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Supergravity and Brane solutions

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Abstract

In this dissertation, we will introduce 11-dim and Type IIB supergravities, both of which allow us to find solutions describing spacetime under the backreaction from branes. These solutions typically contain an AdS factor and hold significant importance in the context of the AdS/CFT correspondence, a topic that will be extensively reviewed in this dissertation. Notably, we will delve into a newly discovered type of brane solution - D3 branes wrapped over spindles. These are different from other well-known wrapped brane scenarios, where supersymmetry is realized through a topological twist.

Contents

Chapter 1

Introduction

Gravity, electromagnetism, weak interactions, and strong interactions are the four fundamental interactions known at present. The Standard Model of particle physics, described by quantum field theory (QFT), properly captures the last three forces, and it is currently our best scientific theory of nature. Gravity, on the other hand, is only described classically by general relativity (GR).

Expanding the Einstein-Hilbert action with the substitution $g_{\mu\nu} = \eta_{\mu\nu} + M_{\mathrm{P}}^{-1} h_{\mu\nu}$ [1], where M_{P} is the Planck mass, leads to massless Fierz-Pauli action plus self-interactions. In particular, the self-coupling term is characterised by M_{P}^{-1}, possessing a mass dimension of -1 and hence non-renormalisable. As a result, developing a consistent (i.e. UV-complete) quantum theory of gravity is a critical subject of study.

String theory is one approach to quantum gravity. In fact, it is sometimes praised as the theory of everything, attempting to unify all fundamental forces, including gravity. Its original formulation (bosonic string theory) contains many drawbacks, like the lack of fermions and the presence of tachyons. These issues are quickly handled by introducing supersymmetry (SUSY), resulting in superstring theories. However, there are five different approaches to include SUSY, hence there are five distinct mathematically consistent superstring theories: Type IIA, Type IIB, Type I, Heterotic $E_8 \times E_8$, and Heterotic $SO(32)$ [**Kiritsis**, 2].

So, which one should we use to describe our physics? Fortunately, we do not have to choose. They are linked by non-perturbative dualities. For example, the physics described by Type IIB in $\mathbb{R}^{1,8} \times S^1$ with radius R is the same one described by Type IIA in $\mathbb{R}^{1,8} \times S^1$

with radius α'/R[1]. This is known as T-duality [3]. As a result, superstring theories are not physically distinct. Indeed, they represent the various limits of M-theory, a quantum theory that exists in 11-dim.

However, the precise formulation of M-theory remains unknown, although certain aspects of it have been well-studied. For instance, its low-energy dynamics can be approximated by 11-dim supergravity. Likewise, the low-energy limits of superstring theories correspond to five distinct 10-dim supergravity theories.

Roughly speaking, supergravity is GR plus local supersymmetry. Hence, we can find metric solutions, as we typically do. In particular, there are solutions that describe the spacetime geometry under the presence of branes — extended objects apparent in both string theory and M-theory. These brane solutions generally have an AdS factor, making them very interesting for studying Maldacena's AdS/CFT[2] conjecture [4].

The AdS/CFT correspondence is a truly amazing equivalency between two theories that seem unrelated. It conjectures that a gravitational (quantum) theory on AdS_{d+1} (e.g. string theory or M-theory) is equivalent to a conformal field theory (i.e. QFT with conformal symmetry) in d-dim. The most well-known example of the correspondence is Type IIB superstring theory on $AdS_5 \times S^5$ being dual to $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills (SYM) theory.

In particular, the low-energy limit of Type IIB superstring on $AdS_5 \times S^5$, i.e. Type IIB supergravity, corresponds to the strong coupling regime of SYM theory, where perturbation theory does not apply. As a result, the AdS side enables us to calculate non-trivial information about the CFT side, which was previously thought to be impossible (and vice versa) [5]. Furthermore, we may be able to gain a better grasp of M-theory by using its dual CFT description.

Many examples of AdS/CFT come from supergravity brane solutions. In fact, the above $AdS_5 \times S^5$ case can be understood from the planar D3-brane solution. Similarly, other brane solutions, like planar M2 or M5-branes, also give AdS/CFT examples. Following that, we will present more complicated brane solutions, such as branes wrapped over calibrated cycles or spindles, all of which have AdS factors.

[1] Here, $\alpha' = l_s^2$ is a constant that characterises the length scale l_s of string theory.
[2] AdS: Anti-de Sitter spacetime, CFT: Conformal Field Theory

Notation

We will work in conventional natural units, i.e. $\hbar = c = 1$, and the Lorentzian signature will be mostly plus. Greek letters μ, ν, ρ, σ typically represent full spacetime indices, although there are exceptions that will be explained in context. While spinor indices are rarely utilised, they are represented by the Greek letters α, β when necessary. For internal symmetry indices, we typically use Latin letters i, j, k, and for vierbein indices, we generally use a, b, c. In most cases, notations should be clear from context, and any specific or unusual notations will be explained as needed.

Outline

The following is the dissertation outline:

- In Chapter 2, we will introduce 11-dim supergravity and Type IIB supergravity, and then explore some solutions. In particular, we will find out that many solutions are asymptotically AdS.

- In Chapter 3, we will introduce the AdS/CFT correspondence. Then, we will motivate the duality between Type IIB in AdS and SYM theory by looking at D3-branes in two perspectives.

- In Chapter 4, we will justify how branes can be wrapped in different spacetime geometries. Then, we will learn that the near horizon geometry describing wrapped branes can be found by solving lower-dim supergravities. In particular, we can wrap a D3-brane over a spindle, which has many new interesting features that are discovered very recently.

Chapter 2

Supergravity

Before we go into supergravity, let us discuss about supersymmetry (SUSY). Typically, the symmetry generators fulfil the commutator relations $[T_a, T_b] \sim T_c$. However, mathematically, there is nothing that prevents us from obtaining symmetry fulfilling the anti-commutator relations $\{T_a, T_b\} \sim T_c$.

This allows us to construct the super-Poincaré algebra (in 4-dim) by introducing the following relations [6]

$$\{Q_\alpha, Q_\beta\} = 2(C\gamma^\mu)_{\alpha\beta}P_\mu, \qquad [Q_\alpha, J_{\mu\nu}] = \frac{1}{2}(\gamma_{\mu\nu})_\alpha{}^\beta Q_\beta \tag{2.1}$$

where C is the charge conjugation matrix, P_μ and $J_{\mu\nu}$ are Poincaré generators, and $\alpha, \beta = 1, ..., 4$ are spinor indices, as the second relation indicates. Q_α in particular is a Majorana spinor. We refer to theories that remain invariant under transformations generated by supercharges Q_α (in this case, there are 4 supercharges) as SUSY theories.

We can already deduce several key consequences of SUSY. First of all, a SUSY theory must have an equal number of fermions and bosons in terms of degrees of freedom. This is because the action of Q_α on a field Ψ alters its spin-statistics, and unequal numbers would result in a different theory, thus not a symmetry.

Subsequently, we can observe that when SUSY is introduced to GR, which is based on general coordinate invariance (i.e. local Poincaré), it must be a local symmetry. Conversely, a local SUSY theory must contain gravity. This is all clear when considering (2.1), i.e. local SUSY = local Poincaré. Hence, such a theory is simply referred to as supergravity [7], which includes a metric and at least one gravitino[1].

In fact, the algebra can be extended to include more supercharges

$$\{Q^i_\alpha, Q^j_\beta\} = 2(C\gamma^\mu)_{\alpha\beta}\delta^{ij}P_\mu \tag{2.2}$$

where $i, j = 1, ..., \mathcal{N}$. So, SUSY theories are characterised by $\mathcal{N}$, which is also the number of gravitini when it is local. We can see that if $Q^i_\alpha \to M^{ij}Q^j_\alpha$, then the algebra is invariant when $M^T M = \mathbb{1}$. This results in an internal symmetry for supercharges known as the R-symmetry. For example, it is $U(1)$ for the $\mathcal{N} = 1$ case, as shown by reformulating the algebra using Weyl spinors

$$\{Q, \bar{Q}\} \sim P_\mu, \quad \{Q, Q\} = 0, \quad \{\bar{Q}, \bar{Q}\} = 0 \tag{2.3}$$

where spinor indices have been omitted for simplicity. This is invariant under $Q \to e^{i\theta}Q$, and so $U(1)$.

The SUSY algebra can be extended further [6]

$$\{Q^i_\alpha, Q^j_\beta\} = 2(C\gamma^\mu)_{\alpha\beta}\delta^{ij}P_\mu + C_{\alpha\beta}U^{ij} + (C\gamma_5)_{\alpha\beta}V^{ij} \tag{2.4}$$

where U^{ij} and V^{ij} are termed central charges, where "central" refers to the centre of our symmetry group, and so they commute with the rest of algebra. These are not included in $\mathcal{N} = 1$ algebra because they are antisymmetric (as are C and $C\gamma_5$). To ensure that the Hilbert space is positive-definite, we must restrict the right-hand side of (2.4), which yields the Bogomol'nyi-Prasad-Sommerfield (BPS) bound, and states that saturate the bound are known as BPS states [8].

We will now go through a quick overview of the SUSY representation theory. We refer to lecture notes [9, 10] or textbooks [11, 12] for a more in-depth discussion. We will start by examining basic facts about Poincaré algebra. Its irreducible representations (irreps) are classified according to their mass (i.e. massive or massless) and spin/helicity. These correspond to momentum squared P^2 (mass) and Pauli-Lubanski squared W^2 (spin)

[1] A gravitino ψ^α_μ is a spin-3/2 field and plays the role of the gauge field for local SUSY. Notably, its gauge parameter is a spinor.

Casimir operators.

However, W^2 is no longer Casimir for SUSY since it requires the same amount of fermions and bosons. So, what are the irreps for SUSY? In general, we take an arbitrary massive or massless state $|\lambda\rangle$ with the spin λ, then apply supercharges to create other states like $|\lambda + 1\rangle$. The irrep is made up of all of these states.

In $\mathcal{N} = 1$ theory, for example, if we start with a massless scalar $|0\rangle$, the resulting irrep is known as the chiral multiplet, which consists of two $|0\rangle$, one $|1/2\rangle$, and one $|-1/2\rangle$. In other words, it contains a massless complex scalar and a Weyl spinor. Likewise, if we start from a $|-1/2\rangle$ state, we obtain the vector multiplet, which is given by a gauge vector and a Weyl spinor. In both scenarios, fields differed by spin-1/2. This is because (2.1) relies on a single Majorana spinor Q_α as the SUSY generator.

Hence, in theories with $\mathcal{N} > 1$, we can expect to find fields distributed across a range of spin values within a multiplet. If we limit our theory at most spin-2 fields[2], we get a matching number for $\mathcal{N}$, which is $\mathcal{N} = 8$, i.e. 32 supercharges. This sets the maximum amount of supercharges that a supergravity can have.

In higher dimensions, the number of components for spinors increases. Specifically, in 11-dim, the Majorana spinor already consists of 32 components, giving rise to a unique supergravity theory. Similarly, in 10-dim, SUSY generators are characterised by Majorana-Weyl spinors, consisting of 16 components. This gives us two distinct maximal (32 supercharges) supergravities, which are recognised as Type IIA and Type IIB.

In this chapter, we will focus on 11-dim supergravity and Type IIB. We shall start with 11-dim supergravity, which in fact looks to be less complicated than Type IIB. Following that, we will discuss some simple vacuum solutions and some planar brane solutions.

2.1 11-dim Supergravity

As stated in the introduction of this chapter, there is only one possible supergravity in 11-dim. This is also a remarkable feature since its uniqueness corresponds to what we assumed to be the unique theory, M-theory.

To begin, we are aware that 11-dim supergravity includes a metric $g_{\mu\nu}$ and a gravitino

[2]Higher-spin fields should be avoided since they yield inconsistent interacting QFTs in Minkowski [13].

ψ_μ^α. The metric has 44 (on-shell[3]) degrees of freedom, while the gravitino exhibits 128. As a result, there is an imbalance in favor of bosonic degrees of freedom. This disparity is reconciled through the introduction of a 3-form field $A_{(3)}$. Therefore, the field content of 11-dim supergravity comprises a metric, a gravitino, and a 3-form.

We will now establish some conventions for spinors in this context. We work with an 11-dim (spacetime) Clifford algebra, which is generated by gamma matrices Γ^a satisfying the relation

$$\{\Gamma^a, \Gamma^b\} = 2\eta^{ab} \tag{2.5}$$

where a and b represent vielbein indices. Alternatively, the antisymmetrised products of gamma matrices, i.e. $\{\Gamma^a, \Gamma^{ab}, \ldots, \Gamma^{012\ldots10}\}$, also generate the Clifford algebra, in which we take the convention $\Gamma^{012\ldots10} = -\mathbb{1}$. Furthermore, it is important to note that $\frac{1}{4}\Gamma^{ab}$ generates the group $Spin(10,1)$. We also define the charge conjugation matrix as Γ^0. Consequently, the Dirac adjoint[4] $\bar{\epsilon} = \epsilon^T\Gamma^0$, where ϵ is a Majorana spinor.

The complete action of 11-dim supergravity was originally derived by Cremmer, Julia, and Scherk back in 1978 [14]. However, we will focus on a simplified version in which only bosonic fields are considered by setting $\psi = 0$. This leads to the following action

$$S = \frac{1}{2\kappa_{11}^2}\int R\star 1 - \frac{1}{2}F_{(4)}\wedge\star F_{(4)} - \frac{1}{6}F_{(4)}\wedge F_{(4)}\wedge A_{(3)}, \qquad 2\kappa_{11}^2 \equiv (2\pi)^8 l_P^9 \tag{2.6}$$

where l_P is the Planck length, R is the Ricci scalar, and $F_{(4)} = dA_{(3)}$ is the field strength. Consequently, the equations of motion are

$$R_{\mu\nu} = \frac{1}{12}(F_{\mu\nu}^2 - \frac{1}{12}g_{\mu\nu}F^2), \qquad d\star F + \frac{1}{2}F\wedge F = 0 \tag{2.7}$$

where $F_{\mu\nu}^2 = F_{\mu\sigma_1\sigma_2\sigma_3}F_\nu{}^{\sigma_1\sigma_2\sigma_3}$ and $F^2 = F_{\sigma_1\sigma_2\sigma_3\sigma_4}F^{\sigma_1\sigma_2\sigma_3\sigma_4}$. The SUSY transformations are, schematically [15]

$$\delta g \sim \epsilon\psi, \qquad \delta A \sim \epsilon\psi, \qquad \delta\psi \sim \hat{\nabla}\epsilon + \epsilon\psi\psi \tag{2.8}$$

[3] In this case, this means they are irreps of the little group $SO(9)$.

[4] The Dirac adjoint is identical to Majorana conjugation since ϵ is a Majorana spinor.

where ϵ is a Majorana spinor that parameterises the transformation, and $\hat{\nabla}$ represents some derivative operator. Since $\psi = 0$, the only non-trivial transformation is

$$\delta\psi_\mu = \left[\nabla_\mu + \frac{1}{288}(\Gamma_\mu{}^{\nu_1\nu_2\nu_3\nu_4} - 8\delta_\mu^{\nu_1}\Gamma^{\nu_2\nu_3\nu_4})F_{\nu_1\nu_2\nu_3\nu_4}\right]\epsilon \tag{2.9}$$

where $\nabla_\mu = \partial_\mu + \frac{1}{4}\omega_{\mu\alpha\beta}\Gamma^{\alpha\beta}$ is the usual Levi-Civita connection.

Therefore, for consistency (i.e. $\psi = 0$), we require $\delta\psi = 0$. The trivial case is just $\epsilon = 0$, meaning no SUSY at all. So, we are looking for non-trivial solutions that preserve SUSY, and such ϵ is referred to as a Killing spinor. This nomenclature is justified if we define $K_\mu \equiv \bar{\epsilon}\Gamma_\mu\epsilon$, then we can show that it is a Killing vector [16]. This can be generalised to K_μ^{ij} for any Killing spinor ϵ^i.

Furthermore, the presence of Killing spinors indicates a highly limited condition that provides crucial insights into the geometry. For example, it has been proved that in circumstances where a timelike Killing spinor exists (i.e. its associated Killing vector is timelike), the Einstein equations are automatically satisfied, subject to the flux satisfying its corresponding equations of motion [16].

Even before delving into the equations of motion, the theory itself can offer valuable insights. Specifically, the 3-form potential must have sources, namely the M2-branes and M5-branes. To illustrate this, let us draw an analogy with electromagnetism coupled to a charged particle, described by the following action

$$S \sim \int d\lambda \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}} + qA_\mu\frac{dx^\mu}{d\lambda} \tag{2.10}$$

where A_μ is a 1-form, and q represents the electric charge. Therefore, we anticipate that a 3-form potential is coupled to a 3-dim worldvolume, specifically originating from the M2-brane. Similarly, we expect the existence of M5-branes that are magnetically coupled to $A_{\mu\nu\rho}$, i.e. coupling to a 6-form potential $\tilde{A}_{\mu_1\dots\mu_6}$ given by $\star F_{(4)} = d\tilde{A}_{(6)}$.

2.2 Type IIB Supergravity

Recall that there are two unique maximal supergravities in 10-dim. This arises from the fact that the SUSY generators are represented by Majorana-Weyl spinors, allowing us to construct two distinct maximal supergravities, i.e. $\mathcal{N} = (1, 1)$ Type IIA or $\mathcal{N} = (2, 0)$ Type IIB.

For our interests, we will concentrate on Type IIB supergravity. However, it is worth mentioning a few words about Type IIA. It can be derived from the Kaluza-Klein reduction of 11-dim supergravity on S^1 (for detailed information, please refer to Appendix A). In this reduction, the 11-dim metric decomposes into a 10-dim metric, a scalar field, and a 1-form potential. Similarly, the 3-form potential in 11-dim also decomposes into a 3-form and a 2-form potentials in 10-dim. One can verify that these fields indeed possess the correct number of degrees of freedom, which amounts to 128, by examining their on-shell components, i.e. irreps of $SO(8)$. Thus, we may already anticipate seeing certain extended objects, such as a D4-brane[5]. Furthermore, the 2-form potential electrically couples to a 1-dim object, which is the fundamental string.

Let us turn our attention to Type IIB supergravity. Again, the on-shell bosonic degrees of freedom must sum up to 128, in which the must-have 10-dim metric accounting for 35 of these degrees. This leaves us with 93 additional degrees of freedom to account for. Just like in Type IIA, Type IIB also features a 2-form potential (coupling to the string), which contributes 28 degrees. Moreover, if they are present, then we also require a scalar for consistency. Recall that Type IIB and Type IIA theories are related by a T-duality transformation. This can swap the boundary conditions of open strings between the two theories. As a result, Dp-branes in Type IIA transform into D$(p\pm1)$-branes in Type IIB [1].

Consequently, certain r-form potentials in Type IIA become $(r \pm 1)$-form potentials in Type IIB. Therefore, we can anticipate the presence of 0-form, 2-form, and 4-form potentials. Indeed, the Type IIB supergravity action is (without fermions) [2]

$$S_{IIB} = S_{NS} + S_R + S_{CS} \tag{2.11}$$

[5]Open strings are subject to boundary conditions at their endpoints, i.e. Dirichlet or Neumann. Specifically, the Dirichlet boundary conditions lead to the definition of hypersurfaces known as Dp-branes, where p denotes the spatial dimension of the brane.

where

$$
\begin{aligned}
S_{NS} &= \frac{1}{2\kappa^2}\int R\star 1 - \frac{1}{2}d\phi\wedge\star d\phi - \frac{1}{2}e^{-\phi}H_{(3)}\wedge\star H_{(3)} \\
S_R &= -\frac{1}{4\kappa^2}\int e^{2\phi}F_{(1)}\wedge\star F_{(1)} + e^{\phi}\tilde{F}_{(3)}\wedge\star\tilde{F}_{(3)} + \tilde{F}_{(5)}\wedge\star\tilde{F}_{(5)} \\
S_{CS} &= -\frac{1}{4\kappa^2}\int A_{(4)}\wedge H_{(3)}\wedge F_{(3)}
\end{aligned}
\tag{2.12}
$$

and we have $H_{(3)} = dB_{(2)}$, $F_{(n)} = dA_{(n-1)}$, and $2\kappa^2 = (2\pi)^7 l_P^8$

$$
\tilde{F}_{(3)} = F_{(3)} - A_{(0)}H_{(3)}, \qquad \tilde{F}_{(5)} = F_{(5)} - \frac{1}{2}A_{(2)}\wedge H_{(3)} + \frac{1}{2}B_{(2)}\wedge F_{(3)} \tag{2.13}
$$

So, the bosonic field content is a 10-dim metric $g_{\mu\nu}$, a scalar (dilaton) ϕ, another scalar (axion) $A_{(0)}$, a 2-form $B_{(2)}$, another 2-form $A_{(2)}$, and a 4-form $A_{(4)}$.

However, these fields result too many degrees of freedom, totaling 163, which exceeds the requirements for supersymmetry. So, what is the issue here? From a group theory perspective, a general 4-form field $A_{(4)}$ in 10 dimensions can be further constrained by demanding that its field strength is self-dual, meaning that $dA_{(4)} = \star dA_{(4)}$. This self-duality condition effectively reduces its degrees of freedom from 70 to 35. Thus, if we impose this self-duality condition

$$
\tilde{F}_{(5)} = \star\tilde{F}_{(5)} \tag{2.14}
$$

the resulting bosonic fields will indeed have the correct number of degrees of freedom.

However, the challenge arises when we attempt to incorporate this condition into the action. The kinetic term for the 4-form potential, under the self-duality condition, trivially vanishes. As a result, it becomes impossible to construct a supersymmetric action for Type IIB supergravity that includes the correct 4-form potential. Fortunately, the equations of motion derived from this action plus the self-duality condition is supersymmetric. In other words, while we cannot formulate a supersymmetric action, we can obtain correct equations of motion for Type IIB.

Recall that to preserve SUSY, we have some Killing spinor equations to be satisfied.

In this case, we have two gravitini ψ and two dilatini λ, which transform like [2]

$$\begin{aligned} \delta\lambda &= \frac{1}{2}(\partial_\mu\phi - ie^\phi\partial_\mu A_{(0)})\Gamma^\mu\epsilon + \frac{1}{4\cdot 3!}(ie^\phi\tilde{F}_{\mu\nu\rho}\Gamma^{\mu\nu\rho} - H_{\mu\nu\rho}\Gamma^{\mu\nu\rho})\epsilon^* \\ \delta\psi_\mu &= (\nabla_\mu + \frac{i}{8}e^\phi F_{\sigma_1}\Gamma^{\sigma_1}\Gamma_\mu + \frac{i}{16\cdot 5!}e^\phi\tilde{F}_{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}\Gamma^{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}\Gamma_\mu)\epsilon \\ &\qquad - \frac{1}{8}(H_{\mu\sigma_1\sigma_2}\Gamma^{\sigma_1\sigma_2} + \frac{i}{3!}e^\phi\tilde{F}_{\sigma_1\sigma_2\sigma_3}\Gamma^{\sigma_1\sigma_2\sigma_3}\Gamma_\mu)\epsilon^* \end{aligned} \tag{2.15}$$

where spinor indices are omitted for simplicity, and ϵ is a left Majorana-Weyl spinor (i.e. $\Gamma^{11}\epsilon = \epsilon$).

If we consider a specific scenario where only D3-branes are present, which are coupled to the 4-form potential, we can set all other potentials to zero. The reduced action becomes [17]

$$S_{IIB} = \frac{1}{2\kappa^2}\int R\star 1 - \frac{1}{2}d\phi\wedge\star d\phi - \frac{1}{2}F_{(5)}\wedge\star F_{(5)} \tag{2.16}$$

and the corresponding equations of motion are

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{4\cdot 4!}F_{\mu\sigma_1\sigma_2\sigma_3\sigma_4}F_\nu{}^{\sigma_1\sigma_2\sigma_3\sigma_4} \\ \Box_{10}\phi &= 0 \\ dF_{(5)} &= 0 \end{aligned} \tag{2.17}$$

while the SUSY transformations are

$$\delta\lambda = \frac{1}{2}(\partial_\mu\phi)\Gamma^\mu\epsilon, \qquad \delta\psi_\mu = \left(\nabla_\mu + \frac{i}{16\cdot 5!}e^\phi F_{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}\Gamma^{\sigma_1\sigma_2\sigma_3\sigma_4\sigma_5}\Gamma_\mu\right)\epsilon \tag{2.18}$$

which are solved by a constant ϕ and a Killing spinor ϵ that satisfies $\delta\psi_\mu = 0$.

2.3 No Flux Solutions

We can consider a scenario where no fluxes are present. In both 11-dim supergravity and Type IIB supergravity, the equations of motion, along with the Killing spinor conditions, simplify to

$$R_{\mu\nu} = 0, \qquad \nabla_\mu\epsilon = 0 \tag{2.19}$$

where μ and ν represent the spacetime indices (10-dim or 11-dim). This further implies

$$[\nabla_\mu, \nabla_\nu]\epsilon \equiv \frac{1}{4}R_{\mu\nu\rho\sigma}\Gamma^{\rho\sigma}\epsilon = 0 \tag{2.20}$$

which means that ϵ is a singlet of the restricted holonomy group generated by $\frac{1}{4}R_{\mu\nu\rho\sigma}\Gamma^{\rho\sigma}$.

In other words, the no flux solutions to 11-dim or Type II supergravity are Ricci-flat manifolds with covariantly constant spinors, and the number of preserved SUSY depends on the corresponding holonomy group. As an example, the simplest solution is Minkowski spacetime, which is Ricci-flat and preserves all 32 supersymmetries since we only require ϵ to be a constant and impose no further restrictions.

However, for our interests, we are looking for solutions of the form $\mathbb{R}^{1,p} \times \mathcal{M}_d$, where $\mathcal{M}_d$ is a (simply connected) Riemannian manifold, and $1+p+d = 10$ or 11. This assumption allows us to utilize Berger's classification, which lists all possible holonomy groups for the Levi-Civita connection for simply connected, irreducible (i.e. not as a product space), and nonsymmetric (i.e. not locally a coset space G/H for Lie groups G and H) Riemannian manifolds [18]. From this list, we can identify all possible candidates for $\mathcal{M}_d$, i.e. Calabi-Yau, G_2, $Spin(7)$, and hyperkähler manifolds.

Now, let us illustrate how to determine the amount of preserved supersymmetry. We will begin by considering the 11-dim case for simplicity. Our spinor is in the spinorial representation $\mathbf{32}$ of $Spin(10,1)$, which decomposes, under our assumption of $\mathbb{R}^{1,10-d} \times \mathcal{M}_d$, as $Spin(10-d,1) \times Spin(d)$.

For example, consider a $Spin(7)$-manifold, which is an 8-dim manifold with the holonomy group $Spin(7)$. The decomposition is as follows [15]

$$\begin{gathered} Spin(10,1) \to Spin(2,1) \times Spin(8) \\ \mathbf{32} \to (\mathbf{2}, \mathbf{8}_+) + (\mathbf{2}, \mathbf{8}_-) \end{gathered} \tag{2.21}$$

where $\mathbf{8}_\pm$ denotes spinorial representations with opposite chiralities.

Following that, we further decompose it under the holonomy group $Spin(7) \subset Spin(8)$, which is

$$\begin{aligned} Spin(10,1) &\to Spin(2,1) \times Spin(7) \\ \mathbf{32} &\to (\mathbf{2}, \mathbf{7}+\mathbf{1}) + (\mathbf{2}, \mathbf{8}) \end{aligned} \tag{2.22}$$

So, if our solution takes the form of $\mathbb{R}^{1,2} \times \mathcal{M}_8$, where $\mathcal{M}_8$ is a $Spin(7)$-manifold, then our covariantly constant spinor ϵ must be in the $(\mathbf{2}, \mathbf{1})$ representation. As a result, ϵ has two real (on-shell) components, which implies that there are two preserved supercharges. This can also be interpreted as preserving $\mathcal{N} = 1$ SUSY in $\mathbb{R}^{1,2}$.

We can replace $\mathcal{M}_8$ with Calabi-Yau 4-folds, denoted as CY_4, which have $SU(4)$ holonomy. In this case, the decomposition of the spinor representation is

$$\begin{aligned} Spin(10,1) &\to Spin(2,1) \times SU(4) \\ \mathbf{32} &\to (\mathbf{2}, \mathbf{6}+\mathbf{1}+\mathbf{1}) + (\mathbf{2}, \mathbf{4}+\bar{\mathbf{4}}) \end{aligned} \tag{2.23}$$

and hence four supercharges are preserved. This can also be interpreted as $\mathcal{N} = 2$ SUSY in $\mathbb{R}^{1,2}$. Similarly, we can also replace $\mathcal{M}_8$ with $CY_2 \times CY_2$, preserving $N = 4$ supersymmetry in $\mathbb{R}^{1,2}$. If $\mathcal{M}_8$ is a hyperkähler manifold, which has $Sp(2)$ holonomy, then it will preserve $N = 3$ SUSY in $\mathbb{R}^{1,2}$. The overall results will be shown in the table below.

In Type IIB supergravity on $Spin(7)$-manifold, the decomposition of the spinor representation is

$$\begin{aligned} Spin(9,1) &\to Spin(1,1) \times Spin(7) \\ \mathbf{16} &\to (\mathbf{1}, \mathbf{7}+\mathbf{1}) + (\mathbf{1}, \mathbf{8}) \end{aligned} \tag{2.24}$$

hence each ϵ in Type IIB has only one real component, resulting in the preservation of two supercharges in total. In other words, this decomposition replaces the spinorial representation of $Spin(2,1)$ in 11-dim supergravity with the trivial representation of $Spin(1,1)$ in Type IIB. Therefore, in both cases, the number of preserved supersymmetries depends solely on the choice of $\mathcal{M}_d$.

$d = \dim \mathcal{M}_d$	Holonomy	SUSY
10	$SU(5)$	2
10	$SU(3) \times SU(2)$	4
8	$Spin(7)$	2
8	$SU(4)$	4
8	$Sp(2)$	6
8	$SU(2) \times SU(2)$	8
7	G_2	4
6	$SU(3)$	8
4	$SU(2)$	16

Table 2.1: This table provides a summary of the preserved SUSY for a given $\mathcal{M}_d$, as presented in [15]. Specifically, the first column indicates the dimension of the respective manifold, the second column specifies the holonomy group, and the last column indicates the number of preserved supercharges.

2.4 Planar Brane Solutions

Consider a D-dim Minkowski spacetime, where D is either 11 or 10. In this background, the introduction of a planar p-brane naturally results in the breaking of the original D-dim Poincaré symmetry into the direct product $ISO(1,p) \times SO(D-p-1)$. As a consequence, the most general metric preserving this remaining symmetry is [19]

$$ds^2 = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B(r)}\delta_{ij}dy^i dy^j \tag{2.25}$$

where x^μ are coordinates of the worldvolume with $\mu = 0, 1, \ldots, p$, and y^i transverse coordinates with $i = 1, 2, \ldots, D-p-1$. We have also defined the radial distance in transverse space as $r^2 = y^i y^i$. To better accommodate the $SO(D-p-1)$ symmetry, it is more convenient to work in spherical polar coordinates [17]

$$ds^2 = e^{2A(r)}\eta_{\mu\nu}dx^\mu dx^\nu + e^{2B(r)}\left[dr^2 + r^2 ds^2(S^{D-p-2})\right] \tag{2.26}$$

where $ds^2(S^{D-p-2})$ denotes the standard metric on S^{D-p-2}.

We can use the vielbein formalism to derive all the necessary information, including spin connections and Ricci tensors. To do so, we define the vielbeins as follows

$$e^a = e^{A(r)}\delta^a_\mu dx^\mu, \qquad e^R = e^{B(r)}dr, \qquad e^m = re^{B(r)}e^{\tilde{m}} \tag{2.27}$$

where $e^{\tilde{m}}$ represents vielbeins on S^{D-p-2}. The subsequent steps involve straightforward calculations of spin connections and curvature 2-forms using Cartan structure equations

$$de^X + \omega^X_{\ \ Y} \wedge e^Y = 0, \qquad R^X_{\ \ Y} = d\omega^X_{\ \ Y} + \omega^X_{\ \ Z} \wedge \omega^Z_{\ \ Y} \tag{2.28}$$

where X, Y, Z represent vielbein indices for the entire spacetime.

The non-vanishing spin connections are summarized as follows

$$\begin{aligned}
\omega^a_{\ \ R} &= A' e^{-B(r)} e^a \\
\omega^m_{\ \ n} &= \omega^{\tilde{m}}_{\ \ \tilde{n}} \\
\omega^m_{\ \ R} &= e^{-B(r)} \left(B' + \frac{1}{r} \right) e^m
\end{aligned} \tag{2.29}$$

where $A' = \frac{dA}{dr}$, and $\omega^{\tilde{m}}_{\ \ \tilde{n}}$ represents the spin connection on S^{D-p-2}. Consequently, the non-vanishing components of the curvature 2-forms are

$$\begin{aligned}
R^a_{\ \ bac} &= -\eta_{bc} A'^2 e^{-2B} \\
R^a_{\ \ RRa} &= e^{-2B} \left(A'' + A'^2 - A'B' \right) \\
R^a_{\ \ mam} &= -e^{-2B} A' \left(B' + \frac{1}{r} \right) \\
R^m_{\ \ npq} &= R^{\tilde{m}}_{\ \ \tilde{n}pq} - e^{-2B} \left(B' + \frac{1}{r} \right)^2 \delta^{[p}_m \delta^{q]}_n \\
R^m_{\ \ RRm} &= e^{-2B} \left(B'' + \frac{1}{r} B' \right)
\end{aligned} \tag{2.30}$$

where the Einstein summation convention is not applied. Then, the Ricci tensors are

$$\begin{aligned}
R_{ab} &= -\eta_{ab} e^{-2B} \left[A'' + dA'^2 + \tilde{d}A'B' + \frac{\tilde{d}+1}{r} A' \right] \\
R_{RR} &= -e^{-2B} \left[d(A'' + A'^2 - A'B') + (\tilde{d}+1)(B'' + \frac{1}{r} B') \right] \\
R_{mn} &= -\delta_{mn} e^{-2B} \left(B'' + dA'B' + \tilde{d}B'^2 + \frac{2\tilde{d}+1}{r} B' + \frac{d}{r} A' \right)
\end{aligned} \tag{2.31}$$

or in terms of spacetime indices (with M and N being coordinates on S^7)

$$
\begin{aligned}
R_{\mu\nu} &= -\eta_{\mu\nu} e^{2A-2B} \left[A'' + dA'^2 + \tilde{d} A'B' + \frac{\tilde{d}+1}{r} A' \right] \\
R_{rr} &= -\left[d(A'' + A'^2 - A'B') + (\tilde{d}+1)(B'' + \frac{1}{r} B') \right] \\
R_{MN} &= -r^2 g_{MN} \left(B'' + dA'B' + \tilde{d} B'^2 + \frac{2\tilde{d}+1}{r} B' + \frac{d}{r} A' \right)
\end{aligned}
\tag{2.32}
$$

where $d = p+1$, and $\tilde{d} = D - d - 2$. It is important to note that one key result is based on the fact that S^{D-p-2} is an Einstein manifold, i.e. its Ricci tensor is proportional to its metric.

M2-brane

Because the M2-brane electrically sources the 3-form potential, we can make the assumption [19]

$$
A_{\mu\nu\rho} = \epsilon_{\mu\nu\rho} e^{C(r)} \implies F_{r\mu\nu\rho} = \epsilon_{\mu\nu\rho} \partial_r e^{C(r)} \tag{2.33}
$$

we then substitute this assumption into the equations of motion, resulting in

$$
d \star F_{(4)} = 0 \implies \nabla^2 C + C'(C' + 6B' - 3A') = 0 \tag{2.34}
$$

where ∇^2 represents the Laplacian in 8-dim Euclidean space, and the condition $F_{(4)} \wedge F_{(4)} = 0$ is trivial due to the ansatz.

With $d = 3$ and $\tilde{d} = 6$, the Einstein equations take the form

$$
\begin{aligned}
A'' + 3A'^2 + 6A'B' + \frac{7}{r} A' &= \frac{1}{3} C'^2 e^{2C-6A} \\
3(A'' + A'^2 - A'B') + 7(B'' + \frac{1}{r} B') &= \frac{1}{3} C'^2 e^{2C-6A} \\
B'' + 3A'B' + 6B'^2 + \frac{13}{r} B' + \frac{3}{r} A' &= -\frac{1}{6} C'^2 e^{2C-6A}
\end{aligned}
\tag{2.35}
$$

and the Killing spinor equations are

$$\begin{aligned}
\delta\psi_\mu &= \left(\partial_\mu - \frac{1}{2}e^{-A-B}\partial_r e^A \gamma_\mu \otimes \sigma_2\sigma_1 \otimes \mathbb{1} - \frac{1}{6}e^{-3A-B}\partial_r e^C \gamma_\mu \otimes \sigma_2 \otimes \mathbb{1}\right)\epsilon \\
\delta\psi_r &= \left(\partial_r - \frac{1}{6}e^{-3A}\partial_r e^C \mathbb{1} \otimes \sigma_1 \otimes \mathbb{1}\right)\epsilon \\
\delta\psi_m &= \left(\tilde{\nabla}_m - \frac{1}{2}\mathbb{1} \otimes \sigma_1 \otimes \tilde{\Sigma}_m - \frac{1}{2}e^{-2B}\partial_r e^B \mathbb{1} \otimes \sigma_1 \otimes \Sigma_m + \frac{1}{12}e^{-3A-B}\partial_r e^C \mathbb{1} \otimes \mathbb{1} \otimes \Sigma_m\right)\epsilon
\end{aligned} \tag{2.36}$$

where gamma matrices are defined as

$$\Gamma_a = \gamma_a \otimes \sigma_1 \otimes \mathbb{1}, \qquad \Gamma_R = \mathbb{1} \otimes \sigma_2 \otimes \mathbb{1}, \qquad \Gamma_m = \mathbb{1} \otimes \sigma_3 \otimes \Sigma_m \tag{2.37}$$

and $(\tilde{\nabla}_m - \frac{1}{2}\mathbb{1} \otimes \sigma_1 \otimes \tilde{\Sigma}_m)\epsilon = 0$ for covariant derivative $\tilde{\nabla}_m$ and gamma matrices $\tilde{\Sigma}_m$ on S^7 [20].

These equations have solutions given by

$$3A = -6B = C, \qquad \epsilon = \epsilon_0 \otimes \eta_0 e^{-\frac{1}{6}C} \tag{2.38}$$

where ϵ_0 is a constant spinor in 3-dim spacetime, and η_0 is a constant spinor in the 8-dim transverse space. Notably, the spinor satisfies the chiral projection condition

$$(\sigma_1 \otimes \mathbb{1}) \cdot \eta = -\eta \tag{2.39}$$

which is equivalent to

$$\Gamma_{012}\epsilon = -\epsilon \tag{2.40}$$

since Γ_{012} squares to the identity and is traceless, it can be used to construct a projection operator that reduces the number of independent components of ϵ by half. Consequently, the number of preserved supersymmetries by this solution is 16.

By substituting these results into the equation (2.34), we obtain

$$\nabla^2 C - C'^2 = 0 \implies \nabla^2 e^{-C} = 0 \tag{2.41}$$

which is solved by harmonic functions

$$e^{-C(r)} \equiv H(r) = 1 + \frac{k}{r^6} \tag{2.42}$$

where k is a constant. Consequently, the metric becomes

$$ds^2 = \left(1 + \frac{k}{r^6}\right)^{-\frac{2}{3}} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{k}{r^6}\right)^{\frac{1}{3}} \left[dr^2 + r^2 ds^2(S^7)\right] \tag{2.43}$$

and the 3-form potential takes the form

$$A_{\mu\nu\rho} = \epsilon_{\mu\nu\rho} \left(1 + \frac{k}{r^6}\right)^{-1} \tag{2.44}$$

which concludes the M2-brane solution.

It is important to note that at $r = 0$, there exists a coordinate singularity, but not a curvature singularity, indicating the presence of a horizon. Just like in the Schwarzschild solution, it is possible to perform an analytical continuation into the horizon, as discussed in [19].

This solution represents the complete 11-dim spacetime geometry when planar M2-branes are introduced into 11-dim Minkowski spacetime. It is evident that the solution is asymptotically flat, which aligns with our expectations since at a far distance, the presence of M2-branes becomes negligible. However, as we examine the near-horizon region, i.e. as $r \to 0$, we observe that $e^{-C} \sim k/r^6$, and by performing a coordinate transformation $\rho = \sqrt{k}/(2r^2)$, the metric takes the form

$$ds^2 = \frac{1}{4} k^2 \left[\frac{1}{\rho^2}(\eta_{\mu\nu} dx^\mu dx^\nu + d\rho^2) + 4 ds^2(S^7)\right] \tag{2.45}$$

which corresponds to $AdS_4 \times S^7$ in Poincaré-AdS coordinates, and this observation holds significant relevance in the context of the AdS/CFT correspondence. Moreover, it is worth noting that $AdS_4 \times S^7$ itself is also a solution to 11-dim supergravity, preserving all 32 supersymmetries. Hence, we can think of the M2-brane solution as interpolating between two maximally supersymmetric solutions, from Minkowski spacetime to $AdS_4 \times S^7$.

M5-brane

Now, let us consider the magnetically charged M5-brane using the ansatz [19]

$$F_{mnpq} = -\epsilon_{mnpqr}\partial_r e^{C(r)} \tag{2.46}$$

and once again, we have $F \wedge F = 0$ and $dF = 0$, which implies

$$d \star F = 0 \implies \nabla^2 e^C = 0 \tag{2.47}$$

where ∇^2 represents the Laplacian in 5-dim Euclidean space. This equation is solved by

$$e^{C(r)} = 1 + \frac{k}{r^3} \tag{2.48}$$

where k is simply a constant.

With $d = 6$ and $\tilde{d} = 3$, the Einstein equations become

$$\begin{aligned} A'' + 6A'^2 + 3A'B' + \frac{4}{r}A' &= \frac{1}{6}C'^2 e^{2C-6B} \\ 6(A'' + A'^2 - A'B') + 4(B'' + \frac{1}{r}B') &= \frac{1}{6}C'^2 e^{2C-6B} \\ B'' + 6A'B' + 3B'^2 + \frac{7}{r}B' + \frac{6}{r}A' &= -\frac{1}{3}C'^2 e^{2C-6B} \end{aligned} \tag{2.49}$$

and the Killing spinor equations are

$$\begin{aligned} \delta\psi_\mu &= \left(\partial_\mu - \frac{1}{2}e^{-A}\partial_r e^A \gamma_\mu\gamma_7 \otimes \Sigma_r - \frac{1}{12}e^{-3B}\partial_r e^C \gamma_\mu \otimes \Sigma_r\right)\epsilon \\ \delta\psi_r &= \left(\partial_r + \frac{1}{12}e^{-3B}\partial_r e^C \gamma_7 \otimes \mathbb{1}\right)\epsilon \\ \delta\psi_m &= \left(\tilde{\nabla}_m - \frac{1}{2}\mathbb{1} \otimes \Sigma_r\tilde{\Sigma}_m + \frac{1}{2}e^{-B}\partial_r e^B \mathbb{1} \otimes \Sigma_m\Sigma_r - \frac{1}{6}e^{-3B}\partial_r e^C \gamma_7 \otimes \Sigma_m\Sigma_r\right)\epsilon \end{aligned} \tag{2.50}$$

with gamma matrices

$$\Gamma_a = \gamma_a \otimes \mathbb{1}, \qquad \Gamma_R = \gamma_7 \otimes \Sigma_R, \qquad \Gamma_m = \gamma_7 \otimes \Sigma_m \tag{2.51}$$

where we have defined $\gamma_7 = \gamma_{012345}$ and $\Sigma_R = \Sigma_{1234}$.

Once again, using the fact that $(\tilde{\nabla}_m - \frac{1}{2}\mathbb{1} \otimes \Sigma_R\tilde{\Sigma}_m)\epsilon = 0$ for covariant derivative $\tilde{\nabla}_m$

and gamma matrices $\tilde{\Sigma}_m$ on S^4 [20]. These are solved by

$$-6A = 3B = C, \qquad \epsilon = \epsilon_0 \otimes e^{-C/12}\eta_0 \tag{2.52}$$

with the projection $\gamma_7\epsilon_0 = \epsilon_0$, i.e. $\Gamma_{012345}\epsilon = \epsilon$ for constant spinors ϵ_0 (in 6-dim) and η_0 (in 5-dim). This implies that the solution preserves 16 supercharges.

Hence, the solution for the planar M5-brane is

$$ds^2 = \left(1 + \frac{k}{r^3}\right)^{-\frac{1}{3}} \eta_{\mu\nu}dx^\mu dx^\nu + \left(1 + \frac{k}{r^3}\right)^{\frac{2}{3}} \left[dr^2 + r^2 ds^2(S^4)\right] \tag{2.53}$$

with the magnetic flux

$$F_{mnpq} = 3k\epsilon_{mnpqr}\frac{1}{r^4} \tag{2.54}$$

and by taking the near-horizon limit with a coordinate transformation $\rho^2 = 4k/r$, the metric becomes

$$ds^2 = 4k^{\frac{2}{3}} \left[\frac{1}{\rho^2}(\eta_{\mu\nu}dx^\mu dx^\nu + d\rho^2) + \frac{1}{4}ds^2(S^4)\right] \tag{2.55}$$

which is precisely the metric for $AdS_7 \times S^4$ in Poincaré coordinates. And, similar to the previous case, $AdS_7 \times S^4$ itself is also a solution to 11-dim supergravity, preserving all 32 supercharges.

D3-brane

Let us recall that the truncated Type IIB action (2.16) is meant to describe the geometry associated with D3-branes. Notably, the D3-brane is dyonically coupled to the potential, meaning it is coupled both electrically and magnetically. This occurs because the flux is self-dual. Consequently, the ansatz for the self-dual flux can be expressed as

$$F_{(5)} = (1 + \star)\, G_{(5)} \tag{2.56}$$

where

$$F_{\mu\nu\rho\sigma r} = G_{\mu\nu\rho\sigma r} = \epsilon_{\mu\nu\rho\sigma}\partial_r e^{C(r)}, \qquad F_{mnijk} = (\star G)_{mnijk} = \epsilon_{mnijkr}e^{4(B-A)}\partial_r e^{C(r)} \tag{2.57}$$

which gives the Einstein equations

$$
\begin{aligned}
A'' + 4A'^2 + 4A'B' + \frac{5}{r}A' &= \frac{1}{4}C'^2 e^{2C-8A} \\
4(A'' + A'^2 - A'B') + 5(B'' + \frac{1}{r}B') &= \frac{1}{4}C'^2 e^{2C-8A} \\
B'' + 4A'B' + 4B'^2 + \frac{9}{r}B' + \frac{4}{r}A' &= -\frac{1}{4}C'^2 e^{2C-8A}
\end{aligned}
\tag{2.58}
$$

and the flux equation is given by

$$
dF_{(5)} = 0 \implies \nabla^2 C + C'(C' + 4B' - 4A') = 0 \tag{2.59}
$$

where ∇^2 represents the Laplacian on 6-dim Euclidean space. Lastly, the Killing spinor equations are

$$
\begin{aligned}
\delta\psi_\mu &= \left[\partial_\mu + \left(\frac{1}{2}e^{-A}\partial_r e^A - \frac{1}{8}e^{-4A}\partial_r e^C\right)\gamma_\mu \otimes \Sigma_r\right]\epsilon \\
\delta\psi_r &= \left(\partial_r + \frac{1}{8}e^{-4A}\partial_r e^C\right)\epsilon \\
\delta\psi_m &= \left(\tilde{\nabla}_m - \frac{1}{2}\mathbb{1} \otimes \tilde{\Sigma}_m\right) + \left(\frac{1}{2}e^{-B}\partial_r e^B + \frac{1}{8}e^{-4A}\partial_r e^C\right)\Sigma_m \Sigma^r \epsilon
\end{aligned}
\tag{2.60}
$$

with gamma matrices

$$
\Gamma_a = \gamma_a \otimes \mathbb{1}, \qquad \Gamma_R = -i\gamma_5 \otimes \Sigma_R, \qquad \Gamma_m = -i\gamma_5 \otimes \Sigma_m \tag{2.61}
$$

where $\gamma_5 = \gamma_{0123}$. They are solved by

$$
4A = -4B = C, \qquad \epsilon = \epsilon_0 \otimes \eta_0 e^{-C/8} \tag{2.62}
$$

with the chiral projections

$$
\gamma_5 \epsilon = i\epsilon_0, \qquad \Sigma_7 \eta_0 = -i\eta_0 \tag{2.63}
$$

or, equivalently, $\Gamma_{0123}\epsilon = \epsilon$. Therefore, each ϵ now has 8 independent components, and as a result, the solution preserves 16 supercharges once again.

As a consequence, the other equations of motion becomes

$$\nabla^2 e^{-C} = 0 \implies e^{-C(r)} = 1 + \frac{k}{r^4} \tag{2.64}$$

and hence the D3-brane solution is

$$ds^2 = \left(1 + \frac{k}{r^4}\right)^{-\frac{1}{2}} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{k}{r^4}\right)^{\frac{1}{2}} \left[dr^2 + r^2 ds^2(S^5)\right] \tag{2.65}$$

which interpolates between 10-dim Minkowski (far horizon) and $AdS_5 \times S^5$ (near horizon). This can seen by taking the limit $r \to 0$ followed by a coordinate transformation $\rho = k^{\frac{1}{2}}/r$

$$ds^2 = k^{\frac{1}{2}} \left[\frac{1}{\rho^2}(\eta_{\mu\nu} dx^\mu dx^\nu + d\rho^2) + ds^2(S^5)\right] \tag{2.66}$$

and once again, $AdS_5 \times S^5$ itself is a solution to Type IIB preserving 32 supersymmetry.

Apex Solutions

In fact, our construction for brane solutions can be extended by replacing the transverse space to other Ricci-flat Riemannian manifolds $\mathcal{M}$ with special holonomy, i.e. $SU(n)$, G_2, $Spin(7)$, or $Sp(n)$. In other words, we can use the following ansatz for the metric

$$ds^2 = e^{2A(r)} \eta_{\mu\nu} dx^\mu dx^\nu + e^{2B(r)} ds^2(\mathcal{M}) \tag{2.67}$$

with the transverse space $\mathcal{M}$ being

$$ds^2(\mathcal{M}) = dr^2 + r^2 ds^2(X) \tag{2.68}$$

where the manifold X depends on the specific nature of $\mathcal{M}$. For instance, if $\mathcal{M}$ is a Calabi-Yau manifold, then X must be Sasaki[6]-Einstein.

By maintaining the same flux ansatz, we can generate a wide range of solutions describing planar branes with transverse spaces represented by special holonomy manifolds. For example, the D3-brane located at the apex of a Calabi-Yau cone (CY_3) can be described

[6]A compact Riemannian manifold X is Sasakian if and only if $dr^2 + r^2 ds^2(X)$ is Kähler [21].

by the following metric

$$ds^2 = \left(1 + \frac{k}{r^4}\right)^{-\frac{1}{2}} \eta_{\mu\nu} dx^\mu dx^\nu + \left(1 + \frac{k}{r^4}\right)^{\frac{1}{2}} \left[dr^2 + r^2 ds^2(SE_5)\right] \tag{2.69}$$

where SE_5 represents a Sasaki-Einstein 5-fold. This particular solution interpolates from $\mathbb{R}^{1,3} \times CY_3$ (far horizon) to $AdS_5 \times SE_5$ (near horizon). In analogy to previous cases, this solution preserves half of the supersymmetry of $\mathbb{R}^{1,3} \times CY_3$. For further in-depth discussions on this topic, we can refer to [22].

Further Discussions

In summary, we have successfully derived planar brane solutions in 11-dim and Type IIB supergravity. Initially, we explored scenarios involving planar M2-branes, M5-branes, and D3-branes in 10/11-dim Minkowski spacetime. These solutions smoothly transition from Minkowski spacetime (far horizon) to $AdS_{p+2} \times S^{(D-p-2)}$ (near horizon) geometries. Furthermore, we extended our analysis to more general solutions that interpolate between $\mathbb{R}^{1,p} \times \mathcal{M}_{D-p-1}$ and $AdS_{p+2} \times X_{D-p-2}$ spacetimes. In both cases, we found that these solutions preserve a portion of supersymmetry. Consequently, our findings provide a rich variety of AdS solutions with preserved supersymmetry that can be used for investigating the AdS/CFT correspondence.

There are a few additional aspects to consider. For instance, we haven not explicitly specified the constant k, but it is directly related to the total number of branes stacked at $r = 0$. This is evident when we calculate the conserved electric/magnetic charges, which are essentially the number of branes

$$Q_{\text{el}} \propto \int_{S^{D-p-2}} \star F_{(p+2)} = \text{Vol}(S^{D-p-2}) k\tilde{d}, \qquad Q_{\text{mag}} \propto \int_{S^{D-p-2}} F_{(p+2)} = \text{Vol}(S^{D-p-2}) k\tilde{d} \tag{2.70}$$

where $\tilde{d}$ is previously defined, and $\text{Vol}(S^{n-1}) = n\pi/\Gamma(1 + \frac{n}{2})$. Therefore, if there are N branes stacked at $r = 0$, then $k \propto Q = N$.

Moreover, it is worth noting that we can accommodate multiple branes at different spatial locations using a multi-centric harmonic function $H(r) = 1 + k\sum_i |r - r_i|^{-\tilde{d}}$, which also satisfies the Laplace equation. In fact, all properties of brane solutions are similar to those of extremal Reissner-Nordström black holes, i.e. $AdS_2 \times S^2$ and multi-centric solutions. Actually, these brane solutions are also known as extremal black p-branes, see [19] for details. In addition, these branes are BPS states, which means they saturate BPS bounds. As a result, the ADM tension/mass of these branes is equivalent to their electric/magnetic charges.

Chapter 3

The AdS/CFT Correspondence

The AdS/CFT correspondence, proposed by Maldacena [4], establishes a duality between conformal field theories (CFT) in d-dim and a gravitational theory in $(d+1)$-dim Anti-de Sitter (AdS) space. The most well-known example of this correspondence is the duality between Type IIB superstring theory on $AdS_5 \times S^5$ and $\mathcal{N}=4$ $SU(N)$ super Yang-Mills (SYM) theory. Before we delve into this specific example, let us clarify what AdS and CFT are, following mainly [23].

3.1 Anti-de Sitter Spacetime

Anti-de Sitter (AdS) spacetime is the maximally symmetric solution to Einstein's equations with a negative cosmological constant. It can be understood as a hypersurface embedded in $\mathbb{R}^{2,d}$ given by

$$U^2 + V^2 - X^i X^i = L^2 \tag{3.1}$$

where $i = 1, \ldots, d$ and the metric is

$$ds^2(\mathbb{R}^{2,d}) = -dU^2 - dV^2 + dX^i dX^i \tag{3.2}$$

and if we change the coordinates by

$$\begin{aligned} U &= \frac{z}{\tau}\left[1+\frac{1}{z^2}\left(L^2+x^a x^a-t^2\right)\right] \\ V &= \frac{L}{z}t \\ X^a &= \frac{L}{z}x^a \\ X^d &= \frac{z}{\tau}\left[1-\frac{1}{z^2}\left(L^2+x^a x^a-t^2\right)\right] \end{aligned} \tag{3.3}$$

where $a = 1, \ldots, d-1$, then the induced metric will become

$$ds^2(AdS_{d+1}) = \frac{L^2}{z^2}\left(-dt^2 + dx^a dx^a + dz^2\right) \equiv \frac{L^2}{z^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right) \tag{3.4}$$

which is called Poincaré patch for AdS, and L is a constant called AdS radius. This is exactly the form we encountered in brane solutions. Note that, this patch covers only a portion of the whole AdS spacetime.

One crucial aspect of AdS spacetime is its boundary, defined at $z = 0$, which is just the conformal compactification of d-dim Minkowski. An intriguing property of AdS spacetime is that the spatial distance from its interior ($z \neq 0$) to the boundary ($z = 0$) is infinite. However, a null curve can reach the boundary in a finite amount of time, allowing it to reflect back in a finite time as well.

Therefore, in the AdS spacetime, it is crucial to specify boundary conditions to properly study the dynamics within the interior. In the context of the AdS/CFT correspondence, different boundary conditions in AdS_{d+1} correspond to introducing sources for the operators in d-dim CFT. This is why it is often stated that the field theory lives at the boundary of AdS [24]. Furthermore, at the boundary ($z = 0$), one can observe a scaling symmetry ($x^\mu \to \lambda x^\mu$ and $z \to \lambda z$), which further suggests that the field theory is a CFT.

3.2 Conformal Field Theory

Conformal transformations are coordinate transformations $x^\mu \to \tilde{x}^\mu$ such that the metric

$$g_{\mu\nu}(x) \to \tilde{g}_{\mu\nu}(x) = \Omega(x) g_{\mu\nu}(x) \tag{3.5}$$

where $\Omega(x)$ is a local positive factor. The interpretation of conformal transformations can vary depending on whether we are dealing with a fixed background metric or a dynamic one.

In the case of a dynamic metric, a conformal transformation corresponds to a diffeomorphism, i.e. a gauge symmetry. However, when dealing with a fixed background metric, the conformal transformation should be viewed as a genuine physical symmetry. In this case, it gives rise to global symmetries with associated conserved currents [1]. This is the interpretation we typically adopt when discussing QFT.

A conformal field theory (CFT) is a QFT (in Minkowski) with conformal symmetry, i.e. the theory is invariant under $\eta_{\mu\nu} \to \Omega(x)\eta_{\mu\nu}$. Consider an infinitesimal transformation

$$x^\mu \to \tilde{x}^\mu = x^\mu + \epsilon^\mu(x) \implies \Omega(x) = 1 - f(x) \tag{3.6}$$

where $\epsilon^\mu(x)$ and $f(x)$ are infinitesimal. This implies

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = f(x)\eta_{\mu\nu} \tag{3.7}$$

known as the conformal Killing equation. After some algebra (e.g. taking the trace and the derivative), we can show that

$$2\partial_\mu \partial_\nu \epsilon_\rho = \eta_{\nu\rho}\partial_\mu f + \eta_{\mu\rho}\partial_\nu f - \eta_{\mu\nu}\partial_\rho f \implies 2\partial^2 \epsilon_\rho = (2-d)\partial_\rho f \tag{3.8}$$

indicating that $d = 2$ CFT is special, which possesses an infinite number of generators.

For $d \neq 2$, we can show that after taking the derivative and some algebra

$$\partial_\mu \partial_\nu f(x) = 0 \tag{3.9}$$

which is solved by $f(x) = A + B_\mu x^\mu$. This implies

$$\epsilon_\mu(x) = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\rho}x^\nu x^\rho \tag{3.10}$$

where a_μ, $b_{\mu\nu}$, and $c_{\mu\nu\rho}$ are infinitesimal constants. Each of these terms represents a different type of conformal transformation:

- a_μ corresponds to spacetime translations.
- $b_{[\mu\nu]}$ corresponds to Lorentz transformations.
- $b_{(\mu\nu)} = \lambda\eta_{\mu\nu}$ are scaling transformations.
- $c_{\mu\nu\rho} = \eta_{\mu\rho}b_\nu + \eta_{\mu\nu}b_\rho - \eta_{\nu\rho}b_\mu$ are special conformal transformations.

As a result, we can show that they form the conformal algebra, which is Poincaré algebra with non-trivial commutators

$$\begin{gathered} [D, P_\mu] = iP_\mu, \quad [D, K_\mu] = -iK_\mu \\ [M_{\mu\nu}, K_\rho] = i(\eta_{\nu\rho}K_\mu - \eta_{\mu\rho}K_\nu), \quad [P_\mu, K_\nu] = -2i(\eta_{\mu\nu}D + M_{\mu\nu}) \end{gathered} \tag{3.11}$$

with P_μ and $M_{\mu\nu}$ as Poincaré generators, D as the scaling generator, and K_μ as the special conformal generator.

It is also worth mentioning that $D, M_{\mu\nu}, P_\mu$ also form a closed algebra, and one might speculate that there are theories that are not conformal but only scale and Poincaré invariant. However, these types of theories appear to be non-unitary and therefore not of physical interest [25]. In our cases, we will assume that scale invariance is sufficient to imply conformal invariance.

We can define the generators as follows

$$\begin{gathered} J_{\mu\nu} = M_{\mu\nu}, \qquad J_{\mu,d} = \frac{1}{2}(P_\mu - K_\mu) \\ J_{\mu,d+1} = \frac{1}{2}(P_\mu + K_\mu), \qquad J_{d,d+1} = D \end{gathered} \tag{3.12}$$

where the comma between indices is used to emphasize two separate indices without any special meaning. Along with the constraint $J_{ab} = -J_{ba}$ for $a = 0, 1, ..., d-1, d, d+1$, we can show that J_{ab} generates the $SO(d, 2)$ Lie algebra, which is also the isometry group of AdS_{d+1}, as easily seen from (3.1). This provides another hint about the AdS/CFT correspondence.

Recall that a CFT is scale-invariant, which means the theory looks the same when we zoom in or out in terms of length or energy. A simple example of a CFT is massless free scalar field theory since it looks the same at all scales. However, not all scale-invariant theories are CFTs. For example, consider a massless ϕ^4 theory in 4-dim, which is scaling invariant. However, there are some subtleties when considering quantum effects. In fact, the beta function of the coupling is non-zero, indicating that it is not scale-invariant at the quantum level.

3.3 The Field-Operator Correspondence

Suppose we have a massive scalar field in AdS, which satisfies

$$\nabla^2\phi(x,z) = m^2\phi^2(x,z) \tag{3.13}$$

where ∇^2 is the Laplacian in AdS with Poincaré patch. We can take a Fourier expansion

$$\phi(x,z) = \int d^4k e^{ik_\mu x^\mu} f(k,z) \tag{3.14}$$

and hence each mode satisfies

$$f'' - \frac{d-1}{z}f' - \left(k^2 + \frac{m^2L^2}{z^2}\right) f = 0 \tag{3.15}$$

where $k = \sqrt{k_\mu k^\mu}$ and $f' = df/dz$. Using $f(k,z) = z^{d/2}h(kz)$, we get

$$\mu^2 h'' + \mu h' - (\mu^2 + \nu^2)h = 0 \tag{3.16}$$

where $\nu^2 = m^2L^2 + d^2/4$ and now $h' = dh/d\mu$. This is just the Bessel equation, whose general solutions are

$$h(\mu) = a(k)K_\nu(\mu) + b(k)I_\nu(\mu) \tag{3.17}$$

where $a(k)$ and $b(k)$ are constants to be determined by boundary conditions, and

$$I_\nu(\mu) = \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+\nu+1)}\left(\frac{\mu}{2}\right)^{2n+\nu}, \qquad K_\nu(\mu) = \frac{\pi}{2}\frac{I_{-\nu}(\mu) - I_\nu(\mu)}{\sin(\pi\nu)} \tag{3.18}$$

We can analyse how $\phi(x,z)$ behave near the conformal boundary $z=0$. In this limit ($z\to 0$), we have $I_\nu \sim \mu^\nu$ and hence $K_\nu \sim \mu^{-\nu}+\mu^\nu$, which results

$$f(k,z) = a(k)z^{d-\Delta}(1+\ldots) + c(k)z^{\Delta}(1+\ldots) \tag{3.19}$$

where $\Delta = \frac{d}{2}+\nu$ and $c(k)$ can be determined from $a(k)$ and $b(k)$. Note that $\ldots$ represents an ascending power series of z^2, which can be seen from $I_\nu(\mu)$. We only need $\Delta \in \mathbb{R}$ which implies

$$m^2 \geq -\left(\frac{d}{2L}\right)^2 \tag{3.20}$$

known as the Breitenlohner-Freedman (BF) bound. This allows the mass to be imaginary (i.e. tachyonic states). Now, we want some boundary conditions to determine two arbitrary constants, which are data at past infinity (i.e. initial condition) and at the conformal boundary (recall that null rays can relate dynamics between the boundary and the interior).

To proceed, we can perform an inverse Fourier transform such that

$$\phi(x,z) = \phi_0(x)z^{d-\Delta}(1+\ldots) + \phi_d(x)z^{\Delta}(1+\ldots) \tag{3.21}$$

where again $\ldots$ represents an ascending power series of z^2, and $\phi_0(x) \sim \int d^4k e^{ikx}a(k)$. So, we can now try fixing a boundary condition. Let us start with the conformal boundary one, i.e. $z=0$. For simplicity $\Delta > d/2$, the leading term is given by $\phi_0(x)$ as $d-\Delta<\Delta$, hence the boundary condition at $z=0$ is fixed by $\phi_0(x)$. This is further justified when $\Delta = d$, and ϕ_0 is exactly the value of $\phi(x,z)$ at $z=0$. Note that, for $\Delta > d$, the leading term is divergent.

To establish its initial condition, we consider the behavior of the fields at past infinity (i.e. $z\to\infty$ and $t\to-\infty$). This implies

$$\phi(x,z) \sim \frac{az^{\frac{d}{2}}}{\sqrt{kz}}e^{ik_\mu x^\mu - kz}(1+\ldots) + \frac{bz^{\frac{d}{2}}}{\sqrt{kz}}e^{ik_\mu x^\mu + kz}(1+\ldots) \tag{3.22}$$

where $k=\sqrt{k_\mu k^\mu}$. For spacelike k (i.e. k is real), we want the field to be regular, and hence we set $b=0$. For timelike k (i.e. k is purely imaginary), we want to set the initial condition such that there are no waves propagating from the past infinity, hence $b=0$. In other words, the initial conditions imply that there are no fields originating from the past infinity, but the fields can propagate into the future due to the conformal boundary.

Therefore, from above conditions, we have

$$f(k,z) = az^{\frac{d}{2}} K_\nu(kz) = a_0 z^{d-\Delta} + ... + a_1 z^{\Delta} + ... \tag{3.23}$$

where one can show that

$$a_1 \propto k^{2\Delta - d} a_0 \tag{3.24}$$

which implies that

$$\phi_d(x) \propto \int d^d y \frac{1}{|x-y|^{2\Delta}} \phi_0(y) \tag{3.25}$$

and hence we can see that $\phi_0(x)$ fixes the conformal boundary and $\phi_d(x)$ is the responce to that condition.

Now, consider the scalar action, which can be expressed after by parts as

$$S = \frac{1}{2} \int_{\mathcal{M}} d^d z dz \sqrt{-g} \phi (\nabla^2 \phi - m^2 \phi^2) - \frac{1}{2} \int_{\partial \mathcal{M}} dS^A \phi \partial_A \phi \tag{3.26}$$

where $A = 0, 1, ..., d-1, z$. If we take the scalar field to be on-shell, then only the boundary term remains

$$S = -\frac{1}{2} \int_{\partial \mathcal{M}} dS^A \phi \partial_A \phi \tag{3.27}$$

and consider the integral evaluated at a fixed z boundary

$$\begin{aligned} S(z) &= \frac{1}{2} \int d^d x \left(\frac{L}{z}\right)^{d-1} \phi \partial_z \phi \\ &= \frac{1}{2} \int d^d x L^{d-1} \left[(d-\Delta) \phi_0^2(x) z^{d-2\Delta} + \ldots + d\phi_0(x) \phi_d(x) + \ldots \right] \end{aligned} \tag{3.28}$$

Therefore, the on-shell action is

$$S = \lim_{z \to 0} S(z) \tag{3.29}$$

whereas the $z \to \infty$ boundary gives no contributions. However, for our assumption $\Delta > d/2$, the on-shell action diverges, but can be regulated by introducing counter terms.

Suppose the scalar action is

$$S = \frac{1}{2}\int_{\mathcal{M}} d^d z dz \sqrt{-g}\phi(\nabla^2\phi - m^2\phi^2) - \frac{1}{2}\int_{\partial\mathcal{M}} dS^A \phi\partial_A\phi + \int_{\partial\mathcal{M}} dS(C\phi^2 + \ldots) \tag{3.30}$$

where C is a constant. Then, the on-shell action is

$$S = -\frac{1}{2}\int_{\partial\mathcal{M}} dS^A \phi\partial_A\phi + \int_{\partial\mathcal{M}} dS(C\phi^2 + \ldots) \tag{3.31}$$

and we can show that for a fixed z boundary

$$\int_{\partial\mathcal{M}} dS(C\phi^2 + \ldots) = C\int d^d x L^{d-1}\left[(d-\Delta)\phi_0^2(x) z^{d-2\Delta} + \ldots + d\phi_0(x)\phi_d(x) + \ldots\right] \tag{3.32}$$

which implies that $C = -\frac{d-\Delta}{2}$. Therefore, the renormalised on-shell action is

$$S = \frac{2\Delta - d}{2}\int d^d x L^{d-1}\phi_0(x)\phi_d(x) \tag{3.33}$$

which implies that

$$S = c\int d^d x d^d y \frac{1}{|x-y|^{2\Delta}}\phi_0(x)\phi_0(y) \tag{3.34}$$

for a constant c depending on d and L.

Now, we can state a more precised version of the AdS/CFT correspondence, which is

$$Z_{AdS}[\phi(x, z=0) = \phi_0(x)] = Z_{CFT}[\phi_0(x)] \tag{3.35}$$

where $Z_{CFT}[\phi_0]$ is the typical generating function for a QFT, in this case it sources some scalar operators $\hat{O}(x)$. For $Z_{AdS}[\phi_0]$, there are more subtleties, even if we can write down some gravitational path integrals. However, we can take the saddle-point approximation when the AdS radius is very large, such that

$$Z_{AdS}[\phi_0] \sim e^{ic\int d^d x d^d y \frac{1}{|x-y|^{2\Delta}}\phi_0(x)\phi_0(y)} \tag{3.36}$$

and hence we can show from the both sides

$$\langle \hat{O}(x)\hat{O}(y)\rangle = \frac{1}{|x-y|^{2\Delta}} \tag{3.37}$$

In other words, the field in AdS acts like a source for the field theory living on the conformal boundary. Moreover, this works for other fields, e.g. the bulk metric in AdS is dual to the stress-energy tensor in CFT, and we could also discuss how the mass of bulk fields is mapped to the scaling dimension of CFT operators.

3.4 D3-brane : Worldvolume Theory

Now, we shall discuss the duality between Type IIB theory in $AdS_5 \times S^5$ and $\mathcal{N} = 4$ $SU(N)$ SYM in 4-dim. First of all, consider a general 3-dim extended object (i.e. 3-brane) in 10-dim Minkowski, whose worldvolume action is Nambu-Goto

$$S = -T_3 \int d^4\sigma \sqrt{-\det h_{ab}} \tag{3.38}$$

where $h_{ab} \equiv \eta_{\mu\nu}\partial_a x^\mu \partial_b x^\nu$ is the induced metric. However, D3-branes are attached by open strings, which can give some excitations on its worldvolume, and hence its effective (not including SUSY, for simplicity) action becomes the Born-Infeld action [26]

$$S = -T_3 \int d^4\sigma \sqrt{-\det\left(h_{ab} + 2\pi\alpha' F_{ab}\right)} \tag{3.39}$$

where F_{ab} is a $U(1)$ field strength, and recall α' characterises the string length scale. We can use the diffeomorphism to fix a static gauge, i.e. $\sigma^a = x^a$ such that $h_{ab} = \eta_{ab} + \partial_a x^i \partial_b x^i$ where x^i are the remaining 6 transverse spatial coordinates. We can now define $\phi^i = 2\pi\alpha' x^i$ and expand the action in small α', which corresponds to low energy in string theory

$$S = -(2\pi\alpha')^2 T_3 \int d^4\sigma \left(\frac{1}{4}F_{ab}F^{ab} + \frac{1}{2}\partial_a\phi^i\partial^a\phi^i + ...\right) \tag{3.40}$$

where we have used the identity $\sqrt{\det M} = e^{\frac{1}{4}\operatorname{Tr}\log\left(MM^T\right)}$ for a given matrix M. So, the bosonic fields on the worldvolume theory are 1 massless vector and 6 massless scalars.

In our earlier discussion, we established that the planar D3-brane solution to supergravity preserves 16 supercharges, corresponding to $\mathcal{N} = 4$ supersymmetry in 4-dim. So, we should introduce a SUSY worldvolume theory. While a rigorous approach involves adding a Wess-Zumino term into the original action and carefully studying it, we can make a reasonable inference based on the fact that the lowest irrep of $\mathcal{N} = 4$ in 4-dim consists of 1 vector, 6 scalars, and 4 Weyl spinors. Therefore, in order to preserve SUSY, we expect to add 4 Weyl spinors to the worldvolume theory.

Furthermore, as previously discussed, when we have N D3-branes stacked together, the endpoints of an open string can attach to different D3-branes. This promotes the $U(1)$ gauge fields to $SU(N)$ gauge fields. Therefore, we conclude that there is an $\mathcal{N} = 4$ $SU(N)$ SYM theory on the worldvolume of the D3-branes. This is a significant finding in the context of the AdS/CFT correspondence, as it establishes a connection between the supergravity solution in the bulk and a gauge theory living on the brane.

Given that $T_3 = (2\pi g_s)^{-1}(2\pi\alpha')^{-2}$ [27], where g_s characterises the string coupling strength, the worldvolume action becomes

$$S = -\frac{1}{2\pi g_s}\int \mathrm{Tr}\left(F_{(2)} \wedge \star F_{(2)} + \dots\right) \tag{3.41}$$

which allows us to identify $g_{YM}^2 = 2\pi g_s$. Moreover, for a $SU(N)$ gauge theory, the 1-loop beta function is given by (when all fields are in the adjoint representation)

$$\beta(g_{YM}) \propto \frac{11}{3} - \frac{2}{3}N_W - \frac{1}{6}N_s = 0 \tag{3.42}$$

where N_W is the number of Weyl spinors and N_s is the number of scalars. This indicates that the theory is scale invariant, i.e. CFT.

3.5 D3-brane : Two Perspectives

The study of D3-branes provides us with two equivalent perspectives, highlighting the duality inherent in the AdS/CFT correspondence [28]. In the first perspective, D3-branes are viewed as the endpoints of open strings, and this perspective is valid only when $g_s \ll 1$

since we are treating strings perturbatively. In this context, the complete action takes the form

$$S = S_{closed} + S_{open} + S_{int} \tag{3.43}$$

where $S_{closed/open}$ represents the effective action describing closed/open strings (including massive ones), which can, in principle, be derived from string scattering amplitudes. The term S_{int} describes the interaction between open and closed strings. S_{open} and S_{int} can be derived from the Born-Infeld action (with a general metric field), along with its Wess-Zumino terms.

However, we are interested in the low-energy limit $\alpha' \to 0$, which leads to

$$S_{closed} \to S_{IIB}, \quad S_{open} \to S_{SYM}, \quad S_{int} \to 0 \tag{3.44}$$

which implies that we have two decoupled theories (i.e. Type IIB supergravity and $\mathcal{N} = 4$ $SU(N)$ SYM theory) Moreover, the gravitational coupling κ is proportional to $\alpha'^2 g_s$, which also approaches zero in this limit. Therefore, S_{IIB} effectively describes a free gravity theory, i.e. Type IIB in 10-dim Minkowski.

In another perspective, D3-branes are regarded as non-perturbative solutions to Type IIB supergravity. This viewpoint is valid only when $Ng_s \gg 1$, since we require the length scale of curvature (e.g. the AdS radius L) to be much larger than the string length scale, i.e. α'. This ensures that supergravity is a valid approximation of superstring theory, as $L^4/\alpha'^2 \propto Ng_s$. In this scenario, we also have two decoupled theories, in which one is Type IIB supergravity in 10-dim Minkowski (far horizon), and the other is Type IIB supergravity in $AdS_5 \times S^5$ (near horizon).

In both perspectives, we have two decoupled theories, with one being Type IIB supergravity in 10-dim Minkowski. This leads to the conjecture that Type IIB supergravity in $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ $SU(N)$ SYM theory. Importantly, we have taken the following limits to establish this correspondence

$$g_s \ll 1, \qquad \alpha' \to 0, \qquad Ng_s \gg 1 \tag{3.45}$$

which also corresponds to the ’t Hooft limit, where $N \to \infty$ while keeping the ’t Hooft coupling $\lambda = N g_{YM}^2 \sim N g_s$ fixed. This is worth mentioning because $SU(N)$ gauge theories can be formulated in terms of the ’t Hooft coupling as follows

$$\mathcal{L} = -\frac{1}{g_{YM}^2} \operatorname{Tr} F^2 = -\frac{N}{\lambda} \operatorname{Tr} F^2 \tag{3.46}$$

and one can perform an expansion of the amplitudes in powers of N in the ’t Hooft limit, which turns out to be related to string theory given by $g_s \sim 1/N$. This further justifies the AdS/CFT correspondence, as discussed in [**ooguri**, 29].

To assess the duality between the two theories, it is essential to align the fields in AdS with operators in the CFT. Nevertheless, for simplicity, let us undertake a simple consistency examination by matching the symmetries on both sides. On the AdS side, we have an isometry group $SO(2,4) \times SO(6)$ arising from $AdS_5 \times S^5$, and this background also preserves all 32 supercharges in Type IIB. On the CFT side, we are dealing with $\mathcal{N} = 4$ $SU(N)$ SYM theory in 4-dim. This SCFT has an R-symmetry group $SU(4) \cong SO(6)$, in addition to the conformal group $SO(2,4)$. However, $\mathcal{N} = 4$ only gives 16 supercharges. In fact, we also require an extra 16 fermionic generators to close out the superconformal algebra [24]. This totals 32 supercharges on both sides. Such consistency strongly suggests that the AdS/CFT correspondence indeed establishes a valid duality.

Chapter 4

Wrapped Brane Solutions

In the previous discussion, we introduced the AdS/CFT correspondence and identified supergravity solutions that include an AdS factor. When studying CFT, we can deform it by introducing certain operators. These operators can trigger a renormalization group (RG) flow, leading to a different CFT in the IR. Consequently, we anticipate observing a similar behavior in the corresponding gravity description.

Indeed, wrapped brane solutions provide a valuable perspective on the renormalization group (RG) flow from planar brane solutions. As an example, M2-branes can be wrapped on a 2-sphere S^2, transforming their worldvolume into $\mathbb{R} \times S^2$. This configuration yields a supergravity solution with a near-horizon geometry of $AdS_2 \times \mathcal{M}_9$. When we zoom in on the worldvolume, according to the definition of a Riemannian manifold, we eventually reach a Euclidean space. Consequently, the worldvolume effectively returns to 3-dim Minkowski space. Therefore, we classify wrapped brane solutions as representing the IR regime, while planar brane solutions are associated with the UV regime. This highlights how different brane configurations can correspond to different phases of the dual field theory, and it plays a crucial role in understanding the AdS/CFT correspondence in various contexts.

When considering brane solutions that preserve SUSY, we must carefully choose the surfaces that can be wrapped on. Our discussion will mainly follow [15, 17]. To begin, we will consider a probe-brane scenario, where the brane is effectively massless and neutral, having no significant influence on the background. This implies that our backgrounds must be Ricci-flat manifolds with covariantly constant spinors, such as the solutions discussed in section 2.3.

Let us examine the worldvolume of M2-branes for simplicity. The worldvolume action is given by

$$S = -T_2 \int d^3\sigma \sqrt{-\det h_{ab}} + \frac{1}{3!}\epsilon^{abc}\partial_a x^\mu \partial_b x^\nu \partial_c x^\rho A_{\mu\nu\rho} \tag{4.1}$$

where h_{ab} represents the induced metric on the brane worldvolume, and we have set the fermionic fields to zero since we are primarily interested in the bosonic configurations that preserve SUSY, similar to our approach in studying supergravity solutions.

To proceed further, let us consider a background metric of the form

$$ds^2 = -dt^2 + g_{ij}dx^i dx^j \tag{4.2}$$

and we choose a gauge where $\sigma^0 = t$. In addition, since we are dealing with a probe-brane scenario (with negligible mass and charges), we set $A_{(3)} = 0$. Consequently, the action is determined by the energy functional

$$E = -T_2 \int d^2\sigma \sqrt{\det m_{pq}}, \quad m_{pq} = g_{ij}\partial_p x^i \partial_q x^j \tag{4.3}$$

where p and q represent the spatial worldvolume coordinates.

Indeed, in the probe-brane scenario, M2-branes minimise their spatial area on-shell. This aligns with our earlier discussion regarding planar M2-branes, which is a minimal surface in Minkowski. This principle holds true for various branes within this probe-brane framework.

Recall that we set the fermionic fields to zero, which means we need to impose constraints to preserve SUSY. By considering the explicit SUSY transformations, we can find the constraint is [17]

$$(1 - \Gamma)\epsilon = 0 \tag{4.4}$$

where ϵ represents the Killing spinor of the background, and

$$\Gamma = \frac{1}{\sqrt{\det m}}\Gamma^0\gamma, \qquad \gamma = \frac{1}{2}\epsilon^{pq}\partial_p x^i \partial_q x^j \Gamma_{ij} \tag{4.5}$$

where $\{\Gamma_i, \Gamma_j\} = 2g_{ij}$. We can also show that Γ is Hermitian and $\Gamma^2 = \mathbb{1}$, which implies

$$\epsilon^\dagger \frac{1-\Gamma}{2}\epsilon = \epsilon^\dagger \frac{1-\Gamma}{2}\frac{1-\Gamma}{2}\epsilon = \left|\frac{1-\Gamma}{2}\epsilon\right|^2 \geq 0 \implies \epsilon^\dagger\epsilon \geq \epsilon^\dagger\Gamma\epsilon \tag{4.6}$$

since we can always normalise $\epsilon^\dagger\epsilon = 1$, hence

$$\epsilon^\dagger\Gamma\epsilon \leq 1 \implies -\bar{\epsilon}\gamma\epsilon \leq \sqrt{\det m} \tag{4.7}$$

So, if we define a 2-form

$$\varphi = -\frac{1}{2}\bar{\epsilon}\Gamma_{ij}\epsilon dx^i \wedge dx^j \tag{4.8}$$

then we get a 2-form that satisfies (in terms of the components) $\sqrt{\det m} \geq \varphi$. Moreover, it can be shown that the form is closed [15].

In fact, this is known as a calibration, which is a p-form ϕ on a Riemannian manifold $\mathcal{M}$ that satisfies

$$d\phi = 0, \qquad \int_\Sigma \phi \leq \mathrm{Vol}(\Sigma) \tag{4.9}$$

for all p-cycles $\Sigma \subseteq \mathcal{M}$. Consequently, we say that a p-cycle Σ is a calibrated cycle if $\varphi|_\Sigma = \mathrm{Vol}|_\Sigma$. We can now show that if Σ is a calibrated cycle, then it has the minimal surface in its homology class. Consider another cycle $\tilde{\Sigma}$ such that $\tilde{\Sigma} - \Sigma$ is a boundary, then

$$\begin{aligned}
\mathrm{Vol}(\Sigma) &= \int_\Sigma \phi \\
&= \int_{\partial R} \varphi + \int_{\tilde{\Sigma}} \phi \\
&= \int_R d\phi + \int_{\tilde{\Sigma}} \phi \\
&= \int_{\tilde{\Sigma}} \phi \\
&\leq \mathrm{Vol}(\tilde{\Sigma})
\end{aligned} \tag{4.10}$$

which proves that $Vol(\Sigma)$ is the minimum area.

Consider backgrounds in section 2.3, in which a calibration can be constructed using bispinors. This is because covariantly constant spinors contain a lot of information about the background geometry. For our interests, we will consider Calabi-Yau manifolds, in which there are two kinds of calibrations. One is constructed from the Kähler 2-form J and the other from the real part of the holomorphic n-form Ω, which are given by

$$J_{mn} = i\rho^\dagger \gamma_{mn}\rho, \qquad \Omega_{m_1\ldots m_{2n}} = \rho^T \gamma_{m_1\ldots m_{2n}}\rho \tag{4.11}$$

where ρ and its conjugation are both covariantly constant spinors. Therefore, we can construct calibrated cycles, i.e. Kähler $2n$-cycles from J^n and special Lagrangian (SLAG) n-cycles from $\mathrm{Re}(\Omega)$.

Back to our M2-brane example, where we have constructed a 2-form calibration φ. So, the M2-brane can only wrap on 2-cycles. If we take the background as $\mathbb{R} \times CY_5$, which preserves 2 supercharges, i.e. the covariantly constant spinor ϵ is subject to chiral projections (in an appropriate vielbein basis)

$$\Gamma^{1234}\epsilon = \Gamma^{3456}\epsilon = \Gamma^{5678}\epsilon = -\Gamma^{78910}\epsilon = -\epsilon \tag{4.12}$$

which also implies that $\Gamma^{012}\epsilon = \epsilon$. Using these spinors, one can show that the Kähler form is

$$J = e^{12} + e^{34} + e^{56} + e^{78} - e^{910} \tag{4.13}$$

where e^a are vielbeins on CY_5. Thus, we can define a Kähler 2-cycle Σ, and consider the M2-brane being wrapped on Σ, i.e. its worldvolume is $\mathbb{R} \times \Sigma$. For example, we can take Σ such that $\mathrm{Vol}(\Sigma) = e^{12}|_\Sigma$, and the constraint (4.4) implies that $\Gamma^{012}\epsilon = \epsilon$. Thus, in this case, there are no further restrictions, and an M2-brane can be wrapped on a Kähler 2-cycle in $\mathbb{R} \times CY_5$ without breaking any SUSY.

If we consider the background to be $\mathbb{R}^{1,2} \times CY_4$ (preserves 4 supercharges), then an M2-brane satisfies

$$\Gamma^{1234}\epsilon = \Gamma^{3456}\epsilon = \Gamma^{5678}\epsilon = \mp\Gamma^{78910}\epsilon = -\epsilon \tag{4.14}$$

which is equivalent to $\Gamma^{012}\epsilon = \pm\epsilon$. One can show that two supercharges satisfy $\Gamma^{012}\epsilon = \epsilon$ and the other two satisfy $\Gamma^{012}\epsilon = -\epsilon$. So, after substitution, we arrive at two Kähler 2-forms

$$J = e^{12} + e^{34} + e^{56} + e^{78} \mp e^{910} \tag{4.15}$$

and hence we can choose to wrap the cycle with $\mathrm{Vol}(\Sigma) = e^{12}|_\Sigma$, which implies $\Gamma^{012}\epsilon = \epsilon$ by computing (4.4). Thus, only two out of four supersymmetries are preserved. The overall results are summarised in the table below. For other branes, they can be wrapped in higher-dim calibrated cycles, allowing for different types of calibration, e.g. Caylay calibrations. A comprehensive table summarizing D3 and M5-branes wrapping on calibrated cycles can be found in [15].

Worldvolume	Supersymmetry
$\mathbb{R} \times (\Sigma_2 \subset CY_2)$	8
$\mathbb{R} \times (\Sigma_2 \subset CY_3)$	4
$\mathbb{R} \times (\Sigma_2 \subset CY_4)$	2
$\mathbb{R} \times (\Sigma_2 \subset CY_5)$	2

Table 4.1: Summary of M2-branes wrapped on Kähler 2-cycles [15].

Recall our discussion on the AdS/CFT correspondence in Chapter 3, where we delved into the realization of a QFT (i.e. SYM theory) on the D3-brane worldvolume. Similarly, we also have QFT on the worldvolumes of M2 and M5-branes. To make them supersymmetric, we need to establish the presence of a constant spinor on the worldvolume. This is straightforward in the Minkowski case. However, on calibrated cycles, spinors are generally not constant. Fortunately, we can resolve this issue through a technique known as topological twist.

To illustrate, let us consider the worldvolume of a planar M5-brane, which corresponds to $\mathbb{R}^{1,5}$. By fixing a static gauge, we can see that the worldvolume theory consists of 5 scalars (plus extra fermions) representing the 5 transverse coordinates. This introduces an internal $SO(5)$ R-symmetry, under which fermions will transform. Consequently, we can introduce a $SO(5)$ gauge field, and the covariant derivative acting on the spinor can be schematically written as

$$D\epsilon \sim (\partial + \omega - A)\epsilon \tag{4.16}$$

where ω (although $= 0$ in Minkowski) represents the $SO(5)$ spin connection, and A is the $SO(5)$ gauge field.

Now, let us consider the scenario where the M5-brane worldvolume is wrapped, e.g $\mathbb{R}^{1,2} \times \Sigma_3$ where Σ_3 is a SLAG 3-cycle. In this case, ω corresponds to the $SO(3)$ spin connection on Σ_3. We can decompose $SO(5) \to SO(2) \times SO(3)$, and choose to turn on only the $SO(3)$ gauge field. Specifically, we can set $A = \omega$ to ensure that we have some notion of constant spinors. This is how supersymmetry is realized for wrapped branes, and it is referred to as the topological twist [30].

4.1 D3-brane Wrapped on Kähler 2-Cycles

Up to this point, we have mainly considered the scenario of probe-branes and investigated the types of backgrounds they can be wrapped on. Now, we can delve into understanding how these wrapped branes affect the spacetime itself. In other words, we aim to find Type IIB supergravity solutions that account for the presence of wrapped branes. While one might initially think this can be achieved in a manner similar to the one used previously, which involved writing an ansatz like (2.25), there is an alternative approach to finding these solutions.

It is possible to obtain lower-dim truncated theories through Kaluza-Klein reduction (see Appendix A). In our cases, these are typically consistent truncations, meaning that any solutions to the truncated theory are also solutions to the original theory. For instance, reducing Type IIB supergravity on S^5 leads to $\mathcal{N} = 8$ $SO(6)$ gauged supergravity in 5 dim. This can be further truncated by considering its Cartan subgroups, resulting in $\mathcal{N} = 2$ $U(1)^3$ supergravity in 5-dim. Even further truncations can yield $\mathcal{N} = 2$ $U(1)$ supergravity in 5-dim (which is the minimal gauged supergravity in 5-dim), whose equations of motion are [17]

$$R_{\mu\nu} = -4g_{\mu\nu} + \frac{2}{3}F^2_{\mu\nu} - \frac{1}{9}F^2 g_{\mu\nu}, \qquad d \star F_{(2)} = -\frac{2}{3}F_{(2)} \wedge F_{(2)} \tag{4.17}$$

with appropriate Killing spinor equations

$$\left[\nabla_\mu - \frac{i}{12}(\gamma_\mu{}^{\alpha\beta} - 4\delta^\alpha_\mu \gamma^\beta)F_{\alpha\beta} - \frac{1}{2}\gamma_\mu - iA_\mu\right]\epsilon = 0 \tag{4.18}$$

Suppose we have a D3-brane wrapped on H^2 (which is a Kähler 2-cycle), then we can assume that the metric takes the form

$$ds^2_5 = P ds^2(AdS_3) + Q ds^2(H^2) \tag{4.19}$$

where P and Q are to be determined

$$ds^2(H^2) = \frac{1}{y^2}(dy^2 + dz^2) \tag{4.20}$$

which is just 2-dim hyperbolic space. In fact, this is just the Euclidean AdS_2 using the Poincaré patch.

By substituting into equations of motion, we have

$$ds_5^2 = \frac{4}{9}ds^2(AdS_3) + \frac{1}{3}ds^2(H^2) \tag{4.21}$$

and Killing spinor equations are solved by the chiral projection

$$\gamma_{ZY}\epsilon = -i\epsilon \tag{4.22}$$

and

$$\partial_y\epsilon = \partial_z\epsilon = 0 \tag{4.23}$$

which means there are constant spinors on the calibrated cycle. This is exactly how SUSY is preserved as discussed before (i.e. topological twist).

Therefore, we can uplift this solution to Type IIB solution by

$$ds_{IIB}^2 = \frac{4}{9}ds^2(AdS_3) + \left[\frac{1}{3}ds^2(H^2) + \left(\frac{1}{3}d\psi + \sigma + \frac{1}{3y}dz\right)^2 + ds^2(KE_4)\right] \tag{4.24}$$

which is $AdS_3 \times Y_7$ with Y_7 being SE_5 fibered over H^2. In the context of the AdS/CFT correspondence, we expect to see a dual CFT in 2-dim, which should be the IR regime of the RG flow where AdS_5 is in the UV regime.

In fact, we can also wrap D3-brane on S^2 (which is also a Kähler 2-cycle). This means that we can try the ansatz

$$ds_5^2 = Pds^2(AdS_3) + Qds^2(H^2) \tag{4.25}$$

where P and Q are, once again, to be determined by equations of motion, and

$$ds^2(S^2) = \frac{4}{(1+y^2+z^2)^2}(dy^2+dz^2) \tag{4.26}$$

which is the metric on S^2 in stereographic projection. After substitution, we get

$$ds_5^2 = \frac{1}{6}ds^2(AdS_3) + \frac{1}{3}ds^2(H^2) \tag{4.27}$$

but we can show that it fails the Killing spinor conditions. So, this will not be a SUSY solution. However, this does not mean that D3-brane on S^2 is not a SUSY solution in Type IIB. We have just shown that it is not a SUSY solution to the minimal gauged supergravity in 5-dim. In fact, it is a SUSY solution of Type IIB, but uplifted from the STU model (i.e. $\mathcal{N}=2$ $U(1)^3$ gauged supergravity in 5-dim).

4.2 D3-branes on a Spindle

In the previous section, we discussed how D3-branes can be wrapped on calibrated cycles, with supersymmetry realised through the topological twist. Recently [31], it has been discovered that branes can wrap on spindles $\Sigma = \mathbb{WCP}^1_{[n_-,n_+]}$, which are topologically S^2 with conical deficit angles $2\pi(1-\frac{1}{n_\mp})$ at the poles.

In this case, supersymmetry is not realised via topological twists, introducing new ways to study wrapped brane solutions. In fact, these are also solutions to minimal gauged supergravity in 5-dim (4.17), which are given by

$$ds_5^2 = \frac{4y}{9}ds^2(AdS_3) + ds^2(\Sigma), \quad A = \frac{1}{4}\left(1-\frac{a}{y}\right)dz \tag{4.28}$$

where the metric of the spindle Σ

$$ds^2(\Sigma) = \frac{y}{q(y)}dy^2 + \frac{q(y)}{36y^2}dz^2 \tag{4.29}$$

which is characterised by a cubic function

$$q(y) = 4y^3 - 9y^2 + 6ay - a^2 \tag{4.30}$$

where a is a constant, which is assumed to be $\in (0,1)$ such that all three roots y_i are real and positive. For $y_1 < y_2 < y_3$, we will take $y \in [y_1, y_2]$ such that the metric on Σ is positive definite. To study its conical deficit, consider $y = y_1 + \tilde{y}$ for small $\tilde{y}$, which implies

$$ds^2(\Sigma) \approx \frac{y_1}{q'(y_1)\tilde{y}} d\tilde{y}^2 + \frac{q'(y_1)\tilde{y}}{36(y_1+\tilde{y})^2} dz^2 \tag{4.31}$$

where $q'(y) = 12y^2 - 18y + 6ay$. By taking $\tilde{y} = \frac{1}{4}\rho^2$, we get

$$ds^2(\Sigma) \approx \frac{y_1}{q'(y_1)}(d\rho^2 + \frac{q'^2(y_1)}{144y_1^3}\rho^2 dz^2) \tag{4.32}$$

Similarly, for $y = y_2 + \tilde{y}$ perturbation, we can show that

$$ds^2(\Sigma) \approx \frac{y_1}{q'(y_i)}(d\rho_i^2 + c_i^2\rho_i^2 dz^2), \qquad c_i = \frac{|q'(y_i)|}{12y_i^{3/2}} \tag{4.33}$$

for $i = 1, 2$. So, we can read off the period as

$$\Delta z = \frac{2\pi}{c_1 n_+} = \frac{2\pi}{c_1 n_-} \tag{4.34}$$

where $n_- > n_+$ since $y_2 > y_1$.

Using another function

$$g(y) = \frac{a-y}{2(y^2 - 2y + a)} \tag{4.35}$$

which has some algebraic relations with $q(y)$, see [32], gives the following

$$y_1 = \frac{(n_+ - n_-)^2(2n_+ + n_-)^2}{4(n_+^2 + n_-n_+ + n_-^2)^2}, \qquad y_2 = \frac{(n_+ - n_-)^2(n_+ + 2n_-)^2}{4(n_+^2 + n_-n_+ + n_-^2)^2} \tag{4.36}$$

which implies

$$\Delta z = \frac{2(n_-^2 + n_-n_+ + n_+^2)}{3n_-n_+(n_- + n_+)} 2\pi \tag{4.37}$$

Therefore, we can compute

$$\frac{1}{2\pi}\int_\Sigma F = \frac{a}{8\pi}\frac{y_2-y_1}{y_1y_2}\Delta z = \frac{n_- - n_+}{2n_-n_+} \tag{4.38}$$

where we have also used the fact that

$$a = \frac{(n_- - n_+)^2(2n_- + n_+)^2(n_- + 2n_+)^2}{4(n_-^2 + n_-n_+ + n_+^2)^3} \tag{4.39}$$

This is different to

$$\chi(\Sigma) = \frac{1}{4\pi}\int_\Sigma R_\Sigma \mathrm{vol}_\Sigma = \frac{n_- + n_+}{n_-n_+} \tag{4.40}$$

which implies that this is not a topological twist (recall $A \sim \omega$ to cancel each other).

Indeed, to solve Killing spinor equations, we write $\Gamma^a = \gamma^a \otimes \sigma^3$ for $a = 0, 1, 2$ with $\gamma^0 = -i\sigma^2$, $\gamma^1 = \sigma^1$, $\gamma^2 = \sigma^3$ and $\Gamma^3 = \mathbb{1} \otimes \sigma^2$, $\Gamma^4 = \mathbb{1} \otimes \sigma^1$, where σ^i are Pauli matrices. We also write $\epsilon = \vartheta \otimes \chi$ with ϑ being the Killing spinor for AdS_3 that satisfies $\nabla_a\vartheta = \frac{1}{2}\gamma_a\vartheta$. Therefore, the remaining spinor is solved by

$$\chi = \left(\frac{\sqrt{q_1(y)}}{\sqrt{y}}, i\frac{\sqrt{q_2(y)}}{\sqrt{y}}\right) \tag{4.41}$$

where

$$q_1(y) = -a + 2y^{3/2} + 3y, \qquad q_2(y) = a + 2y^{3/2} - 3y \tag{4.42}$$

which satisfies $q(y) = q_1(y)q_2(y)$.

Now, we can uplift to Type IIB solution just like (4.24) by replacing $H^2 \to \Sigma$. A truly remarkable feature is that even if Σ is singular at poles, but SE_5 fibered over Σ is regular. Making this to $AdS_3 \times \mathcal{M}_7$. Therefore, this suggests new interesting AdS/CFT examples, in which CFT are now taken on a worldvolume with singularities (i.e. $\mathbb{R}^{1,1} \times \Sigma$). Indeed, one can compute the central charge of both sides to check the duality, which has been done in [31].

Chapter 5

Conclusion

In this dissertation, we have introduced 11-dim supergravity and Type IIB supergravity. Then, we found the solutions with no fluxes, which are Ricci-flat manifolds with covariantly constant spinors. In fact, these can also be interpreted as probe-brane solutions (since we assume branes are negligible). Following that, we considered planar (i.e. flat worldvolume) brane solutions to supergravity, which describe the geometry when M2, M5, or D3-branes are present. In particular, these solutions have near horizon geometry as $AdS_{p+2} \times S^{D-p-2}$. Therefore, we introduced the AdS/CFT correspondence to implicitly see how the well-known example (i.e. Type IIB in $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ SYM theory) arises. This can be generalised to other branes. Finally, we recalled that CFT can have RG flows, which motivates us to consider wrapped brane solutions. In particular, we first discovered that the brane configuration minimises its surface area, which are calibrated cycles. Then, we argued that we would like there to be some notion of constant spinors on the worldvolume theory, which is realised by the topological twist. We can also explicitly find the solutions and their corresponding supergravity solutions, i.e. those describing the near horizon geometry of a wrapped brane, by using lower-dim supergravities, which are derived from higher-dim ones after Kaluza-Klein reductions. Recently, there has been a new way to realise SUSY on wrapped D3-branes without the topological twist. In particular, we have D3-branes wrapped on a spindle, which is topologically equivalent to S^2 but with conical singularities at the poles. However, one remarkable feature is that the singularity vanishes as we uplift the solution to the whole Type IIB solution. Therefore, this provides new interesting questions that can be studied, e.g. how CFT is defined when the background has conical singularities.

To reflect on this dissertation, my initial plan was to provide more details for each interesting topic that we encountered. For example, we could say more about SUSY and supergravity, like their representations and how supergravities are built (i.e. to derive their SUSY transformations from the first principle). Or, we would like to say in details about the Clifford algebra across any dimensions and in both signatures. Also, we could say more about their geometries (e.g. complex geometry and fibre bundles), which provides another perspective that is very interesting to learn. However, these were not done due to my bad time organisation, as we can probably see that the later this dissertation goes, the less detailed it is. For example, we only talked about D3-branes on spindles, but there are definitely M2 or M5-branes on spindles, which provide different properties compared to the D3-brane case. In particular, there is a whole spacetime solution that describes M2-branes on a spindle, whereas all our wrapped brane solutions only describe their near horizon geometry. Nevertheless, this dissertation should be enough as an introduction to this field (i.e. AdS/CFT and wrapped branes).

Appendix A

Kaluza-Klein Reduction on S^1

The original idea of Kaluza-Klein theory was to unify gravity and electromagnetism by formulating GR in 5-dim, with the fifth one being compact and periodic, i.e. S^1. When the radius of this circle is small, it can be effectively assumed that fields in 4-dim have no dependence on this extra dimension. Consequently, a metric in 5-dim $G_{MN}(x, y)$ can be decomposed into a metric in 4-dim $g_{\mu\nu} \sim G_{\mu\nu}$, a vector field $A_\mu \sim G_{\mu 4}$, and a scalar field $\phi \sim G_{44}$. This decomposition results in the Einstein equations in 5-dim breaking down into equations of motion in 4-dim, which resemble the 4-dim Einstein-Maxwell equations but have the addition of a scalar field. While it might seem ideal to set $\phi = 0$ this approach leads to certain inconsistencies. As a result, the Kaluza-Klein idea was temporarily abandoned.

In recent times, the Kaluza-Klein idea has been revisited, primarily due to developments in string theory, which is formulated with extra dimensions. So, for phenomenological purposes, understanding how to extract lower-dim information from a higher-dim theory has become crucial. Moreover, Kaluza-Klein reductions can sometimes lead to consistent truncations, which mean that any solution to the truncated theory remains a valid solution to the original theory. This provides a powerful mathematical framework for simplifying theories, making it a valuable tool in theoretical physics.

As an example, consider two scalar fields in 4-dim Minkowski, described by the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\phi\Box\phi - \frac{1}{2}\chi\Box\chi - \frac{1}{2}g\phi\chi^2 \tag{A.1}$$

which gives the following equations of motion

$$\Box\phi = \frac{1}{2}g\chi^2, \qquad \Box\chi = g\phi\chi \tag{A.2}$$

where g is a coupling constant. There are two possibilities for truncation: $\phi = 0$ or $\chi = 0$. If $\phi = 0$, the truncated theory is inconsistent because

$$\Box 0 = \frac{1}{2}g\chi^2, \qquad \Box\chi = 0 \tag{A.3}$$

i.e. χ sources ϕ, hence $\phi = 0$ truncation is inconsistent. If $\chi = 0$, then

$$\Box\phi = 0, \qquad 0 = 0 \tag{A.4}$$

which is consistent. So, why is this the case? The group theory is responsible for this. In the Lagrangian, there is a $\mathbb{Z}_2$ symmetry in which ϕ transforms like a $\mathbb{Z}_2$ "scalar", i.e. $\phi \to \phi$, whereas $\chi \to -\chi$ transforms like a $\mathbb{Z}_2$ "vector". Therefore, if we only keep the group scalars, i.e. ϕ in this case, then the truncation will be guaranteed to be consistent, since scalars can only generate scalars.

In this appendix, we will mainly follow the lecture note given by Christopher Pope [33]. In particular, we will show how to perform a Kaluza-Klein reduction on S^1 for tensors, and showing that Type IIA is just 11-dim supergravity Kaluza-Klein reduced on S^1.

A.1 Scalar

To begin, consider a scalar field $\phi(x, y)$ in $\mathbb{R}^{1,3} \times S^1$ for simplicity, with x^μ being the coordinates on $\mathbb{R}^{1,3}$ and y being the coordinate on S^1, which satisfies

$$\Box_5\phi = 0 \tag{A.5}$$

where $\Box_5 \equiv \Box_4 + \partial_y^2$ is the d'Alembertian in 5-dim. The periodicity of the extra dimension enables us to perform a Fourier expansion

$$\phi(x, y) = \sum_{n=-\infty}^{\infty} \phi^{(n)}(x) e^{iny/L} \tag{A.6}$$

where L denotes the radius of S^1. Each Fourier mode $\phi^{(n)}(x)$ can be interpreted as scalar fields living in 4-dim Minkowski, and their equations of motion are

$$\left(\Box_4 - \frac{n^2}{L^2}\right)\phi^{(n)} = 0 \tag{A.7}$$

which is derived by Fourier expanding (A.5).

So, a massless scalar field in 5-dim can be thought of as equivalent to an infinite set of scalar fields in 4-dim, each with a mass $|n|/L$. From a physical standpoint, when the radius L of the extra dimension approaches zero (i.e. $L \to 0$), the massive fields become too heavy to be excited. Consequently, only the massless mode (i.e. $n = 0$) remains observable in 4-dim. In this sense, the truncation provides an approximation of the 5-dim theory or describes its low-energy dynamics.

This, in fact, is not just an approximation; it is a consistent truncation. In other words, any solution satisfying the following

$$\Box_4 \phi^{(0)} = 0, \qquad \phi^{(n\neq 0)} = 0 \tag{A.8}$$

is a valid solution to (A.5). Again, this is understandable in terms of group theory. Consider a translation along S^1, which is a $U(1)$ transformation on Fourier modes, only $\phi^{(0)}$ does not transform, i.e. a $U(1)$ scalar. So, if we just keep the massless modes, we can be sure of a consistent truncation.

A.2 Metric

Consider Einstein gravity in $(D+1)$-dim, where the spacetime is assumed to be $\mathcal{M}_D \times S^1$ for some D-dim Lorentzian manifold $\mathcal{M}_D$. This is described by the following Lagrangian

$$\mathcal{L} = \sqrt{-\hat{g}}\hat{R} \tag{A.9}$$

where we use hats to emphasise objects in $(D+1)$-dim, $\hat{g}$ represents the determinant of the metric $\hat{g}_{MN}$, and $\hat{R}$ is the Ricci scalar defined with the Levi-Civita connection (i.e. Christoffel).

First, we shall clarify our index conventions. Uppercase Latin letters like $M, N, \ldots \in 0, 1, \ldots, D-1, z$ denote $(D+1)$-dim spacetime indices, where z represents the coordinate on S^1. Lowercase Greek letters like $\mu, \nu, \ldots \in 0, 1, \ldots, D-1$ denote D-dim spacetime indices.

Similar to scalar fields, we can Fourier expand the metric $\hat{g}_{MN}$ as follows

$$\hat{g}_{MN}(x, z) = \sum_{n=-\infty}^{\infty} \hat{g}^{(n)}_{MN}(x) e^{inz/L} \tag{A.10}$$

where, again, L denotes the radius of S^1. By keeping the $U(1)$ singlet, we ensure a consistent truncation. Henceforth, we will omit the superscript for simplicity, i.e. $\hat{g}_{MN}(x) \equiv \hat{g}^{(0)}_{MN}(x)$.

Unlike scalars, we must also pay attention to spacetime indices. The metric $\hat{g}_{MN}$ decomposes into $\hat{g}_{\mu\nu}$, $\hat{g}_{\mu z}$, and $\hat{g}_{zz}$. These components can be used to define fields living in D-dim, such as a metric $g_{\mu\nu} = \hat{g}_{\mu\nu}$, a vector $\mathcal{A}_\mu = \hat{g}_{\mu z}$, and a scalar $\phi = \hat{g}_{zz}$. However, these seemingly straightforward choices result in cumbersome equations of motion in D-dim, which are highly inconvenient. The underlying reason is that this parameterisation does not take into account the inherent symmetries of the theory.

Nevertheless, this decomposition still provides the lower-dimensional field contents: a D-dimensional metric $g_{\mu\nu}$, a vector $\mathcal{A}_\mu$, and a scalar ϕ. A more suitable parameterization is given by

$$d\hat{s}^2 = e^{2\alpha\phi} g_{\mu\nu} dx^\mu dx^\nu + e^{2\beta\phi}(dz + \mathcal{A}_\mu dx^\mu)^2 \tag{A.11}$$

where α and β are some constants to be determined. In other words, we have

$$\hat{g}_{\mu\nu} = e^{2\alpha\phi} g_{\mu\nu} + e^{2\beta\phi} \mathcal{A}_\mu \mathcal{A}_\nu, \quad \hat{g}_{\mu z} = e^{2\beta\phi} \mathcal{A}_\mu, \quad \hat{g}_{zz} = e^{2\beta\phi} \tag{A.12}$$

Now, the rest of the story is just a tedious computation using vielbeins, i.e.

$$\hat{e}^a = e^{\alpha\phi} e^a, \qquad \hat{e}^z = e^{\beta\phi}(dz + \mathcal{A}) \tag{A.13}$$

where $\hat{e}^a$ and $\hat{e}^z$ are vielbeins in 5-dim and e^a are vielbeins on $\mathcal{M}_D$ (i.e. $g_{\mu\nu}$). We begin by calculating the spin connections using the Cartan structure equations, resulting

$$\begin{aligned} \hat{\omega}^{ab} &= \omega^{ab} + \alpha e^{-\alpha\phi}(\partial^b \phi \hat{e}^a - \partial^a \phi \hat{e}^b) - \frac{1}{2}\mathcal{F}^{ab} e^{(\beta-2\alpha)\phi} \hat{e}^z \\ \hat{\omega}^{az} &= -\beta e^{\alpha\phi} \partial^a \phi \hat{e}^z - \frac{1}{2}\mathcal{F}^a{}_b e^{(\beta-2\alpha)\phi} \hat{e}^b \end{aligned} \tag{A.14}$$

where $\partial_a \equiv e^\mu_a \partial_\mu$ represents the dual basis of vielbeins, and $\mathcal{F}_{ab}$ are the vielbein components of $\mathcal{F} = d\mathcal{A}$.

Next, we will move on to curvature 2-forms. If α and β remain unfixed, the expressions can become very complicated. So, we will not present the full expression without constraining α and β. However, for a first-time calculation, it is worth attempting to compute them. Here, we will outline a few key steps during the calculation. As before, we will use the Cartan structure equation to obtain curvature 2-forms, then carefully read off their components, paying attention to antisymmetric indices. Once the curvature 2-forms are computed, determining the Ricci scalar will be straightforward.

Additionally, do not forget $\sqrt{-\hat{g}} = \det \hat{e}^A_M$ and $\sqrt{-g} = \det e^a{}_\mu$, where A represents the vielbein indices in $(D+1)$-dim. This implies $\sqrt{-\hat{g}} = e^{(D\alpha+\beta)\phi}\sqrt{-g}$, and therefore

$$\mathcal{L} = \sqrt{-\hat{g}}\hat{R} \;\sim\; e^{(\beta+(D-2)\alpha)\phi}\sqrt{-g}R + \dots \tag{A.15}$$

hence, a natural choice is to set $\beta = -(D-2)\alpha$ such that the lower-dim Einstein-Hilbert term is in its canonical form. Similarly, the scalar field sector looks like (after fixing β)

$$\mathcal{L} = \sqrt{-\hat{g}}\hat{R} \;\sim\; -\sqrt{-g}\,\alpha^2(D-1)(D-2)\,\partial_\mu\phi\partial^\mu\phi + \dots \tag{A.16}$$

hence, for conventional normalization, we have $\alpha^2 = \frac{1}{2(D-1)(D-2)}$. With these choices, we can write down the Lagrangian

$$\mathcal{L} = \sqrt{-g}(R - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4}e^{-2(D-1)\alpha\phi}\mathcal{F}^2) \tag{A.17}$$

which gives the following equations of motion

$$\begin{aligned} R_{\mu\nu} &= \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}e^{-2(D-1)\alpha\phi}(\mathcal{F}^2_{\mu\nu} - \frac{1}{2(D-2)}\mathcal{F}^2 g_{\mu\nu}) \\ \nabla^\mu(e^{-2(D-1)\alpha\phi}\mathcal{F}_{\mu\nu}) &= 0 \\ \Box\phi &= -\frac{1}{2}(D-1)\alpha e^{-2(D-1)\alpha\phi}\mathcal{F}^2 \end{aligned} \tag{A.18}$$

where $\mathcal{F}^2_{\mu\nu} = \mathcal{F}_{\mu\rho}\mathcal{F}_\nu{}^\rho$ and $\mathcal{F}^2 = \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$, and one can immediately see setting $\phi = 0$ is inconsistent since it is sourced by $\mathcal{F}^2$.

A.3 Vector

Now, we begin considering gauge fields. We assume that the reduction of the potential is

$$\hat{A}_{(n-1)}(x,z) = A_{(n-1)}(x) + A_{(n-2)}(x) \wedge dz \tag{A.19}$$

which implies

$$\hat{F}_{(n)} = dA_{(n-1)} + dA_{(n-2)} \wedge dz \tag{A.20}$$

Just like in the metric case, the simple-looking choice

$$F_{(n)} = dA_{(n-1)}, \qquad F_{(n-1)} = dA_{(n-2)} \tag{A.21}$$

is not convenient. This is because the basis in our metric convention is $dz + \mathcal{A}_{(1)}$, but only dz is used here. So, we consider

$$\begin{aligned}\hat{F}_{(n)} &= dA_{(n-1)} + dA_{(n-2)} \wedge (dz + \mathcal{A}_{(1)} - \mathcal{A}_{(1)}) \\ &= dA_{(n-1)} - dA_{(n-2)} \wedge \mathcal{A}_{(1)} + dA_{(n-2)} \wedge (dz + \mathcal{A}_{(1)}) \\ &= F_{(n)} + F_{(n-1)} \wedge (dz + \mathcal{A}_{(1)})\end{aligned} \tag{A.22}$$

in other words, we have

$$F_{(n)} = dA_{(n-1)} - dA_{(n-2)} \wedge \mathcal{A}_{(1)}, \qquad F_{(n-1)} = dA_{(n-2)} \tag{A.23}$$

Indeed, we can observe this convenience by looking at their components

$$\begin{aligned}\hat{F}_{(n)} &= \frac{1}{n!}\,\hat{F}_{A_1\ldots A_n}\,\hat{e}^{A_1\ldots A_n} \\ &= \frac{1}{n!}e^{n\alpha\phi}\,\hat{F}_{a_1\ldots a_n}\,e^{a_1\ldots a_n} + \frac{1}{(n-1)!}e^{((n-1)\alpha+\beta)\phi}\,\hat{F}_{a_1\ldots a_{n-1}z}\,e^{a_1\ldots a_{n-1}} \wedge (dz + \mathcal{A}_{(1)}) \\ &= \frac{1}{n!}\,F_{a_1\ldots a_n}\,e^{a_1\ldots a_n} + \frac{1}{(n-1)!}\,F_{a_1\ldots a_{n-1}}\,e^{a_1\ldots a_{n-1}} \wedge (dz + \mathcal{A}_{(1)})\end{aligned} \tag{A.24}$$

where $e^{a\ldots b} = e^a \wedge \ldots \wedge e^b$ and we can read off

$$\hat{F}_{a_1\ldots a_n} = e^{-n\alpha\phi}F_{a_1\ldots a_n}, \qquad \hat{F}_{a_1\ldots a_{n-1}z} = e^{(D-n-1)\alpha\phi}F_{a_1\ldots a_{n-1}} \tag{A.25}$$

Therefore, the Lagrangian

$$\mathcal{L} = -\frac{1}{2n!}\sqrt{-\hat{g}}\hat{F}^2_{(n)} = \sqrt{-g}\left(-\frac{1}{2n!}e^{-2(n-1)\alpha\phi}F^2_{(n)} - \frac{1}{2(n-1)!}e^{2(D-n)\alpha\phi}F^2_{(n-1)}\right) \tag{A.26}$$

which can be also expressed as

$$\mathcal{L} = -\frac{1}{2}e^{-2(n-1)\alpha\phi}F_{(n)} \wedge \star F_{(n)} - \frac{1}{2}e^{2(D-n)\alpha\phi}F_{(n-1)} \wedge \star F_{(n-1)} \tag{A.27}$$

A.4 Type IIA Supergravity

Recall the Lagrangian of 11-dim supergravity is

$$\mathcal{L}_{11} = \hat{R}\,\hat{\star}\,1 - \frac{1}{2}\hat{F}_{(4)} \wedge \hat{\star}\hat{F}_{(4)} - \frac{1}{6}\hat{F}_{(4)} \wedge \hat{F}_{(4)} \wedge \hat{A}_{(3)} \tag{A.28}$$

which becomes (after Kaluza-Klein reduced on S^1)

$$\begin{aligned}\mathcal{L}_{IIA} = R \star 1 &- \frac{1}{2}d\phi \wedge \star d\phi - \frac{1}{2}e^{\frac{3}{2}\phi}\mathcal{F}_{(2)} \wedge \star\mathcal{F}_{(2)} \\ &- \frac{1}{2}e^{\frac{1}{2}\phi}F_{(4)} \wedge \star F_{(4)} - \frac{1}{2}e^{-\phi}F_{(3)} \wedge \star F_{(3)} - \frac{1}{2}dA_{(3)} \wedge dA_{(3)} \wedge A_{(3)}\end{aligned} \tag{A.29}$$

which is exactly the bosonic sector of Type IIA supergravity. We can also obtain the SUSY variations in Type IIA from 11-dim supergravity (see [2] for the details).

Bibliography

1. Tong, D. *Lectures on String Theory* 2012. arXiv: 0908.0333 [hep-th].
2. Becker, K., Becker, M. & Schwarz, J. H. *String theory and M-theory: A modern introduction* (Cambridge University Press, Dec. 2006).
3. Blumenhagen, R., Lüst, D. & Theisen, S. *Basic concepts of string theory* (Springer, Heidelberg, Germany, 2013).
4. Maldacena, J. M. The Large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.* **2,** 231–252. arXiv: hep-th/9711200 (1998).
5. D'Hoker, E. & Freedman, D. Z. *Supersymmetric Gauge Theories and the AdS/CFT Correspondence* 2002. arXiv: hep-th/0201253 [hep-th].
6. Nastase, H. *Introduction to the ADS/CFT Correspondence* (Cambridge University Press, Sept. 2015).
7. Trigiante, M. Gauged supergravities. *Physics Reports* **680,** 1–175 (Mar. 2017).
8. De Wit, B. *Supergravity* 2002. arXiv: hep-th/0212245 [hep-th].
9. Tong, D. *Supersymmetric Field Theory* 2023.
10. Closset, C. *Lecture notes on supersymmetry and supergravity* 2020.
11. Müller-Kirsten, H. J. W. & Wiedemann, A. *Introduction to Supersymmetry* 2nd. eprint: https://www.worldscientific.com/doi/pdf/10.1142/7594 (WORLD SCIENTIFIC, 2010).
12. Freedman, D. Z. & Van Proeyen, A. *Supergravity* (Cambridge Univ. Press, Cambridge, UK, May 2012).

13. Weinberg, S. Photons and Gravitons in S-Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass. *Physical Review* **135,** 338–345 (1964).

14. Cremmer, E., Julia, B. & Scherk, J. Supergravity Theory in Eleven-Dimensions. *Phys. Lett. B* **76,** 409–412 (1978).

15. Gauntlett, J. P. *Branes, Calibrations and Supergravity* 2003. arXiv: hep-th/0305074 [hep-th].

16. Gauntlett, J. P. & Pakis, S. The geometry of $D = 11$ Killing spinors. *Journal of High Energy Physics* **2003,** 039–039 (Apr. 2003).

17. Jiao, Y. *Supergravity and Branes on Curved Manifolds* MA thesis (Imperial Coll., London, 2022).

18. Hull, C. M. *Lecture notes on Differential Geometry* 2022.

19. Stelle, K. S. *BPS Branes in Supergravity* 2009. arXiv: hep-th/9803116 [hep-th].

20. Lü, H., Pope, C. N. & Rahmfeld, J. A construction of Killing spinors on Sn. *Journal of Mathematical Physics* **40,** 4518–4526 (Sept. 1999).

21. Sparks, J. *Sasaki-Einstein Manifolds* 2010. arXiv: 1004.2461 [math.DG].

22. Acharya, B., Figueroa-O'Farrill, J., Hull, C. & Spence, B. *Branes at conical singularities and holography* 1999. arXiv: hep-th/9808014 [hep-th].

23. Wiseman, T. *An introduction to AdS-CFT* 2023.

24. DeWolfe, O. *TASI Lectures on Applications of Gauge/Gravity Duality* 2018. arXiv: 1802.08267 [hep-th].

25. Herzog, C. *Conformal field theory with boundaries and defects* 2021.

26. Fradkin, E. S. & Tseytlin, A. A. Effective Field Theory from Quantized Strings. *Phys. Lett. B* **158,** 316–322 (1985).

27. De Alwis, S. A note on brane tension and M-theory. *Physics Letters B* **388,** 291–295 (Nov. 1996).

28. Ammon, M. & Erdmenger, J. *Gauge/gravity duality: Foundations and applications* (Cambridge University Press, Cambridge, Apr. 2015).

29. Ramallo, A. V. *Introduction to the AdS/CFT correspondence* 2013. arXiv: 1310.4319 [hep-th].

30. Bershadsky, M., Vafa, C. & Sadov, V. D-branes and topological field theories. *Nuclear Physics B* **463,** 420–434 (Mar. 1996).

31. Ferrero, P., Gauntlett, J. P., Ipiña, J. M. P., Martelli, D. & Sparks, J. D3-Branes Wrapped on a Spindle. *Physical Review Letters* **126** (Mar. 2021).

32. Gauntlett, J. P., Conamhna, O. A. P. M., Mateos, T. & Waldram, D. Supersymmetric 3D Anti–de Sitter Space Solutions of Type IIB Supergravity. *Physical Review Letters* **97** (Oct. 2006).

33. Pope, C. *Kaluza-Klein Theory*