1}{6}g_{MN}(F^i_{[2]})^2) \\
& + \frac{1}{2}X^{-2}(\bar{A}_{(M}{}^P A_{N)P} - \frac{1}{6}g_{MN}\left|A_{[2]}\right|^2),
\end{aligned}
\tag{3.23}
$$

with $F_{[3]} = \mathcal{D}A_{[2]} \equiv dA_{[2]} - igB_{[1]} \wedge A_{[2]}$.

The dimensional reduction ansatz is

$$
ds_5^2 = e^{-2k|z|}ds_4^2 + dz^2, \quad A_{[2]} = \frac{1}{\sqrt{2}}e^{-k|z|}(\mathcal{F}_{[2]} - i * \mathcal{F}_{[2]}),
\tag{3.24}
$$

in which unmentioned fields are set to zero and $k < 0$, z is the coordinate of the fifth dimension ds_4^2 is the worldvolume metric, $\mathcal{F}_{[2]}$ is the Maxwell field in the 4-dimensional $\mathcal{N} = 2$ supergravity, and $*$ donates the 4-dimensional Hodge dual. Here, the absolute value of z on the exponential is the characteristic of the Randall-Sundrum model which is an orbifold. Different from the general Kaluza-Klein dimensional reduction discussed in sec.2.4, in which the surviving fields are independent of the reduced dimensions, the reduction ansatz of both the metric and 2-form gauge fields depends on the fifth dimension z.

The first three equations in Eq.(3.23) are satisfied trivially because $\bar{A}_{[2]} \wedge A_{[2]} = 0$ and $\tilde{*}A_{[2]} = A_{[2]} \wedge dz$ using $*\mathcal{F}_{[2]} \wedge *\mathcal{F}_{[2]} = -\mathcal{F}_{[2]} \wedge \mathcal{F}_{[2]}$, where the minus sign appears because of the Minkowski signature of the spacetime. Substituting the ansatz Eq.(3.24) into the $A_{[2]}$ equation, we find

$$
d\mathcal{F}_{[2]} = 0, \quad d * \mathcal{F}_{[2]} = 0, \quad g = \begin{cases} +k, & z > 0, \\ -k, & z < 0. \end{cases}
\tag{3.25}
$$

With the non-vanishing component of Ricci tensor given by

$$
\begin{aligned}
R^{(5)}_{\mu\nu} &= e^{2k|z|}R^{(4)}_{\mu\nu} - 4k^2\eta_{\mu\nu} + 2k\delta(z)\eta_{\mu\nu} \\
R^{(5)}_{zz} &= -4k^2 + 8k\delta(z),
\end{aligned}
\tag{3.26}
$$

Einstein's equations provide us

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{1}{2}(\mathcal{F}_{\mu\rho}\mathcal{F}_{\nu}{}^{\rho} - \frac{1}{4}\mathcal{F}^2 g_{\mu\nu}), \quad k^2 = g^2. \tag{3.27}$$

Eq.(3.25) and Eq.(3.27) are the equations of motion for the Einstein-Maxwell theory except for the domain wall $z = 0$. On the domain wall, we need an external δ-function source. That is, with a subtle construction of worldvolume and the transverse space, we get a consistent dimensional reduction from a 5D $N = 4$ supergravity to 4D $N = 2$ supergravity whose bosonic sector is exactly the metric and the Maxwell gauge field. The geometry requires that the Yang-Mills coupling constant g has opposite signs on the two sides of the domain wall at $z = 0$, equal to the inverse of the AdS radius.

3.3.2 Reissner–Nordström black holes on the branes

The Einstein-Maxwell theory, Eq.(3.25) and Eq.(3.27), admits a class of Reissner–Nordström black holes

$$ds^2 = -\frac{\Delta}{r^2}dt^2 + \frac{r^2}{\Delta}dr^2 + r^2 d\Omega^2, \quad \mathcal{A}_0 = -\frac{q}{r} \tag{3.28}$$

where $\Delta = r^2 - 2Mr + q^2/4$ and is irrelevant to the Δ in Eq.(2.31). Here, we are only interested in the extremal black holes with $M = |q|/2$,

$$ds_4^2 = -H^{-2}dt^2 + H^2 dy^i dy^i, \quad \mathcal{F}_{[2]} = 2dt \wedge dH^{-1}, \quad \nabla^2_{(y)}H = 0. \tag{3.29}$$

With the dimensional reduction ansatz Eq.(3.24), we can lift the extremal RN black holes solution straightforwardly to

$$\begin{aligned} ds_5^2 &= e^{-2k|z|}(-H^{-2}dt^2 + H^2 dy^i dy^i) + dz^2, \quad \nabla^2_{(y)}H = 0 \\ A_{[2]} &= \sqrt{2}e^{-k|z|}(dt \wedge dH^{-1} + \frac{i}{2}\varepsilon_{ijk}\partial_i H dy^j \wedge dy^k). \end{aligned} \tag{3.30}$$

Like the black string solution, the sources are spread out in the transverse dimension. It can be viewed as strings coupling to the 2-form gauge field $A_{[2]}$.

The ansatz of Kaluza-Klein reduction from type IIB supergravity to 5D $\mathcal{N} = 4$ gauged supergravity with a S^5 was derived in ref.[28]. To generate the $\mathrm{SU}(2) \times \mathrm{U}(1)$ gauge symmetry, we need a $S^3 \times S^1$ foliation of the S^5 given by

$$d\Omega_5^2 = d\xi^2 + \sin^2\xi d\tau^2 + \cos^2\xi d\Omega_3^2. \tag{3.31}$$

Here, $0 \leq \xi \leq \frac{\pi}{2}$, $0 \leq \tau \leq 2\pi$, and $d\Omega_3^2$ is the metric on the unit S^3. After being lifted to type IIB supergravity, both SU(2) × U(1) gauge fields and the scalar vanish. We have the ansatz

$$\begin{aligned} d\hat{s}_{10}^2 &= e^{-2k|z|}ds_4^2 + dz^2 + g^{-2}\left(d\xi^2 + \sin^2\xi d\tau^2 + \cos^2\xi d\Omega_3^2\right), \\ \hat{H}_{[5]} &= 4g\epsilon_{[5]} + 4g^{-5}\sin\xi\cos^3\xi d\xi \wedge d\tau \wedge \Omega_{[3]}, \\ \hat{A}_{[2]} &= -\frac{1}{2}g^{-1}\sin\xi e^{-k|z|-i\tau}(\mathcal{F}_{[2]} - i * \mathcal{F}_{[2]}). \end{aligned} \tag{3.32}$$

Here, we label 10-dimensional variables with hat, $\epsilon_{[5]} = e^{-4k|z|}\epsilon_{[4]} \wedge dz$, with $\epsilon_{[4]}$ being the volume form of the 4-dimensional metric ds_4^2, and $\Omega_{[3]}$ is the volume form of the unit S^3. $\hat{A}_{[2]}$ is a complex 2-form defined by

$$\hat{A}_{[2]} = \hat{A}_{[2]}^{RR} + i\hat{A}_{[2]}^{NS}, \tag{3.33}$$

where $\hat{A}_{[2]}^{RR}$ and $\hat{A}_{[2]}^{NS}$ are the R-R and NS-NS 2-form potentials of the type IIB supergravity. $\hat{H}_{[5]}$ is the self-dual 5-form field strength in the R-R sector, and the dilaton and axion are set to zero.

Then, we can uplift the RN black hole on the branes solution to 10-dimensional as

$$\begin{aligned} d\hat{s}_{10}^2 &= e^{-2k|z|}(-H^{-2}dt^2 + H^2dy^idy^i) + dz^2 + g^{-2}\left(d\xi^2 + \sin^2\xi d\tau^2 + \cos^2\xi d\Omega_3^2\right), \\ \hat{H}_{[5]} &= 4ge^{-4k|z|}H^2dt \wedge d^3y \wedge dz + 4g^{-5}\sin\xi\cos^3\xi d\xi \wedge d\tau \wedge \Omega_{[3]}, \\ \hat{A}_{[2]} &= -g^{-1}\sin\xi e^{-k|z|-i\tau}(dt \wedge dH^{-1} - \frac{i}{2}\epsilon_{ijk}\partial_i H dy^j \wedge dy^k), \quad \nabla_{(y)}^2 H = 0, \end{aligned} \tag{3.34}$$

with $g = k$ for $z > 0$ and $g = -k$ for $z < 0$. It's a D3-brane solution coupling to the self-dual field strength $H_{[5]}$. Hence, we can interpret it as an open string ending on a D3-brane.

No matter the brane solutions in 5 dimensions, Eq.(3.30), or in 10 dimensions, Eq.(3.34), the gravitational effect is only within the worldvolume and the transverse space dependence is fixed by $e^{-2k|z|}$ as we discuss in the section 3.1. Both of them admit a deformed consistent truncation, which depends on the transverse space, to the worldvolume gravity. The crucial structure in the embedding is the non-trivial Randall-Sundrum type of transverse space geometry. Further, the AdS_5 singular structure changes due to the incorporation of the worldvolume gravity. With a similar investigation of the curvature variables along the null and timelike geodesics, we can also find the AdS horizon becomes singular. Fortunately, for our extremal BPS solution, the RN black holes on branes solutions do not suffer the Gregory-Laflamme instability [27] as the black-string solution.

3.3.3 Worldvolume Supergravity on D3-branes

$\mathrm{AdS}_5 \times S^5$ spacetime is the asymptotic near-brane geometry of a D3-brane in type IIB supergravity which has 16 unbroken supersymmetries. However, the worldvolume Einstein-Maxwell 4D $\mathcal{N}=2$ supergravity constructed in ref.[14] has only 8 supersymmetries. Noting the difference, ref.[15] suggested that it is possible to embed a 4D $\mathcal{N}=4$ supergravity, which has 16 supersymmetries, on the worldvolume of D3-brane.

Type IIB supergravity admits N parallel D3-branes solution, as we mentioned in subsection 2.2.3,

$$\begin{aligned} d\hat{s}^2 &= H^{-1/2}\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2}(dr^2 + r^2 d\Omega_5^2), \\ \hat{F}_{[5]} &= 16\pi N(1+\hat{*})\mathrm{vol}_5, \quad H = 1 + \frac{4\pi N}{r^4}. \end{aligned} \tag{3.35}$$

with vol_5 is the volume form of the S^5. This D3-brane solution is the skeleton used to embed the 4D $\mathcal{N}=4$ supergravity whose Lagrangian of the bosonic part is

$$\mathcal{L}_{4DN4} = R * \mathbf{1} - \frac{1}{2}d\phi \wedge *d\phi - \frac{1}{2}e^{2\phi}d\chi \wedge *d\chi - \frac{1}{2}e^{-\phi}\mathcal{F}^\Lambda_{[2]} \wedge *\mathcal{F}^\Lambda_{[2]} - \frac{1}{2}\chi\mathcal{F}^\Lambda_{[2]} \wedge \mathcal{F}^\Lambda_{[2]}. \tag{3.36}$$

It is composed of the metric $g_{\mu\nu}$, 6 U(1) vectors $\mathcal{A}^\Lambda_{[1]}$ with field strength $\mathcal{F}^\Lambda_{[2]} = d\mathcal{A}^\Lambda_{[1]}$, and one complex scalar $\tau = \chi + ie^{-\phi}$, which parametrizes the coset space $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2)$.

The embedding of the 4D $\mathcal{N}=4$ supergravity on the worldvolume of skeleton D3-branes is given in ref.[15] as

$$\begin{aligned} d\hat{s}^2 &= H^{-1/2}g_{\mu\nu}(x)dx^\mu dx^\nu + H^{1/2}dy^\Lambda dy^\Lambda, \quad \hat{\Phi} = \phi(x), \quad \hat{C}_0 = -\chi(x), \\ \hat{H}_{[3]} &= \frac{1}{\sqrt{2}}\mathcal{F}^\Lambda_{[2]} \wedge dy^\Lambda, \quad F^\Lambda_{[3]} = -\frac{1}{\sqrt{2}}e^{-\phi} * \mathcal{F}^\Lambda_{[2]} \wedge dy^\Lambda, \\ \hat{F}_{[5]} &= H^{-2}\mathrm{vol}_4 \wedge dH - *_6 dH, \quad H = 1 + \frac{4\pi N}{r^4}, \end{aligned} \tag{3.37}$$

in which $\Lambda \in \{1, \cdots, 6\}$, $*$ and the volume form vol_4 are both associated with $g_{\mu\nu}$, $*_6$ is the Hodge dual on the $\mathbb{R}^6$ transverse space. The type IIB supergravity comprises a dilaton $\hat{\Phi}$, a NS-NS 3-form field strength $\hat{H}_{[3]}$, and R-R sector field strengths $\hat{F}_{[1]}$, $\hat{F}_{[3]}$ and $\hat{F}_{[5]}$.

3.3.4 Stationary Black Holes in 4D $\mathcal{N}=4$ worldvolume supergravity

Due to the various field contents in 4D $\mathcal{N}=4$ supergravity, there could be more possible solutions on the worldvolume. To find stationary black hole solutions, we can utilize the trick developed in ref.[29, 30]. Note that, a 1-form gauge field is dual to a scalar field(axion) in 3 dimensions. After reducing the 4-dimensional supergravity along a timelike U(1) isometry and obtaining a 3-dimensional Euclidean theory, we can make a dualization turning all the gauge fields into scalars. The resulting theory comprises the metric and scalars, which can be described using a non-linear sigma model with target space $\mathcal{M}_s$. The null geodesics on $\mathcal{M}_s$ correspond to certain classes of black holes.

With the reduction ansatz[3]

$$\begin{aligned} &ds_4^2 = -e^{\rho(x)}(dt + A_{[1]})^2 + e^{-\rho(x)}h_{mn}(x)dx^m dx^n, \ \phi = \phi(x), \ \chi = \chi(x) \\ &\mathcal{A}^\Lambda_{[1]} = a^\Lambda(x)(dt + A_{[1]}) + \mathcal{B}^\Lambda_{[1]}, \end{aligned} \tag{3.38}$$

we get the reduced 3-dimensional Euclidean Lagrangian from the Lagrangian Eq.(3.36)

$$\begin{aligned} \mathcal{L}_3 =& R * \mathbf{1} - \frac{1}{2}d\phi \wedge *d\phi - \frac{1}{2}e^{2\phi}d\chi \wedge *d\chi - \frac{1}{2}d\rho \wedge *_3 d\rho + \frac{1}{2}e^{-\rho-\phi}da^\Lambda \wedge *_3 da^\Lambda \\ &+ \frac{1}{2}e^{2\rho}F_{[2]} \wedge *_3 F_{[2]} - \frac{1}{2}e^{\rho-\phi}\mathcal{H}^\Lambda_{[2]} \wedge *_3\mathcal{H}^\Lambda_{[2]} + \chi da^\Lambda \wedge \mathcal{H}^\Lambda_{[2]}, \end{aligned} \tag{3.39}$$

where R is the Ricci sacalar associate to h_{mn}, $\mathcal{H}^\Lambda_{[2]} = d\mathcal{B}^\Lambda_{[1]} + a^\Lambda F_{[2]}$, and $F_{[2]} = dA_{[1]}$. The Bianchi identities and equations of motion of $\mathcal{H}_{[2]}$ and $F_{[2]}$ are

$$\begin{aligned} &d\mathcal{H}^\Lambda_{[2]} = da^\Lambda \wedge F_{[2]}, \quad d(e^{-\phi+\rho} *_3 \mathcal{H}^\Lambda_{[2]} - \chi da^\Lambda) = 0, \\ &dF_{[2]} = 0, \quad d(e^{2\rho} *_3 F_{[2]} - e^{-\phi+\rho}a^\Lambda *_3 \mathcal{H}^\Lambda_{[2]} + \chi a^\Lambda da^\Lambda) = 0. \end{aligned} \tag{3.40}$$

To make a dualization, it is equivalent to introducing a set of new variables that interchange the roles of Bianchi identities and equations of motion. We can make the choice

$$\begin{aligned} &\mathcal{H}^\Lambda_{[2]} = e^{\phi-\rho} *_3 (dh^\Lambda + \chi da^\Lambda) \equiv e^{\phi-\rho} *_3 \mathcal{P}^\Lambda_{[1]}, \\ &F_{[2]} = -e^{-2\rho} *_3 (df + \frac{1}{2}h^\Lambda da^\Lambda - \frac{1}{2}a^\Lambda dh^\Lambda) \equiv -e^{-2\rho} *_3 \mathcal{Q}_{[1]}, \end{aligned} \tag{3.41}$$

where the scalar fields h^Λ and f are the axions dual to $\mathcal{H}^\Lambda_{[2]}$ and $F_{[2]}$, with $\mathcal{P}^\Lambda_{[1]}$ and $\mathcal{Q}_{[1]}$ being their field strength. Redefine ρ and ϕ as

$$\rho = -\frac{1}{\sqrt{2}}(\phi_1 + \phi_2), \quad \phi = -\frac{1}{\sqrt{2}}(\phi_1 - \phi_2). \tag{3.42}$$

With the integration by part using Eq.(3.41), the Lagrangian in terms of new variables, 16 scalars and axions in total, are

$$\begin{aligned} \mathcal{L}_3 =& R * \mathbf{1} - \frac{1}{2}d\phi_1 \wedge *d\phi_1 - \frac{1}{2}d\phi_2 \wedge *_3 d\phi_2 - \frac{1}{2}e^{-\sqrt{2}\phi_1+\sqrt{2}\phi_2}d\chi \wedge *d\chi \\ &+ \frac{1}{2}e^{\sqrt{2}\phi_1}da^\Lambda \wedge *_3 da^\Lambda + \frac{1}{2}e^{\sqrt{2}\phi_2}\mathcal{P}^\Lambda_{[1]} \wedge *_3\mathcal{P}^\Lambda_{[1]} - \frac{1}{2}e^{\sqrt{2}\phi_1+\sqrt{2}\phi_2}\mathcal{Q}_{[1]} \wedge *_3\mathcal{Q}_{[1]}. \end{aligned} \tag{3.43}$$

[3]The reduction ansatz for gauge field is slightly different from that in subsec.2.4.1

In terms of the non-linear sigma model, the target space is the closet space SO(8, 2)/(SO(6, 2) × SO(2)) and the Lagrangian becomes [15]

$$\mathcal{L}_3 = R * \mathbf{1} + \frac{1}{4}\mathrm{tr}(dM^{-1} \wedge *_3 dM), \tag{3.44}$$

with the matrix $M = \mathcal{V}^T W_0 \mathcal{V}$ where

$$\mathcal{V} = \exp\left\{\frac{1}{2}\phi_1 H_1 + \frac{1}{2}\phi_2 H_2\right\} \exp\left(-\chi E_1{}^2\right) \exp(-f V^{12}) \exp(a^\Lambda U_\Lambda{}^1 + h^\Lambda U_\Lambda{}^2) \tag{3.45}$$

is the coset representative, (H_1, H_2) and $(E_1{}^2, V^{12}, U_\Lambda{}^1, U_\Lambda{}^2)$ are the 2 non-compact Cartan generators and the 14 positive-root generators of $\mathbf{so}(2, 8)$, and the fiducial matrix $W_0 = \mathrm{diag}(-1, -1, 1, 1, 1, 1, 1, 1, -1, -1)$, which determines the denominator group of the coset to be SO(6, 2) × SO(2). Ref.[24] provided an elementary introduction to constructing the coset manifold $\mathcal{V}$ based on the group theory. Varying the Larganian, we get the equations of motion

$$R_{mn} = -\frac{1}{4}\mathrm{tr}(\partial_m M \partial_n M^{-1}), \quad \nabla_m(M^{-1}\nabla^m M) = 0, \tag{3.46}$$

with ∇ associated to h_{mn}.

Let's assume that all the scalars are taken to depend on the spatial coordinates through a single harmonic function $\sigma(x)$, *i.e.* $M(\sigma)$. Then, the scalar equation of motion becomes

$$\frac{d}{d\sigma}\left(M^{-1}\frac{dM}{d\sigma}\right) = 0, \tag{3.47}$$

with solutions

$$M = A\exp(\sigma B), \tag{3.48}$$

where the constant matrix B gives the velocity for the geodesic parametrized by σ, while the constant matrix A specifies the initial point of the geodesic. Then Einstein's equation reduced to

$$R_{mn} = -\frac{1}{4}\mathrm{tr}(B^2)\partial_m\sigma\partial_n\sigma. \tag{3.49}$$

As the black hole solutions correspond to the null geodesics on the target space $\mathcal{M}_s$, we need to consider the metric on geodesics pulled back from the target space, which takes the form

$$ds^2 = -\frac{1}{2}\mathrm{tr}(dM^{-1}dM) = \frac{1}{2}\mathrm{tr}(B^2)\, d\sigma^2. \tag{3.50}$$

That is, the stationary black hole solutions on the worldvolume have Ricci-flat spatial geometries. Thus, the harmonic function $\sigma : x \mapsto \sigma(x)$ maps 3-dimensional Ricci-flat space to null geodesics on the target space. The extension to involve multiple harmonic maps $\sigma_a(x)$ is straightforward.

After being lifted back to 10 dimensions with Eq.(3.37), the solutions describe charged black holes on the worldvolume of D3-branes or string ending on the D3-branes. Note the structure of Eq.(3.37). The supergravity is confined to the world-volume, and the transverse space dependence is fixed by the factor $H^{-1/2}(y)$, which is exactly the characteristic of type I effective gravity. And the sources, *i.e.* black holes, spread out in the transverse space similar to the black-spoke solutions.

Chapter 4

Type III - Effective Theory near branes

In chapter 3, we explored the type I construction of lower-dimensional effective gravity through embedding Einstein's gravity or supergravity on the world volume. This results in gravitational effects that are only along the world volume, with a fixed transverse space dependence factor. Instead of being localized at a point, the source spreads out in the transverse space. This ingenious construction allows for a consistent truncation to a world volume gravity and satisfies all of the criteria for an effective 4-dimensional gravity, except for a reasonable Newton constant.

In this chapter, we will discuss the type III construction classified in [11], which is rather different. In type III construction, we consider a δ-function type of source sitting at one point in the spacetime. By carefully choosing the geometry and the boundary condition of the eigenvalue problem on the transverse space, we can have a transverse spectrum containing a zero eigenvalue with a non-constant eigenfunction, *i.e.* non-zero c_2 and non-trivial $\zeta_{(0)}$ in Eq.(3.7). The zero mode admits the massless graviton while the non-constant eigenfunction is vital in localizing the gravity near the brane on which the source sits. The dominant contribution to gravity in the Newtonian limit comes from the zero modes with a $1/r$ behavior, which are corrected by higher modes that decrease rapidly at large distances and are suppressed by a scale factor. In different models, the transverse space spectra are different, and they will have different mechanisms to keep the lower-dimensional behavior stable.

In contrast to type III, the type II construction employs a Neumann near-source boundary condition. This results in the absence of a zero eigenvalue and a massless graviton in type II, which is different from type III where the boundary condition used is Robin.

In this chapter, we focus on the SS-CGP model, which is based on the ground-state solution of 6-dimensional supergravity found by Salam and Sezgin [31] that was lifted into type IIA supergravity by Cvetic, Gibbons and Pope [32]. We end this chapter with the Randall-Sundrum model and compare it to the SS-CGP model as it has a distinct spectrum. This discussion also complements the Randall-Sundrum structure that we used in the type I construction.

4.1 Effective gravity based on SS-CGP model

4.1.1 The SS-CGP Background

The bosonic sector of the 6-dimensional Salam-Sezgin theory is given by

$$\bar{\mathcal{L}}_6 = \bar{R}\bar{*}\mathbf{1} - \frac{1}{4}d\bar{\phi}\wedge\bar{*}d\bar{\phi} - \frac{1}{2}e^{\frac{1}{2}\bar{\phi}}\bar{F}_{[2]}\wedge\bar{*}\bar{F}_{[2]} - \frac{1}{2}e^{\bar{\phi}}\bar{H}_{[3]}\wedge\bar{*}\bar{H}_{[3]} - 8\bar{g}^2e^{-\frac{1}{2}\bar{\phi}}\bar{*}\mathbf{1}, \tag{4.1}$$

where we put a bar on all quantities in the 6 dimensions in this chapter and $d\bar{H}_{[3]} = \frac{1}{2}\bar{F}_{[2]}\wedge\bar{F}_{[2]}$, $\bar{F}_2 = d\bar{A}_{[1]}$. There exists a ISO$(1,3)\times S^2$ vacuum solution

$$\begin{aligned} d\bar{s}_6^2 &= \eta_{\mu\nu}dx^\mu dx^\nu + \frac{1}{8\bar{g}^2}\left(d\theta^2+\sin^2\theta d\varphi^2\right), \\ \bar{A}_{[1]} &= -\frac{1}{2\bar{g}}\cos\theta d\varphi, \quad \bar{H}_{[3]}=0, \quad \bar{\phi}=0. \end{aligned} \tag{4.2}$$

Cvetic, Gibbons and Pope [32] showed that the Salam-Sezgin theory can be uplifted to the type I supergravity via a $\mathbb{R}\times\mathcal{H}^{(2,2)}$. The 3-dimensional hyperbolic space $\mathcal{H}^{(2,2)}$ is defined as the surface $X_1^2+X_2^2-X_3^2-X_4^2=1$ embedded in the Euclidean space $\mathbb{E}^4$ with U(1) × U(1) isometry. In terms of a new set of parameters

$$X_1=\cosh\rho\cos\alpha,\ X_2=\cos\rho\sin\alpha,\ X_3=\sinh\rho\cos\beta,\ X_4=\cos\rho\sin\beta, \tag{4.3}$$

we can write the metric on $\mathcal{H}^{(2,2)}$ as

$$ds_3^2 = \cosh 2\rho\ d\rho^2 + \cosh^2\rho\ d\alpha^2 + \sinh^2\rho\ d\beta^2, \tag{4.4}$$

where $\rho \geq 0$ and $0 \leq \alpha < 2\pi$, $0 \leq \beta < 2\pi$. For our purposes, let's introduce coordinates $\psi \in [0, 4\pi)$ and $\chi \in [0, 2\pi)$ defined as

$$\psi = \alpha + \beta, \quad \chi = \alpha - \beta. \tag{4.5}$$

The embedding of the Salam-Sezgin theory in type I supergravity in terms of these parameters was given in ref.[10] as

$$\begin{aligned}
ds_{10}^2 &= (\cosh 2\rho)^{1/4} \Big[e^{-\frac{1}{4}\bar{\phi}} d\bar{s}_6^2 + e^{\frac{1}{4}\bar{\phi}} dy^2 + \frac{1}{2}\bar{g}^{-2} e^{\frac{1}{4}\bar{\phi}} \big(d\rho^2 \\
&+ \frac{1}{4}[d\psi + \mathrm{sech}2\rho(d\chi - 2\bar{g}\bar{A})]^2 + \frac{1}{4}\,(\tanh 2\rho)^2 (d\chi - 2\bar{g}\bar{A})^2 \big) \Big], \\
F_{[3]} &= \bar{H}_{[3]} - \frac{\sinh 2\rho}{4\bar{g}^2 (\cosh 2\rho)^2} d\rho \wedge d\psi \wedge (d\chi - 2\bar{g}\bar{A}_{[1]}) \\
&+ \frac{1}{4\bar{g}^2 \cosh 2\rho} \bar{F}_{[2]} \wedge [d\psi + \cosh 2\rho (d\chi - 2\bar{g}\bar{A}_{[1]})], \\
e^{\phi} &= (\cosh 2\rho)^{-1/2} e^{-\frac{1}{2}\bar{\phi}}.
\end{aligned} \tag{4.6}$$

In the 10-dimensional metric, the $y \in [0, l_y)$ origins from the $\mathbb{R}$ while the rest are the embedding of $\mathcal{H}^{(2,)}$. We can verify this reduction ansatz by substituting it into the bosonic Lagrangian of type I supergravity

$$\mathcal{L}_{10} = R * \mathbf{1} - \frac{1}{2} d\phi \wedge * d\phi - \frac{1}{2} e^{-\phi} F_{[3]} \wedge * F_{[3]}, \tag{4.7}$$

and obtaining the bosonic equations of motion for the Salam-Sezgin theory, which follow from Eq.(4.1).

Then, the uplifted vacuum solution Eq.(4.2) in type I supergravity becomes

$$\begin{aligned}
ds_{10}^2 = \; & \mathcal{H}^{-1/4} \left(\eta_{\mu\nu} dx^\mu dx^\nu + dy^2 + \frac{1}{4g^2} (d\psi + \mathrm{sech}2\rho (d\chi + \cos\theta d\varphi))^2 \right) + \frac{1}{g^2} \mathcal{H}^{3/4} ds_{EH}^2, \\
B_{[2]} = \; & \frac{1}{4g^2} (d\chi + \mathrm{sech}2\rho \; d\psi) \wedge (d\chi + \cos\theta d\varphi), \quad e^{2\phi} = \mathcal{H}, \quad \mathcal{H} = \mathrm{sech}2\rho,
\end{aligned} \tag{4.8}$$

where $F_{[3]} = dB_{[2]}$, $g = \sqrt{2}\bar{g}$ and ds_{EH}^2 is the 4-dimensional Eguchi-Hanson metric

$$ds_{EH}^2 = \cosh 2\rho \left(d\rho^2 + \frac{1}{4} (\tanh 2\rho)^2 (d\chi + \cos\theta d\varphi)^2 + \frac{1}{4} (d\theta^2 + \sin^2\theta d\varphi^2) \right). \tag{4.9}$$

The coordinates take values in $x^\mu \in \mathbb{R}^{1,3}$, $y \in [0, l_y), \chi, \varphi \in [0, 2\pi), \psi \in [0, 4\pi), \theta \in [0, \pi]$ and $\rho \in [0, \infty)$. As opposed to 4π, χ has a period of 2π suggests that the boundary of the Eguchi-Hanson space at $\rho \to \infty$ is given by $\mathbb{RP}^3 \simeq S^3/\mathbb{Z}_2$, where the S^3 is realized as a Hopf fibration over S^2, with χ being the fiber coordinate. While in the $\rho \to 0$ limit, the Eguchi-Hanson space becomes $\mathbb{R}^2 \times S^2$ with (ρ, χ) acting as the polar coordinate to comprise the $\mathbb{R}^2$ [33].

The SS-CGP solution, Eq.(4.8), takes the form of an NS5-brane wrapping on the $(y, \psi) \in T^2$ with the transverse space being the Eguchi-Hanson space. Indeed, similar to the brane solutions we discuss in section 2.2, the factor $\mathcal{H} = \mathrm{sech}2\rho$ is the harmonic function on the transverse space

$$\Delta_{EH}\mathcal{H} = -\frac{g^2}{2}(\mathcal{F}_{[2]})^2, \tag{4.10}$$

with Δ_{EH} being the Laplacian on the Eguchi-Hanson space. The appearance of the anti-self-dual field strength $\mathcal{F}_{[2]} = d\mathcal{A}_{[1]}$, defined as

$$\mathcal{F}_{[2]} = d\mathcal{A}_{[1]}, \quad \mathcal{A}_{[1]} = \mathrm{sech}2\rho(d\chi + \cos\theta d\varphi), \tag{4.11}$$

is the result of the transgression [10, 34].

The supersymmetry transformation rules in type I supergravity are given by

$$\delta\psi_M = \nabla_M\epsilon - \frac{1}{8}F_{MNP}\Gamma^{NP}\Gamma_{11}\epsilon, \quad \delta\lambda = \Gamma^M\partial_M\phi\epsilon - \frac{1}{12}F_{MNP}\Gamma^{MNP}\Gamma_{11}\epsilon. \tag{4.12}$$

The spinors are written in the form of $\epsilon = e^{-\frac{1}{2}\chi\Gamma_{89}}\eta$, where η is any constant spinor satisfying the two projection conditions

$$\Gamma_{11}\eta = -\eta, \quad \Gamma_{67}\eta = \Gamma_{89}\eta. \tag{4.13}$$

Here, the $6, 7, 8$ and 9 vielbein indices refer to the 4 transverse directions, with

$$\begin{aligned} \hat{e}^6 &= \frac{1}{2}\sinh 2\rho(\cosh 2\rho)^{-1/2}(d\chi + \cos\theta d\varphi), \quad \hat{e}^7 = (\cosh 2\rho)^{1/2}d\rho, \\ \hat{e}^8 &= \frac{1}{2}(\cosh 2\rho)^{1/2}d\theta, \quad \hat{e}^9 = \frac{1}{2}(\cosh 2\rho)^{1/2}\sin\theta d\varphi. \end{aligned} \tag{4.14}$$

Thus, the SS-CGP model exists $\frac{1}{2} \times \frac{1}{2} \times 32 = 8$ Killing spinors and preserves 8 supersymmetry.

It is available to include an additional NS5-brane [10] into the SS-CGP solution without breaking any more supersymmetry. After embedding the SS-CGP solution into the type IIA supergravity, we only need to add to the special solution $\mathcal{H} = \mathrm{sech}2\rho$ of Eq.(4.10) homogeneous terms as

$$\mathcal{H} = \mathrm{sech}2\rho + \tilde{\mathcal{H}}, \quad \tilde{\mathcal{H}} = -k\log\tanh\rho. \tag{4.15}$$

Here, k is positive to obtain a well-behaved positive-tension brane solution. Accordingly, the NS-NS 2-from is modified to be

$$B_{[2]} = \frac{1}{4g^2}\left((1+k)d\chi + \mathrm{sech}2\rho d\psi\right) \wedge (d\chi + \cos\theta d\varphi). \tag{4.16}$$

4.1.2 Geodesics, Perturbations and Newtonian Potential

A well-behaved effective gravity needs to possess a $1/r$ gravitational potential in the Newtonian limit. We can determine the gravitational potential of the effective gravity by measuring the response of a test particle to a known source mass, *i.e.* the distortion of timelike geodesics due to the gravitational perturbations.

Let's donate the coordinates of the geodesics as $Z^M = (X^\mu, Y, P, \Theta, \Phi, \Sigma, \Psi)$ instead of $(x^\mu, y, \rho, \theta, \varphi, \chi, \psi)$ to avoid confusion with the global coordinates. The timelike geodesics equation with proper time τ serving as the parameter is

$$\frac{d^2 Z^M}{d\tau^2} + \Gamma^M_{KL}(Z)\frac{dZ^K}{d\tau}\frac{dZ^L}{d\tau} = 0. \tag{4.17}$$

Let's find the unperturbed geodesics in the SS-CGP background Eq.(4.8). Firstly, its isometry group is $\mathrm{ISO}(1,3) \times \mathrm{U}(1)^3 \times \mathrm{SO}(3)$ resulting from the Minkowski coordinates x^μ, 3-torus coordinates (y, χ, ψ) and the S^2 coordinates (θ, φ). With these isometries, we can choose geodesics with $Y = \Phi = \Sigma = \Psi = 0$ and $\Theta = \pi$ without losing generality. Then, the equation of motion for the transverse radius $P(\tau)$ reduces to

$$\frac{d^2 P}{d\tau^2} + \frac{1}{4}\tanh(2P)\left(\frac{dP}{d\tau}\right)^2 - \frac{g^2}{4}\tanh(2P)\eta_{\mu\nu}\frac{dX^\mu}{d\tau}\frac{dX^\nu}{d\tau} = 0, \tag{4.18}$$

while the equation for X^μ are the usual geodesic equations on $\mathbb{R}^{1,3}$. Thus, we can find a static timelike geodesic on the 4-dimensional Minkowski subspace as

$$X^0 = \tau, \quad X^i = 0, \quad P = 0, \quad \Theta = \pi, \quad \Phi = 0, \quad \Sigma = 0, \quad \Psi = 0. \tag{4.19}$$

Let's find the distortion of the static geodesic Eq.(4.19) under perturbations H_{MN} given by

$$\hat{g}_{MN} = (\cosh 2\rho)^{1/4}(\bar{g}_{MN} + H_{MN}), \tag{4.20}$$

where $\bar{g}_{MN}$ is the string-frame metric on the SS-CGP background related to the Einstein-frame with the Weyl transformation $\bar{g}_{MN} = e^{\phi/2} g_{MN}$. To check the 4-dimensional gravitational potential in the Newtonian limit, we only need to consider the time-independent perturbations with components along the x^μ and ρ and $H_{0i} = 0$. Further, we assume the perturbations only depend on coordinates x^μ and ρ for simplicity. In this case, the redisual isometry is $\mathrm{U}(1)^3 \times \mathrm{SO}(3)$, and we can choose $Y = \Phi = \Sigma = \Psi = 0$ and $\Theta = \pi$. Before calculating the perturbed geodesic equations, we write the deviation of the original static geodesic as

$$X^0 = \tau + \delta X^0, \quad X^i = \delta X^i, \quad P = \delta P. \tag{4.21}$$

Then, the perturbed geodesic equations up to the first order are

$$\frac{d^2\delta X^0}{d\tau^2}=0,\quad \frac{d^2\delta X^i}{d\tau^2}=\frac{1}{2}\frac{\partial}{\partial\delta X^i}H_{00},\quad \frac{d^2\delta P}{d\tau^2}+\frac{g^2}{2}\delta P-\frac{g^2}{2}\frac{\partial}{\partial\delta P}H_{00}=0. \tag{4.22}$$

In the calculation, we need to use $H_{0i}=0$ and time independence of the perturbation. The full geodesics equation under the perturbations Eq.(4.20) can be found in ref.[11]. The δX^0 equation allows us to set $\delta X^0=0$ and $X^0=\tau$ in the linear regime. Thus, we can take τ as the Newtonian time t. The δX^i equation is exactly the Newtonian equation of motion with the gravitational potential

$$V_N(\delta X^i,\delta P)=-\frac{1}{2}mH_{00}(\delta X^i,\delta P), \tag{4.23}$$

where m is the small mass of the test particle following the geodesic. In summary, the perturbed geodesics are

$$X^0=t,\quad m\frac{d^2X^i}{dt^2}=-\frac{\partial}{\partial X^i}V_N,\quad m\left(\frac{d^2P}{dt^2}+\frac{g^2}{2}P\right)=-g^2\frac{\partial}{\partial P}V_N. \tag{4.24}$$

We achieve one of the main conclusions that the gravitational potential is determined by the perturbations about the SS-CGP background via $V_N\propto H_{00}$ at the leading order. The result is similar to the Newtonian limit of general relativity in the Newton gauge, in which $g_{00}=1+2\Phi$.

We need to find the H_{00} perturbation by solving the equations of motion in the SS-CGP background. Because we are interested in the perturbations depend on x^μ and ρ, we can solve the perturbation problem easier in the 5-dimensional theory reduced from the type I supergravity on $T^3\times S^2$, which corresponds to the residual isometry $\mathrm{U}(1)^3\times\mathrm{SO}(3)$. The reduced 5-dimensional Lagrangian is [11]

$$\begin{aligned}\mathcal{L}_5&=R*\mathbf{1}-\frac{1}{2}d\Phi_i\wedge *d\Phi_i-\frac{1}{2}e^{\sqrt{2}\Phi_1}d\sigma\wedge *d\sigma-V*\mathbf{1},\quad i=1,2,3,\\ V&=2g^2e^{\sqrt{\frac{2}{5}}\Phi_2-\frac{8}{\sqrt{15}}\Phi_3}\left(e^{-\sqrt{2}\Phi_1}+\sigma^2+\frac{1}{4}e^{\sqrt{2}\Phi_1}(\sigma^2-2)^2-4e^{-\sqrt{\frac{2}{5}}\Phi_2+\sqrt{\frac{3}{5}}\Phi_3}\right),\end{aligned} \tag{4.25}$$

and the reduced Salam-Sezgin vacuum solution is

$$\begin{aligned}&ds_5^2=(\sinh 2\rho)^{\frac{2}{3}}\left(\eta_{\mu\nu}dx^\mu dx^\nu+\frac{1}{g^2}d\rho^2\right),\quad e^{-\sqrt{2}\Phi_1}=(\tanh 2\rho)^2\\ &e^{\sqrt{10}\Phi_2}=e^{\sqrt{15}\Phi_3}=(\sinh 2\rho)^2,\quad \sigma=\sqrt{2}\mathrm{sech}2\rho.\end{aligned} \tag{4.26}$$

Consider perturbations around the reduced SS-CGP background as

$$g_{MN}=(\sinh 2\rho)^{\frac{2}{3}}(\bar{g}_{MN}+H_{MN}),\quad \Phi_i=\bar{\Phi}_i+\phi_i,\quad \sigma=\bar{\sigma}+\Sigma \tag{4.27}$$

where we put bars on the background variables, and $d\bar{s}_5^2 = \bar{g}_{MN}dX^M dx^N = \eta_{\mu\nu}dx^\mu dx^\nu + \frac{1}{g^2}d\rho^2$ so that the perturbation is exactly the one in Eq.(4.20).

Varying the Lagrangian Eq.(4.25) and substituting the perturbation around the SS-CGP background, we can get the equations of motion up to the first order. Firstly, for the scalar part,

$$\begin{aligned}
\Sigma:\ & (\Delta_5 - 8g\text{csch}(4gz)\partial_z - 8g^2(\text{sech}(2gz))^2)\Sigma = -2g\,\text{sech}(2\text{gz})\tanh(2gz) \\
& \qquad\qquad\qquad\qquad \times \big(\sqrt{2}\text{G}_z - 2(\partial_z\phi_1 - 2g\tanh(2gz)\phi_1)\big), \\
\phi_1:\ & (\Delta_5 - 8g^2)\phi_1 = -4g\text{csch}(4gz)\big(\sqrt{2}\mathcal{G}_z + 2\cosh(2gz)(\partial_z\Sigma + 2g\tanh(2gz)\Sigma)\big), \\
\phi_2:\ & \Delta_5\phi_2 - \frac{8g^2}{5}\phi_2 + \frac{32}{5}\sqrt{\frac{2}{3}}g^2\phi_3 = 2\sqrt{\frac{2}{5}}g\big(\coth(2gz)\mathcal{G}_z + 2gH_{zz}\big), \\
\phi_3:\ & \Delta_5\phi_3 - \frac{56g^2}{15}\phi_3 + \frac{32}{5}\sqrt{\frac{2}{3}}g^2\phi_2 = \frac{4g}{\sqrt{15}}\big(\coth(2gz)\mathcal{G}_z + 2gH_{zz}\big)
\end{aligned} \tag{4.28}$$

where $z(\rho) = \rho/g$, $\Delta_5 = \eta^{\mu\nu}\partial_\mu\partial_\nu + g^2(\partial_\rho^2 + 2\coth(2\rho)\partial_\rho)$ and $\mathcal{G}_z = \partial^M H_{Mz} - \frac{1}{2}\partial_z H_N^{\ N}$. We can solve them by first requiring

$$\Sigma = \sinh(2gz)\tanh(2gz)\phi, \quad \phi_3 = \sqrt{\frac{2}{3}}\phi_2, \tag{4.29}$$

and set $\mathcal{G}_z = 0$ as part of the de Donder gauge. The equations become

$$\begin{aligned}
\phi_1:\ & (\Box_5 + 2g\text{csch}(4gz)(3\cosh(4gz) - 1)\partial_z + 8g^2)\phi_1 = 0, \\
\phi_2:\ & \Delta_5\phi_2 - \frac{8g^2}{3}\phi_2 - 4\sqrt{\frac{2}{5}}g^2 H_{zz} = 0.
\end{aligned} \tag{4.30}$$

where $\Box_5 = \eta^{MN}\partial_M\partial_N$. Since the equation of ϕ_1 decouples from the perturbation H_{MN}, we can simply set $\phi_1 = 0$. Then, the Einstein Equations with the de Donder gauge $\mathcal{G}_N = \partial^M H_{MN} - \frac{1}{2}\partial_N H_M^{\ M} = 0$ are

$$zz:\ \Delta_5 H_{zz} - \frac{8g^2}{3}H_{zz} - \frac{8g^2}{3}H_{zz} + \frac{8\sqrt{10}}{9}g^2\phi_2 = 4g\coth(2gz)\big(\partial_z H_{zz} - \frac{\sqrt{10}}{3}\partial_z\phi_2\big) \tag{4.31}$$

which can solve them by introducing $H_{zz} = \frac{1}{\sqrt{2}}\phi + \varphi$ and $\phi_2 = \frac{3}{2\sqrt{5}}\phi$. Integrating the scalars and Einstein equation, we have

$$\begin{aligned}
\phi:\ & \Delta_5\phi = \frac{8\sqrt{2}}{3}g^2\varphi, \\
\varphi:\ & (\Box_5 - 2g\coth(2gz)\partial_z)\varphi = 0 \\
\mu z:\ & \Box_5 H_{\mu z} = 2g\coth(2gz)\big(\partial_\mu\varphi - \frac{1}{\sqrt{2}}\partial_\mu\phi\big) \\
\mu\nu:\ & \Delta_5 H_{\mu\nu} = 4g\coth(2gz)\partial_{(\mu}H_{\nu)z} + \frac{8g^2}{3}\varphi\eta_{\mu\nu}
\end{aligned} \tag{4.32}$$

As Δ_5 is a linear operator, we can further split $H_{\mu\nu} = \mathcal{H}_{\mu\nu} + K_{\mu\nu} + J\eta_{\mu\nu}$ where

$$\Delta_5 \mathcal{H}_{\mu\nu} = 0, \quad \Delta_5 K_{\mu\nu} = 4g\coth(2gz)\partial_{(\mu}H_{\nu)z}, \quad \Delta_5 J = \frac{8g^2}{3}\varphi. \tag{4.33}$$

The ϕ_2 equation in Eq.(4.32) restrict $\phi_2 = J$.

Recall that we are interested in finding the Newtonian potential H_{00} in the time-independent perturbations. Immediately, we have $H_{0z} = 0, K_{00} = 0$, and $H_{00} = \mathcal{H}_{00} - J$. That is, the gravitational potential is only determined by the equations.

$$\Delta_5 \mathcal{H}_{00} = 0, \quad \Delta_5 J = \frac{8g^2}{3}\varphi, \quad (\Box_5 - 2g\coth(2gz)\partial_z)\varphi = 0. \tag{4.34}$$

For simplicity, we only consider perturbations with spherical symmetric in $\mathbb{R}^{1,3}$ and $\varphi = 0$. Then, the only condition for H_{00} should be the harmonic function as

$$\Delta_5 H_{00} = \partial_r^2 + \frac{2}{r}\partial_r + g^2(\partial_\rho^2 + 2\coth 2\rho\,\partial_\rho)H_{00} = 0. \tag{4.35}$$

We achieve the conclusion of the subsection that the gravitational potential is determined by the perturbations about the SS-CGP background via $V_N \propto H_{00}$ at the leading order, while H_{00} is the Green's function associated with Δ_5, the CPS operator [10]. We write the transverse part of the CPS operator as

$$\Delta = \partial_\rho^2 + 2\coth 2\rho\,\partial_\rho, \tag{4.36}$$

and name it the transverse operator.

4.1.3 Green's Function for the CPS Operator Δ_5

We suppose the Green's function G may be written formally

$$G(r,\rho) = \int_{\mathcal{I}} f^\omega(r)\zeta_\omega(\rho)d\omega, \tag{4.37}$$

where $\mathcal{I}$ is the spectrum of the transverse operator Δ and the functions are eigenfunctions of the worldvolume or transverse differential operators away from $r = \rho = 0$,

$$\begin{aligned} \left(\partial_r^2 + \frac{2}{r}\partial_r\right) f^\omega(r) &= g^2\omega^2 f^\omega(r), \\ (\partial_\rho^2 + 2\coth(2\rho)\partial_\rho)\zeta_\omega(\rho) &= -\omega^2\zeta_\omega(\rho). \end{aligned} \tag{4.38}$$

We find that the mass $g^2\omega^2$ of a worldvolume function is determined by the transverse space eigenfunction it couples to. To have a massless graviton, we want the transverse spectrum containing a zero eigenvalue. Let's investigate how many bound states the transverse operator admits.

In order to write the Green's function in the Eq.(4.37), we require that any bound states ζ_i with eigenvalue ω_i be Kronecker delta orthonormalized and any scattering states ζ_ω be Dirac delta distribution orthonormalized

$$\int_0^\infty \sinh(2\rho)\zeta_i(\rho)\zeta_j(\rho)d\rho = \delta_{ij}, \quad \int_0^\infty \sinh(2\rho)\zeta_\omega(\rho)\zeta_\tau(\rho)d\rho = \delta(\omega-\tau). \tag{4.39}$$

The measure $\mu(\rho) = \sinh(2\rho)$ can be seen from a consideration of the $H^2_{\mu\nu}$ terms in the perturbative action or simply as the ρ dependence part of the background measure. The orthonormalized basis needs to comprise a self-adjoint domain $\mathcal{D}$ of the transverse operator. By definition, any two functions ζ_ω and ζ_τ in the self-adjoint domain $\mathcal{D}$ must satisfy

$$\langle\zeta_\omega, \Delta(\zeta_\tau)\rangle = \langle\Delta(\zeta_\omega), \zeta_\tau\rangle\,. \tag{4.40}$$

With $\Delta = \frac{1}{\mu(\rho)}\partial_\rho\big(\mu(\rho)\partial_\rho\big)$, we can integrate by part and get

$$\begin{aligned}
\langle\zeta_\omega, \Delta(\zeta_\tau)\rangle &\equiv \int_0^\infty \mu(\rho)\zeta_\omega\frac{1}{\mu(\rho)}\partial_\rho\big[\mu(\rho)\partial_\rho\zeta_\tau\big] \\
&= \int_0^\infty \mu(\rho)\frac{1}{\mu(\rho)}\partial_\rho\big[\mu(\rho)\partial_\rho\zeta_\omega\big]\zeta_\tau + \left(\mu(\rho)\zeta_\omega\partial_\rho\zeta_\tau - \mu(\rho)\partial_\rho\zeta_\omega\right)\Big|_0^\infty \\
&= \langle\Delta(\zeta_\omega), \zeta_\tau\rangle + \left(\mu(\rho)\zeta_\omega\partial_\rho\zeta_\tau - \mu(\rho)\partial_\rho\zeta_\omega\right)\Big|_0^\infty
\end{aligned} \tag{4.41}$$

That is, the self-adjointness of an operator is not always defined and depends on the boundary conditions [35].

There is no problem for bound states at infinity $\rho\to\infty$, where the functions and their derivatives can be safely taken to vanish with physical requirements. The boundary condition at $\rho\to 0$ needs more attention. We can solve the transverse operator in the $\rho\to 0$ limit with the Frobenius method. We first write $\zeta_\omega = \rho^s\sum_{j=0}^\infty a_j\rho^j,\ a_0\neq 0$, substitute it into the transverse operator in Eq.(4.38) and solve the coefficient for each order of the polynomial. We can find the relations

$$s=0, \quad a_1 = 0, \quad a_{j+2} = -\frac{\omega^2}{(j+2)^2}a_j. \tag{4.42}$$

Take care of the $s=0$, which implies the $\log\rho$ contribution. Thus, in the $\rho\to 0$ limit, the leading order is

$$\zeta_\omega(\rho) \to a_\omega + b_\omega\log\rho, \tag{4.43}$$

with constant a_ω and b_ω. Then, the self-adjointness and $\rho \to 0$ boundary condition require

$$a_\omega b_\tau - a_\tau b_\omega = 0. \tag{4.44}$$

As a_ω / b_ω is a single-valued function of ω, we can only have a unique bound state.

We can view the result from the perspective of the Schrodinger equation. With the rescaling $\Psi_\omega = \sqrt{\sinh(2\rho)}\zeta_\omega(\rho)$, We can rewrite the Δ eigenvalue problem as a Schrodinger equation

$$-\frac{d^2\Psi_\omega}{d\rho^2} + V(\rho)\Psi_\omega = \omega^2\Psi_\omega, \quad V(\rho) = 2 - (\coth 2\rho)^2. \tag{4.45}$$

The Schrodinger equation possesses a potential of Poschl-Teller type and the corresponding Sturm-Liouville problem is integrable. As we need to take care of only the $\rho \to 0$ behavior of the bound state, in which limit the potential takes the form

$$V(\rho) = -\frac{1}{4\rho^2}. \tag{4.46}$$

A review of the $1/x^2$ potential quantum mechanical problem is given in ref.[36]. The special character of the transverse operator is that the potential in the $\rho \to 0$ has the factor $-\frac{1}{4}$, which is a critical value. For potential $V = \gamma/\rho^2$, there is an infinity discrete L^2 normalizable bound states appear when $\gamma < -\frac{1}{4}$, no bound state for $\gamma > -\frac{1}{4}$, and only one bound state when $\gamma = -\frac{1}{4}$.

The general solution of the transverse operator is given by Legendre functions

$$\zeta_\omega(\rho) = a_\omega \mathcal{P}_{-\frac{1}{2}+\frac{\sqrt{1-\omega^2}}{2}}(\cosh(2\rho)) + b_\omega \mathcal{Q}_{-\frac{1}{2}+\frac{\sqrt{1-\omega^2}}{2}}(\cosh(2\rho)). \tag{4.47}$$

Considering a massless graviton is required and there is only one bound state, we need to carefully choose a boundary condition of the transverse operator to process a bound state with the zero eigenvalue $\omega = 0$. After normalized, the zero mode is

$$\zeta_0 = \pm\frac{2\sqrt{3}}{\pi} \log\tanh\rho, \tag{4.48}$$

which can be chosen by boundary conditions

$$(\sinh(2\rho)\log\tanh\rho\, \partial_\rho - 2)\zeta_\omega(\rho)\Big|_{\rho=0} = 0, \quad \sqrt{\sinh(2\rho)}\zeta_\omega(\rho)\Big|_{\rho\to\infty} < \infty. \tag{4.49}$$

The boundary condition at infinity promises the eigenfunctions are normalizable. In the $\rho \to 0$ limit, $\zeta_0 \sim \log\rho$ implies that the transfer space has the structure $\mathbb{R}^2 \times \{\text{compact}\}$. Notice the in the $\rho \to 0$ limit, the Eguchi-Hanson space becomes $\mathbb{R}^2 \times S^2$

with (ρ, χ) comprise the $\mathbb{R}^2$ in Eq.(4.9). We know that the non-compact part of the transverse space is spanned by ρ and χ. We will see that the non-constant zero mode makes possible the integrals over products of the zero-mode transverse, which allows for the study of lower-dimensional effective field theory beyond linear order.

Given the boundary conditions, we find that the scattering states are given by

$$\zeta_\omega(\rho) = \mathcal{M}_\omega \mathcal{Q}_{-\frac{1}{2}+\frac{\sqrt{1-\omega^2}}{2}}(\cosh(2\rho)) + c.c, \quad \omega > 1. \tag{4.50}$$

with normalization constant $\mathcal{M}_\omega$. That is, the spectrum is composed of a discrete state at $\omega = 0$ and continuum states $\omega > 1$ [11]. The mass gap guarantees the stability of the lower-dimensional gravity. The magnitude g of the mass gap depends on the geometry of the SS-CGP background geometry in Eq.(4.8). When the energy level is low or the length scale is large enough, the gravitational behavior will be low-dimensional.

The basis Eq.(4.48) and Eq.(4.50) satisfies the resolution of identity

$$\int_{\mathcal{I}} \zeta_\omega(\rho)\zeta_\omega(\eta)d\omega = \frac{\delta(\rho-\eta)}{\mu(\rho)}. \tag{4.51}$$

It is easy to find the solutions of the worldvolume differential equation

$$\left(\partial_r^2 + \frac{2}{r}\partial_r - g^2\omega^2\right) f^\omega(r) = \frac{1}{4\pi r^2}\delta(r) \tag{4.52}$$

as

$$f_\omega = -\frac{\exp(-g\omega r)}{4\pi r}. \tag{4.53}$$

We can find the Green's function of the CPS operator

$$\Delta_5 G(r, \rho-\eta) = \frac{g\hat{\kappa}^2 M\delta(r)\delta(\rho)}{4\pi r^2 \sinh 2\rho}, \tag{4.54}$$

by invoking a resolution of the identity as

$$G(r, \rho-\eta) = -\frac{g\hat{\kappa}^2 M}{4\pi r}\zeta_0(\rho)\zeta_0(\eta) - \int_1^\infty \frac{g\hat{\kappa}^2 M \exp(-g\omega r)}{4\pi r}\zeta_\omega(\rho)\zeta_\omega(\eta)d\omega. \tag{4.55}$$

Here, $\hat{\kappa}^2$ is the five-dimensional Newton constant and M is the mass of the source. We can check that Green's function inherits the Robin boundary condition at $\rho = 0$ and satisfy the boundary condition at $\rho \to \infty$

$$(\sinh(2\rho)\log\tanh\rho\,\partial_\rho - 2)G(r, \rho-\eta)\Big|_{\rho=0} = 0, \quad G(r, \rho-\eta)\Big|_{\rho\to} = 0. \tag{4.56}$$

We can get the coefficient on the right-hand side of Eq.(4.54) with

$$\Delta(g)G(\mathbf{x}) = \frac{1}{\sqrt{-g}}\delta^{(5)}(\mathbf{x}) \tag{4.57}$$

where $\Delta(g) = \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}g^{\mu\nu}\partial_\nu)$ is the harmonic operator defined with the metric in Eq.(4.26).

Let's check the asymptotic behavior of the Green's function to understand the effective gravity in different regimes [11]. Firstly, $\eta \ll 1$ and $R = \sqrt{g^2r^2 + \rho^2} \ll 1$, the leading behavior is given by

$$G(r, \rho - \eta) = -\frac{g^4\hat{\kappa}^2 M}{2\pi(g^2r^2 + (\rho - \eta)^2)^{\frac{3}{2}}} + \mathcal{O}(\frac{1}{R^2}). \tag{4.58}$$

The gravitational behavior returns to 5-dimensional in the near-source regime. We do not need to worry about it. In the regime close to the source sitting at $\rho = \eta$, *i.e.* the small distance scale, it must be a high energy regime where all the massive gravitons modes are excited.

The other interesting regime is the near brane large radius one with $\eta \ll 1$, $r \gg 1$ and $\rho \ll 1$. The zero mode has the form

$$G(r, \rho - \eta) = -\frac{g^4\hat{\kappa}^2 M}{4\pi r}\zeta_0(\rho)\zeta_0(\eta) + \mathcal{O}(\exp(-gr)). \tag{4.59}$$

The leading contribution comes from the zero mode, acting as a 3-dimensional potential with $1/r$, while contributions from higher modes are suppressed by $\exp(-g\omega r)$ at the $r \gg 1$. We can see that, in this type III construction, the Greens function is effectively 3-dimensional at large r, but also that it diverges logarithmically when ρ or η approaches zero. That is, instead of effective gravity *on* brane, we only have a well-behavior gravity *near* the brane at $\rho \sim \eta \sim 0$.

4.1.4 Newton Constant

We also need to check the 4-dimensional Newton constant. Different from the type I construction, in which there is no well-defined Newton constant due to the trombone symmetries, the construction based on SS-CGP background does possess a reasonable Newton constant. According to Eq.(4.24), in the radial coordinates $R^2(t) = X^i(t)X^i(t)$ on the worldvolume, we have [11]

$$\begin{aligned} R''(t) - \frac{l_W^2}{R(t)^3} &= -\frac{6g\hat{\kappa}^2 M}{\pi^3 R(t)^2}\log\tanh(P(t))\log\tanh(\eta(t)) + \mathcal{O}(R(t)^{-3}) \\ P''(t) - \frac{g^2}{2}P(t) &= \frac{12g^3\hat{\kappa}^2 M}{\pi^3 R(t)}\frac{\log\tanh(\eta)}{\sinh(2P(t))} + \mathcal{O}(R(t)^{-2}), \end{aligned} \tag{4.60}$$

in the $r \gg 1$ and $\rho \ll 1$ regime with l_W^2 the worldvolume angular momentum.

One method to get the lower-dimensional Nowton constant is finding a fixed transverse plane P. To find a fixed transverse point, we need to include the coordinate angular coordinate χ which pairs with ρ to form an $\mathbb{R}^2$ [10] and add an angular momentum term to the geodesic equation as

$$P''(t) - \frac{g^2}{2}P(t) - \frac{l_T^2}{P(t)^3} = \frac{12g^3\hat{\kappa}^2 M}{\pi^3 R(t)}\frac{\log\tanh(\eta)}{\sinh(2P(t))} + \mathcal{O}(R(t)^{-2}), \tag{4.61}$$

We can find a constant P solution as

$$P = 2^{\frac{1}{4}}\sqrt{\frac{l_T}{g}} + \frac{3g^{\frac{3}{2}}\hat{\kappa}^2 M\log(\eta)}{2^{\frac{1}{4}}\pi^3 R(t)\sqrt{l_T}} + \mathcal{O}(R(t)^{-2}). \tag{4.62}$$

At the leading order, there is a stable circular orbit with $P = 2^{1/4}\sqrt{l_T/g}$, and the potential is attractive.

If we suppose there is some minimum non-zero transverse angular momentum l_T as the Bohr-Sommerfeld quantization condition which restricts the minimum radius P. Considering a similar interpretation for η where mass M source is located and substituting the stable circular orbit into the worldvolume radial equation in Eq.(4.60), we have

$$R''(t) - \frac{l_W^2}{R(t)^3} = -\frac{6g\hat{\kappa}^2 M}{\pi^3 R(t)^2}\left(\log\tanh\left(2^{\frac{1}{4}}\sqrt{\frac{l_T}{g}}\right)\right)^2 \approx -\frac{6g\hat{\kappa}^2\log\left(\sqrt{2}g/l_T\right)^2 M}{4\pi^3 R(t)^2}. \tag{4.63}$$

Comparing it with the usual radial geodesic equation in 4 dimensions

$$r''(t) - \frac{l_W^2}{r(t)^3} = -\frac{\kappa^2 M}{4\pi^3 r^2}, \tag{4.64}$$

we have the effective 4-dimensional Newton constant

$$\kappa = \frac{\sqrt{6g}}{\pi}\left|\log\left(\frac{\sqrt{2}l_T}{g}\right)\right|\hat{\kappa}. \tag{4.65}$$

That is, the Newton constant depends on the 5-dimensional Newton constant $\hat{\kappa}^2$, minimum transverse angular momentum l_T and the geometry factor g in the SS-CGP background Eq.(4.8).

We can also achieve a similar result with the quantum localization method or smeared transverse expectation values method, which are presented in ref.[11]. The numerical approximation is about

$$\kappa \approx 1.7\sqrt{g}\hat{\kappa} \tag{4.66}$$

if we set $\hbar = g$. To get the same result, we need the minimum non-zero transverse angular momentum $l_T \approx 6.5\hbar$.

4.1.5 The Effective Field Theories of the SS-CGP background

Since the products of non-constant zero modes Eq.(4.48) are integrable, it opens the way to study the effective field theories on the worldvolume by expanding the action into more than quadratic order.

Ref.[37] introduced the *convert symmetry breaking* as the phenomenon that the interaction coefficient at the quartic order in the action is different from the square of the cubic order expansion constant as expected in gauge theories. Instead of studying the theories in the SS-CGP background, ref.[37] studied the simpler d-dimensional scalar QED model reduced from $(d+1)$ dimensions with geometry $\mathbb{R}^{1,d} \times I$, with $I = [0, 1]$. The scalar QED model rather captures the same characteristic.

When the Maxwell theory does not couple to complex scalar fields, the non-constant transverse zero mode induces the Stueckelberg fields in the zero-level sector whose presence indicates the U(1) symmetry associated with the zero-level sector being non-linearly realized.

After coupling to the complex scalar fields, ref.[37] obtained the effective field theory by integrating out all the fields whose mass is greater than the least massive gauge field. Again, the non-linear realization of the U(1) gauge symmetry appears. At the same time, the mismatch of the coupling constants present at the quartic term is explained by the Stueckelberg field. It was shown that the effective theory, however, keeps the gauge invariant due to the joint action of the unusual quartic coupling and the non-linear realization.

As the non-constant zero mode Eq.(4.48) appears in the SS-CGP model, the analysis in ref.[37] implies that the convert symmetry breaking will appear in the effective field theories in the SS-CGP background as a consequence of the underlying gauge symmetry, four-dimensional diffeomorphisms, being non-linearly realized.

4.2 Effective gravity based on RS model

Randall and Sundrum provided a type III effective gravity construction based on two domain walls in a 5-dimensional AdS background with $\mathbb{Z}_2$ symmetry [17]. Putting one of them to infinity, they obtained an effective one-brane construction, on which the lower-dimensional gravity is localized. This construction has a rather different transverse spectrum compared with the effective gravity based on the SS-CGP background.

4.2.1 The geometry of Randall-Sundrum model

The Randall-Sundrum background is constructed by making the fifth dimension y of the AdS_5 to be periodic with range $[-r_c\pi, r_c\pi)$ possessing a $\mathbb{Z}_2$ symmetry $y \leftrightarrow -y$. The action of the Randall-Sundrum model takes the form

$$\begin{aligned} S &= S_{grav} + S_{brane1} + S_{brane2} \\ S_{grav} &= \int d^4x \int dy \sqrt{-G}\{-\Lambda + 2M^3 R(G)\} \\ S_{brane} &= \int d^4x \sqrt{-g}\{\mathcal{L} + V_{brane}\} \end{aligned} \tag{4.67}$$

where the two domain walls locate at $y = 0$ and $y = r_c\pi$ in the background, R is the 5-dimensional Ricci scalar and Λ and V_{brane} are cosmological terms in the background and branes respectively. The explicit expression of the bosonic part of the 3-brane action is [4]

$$S_{brane} = \int d^4x \sqrt{-g}\{V_{brane} + \frac{1}{2}g^{\mu\nu}D_\mu\phi D_\nu\phi - U(\phi) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \cdots\}. \tag{4.68}$$

Substituting the $\mathbb{Z}_2$ symmetric metric ansatz

$$ds^2 = e^{-2k|y|}\eta_{\mu\nu}dx^\mu dx^\nu + dy^2, \quad 0 \le y \le \pi r_c \tag{4.69}$$

into the 5-dimensional Einstein's equations of the action, we get the relation between the cosmological terms

$$V_{brane1} = -V_{brane2} = 24M^3k, \quad \Lambda = -24M^3k^2, \tag{4.70}$$

required by the orbifold symmetry $y \to -y$. Finally, we achieve an effective one-brane model by putting the second brane into the infinity by choosing the limit $r_c \to \infty$. According to the Randall-Sundrum model, the 3-brane located at $y = 0$ is where we live.

In this construction, we obtain an effectively non-compact r_c transverse space. However, it must be essentially different from the true non-compact transverse space in the SS-CGP background. Nevertheless, the orbifolded domain wall structure is useful in constructing the embedding of gravity as we have seen in the chapter 3.

4.2.2 Effective Gravity on the Randall-Sundrum background

Let's check the effective gravity on the $y = 0$ domain wall result from the Randall-Sundrum construction. Firstly, we need to check the Newton constant or equivalently Planck mass. The Planck mass can be seen by substituting into Eq.(4.67) the perturbed metric with $\eta_{\mu\nu}$ replaced by $g_{\mu\nu}$ in Eq.(4.69),

$$S_{eff} = \int d^4x \int_0^{\pi r_c} dy \{2M^3 e^{-2k|y|}\sqrt{g}R(g) - \Lambda\} \tag{4.71}$$

and integrating out the fifth dimension y. The resulted Planck mass is

$$M_{pl}^2 = 2M^3 \int_0^{\pi r_c} dy e^{-2k|y|} = \frac{M^3}{k}\left(1 - e^{-2k\pi r_c}\right), \tag{4.72}$$

which will be M^3/k as we put the auxiliary brane to infinity. k can be identified as the AdS radius. It is the factor $\exp(-2k|y|)$ that saves the Planck mass. In a general non-compact transverse space, the integral over the transverse space will result in the relation $M_{pl}^2 = 2M^n \mathrm{V}_{n-2}$ where the volume of the transverse space V_{n-2} diverges.

Then, the spectrum of the transverse space. Similarly, we separate the perturbations $h_{\mu\nu}$, defined by $G_{\mu\nu} = e^{-2k|y|}\eta_{\mu\nu} + h_{\mu\nu}(x, y)$, into $h_{\mu\nu} = \int_{\mathcal{I}} \psi_\omega(y)\tilde{h}^\omega_{\mu\nu}(x)d\omega$. Then, the transverse space equations of motion become

$$\left[\frac{1}{2}\partial_y^2 + 2k\delta(y) - 2k^2 + \frac{\omega^2}{2}e^{2k|y|}\right]\psi_\omega(y) = 0. \tag{4.73}$$

With a change of coordinate $z \equiv \mathrm{sgn}(y)(e^{k|y|} - 1)/k$, the differential equation is transformed into the Schrodinger equation

$$\left[-\frac{1}{2}\partial_z^2 + V(z)\right]\psi_\omega(y) = 0, \quad V(z) = \frac{15k^2}{8(k|z| + 1)^2} - \frac{3k}{2}\delta(z). \tag{4.74}$$

The δ-function appears here due to the $\mathbb{Z}_2$ symmetry, which supports a single normalizable bound state. The positive definite part of potential admits continuum modes. As the potential falls off to zero as $|z| \to \infty$, we expect that the scattering modes behave like plane waves in the asymptotically faraway. The spectrum is composed of a bound state with zero eigenvalue with continuum modes right down to zero. There is a zero mode corresponding to the massless graviton but no mass gap. With the boundary condition required by the δ-function at $z = 0$, we can solve the transverse problem with Bessel functions [17]. For small ω, the wavefunction is

$$\psi_\omega \sim N_\omega(|z| + 1/k)^{1/2}\left[Y_2(\omega(|z| + 1/k)) + \frac{4k^2}{\pi\omega^2}J_2(\omega(|z| + 1/k)\right], \tag{4.75}$$

and zero mode

$$\psi_0(z) = \frac{1}{k(k|z|+1)^{3/2}}. \tag{4.76}$$

Here, N_ω is a normalization constant.

The gravitational potential in the Newtonian limit comprises the exchange of the zero-mode and continuum massive mode propagators. The gravitational potential on the domain wall $z = 0$ of a source with mass M is

$$V(r) \sim \frac{G_N M}{r} + \int_0^\infty d\omega \frac{G_N}{k} \frac{M e^{-\omega r}}{r} \frac{\omega}{k} = \frac{G_N M}{r} \left(1 + \frac{1}{r^2 k^2}\right). \tag{4.77}$$

in which the Newton constant is $G_N = k/M^2$ resulted from Eq.(4.72). The ω/k suppression in the integral is a result of continuum wave-functions at $z = 0$ in Eq.(4.75). Each massive mode contributes a Yukawa type of potential, while the overall contributions of massive modes is a type of $1/r^3$ potential suppressed by the AdS radius $1/k$. In large r, the gravity is effectively lower-dimensional. If we take the AdS radius to be small enough, like the Planck scale, the correction from the massive mode is hard to observe.

As there is no mass gap, we would be afraid of the excitation of higher modes inducing energy loss. Roughly, the excitation of massive modes for small ω is suppressed by $(\omega/k)^2$ due to the continuum wavefunction suppression, which serves effectively as a mass gap. The energy loss can happen when the the coupling of zero mode to massive modes does not ultimately couple back to the matter on the brane. We need to take care that the gravitational self-coupling gets large at a large value of z, as the blue shift of the AdS space. However, ref.[17] showed that the probability for the massless mode propagating to large z is too small to incur a problem. Besides, the gravitational coupling to the matter does not change significantly because the interaction is dominated by the massless graviton while the interaction between the massive modes and matter is suppressed by ω/k.

In summary, we can find that the $\mathbb{Z}_2$ symmetry and the factor $e^{-2k|y|}$ are vital in the construction. It provides a sensible Newton constant, a massless graviton, and an effective mass gap that suppresses the excitation of massive modes. The Randall-Sundrum effective gravity is an example without a mass gap. Ref.[11] showed that if the transverse space geometry can be factorized into $\mathbb{R}^b \times \{\text{compact}\}$, the effective gravity will behave with $1/r$ potential at large r spontaneously for $b \geq 3$.

Chapter 5

Conclusion and Outlook

In this dissertation, we first introduced supergravity and brane solutions in chapter 2. We took the 11-dimensional supergravity as an instance and introduced its equations of motion and supersymmetry. Following that, we discussed the brane solution of the single-charge action, which can be classified into electric, magnetic and dyonic brane solutions. To provide a complete story, we briefly discussed the super p-brane action, which is the source of the brane solution. Finally, we presented the important and classic Kaluza-Klein dimensional reduction and showed the reduction from $(D+1)$ to D dimensions explicitly. The KK dimensional reduction allows us to connect different supergravity theories and different brane solutions.

We started our main topic of effective theories on or near branes in the rest chapters. In chapter 3, we illustrate the type I construction. The type I theories are constructed by embedding the gravitational theories into the worldvolume of brane solutions. Hence, they admit natural truncations to the lower-dimensional worldvolume gravity. The worldvolume gravity can be general relativity or supergravity. We provided some worldvolume black hole solutions including the black-spoke solution for the general relativity scenario, the extremal RN black solution and a class of more general stationary black hole solutions for the supergravity cases. Because the type I effective gravity theory results from embedding, they almost satisfy all the criteria that we want for a lower-dimensional gravity, except for a sensible Newton constant because of the trombone symmetries. The lower-dimensional behavior is stable because the gravity is only along the worldvolume directions and dressed by a fixed factor depending on the transverse space. Thus, the type I effective theory is gravity on brane. In the perturbative description, the type I construction corresponds to a sort of Dirichlet boundary condition with the perturbation fixed by a specific factor depending on the transverse space.

In chapter 4, we moved to the type III effective gravity near branes. We concentrate on the effective theories that are based on the SS-CGP background. We found that the gravitational potential in the Newtonian limit depends on only the 00-component of the perturbations H_{00}, which is the Green's function of the CPS operator Δ_5. The eigenproblem of the CPS operator can be separated into worldvolume and transverse parts. The transverse differential equation is a Sturm–Liouville problem and can be transformed into a Schrodinger equation with Poschl-Teller potential, in which the self-adjointness restricts the spectrum to only a unique bound state. We set the bound state to be the eigenstate of the zero eigenvalue by carefully choosing a Robin boundary condition. The continuum spectrum and the bound state of the transverse operator are separated by a mass gap with the magnitude of g. It is the mass gap that provides us with a low-energy effective gravity. The effective gravity is essentially 5-dimensional if we consider the near source regime, while it is 3-dimensional when we come to the large r regime. It is a result of the transverse spectrum. When we consider the near-source area, we are considering high-energy physics, in which all the massive modes are excited. To provide a complete discussion about the SS-CGP-based construction, we provided a short comment on the convert symmetry breaking in the effective field theories resulting from the non-constant transverse zero mode.

Finally, we discussed the Randall-Sundrum model briefly. The Randall-Sundrum transverse problem does not possess a true mass gap compared with the SS-CGP construction. The $\mathbb{Z}_2$ symmetry and the factor $e^{-2k|y|}$ are vital in the Randall-Sundrum theory. They provide a reasonable Newton constant and an effective mass gap. What's more, the Randall-Sundrum geometry is widely used in constructing the embedding in the type I theories as we presented.

There is still much work can be done on related topics. For the worldvolume supergravity in type I constructions, recent literature only concerns the bosonic sector. We can provide a more general construction including the fermionic part. In the type III construction, the zero mode $\log \tanh \rho$ implies the transverse space includes a 2-dimensional non-compact subspace. We need to understand the 2-dimensional structure better because it seems to be general for systems that successfully achieve gravity localization. Besides, we can explore further the convert symmetry breaking of the effective field theory to understand the gravity self-coupling and the coupling to matter fields.

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