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Quantum Fields and Fundamental Forces

Brane webs and 5d $\mathcal{N} = 1$ supersymmetric field theories

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Abstract

We study 5d $\mathcal{N} = 1$ supersymmtric field theories using brane configurations in Type IIB string theory called *brane webs*. We introduce the basic features and demonstrate how field theoretic quantities can be computed simply from the brane webs. We then demonstrate how brane web decompositions can be used to find combinatorial objects called *magnetic quivers* which encode the Higgs branch of the theory, in particular when the coupling is infinite. Finally we compute magnetic quivers for two families of field theories.

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Chapter 1

Introduction

This thesis concerns the application of brane configurations in Type IIB string theory to the understanding of 5d $\mathcal{N} = 1$ supersymmetric gauge theories. The study of supersymmetric quantum field theories is partially motivated by the difficulty presented by ordinary quantum field theory. Quantum field theories without supersymmetry, such as Quantum Chromodynamics (QCD) become more or less intractable in regimes where the coupling strength is large and the physics can no longer be treated perturbatively. As a consequence, non-perturbative effects such as confinement cannot be explored and understood in any straightforward fashion. Even in the best case scenario where the coupling is small and perturbative technieques can be used an asymptotic expansion often the best one can do for any physical quantity. In supersymmetric theories however, the situation is much improved and many exact results can be obtained. Non-renormalisation theorems can be proved [1] which ensure exact potentials can be obtained which do not run with scale. Phenomena such as confinement can also be understood in an exact fashion [2]. The study of supersymmetric theories can also lead to developments in mathematics [3]. Supersymmetric theories can then be viewed (notwithstanding any potential phenomenological applications) as useful toy models of quantum field theory that capture many of the key features in a setting where we have a chance to understand them. The study of toy models has historically been a very fruitful pursuit in theoretical physics. Supersymmetric quantum field theories are also important in string theory.

Supersymmetry places many constraints on the kind of theories that can exist. Particles and fields must come in groups called *supermultiplets*. In the 5d $\mathcal{N} = 1$ supersymmetry we will consider this means each gauge boson comes with a scalar and one spinor. Each matter fermion must come with four real scalars. This greatly constrains the possible theories. In fact once the gauge group and matter content are decided the interactions are completely determined. Supersymmetric theories come with many interesting features. Of particular interest in is the *moduli space*, which is the space of all vacua of the quantum theory. This space is parameterised by the vacuum expectation values (vevs) of scalars. In theories with 8 supercharges, which includes 5d $\mathcal{N} = 1$, the moduli spaces have a rich mathematical structure and take the form of *symplectic singularities* [4], mathematical objects which have their natural setting in algebraic geometry [5]. Much progress has been made in computing these moduli spaces in recent years [6, 7, 8, 9, 10, 11, 12]

Supersymmetric theories in 5d are particularly interesting as they provide examples of superconformal field theories (SCFTs) in dimensions greater than 4. These were not known to exist until they were found using string theory considerations in [13]. Understanding the space of SCFTs in five dimensions has been a recent focus [14, 15]

In this work our focus will be on using brane configurations, called brane webs, from Type IIB string theory to understand supersymmetric field theories. This approach allows us to use the tools and dualities of string theory to understand supersymmetric field theory. This approach was pioneered in [16, 17, 18]. The use of brane webs allows us to represent a supersymmetric gauge theory as 2d graph subject to certain rules Phase transitions in the field theory can be seen as geometric transitions on the graph. The use of brane webs is particularly fruitful for understanding non-perturbative effects. In the opposite direction, ideas from field theory can be used to help us understand features of string theory, which is a good motivation in its own right. This is a fruitful direction of reseach which makes possible computations that would be much more difficult using field theory techniques, for example theories without a lagrangian description are just as easy to analyse with brane webs as lagrangian theories. All the while, the understanding of string theory is advanced.

This work will be organised as follows. In 2 we will review the basic features of 5d $\mathcal{N} = 1$ gauge theories from the field theory perspective. In 3 we will introduce brane webs of (p,q) 5-branes and demonstrate their use in making computations for 5d theories. In 4 we will discuss [p,q] 7-branes and demonstrate the use of brane webs and objects called magnetic quivers in computing moduli spaces. In 5 we will use brane webs to compute magnetic quivers for two families of theories.

Chapter 2

Basic properties of supersymmetric gauge theories in five dimensions

In this chapter we will introduce the basic concepts of 5d $\mathcal{N} = 1$ supersymmetric field theories that we will require. We will assume throughout a a knowledge of representations of Lie groups and Lie algebras as well as a basic knowledge of 4d $\mathcal{N} = 1$ supersymmetry. We direct the reader to [19] for a reference on supersymmetry.

2.1 Supersymmetry algebra in five dimensions.

Here we will review the supersymmetry algebra in five dimensions The Lorentz group in five dimensions is SO(4,1). The spinor representation of this group is four dimensional and pseudoreal. To form the minimal $\mathcal{N} = 1$ supersymmetry algebra we need two supercharge spinors $Q_\alpha, \tilde{Q}_\beta$ $\alpha, \beta = 1, 2, 3, 4$ which combine to form the supersymmetry algebra in five dimensions:

$$\{Q_\alpha, \tilde{Q}_\beta\} = 2\gamma^\mu_{\alpha\beta} P_\mu$$

(2.1)

Where the γ^μ are the five dimensional gamma matrices with $\mu = 0, 1, \ldots 5$.

We have two spinors with 4 components leading to a total of 8 supercharges in 5d $\mathcal{N} = 1$. This is in contrast the 4 supercharges of 4d $\mathcal{N} = 1$. Other theories with 8 supercharges include 3d $\mathcal{N} = 4$, 4d $\mathcal{N} = 2$ and 6d $\mathcal{N} = (1, 0)$ supersymmetry. There are many common features amongst theories with 8 supercharges, in particular the strucure of their moduli space of vacua. Particles and fields in a supersymmetric theory must lie in representations - called *supermultiplets* - of the super Poincaré algebra, which is the combination of the supersymmetry algebra (2.1) and the Poincaré algebra. Recall in ordinary quantum field theory particles and fields must lie in represntations of the Poincaré algebra. Physically this constrains a particle state to be described by a momentum and a spin. In supersymmetry, the fact that particles and fields come in representations of the super Poincaré algebra leads to more constraints. For 5d $\mathcal{N} = 1$ theories, there are two supermultiplets, the vector multiplet and the hypermultiplet. The vector multiplet consists of one vector field, one spinor field and one real scalar and the hypermultiplet of four real scalars and a spinor [13]. In a supersymmetric gauge theory, the gauge fields will come in vector multiplets, and the matter fields in hypermultiplets. For instance, a theory with a $U(1)$ gauge group and one matter flavour, which we could call super QED, contains one vector mutiplet and one hypermultiplet. Supersymmetry constrains completely the types of interactions that are possible between these fields. As a consequence, once we specify the gauge group, the matter content which consists of hypermultiplets in representations of the gauge group and the possible global flavour symmetries under which the matter transforms we have completely specified a theory. Given this it is useful to represent gauge theories diagrammatically using a diagram known as a *quiver*. The quivers we will use consist of nodes which are either circles or squares. Numbers adjacent to the nodes are called ranks. A circular node represents a factor of the gauge group, a square node a factor of the global flavour symmetry group. For our purposes, the only gauge groups we will consider are special unitary gauge groups $SU(N)$ so a circular node of rank N represents a factor $SU(N)$ in the gauge group. A square node of rank N represents a factor of $SU(N)$ in the global symmetry. A line joining two nodes represents a hypermultiplet transforming in the bifundamental representation of the groups represented by the nodes. For example, 2.1 shows a gauge theory with gauge group $SU(N_c)$ and global symmetry $SU(N_f$. The line between the two nodes means we have have a hypermultiplet transforming in bifundamental representation of $SU(N_c)$ and $SU(N_f)$, which is just N_f flavours in the fundamental representation of

$SU(N_c)$. This is Supersymmetric Quantum Chromodynamics or SQCD with N_c colours and N_f flavours. Ordinarily the global symmetry would only be present if these flavours have the same mass, however if we allow the masses to transform in the adjoint representation of $SU(N_f)$ then this is a valid symmetry for arbitrary flavour masses.

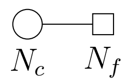


Figure 2.1: Quiver diagram for a theory with gauge group $SU(N_c)$ with N_f flavours transforming in the fundamental representation of the gauge group.

Another example is shown in 2.2, which represents a gauge theory with gauge group $SU(2) \times SU(2)$. There are two flavours charged under each gauge group factor, as well as a bifundamental hypermultiplet charged under both gauge groups.

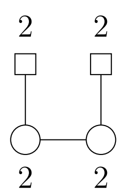


Figure 2.2: Quiver diagram for an $SU(2) \times SU(2)$ gauge theory with two flavours charged under each guage group factor and hypermultiplet in the bifundamental of the gauge group.

2.2 5d $\mathcal{N} = 1$ field theories

Here we will review the basic properties of 5d N = 1 supersymmetric gauge theories. The results we present here were first discussed in [13, 20, 21]. Of general interest in supersymmetric gauge theories is the *moduli space of vacua* which is the space of all allowed vacuum states of the theory. The moduli space is parameterised by the vacuum expectation values (vevs) of the scalar fields in the theory. Classically it is obtained by solivng for zeros of the potential energy, however the moduli space is generally changed by quantum effects. Generically, the moduli space contains two main brances, the Coulomb branch and the Higgs branch. The Coulumb branch is defined as the space of vacua where only the scalar fields in the vector multiplets associated to the gauge fields have a vev. The Higgs branch contains the vacua where only the scalar fields in the matter hypermultiplets have a vev. There may also exist a mixed branch where both vector multiplet and hypermultiplet scalars have a vev. For the rest of this section we will focus on the Coulomb branch.

Recall the scalar fields in the vector multiplets are in the adjoint representation of the gauge group G. On the Coulomb branch they are constrained to take values in the Cartan subalgebra of G. Since the scalars are in the adjoint representation the unbroken gauge group is generated by those generators which commute with the cartan subalgebra, which is precisely the cartan subalgebra. Since the unbroken generators commute, we the unbroken gauge group is $U(1)^r$, where r is the rank of G. The Weyl W group of G still has a non-trivial action on the cartan subalgebra. Since we should not distinguish points on the moduli space differing by a gauge transformation, the Coulomb branch is correctly described by allowing the scalars to take vevs in a single Weyl chamber of the cartan subalgebra. In practice, this means for $SU(2)$ with a single scalar ϕ we take $\phi \geq 0$. For $SU(N)$ which has rank $N-1$ and so $N-1$ many scalars $\phi_1 \dots \phi_{N-1}$ we restrict ourselves to $0 \leq \phi_{N-1} \leq \dots \phi_1$. This then completely describes the physical points of the Coulomb branch.

It is known [13] that the low energy dynamics on the Coulomb branch are captured in a single holomorphic function of the vevs of the vector multiplet scalars $\mathcal{F}(\phi)$, known as the *prepotential*. The prepotential is constrained to be a cubic function of the scalars, it receives perturbative corrections that are exact after a one-loop calculation. The full perturbative prepotential given an arbitrary gauge group with multiple factors $G = \prod_i G_i$ and matter hypermultiplets in representations ρ is [21]:

$$\mathcal{F} = \sum_{G_k} \frac{1}{g_k^2} h_{ij}\phi^i\phi^j + \frac{c}{3} d_{ijk}\phi^i\phi^j\phi^k + \frac{1}{6}\left(\sum_{\mathbf{R}} |\mathbf{R}\cdot\phi|^3 - \sum_{\rho}\sum_{\mathbf{w}\in\mathbf{W}_\rho} |\mathbf{w}\cdot\phi + m|^3\right) \tag{2.2}$$

In the above $\frac{1}{g_i^2}$ is the classical gauge coupling of each factor G_i $h_{ij} = Tr(T_iT_j)$, $d_{ijk} = \frac{1}{2}Tr(T_i\{T_j, T_k\})$, where T_i are the generators of the Cartan subalgebra of the fundamental representation, $\mathbf{R}$ refer to the roots of G and $\mathbf{w}, \mathbf{W}_\rho$ the weights and weight spaces respectively of the hypermultiplet representations ρ. c is the classical chern simons level, it will be zero all our computations. The first terms represent the classical contributions and the terms in brackets represent one loop corrections. We note the right hand side of this formula is a factor of two larger than its usual form in the literature in order to provide agreement with our brane web calculations later. This is equivalent to multiplying the lagrangian by a constant. Let us calculate $\mathcal{F}(\phi)$ for a pure $SU(2)$ gauge theory with no matter flavours. The Cartan subalgebra in this case is one dimensional so our Coulomb branch is parameterised by one scalar ϕ, and the gauge group is broken to $U(1)$. In this case, quotienting by Weyl symmetry is equivalent to taking $\phi > 0$. Taking our generator T to be $\frac{1}{2}diag(1, -1)$ we find $h = \frac{1}{2}$, $d_{ijk} = 0$. For the one loop correction we note $SU(2)$ has roots ± 1 in our normalisation. Thus we find the prepotential

$$\mathcal{F} = \frac{1}{2}\frac{\phi^2}{g^2} + \frac{1}{3}\phi^3 \tag{2.3}$$

From the prepotential we can calculate the metric on the moduli space $\tau(\phi)_{ij}d\phi^i d\phi^j$ from $\tau(\phi)_{ij} = \frac{\partial^2 \mathcal{F}(\phi)}{\partial\phi^i \partial\phi^j}$. This determines the kinetic terms for the vector multiplet scalars and the gauge couplings for the gauge fields. For the pure $SU(2)$ theory this is given by $\tau(\phi) = \frac{1}{g^2} + 2\phi$. This gives us the effective gauge coupling for the photon of the unbroken gauge $U(1)$ at a point on the Coulomb branch. At the origin of the coulomb branch this reduces to the classical coupling. We note the prepotential does not account for non-perturbative effects such as instantons.

A key feature of 5d theories is that there is an additional conserved current given by $j = Tr(*F \wedge F)$, often called a topological symmetry. Note this expression gives us a vector uniquely in five dimensions. We get associated to this current a global $U(1)_I$ symmetry whose charge is the instanton number I. For a gauge group with multiple factors, we will get an instanton number I_i for each factor.

2.2.1 BPS spectrum

The 5d theories have a BPS spectrum consisting of particles as well as magnetic monopole strings [13]. The masses of these states are determined by the their charges under abelian $U(1)$ symmetries, both global and gauged. The relevant quantum numbers are the electric charges n_{e_i}, $i = 1 \ldots r$ under the

$U(1)^r$ residual gauge group on the coulomb branch, the instanton number I giving the charge under the topological symmetry, as well as flavour charges Q_{f_i} under the $U(1)$ factors of any flavour symmetry groups. The mass of BPS states is then given by [17]:

$$M = |n_{e_i}\phi^i + \frac{I_i}{g_i^2} + Q_{f_i}m_i| \tag{2.4}$$

where we sum over repeated indices. m_i is the bare mass of the flavour f_i. In five dimensions we have magnetic monopole strings [13] whose tensions can be calculated from the prepotential via the quantities $\phi_{D_i} = \frac{\partial \mathcal{F}}{\partial \phi^i}$. A monopole string is characterised by r integer quantum numbers n_{m_i} which we call the magnetic fluxes and has tension

$$T_M = n_{m_i}\phi_{D_i} \tag{2.5}$$

For example, the $SU(2)$ theory described above has one type of monopole with tension $T_M = \frac{\phi}{g^2} + \phi^2$
We have described the basic field theory results we will use. We will not pursue the field theory approach any further and will instead make use of the simpler and more elegant calculations using branes.

Chapter 3

Introduction to Brane Webs

We now begin the application of particular brane configurations in Type IIB string theory, called *brane webs* to the analysis of 5d theories. The application of the brane constructions we will use here to 5d theories was pioneered in [16, 17, 22]. We will find that that a 5d theory can be represented by a 2d planar graph, called a brane web which represents a configuration of branes in type IIB string theory. Supersymmetry is enforced by a condition on the slopes of the branes, which provides geometric constraints on the construction of the brane webs. The gauge group matter content, gauge couplings, hypermultiplet bare masses can all be read directly from the brane web. Various transitions in the theory can also be understood from transitions due to deformations on the brane web, characterised for instance by a brane undergoing a 90 degree rotation suddenly as a parameter is varied. In addition, ideas from string theory such as $SL(2, Z)$ duality can be expoited to discover relations between different field theories. We begin by introducing the basic notions from string theory that we will require. We will then show how to construct a brane web for a given theory, and how to calculate the prepotential and identify the BPS spectrum purely from features of the web. We will attempt to give a somewhat self containted description of all the string theory results we use however we will present without proof results from string theory. For a thorough description of these results we direct the reader to the many string theory texts available. A detailed understanding will not be necessary however to grasp the basic ideas presented here

3.1 Branes in Type IIB String Theory

The fundamental object we will need to construct our brane configurations is the (p,q) 5-brane.

We recall in string theory a Dp-brane is a $p+1$ dimensional object which fills out 1 time and p spatial dimensions in which the ends of open strings are constrained to move. One can consider the worldvolume theory of a Dp brane or collection of Dp branes, which is a $p+1$ dimensional quantum field theory describing the low energy dynamics of the string theory in the presence of the branes. The D5 brane has a dual object, the Neveu-Schwarz or NS5-brane. This is a solitonic object which is dual to the D5-brane under the strong-weak duality of Type IIB. It is known that a collection of p NS5-branes and q D5-branes can form a bound state and form a single object known as (p,q) 5-brane. (p,q) 5-branes will serve as the building blocks of the brane configurations we will use. We now describe the general setup.

In the 10 dimensional Minkowski spcae of Type IIB, we take the time direction to be x^0 and $x^1, \ldots x^9$ to be the spatial directions. We let the (p,q) 5-branes be extended in dimensions x^0, x^1, x^2, x^3, x^4, and let it occupy a 1-dimensional subspace of the (x^5, x^6) plane. The degree of freedom in the (x^5, x^6) plane is what allows us to construct the brane configuration . It was shown in [17] that we may position (p,q) 5-branes in the (x^5, x^6) plane subject to the following two rules:

- A (p,q) 5-brane forms a straight line in the (x^5, x^6) plane with slope $\frac{q}{p}$.
- Multiple (p,q) 5-branes may end at a common vertex if the charge is conserved: $\sum_i p_i = \sum_i q_i = 0$.

We will refer to the 2d cross section of a brane configuration in the (x^5, x^6) as a *brane web.* Recall Type IIB string theory has 32 supersymmetries. The first condition above ensures the brane configuration breaks exactly one quarter of these supersymmmetries leaving 8 remaining supercharges, precisely the right amount for a 5d $\mathcal{N} = 1$ theory. We note according to this, a D5-brane or (1,0) 5-brane forms a horizontal line in the (x^5, x^6), and an NS5-brane or (0,1) 5-brane forms a vertical line. A 5d $\mathcal{N} = 1$ theory can take the form of the worldvolume theory of a collection of D5-branes with finite extent in the (x^5, x^6) plane, ie a brane web with a collection of finite horizontal lines. In this case, the fifth spatial direction is compact and can be considered non-dynamical, analagously for instance to Kaluza-Klien models, giving a

5d theory. In general, a brane web with n finite coincident D5-branes will represent a theory with gauge group $SU(n)$. If the D5-branes are separated this corresponds to spontaneous breaking the gauge symmetry.

3.2 Example: $SU(2)$

We present the simplest non-trivial example of a brane web in 3.1, which represents the Coulomb branch of a pure $SU(2)$ theory with no matter. To draw this brane web, we first place two finite D5-branes to give as an $SU(2)$ gauge group. To be finite, these D5-branes must end at a vertex which conserves charge, which leads us to the configuration of 3.1. For instance the top right vertex has an (0,1) and a (1,0) 5-brane going into the vertex, and a (1,1) brane going out, or equivalently a (-1,-1) brane going into the vertex, leading to charge conservation. This brane web represents the theory in the Coulomb branch phase, where the gauge group is broken to $U(1) \subseteq SU(2)$. The fact that the gauge group is broken can be understood from the fact that the two D5-branes are separated. In string theory, one generally expects the gauge symmetry is enhanced when multiple D-branes are made to coincide. We suppose then that the symmetry breaking parameter, in this case the vev ϕ which parameterises the one dimensional cartan subalgebra of $SU(2)$(see previous section) should be proportional to the separation between the two D5-branes. In fact, if one considers the lowest level excitation of a fundamental string stretched between the two D5-branes, it represents a massive vector boson with mass proportional to the string length which is the D5-brane separation. The only such state that should exist in an $SU(2)$ theory broken to $U(1)$ is the W-boson associated to the symmetry breaking. We know from field theory the W-boson mass is proportional to the vev of the symmetry breaking higgs field, which is precisely ϕ. We may rescale our units/redefine the string tension so that both the mass and length are equal to ϕ Thus we conclude: *The distance between the finite D5-branes is equal to* ϕ. We have seen the first example of a quantity of the field theory that can be read directly from the corresponding brane web.

Another state that can be realised on the brane web is the magnetic monopole string. This is realised as D3-brane spanning the finite rectangular face the brane web [17]. The tension of this state is proportional to the area of the face. We know from our calculations of the previous section this monopole tension should be $\frac{\phi}{g^2} + \phi^2$. We deduced the vertical length of the NS5-branes

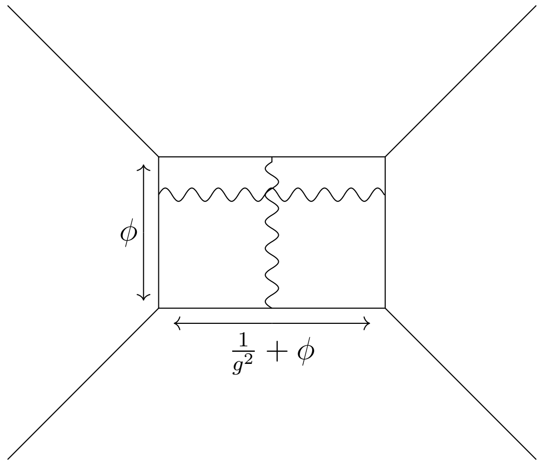


Figure 3.1: Brane web of pure $SU(2)$. The vertical and horizontal squiggly lines represent fundamental and D string exciations respectively, which correspond to W boson and instanton states respectively.

is ϕ, thus by comparison with the field theory result we conclude the length of the horizontal D5-branes is $\frac{1}{g^2}+\phi$. Thus we have determined how to construct a brane web corresponding to any value of the gauge coupling and any value of ϕ. A third state we may realise is due to a D1-brane or D string stretched between the two NS5-branes. According to our normalisation its mass is equal to its length which is $\frac{1}{g^2} + \phi$. Recalling the BPS mass formula (2.4), we see this is charged under the $U(1)_I$ and has instanton number $I = 1$. We emphasise how the analysis of the non-perturbative instanton was no more difficult in the brane web picture than the the analysis of the perturbative W-boson. Using only the tools of field theory the non-perturbative features would generally be much more difficult to analyse.

3.3 Deformations and symmetries of brane webs

We now discuss the different kind of deformations we can perform on the brane web and their relation to the field theory parameters. We define a deformation as any change to the brane web which does not alter the number of external semi-infinite brane webs which we will refer to as the external legs. We can identify two types of deformation. First, a *local* deformation is one which does not move any of the external legs, a *global* deformation is one which moves the external legs. Examples of local and global deformations are seen in 3.2 and 3.3. Since the global deformations involve moving the infinite external legs they require cost an infinite amount of energy and are associated to an infinite mass. Consequently we associate them with the changing of non-dynamical parameters such as gauge couplings and hypermultiplet masses. The local deformations only move finite branes, and so are associated with a finite mass. It is natural then to associate these deformations with the channging the vevs of the hypermultiplet scalars.

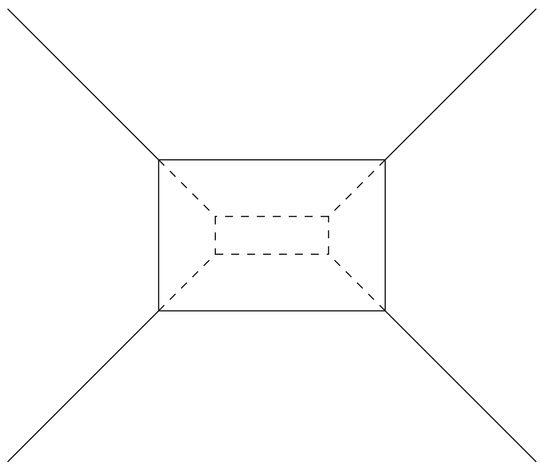


Figure 3.2: Local deformation corresponding to a change in ϕ_1, the dashed lines show the brane web after the deformation.

We can see in 3.2, the correct way to represent moving along the Coulomb branch of the $SU(2)$ theory is to expand or contract diagonally along the external legs, avoiding moving them. This agrees with the conclusions of the previous section. By deforming according to 3.2 the W-boson and instanton masses scale as they should with a change in ϕ. If one simply increased the vertical length of the NS5-branes, the W-boson mass would increase and the instanton mass would stay constant. Clearly from the mass formulae this cannot be associated with a changing ϕ only. The theory has one independent global deformation associated to the gauge coupling. This is shown in 3.3. This must correspond to changing the gauge coupling as it is the only parameter in the field theory. This also agrees with our previous deduction that the D5-brane length is $\frac{1}{g^2} + \phi$. We see here concretely the gauge coupling corresponds to a geometric parameter of the brane web, the difference

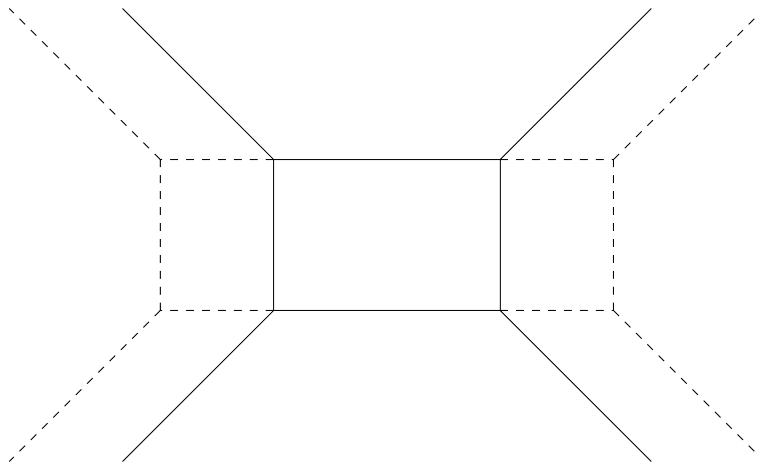


Figure 3.3: Global deformation corresponding to a change in $\frac{1}{g_2}$.

in length between the horizontal and vertical sides of the rectangular face. We can see $\frac{1}{g^2}$ corresponds to the length of the D5-branes when $\phi = 0$. Associated with the gauge coupling is the special transition point where the gauge coupling goes to infinity: $\frac{1}{g^2} = 0$. This is associated with a transition on the brane web where the rectangular face becomes square. This is an example of a special transition in the parameter space of the field theory having a corresponding transition in the brane web, the transition from rectangle to square. It was found in [13] that at infinite $\frac{1}{g^2} = 0$ there is an enhancement of the global symmetry from the classical $U(1)_I$ to $SU(2)$. We can see this reflected in the brane web. Since the internal face is a square at infinte coupling, the length of the fundamental and D string states - corresponding respectively to the masses W boson and instanton - states become equal. Then the W boson and instanton can form a doublet under the enhanced $SU(2)$ global symmetry.

For a general brane web we can count the number of local and global deformations without calculating all the details. It is straightforward to see a local deformation can only arise from expanding or contracting a finite face. Hence we conclude the number of local deformations is equal to the number of faces in the brane web. Since these deformations are associated to scalar vevs on the Coulomb branch, this in turn means the number of faces counts the rank of the gauge group. The global deformations are associated with moving external legs. Each external leg gives us one deformation however these will not all provide new brane webs. Firstly two deformations are associated with translating the entire brane web in the (x^5, x^6) plane. Further, one we have positioned all but one of the external legs, the final postion is determined by the rules for constructing brane webs, this means three of our global deformations are always redundant. We note that associated to each parameter in our field theory there is an abelian $U(1)$ factor of the global symmetry. For gauge couplings we have the topological $U(1)_I$ symmetry. For the hypermultiplet masses we have flavour symmetries. Hence the number of global deformations of the brane web is equal to the rank of the global symmetry group of the corresponding field theory. We summarise all this as follows:

- $Rank(GaugeSymmetry) = \#(Faces)$
- $Rank(GlobalSymmetry) = \#(External\ Legs) - 3$

Let us compare this to the $SU(2)$ theory. We have four external legs leading to one global deformation. This is associated to the guage coupling paramater $\frac{1}{g^2}$ and the global symmetry $U(1)_I$. There is one face corresponding to the scalar vev ϕ and the $U(1)$ gauge symmetry, in agreement with our precious discussions. We note there is an additonal global symmtry of the brane web: the $SO(3)$ group associated with rotations of the remaining string theory dimensions x^7, x^8, x^9. This can naturally be identified with another global symmetry from the field theory which we have so far omitted to mention, namely the $SU(2)_R$ symmetry. One final symmetry of the brane web we will discuss is the action of of S-duality, an element of the $SL(2, Z)$ symmetry of Type IIB string theory. This symmetry exchanges D5-branes with NS5-branes and more generally exchanges a (p,q) 5-brane with a (q,-p) 5-brane [16]. Simply put, S-duality rotates the brane web by 90 degrees. Since this is a fundamental symmetry of string theory, this tells us rotating our brane web by 90 degrees must give us identical physics. This provides us with a powerful duality between different field theories which we may exploit.

3.4 Product gauge groups and matter

We now discuss how to construct brane webs for theories with matter. To understand this, we will first analyse a brane web for a product gauge group. To do this we glue brane webs together along semi-infinite NS5-branes. In the following we analyse the simplest example of an $SU(2) \times SU(2)$ theory. We will determine the full relation between the brane web and field theory parameters and compute the prepotential, and then analyse the BPS spectrum. Then we will see how taking the limit of zero coupling corresponds to adding matter to our theory.

3.4.1 An $SU(2) \times SU(2)$ theory

3.4 represents a brane web for an $SU(2) \times SU(2)$. This is simply two copies of the $SU(2)$ brane web glued along an infinite NS5-brane. 3.4 shows a point on the Coulomb branch which now is two dimensional, parameterised by the vevs of two scalar fields ϕ_1, ϕ_2 associated to the Cartan generators of each $SU(2)$ factor. The gauge group is broken to $U(1) \times U(1)$. In addition the field theory has a massless hypermultiplet transforming in the bifundamental representation of $SU(2) \times SU(2)$. Let us check this by examining how the parameters on the brane web compare to the field theory. To start we can find the ranks of the global and gauge symmetries using the discussions of the previous section. There are two internal faces corresponding to a rank 2 gauge symmetry in agreement with our gauge group. There are six external legs meaning we should have a rank 3 global symmetry with corresponding parameters.

Comparing to the field theory we have two topological symmetries associated with conserved currents from each $SU(2)$ factor, the corresponding parameters are the two gauge couplings. We then have the flavour symmetry of the bifundmantal hypermultiplet with the corresponding parameter being the bare mass. We could give the hypermultiplet a mass by moving apart the central NS5-branes in the x^5 direction however we will keep this mass zero for our purposes. Thus the number of global and local deformations are in one to one correspondence to the ranks of the global and gauge symmetries of the field theory. Let us identify which deformations correspond to which field theory parameters. As before we get W-boson states from fundamental strings stretched between the two D5-branes on each face, allowing us to identify these distances as corresponding the vevs ϕ_1, ϕ_2 as before. The local deformations have a different form however for each vev. For the smaller vev ϕ_2, the local deformation is the same as before, given by expanding the face along the diagonal lines. For the larger vev ϕ_1, the deformation is different. We cannot move the central external NS5-branes in a local deformation so now the face expands along the diagonal external legs on the right hand side

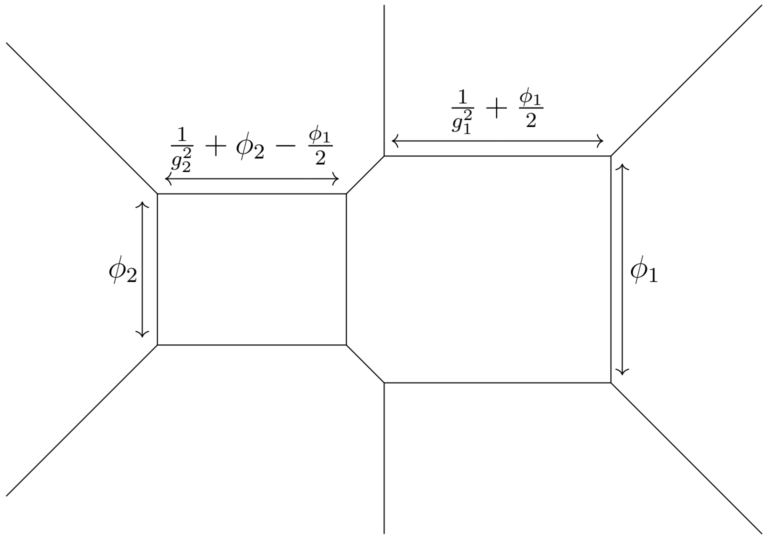


Figure 3.4: Brane web for a $SU(2) \times SU(2)$ theory with a bifundamental hypermultiplet.

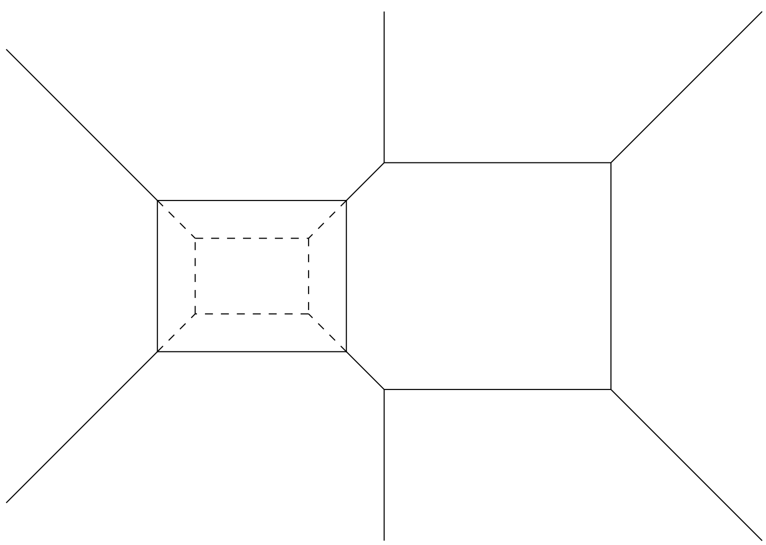


Figure 3.5: The local deformation corresponding to a change in ϕ_2

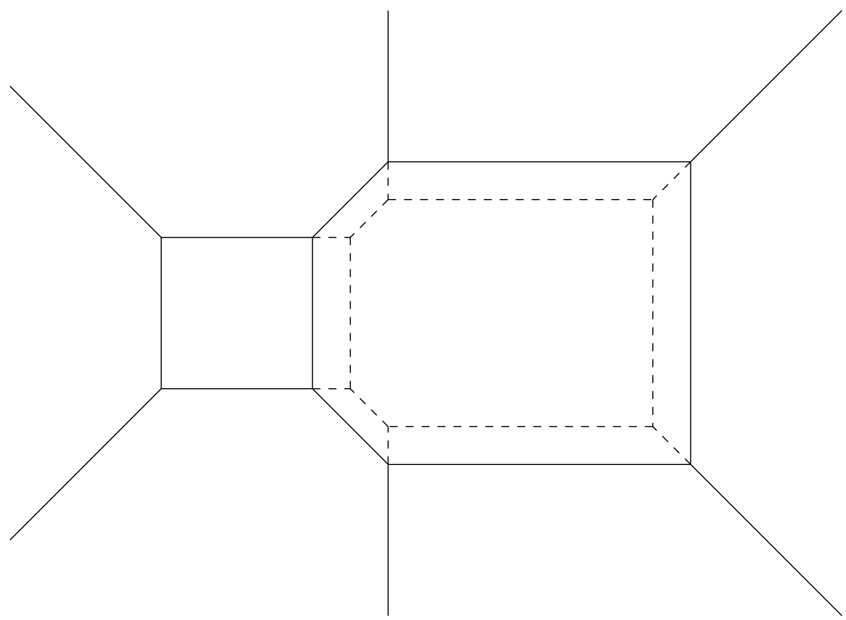


Figure 3.6: Local deformation corrsesponding to a change in ϕ_1.

and vertically along the central NS5-branes, and pushes the left face further to the left. As a consequence of the different form of this deformation, the length of the horizontoal D5-branes bounding the right hand face scales differently with ϕ_1 as the left hand side. With a little trigonometry we can see a local deformation causing a change δ in ϕ_1 cause the the horizontal length of the right face to change by $\frac{\delta}{2}$ and on the right by $-\frac{\delta}{2}$. The local deformation causing a change δ in ϕ_2 causes the horizontal length of the left face to change by δ If as before we associate the D5-brane length at the origin of the Coulomb branch ($\phi = 0$) with the gauge coupling, D5-brane length is on the right equal to $\frac{1}{g_1^2} + \frac{\phi_1}{2}$, and on the left equal to $\frac{1}{g_2^2} + \phi_2 - \frac{1}{2}\phi_1$. We can now identify the global deformations corresponding to the gauge couplings as moving the left/right hand external legs away in the negative/positive x^5 direction. Having found the dimensions of the brane web in term of the field theory quantities we can now calculate the monopole tensions and hence reconstruct the prepotential. Recalling our school knowledge of calculating the areas of 2d shapes, we can split the right hand face into a rectangle with area $\frac{\phi_1}{g_1^2} + \frac{1}{2}\phi_1^2$ and a trapezium with area $\frac{1}{4}(\phi_1^2 - \phi_2^2)$, the left hand side is rectangle, thus we find the monopole tensions:

$$T_{m_1} = \frac{\phi}{g_1^2} + \frac{3}{4}\phi_1^2 - \frac{1}{4}\phi_2^2$$
$$T_{m_2} = \frac{\phi_2}{g_2^2} + \phi_2 - \frac{1}{2}\phi_1\phi_2 \tag{3.1}$$

Knowing from the field theory results these tensions are given by $T_{M_i} = \frac{\partial \mathcal{F}}{\partial \phi_i}$ we can integrate and add these expressions - taking care not to double count the $\phi_1\phi_2$ term - to obtain the prepotential:

$$\mathcal{F} = \frac{\phi_1^2}{2g_1^2} + \frac{\phi_2^2}{2g_2^2} + \frac{1}{4}\phi_1^3 + \frac{1}{3}\phi_2^3 - \frac{1}{4}\phi_1\phi_2^2 \tag{3.2}$$

Indeed we can compare this to the result given by the field theory expression for the prepotential (2.2). The terms given by the pure gauge part of the expression are two copies of the pure $SU(2)$ case (2.3) subbing in ϕ_1, ϕ_2 as we would expect. There is now a matter contribution due to the bifundamental hypermultiplet. The weights of this representation are $(\pm\frac{1}{2}, \pm\frac{1}{2})$, (recall the weights of the funamental representation of $SU(2)$ are $\pm\frac{1}{2}$), this gives a contribution to the prepotential $\frac{1}{3}\left(|\frac{\phi_1}{2} + \frac{\phi_2}{2}|^3 + |\frac{\phi_1}{2} - \frac{\phi_2}{2}|^3\right)$. Doing out the algebra for $\phi_1 \geq \phi_2$ we find agreement with (3.2). The above calculations verify our field theory interpretation of the brane web in 3.4, and also demonstrate a method of calculating the prepotential by simply computing the areas of the internal faces of the brane web.

3.4.2 BPS spectrum: String webs and strings.

Let us now analyse the spectrum of BPS states. As for the pure $SU(2)$ case we can do this by considering fundamental strings and D strings stretched across the faces, calculate their masses by finding their length, and discern their charges under the abeleian gauge and global symmetries by comparing the mass with the BPS formula (2.4). The relevant charges for our theory are the electric charges (n_{e_1}, n_{e_2}) and the instanton charges (I_1, I_2). The first states are simple analogs of the pure $SU(2)$ theory. We have two W-bosons from stretching fundamental strings between the D5-branes bounding the internal faces, these have lengths and masses ϕ_1, ϕ_2 corresponding to states with charges $(1, 0)$ and $(0, 1)$ respectively. In addition we have two instanton states from stretching D strings between the NS5-branes bounding the faces. The lengths of these are $\frac{1}{g_1^2} + \phi_1 - \frac{1}{2}\phi_2$ and $\frac{1}{g_2^2} + \phi_2 - \frac{1}{2}\phi_1$ for the right and left faces respectively. This strings correspond to instanton states charged under both $SU(2)$ factors with respective electric charges $(1, -\frac{1}{2})$ and $(-\frac{1}{2}, 1)$ and respective instanton charges $(1, 0)$ and $(0, 1)$. There are also magnetically charged states with charges (n_{m_1}, n_{m_2}), where the n_{m_i} are integers and corresponding mass $n_{m_1}T_{m_1} + n_{m_2}T_{m_2}$ where the monopole tensions are given by (3.1). We can understand some basic features of the dynamics from these computatoins. The W-bosons being charged separately under the gauge groups should not interact with each other, however the instanton particles carry both types of charges and so should be interacting. There are some missing states however. This theory includes a hypermultiplet with electric charges $(\pm\frac{1}{2}, \pm\frac{1}{2})$. Consequently, using the BPS formula (2.4) we should have two states of mass $\frac{1}{2}\phi_1 \pm \frac{1}{2}\phi_2$. To realise these states in the brane web we must make use of new objects from string theory, "string webs" and "strips". We will simply state the properties of these tools here, for a thorough introduction see [16]. We have been using (p,q) 5-branes, bound states of D5-branes and NS5-branes so far in our brane webs, however in string theory we also have (p,q) strings, bound states of p fundmental strings and q D-strings. In a brane web, a (p,q) string may end on a (p,q) 5-brane, and is constrained to have a slope in the (x^5, x^6) plane of $-\frac{q}{p}$. Simply put, a (p,q) string has slope rotated 90 degrees from the slope of the corresponding (p,q) 5-brane. (p,q) strings may also end at vertices of many (p,q) strings as long as they satisfy charge conservation at the vertex, just like the 5-branes. These conditions mean we may draw webs of strings within a brane web diagram according to the same rules as 5-branes, with the extra feature that the strings must end perpendicularly on branes. Importantly, the tension of a (p,q) string is given by $\sqrt{p^2 + q^2 T_s}$, where T_s is the string tension. In our normalisation where we take the length of a string to equal the mass of the corresponding quantum state, this means a (p,q) string of length l will have mass $\sqrt{p^2 + q^2 l}$. A string web represents a quantum state whose mass is found by adding up the tensions of each segment, which in gemoetric terms is just the sum of all the lengths weighted by the slope in the above way. This agrees with our previous results. A fundamental string stretched between two D5-branes, such as in 3.1, is a string web with one (1,0) string, which has vertical slope and ends on (1,0) 5-branes, having horizontal slope. Similarly the horizontal D-string leading to the instanton state is a string web with one (0,1) string. A non-trivial example is shown in 3.7.

This consists of a (1,1) string that ends in the middle of the (1,1) 5-brane

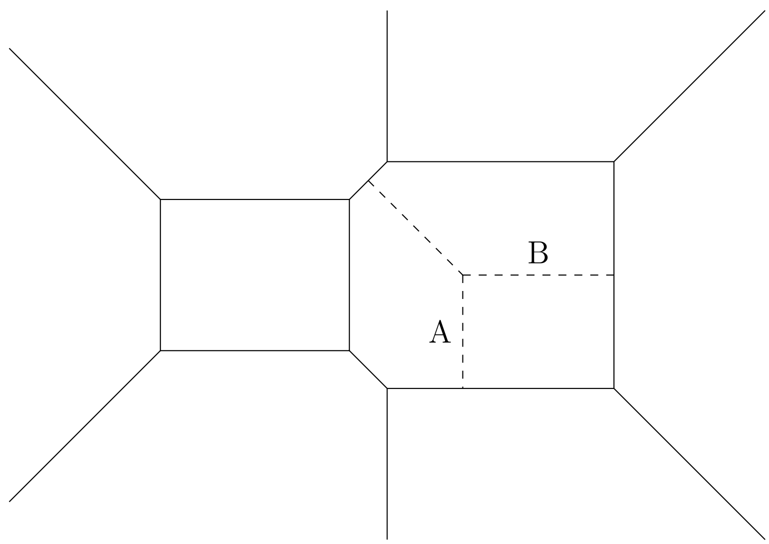


Figure 3.7: String web state indicated by dashed line consisting of a (1,1), (1,0) and (0,1) string. A and B are the lengths of the (0,1) and (1,0) strings respectively

connecting the two faces, splitting into a (1,0) string and a (0,1) string at the vertex. Note (p,q) strings have the same sign ambiguity in their charges as (p,q) 5-branes reflecting two choices of orientation. There is one free parameter for this web which is the length l of the the (1,1) string, which one may naively suppose to give a continuum of states. We can calculate the lengths of the other segments of this string web (A,B in 3.7) in terms of l using some basic trigonometry. They are:

$$\begin{aligned} A &= \frac{3}{4}\phi_1 + \frac{1}{4}\phi_2 - \frac{1}{\sqrt{2}}l \\ B &= \frac{1}{g_1^2} + \frac{3}{4}\phi_1 - \frac{1}{4}\phi_2 - \frac{1}{\sqrt{2}}l \end{aligned} \tag{3.3}$$

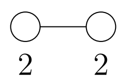
Now the masses of these segments are just equal to the lengths, but the (1,1) string contributes $\sqrt{2}l$ to the sum. Adding all these together we find a state with mass $\frac{1}{g_1^2} + \frac{3}{2}\phi_1$, the l. The properties of (p,q) strings and Pythagoras' theorem conspire to remove the l dependence of the mass, giving us a single well-defined state. This is a new instanton state with electric charge $\left(\frac{3}{2}, 0\right)$. There is no corresponding state with instanton charge $(0, 1)$. Note this state will disappear from the spectrum when $\phi_1 = \phi_2$, as the diagonal (1,1) 5-

brane disappears in this instance. There is clearly some form a transition in the brane web when $\phi_1 = \phi_2$, we have identified this transition has a concrete impact in the quantum theory leading to the destruction of a state. Let us now introduce another object we can use to identify different states, called a "strip" [22]. A (p,q) strip is a string like instanton living inside a (p,q) 5-brane. (p,q) strips can end at the vertices where the (p,q) 5-branes they live inside end. Notably the tension of a (p,q) strip scales as $\frac{T_s}{\sqrt{p^2+q^2}}$ meaning a (p,q) strip of length l will give rise to a state of mass $\frac{l}{\sqrt{p^2+q^2}}$ in the normalisation we use. (p,q) strips form a kind of vertex with (p,q) strings known as a "bend". The general rule for a bend is given b (p,q) strips inside a (p,q) 5-brane, and a pair of integers (s,t) satisfying $pt - qs = 1$, the b strips can join with a $b(s,t) - a(p,q)$ string. Consider for example a (1,1) strip. taking $(s,t) = (0,1)$ we see it satisfies the condition and a single (1,1) strip may form a bend with a (-1,0) string, which is just a fundamental string, for $a = b = 1$ in the above formula. Let us use these facts to find the hypermultiplet states in our brane web. Firstly, consider a (1,1) strip inside the (1,1) 5-brane connecting the two faces, ending at each vertex where the 5-brane ends. This state has mass $\frac{1}{2}\phi_1 - \frac{1}{2}\phi_2$, with the $\sqrt{2}$ factor from the length cancelling with the tension factor as before. This gives us our first hypermultiplet state. The next state can be realised by considering this strip again, however instead of ending at the vertex on the left hand face, it froms a bend with a fundamental string which ends on the opposite D5-brane, which is allowed as we found above. Adding up the tensions of the different segments, noting the fundamental sting has length ϕ_2, we find a state with mass $\frac{1}{2}\phi_1 + \frac{1}{2}\phi_2$. Using strips and string webs, we have identified how the hypermultiplet states arise naturally in the brane web, and also found a new instanton type state. We see the full features of the field theory are encoded in the various geometric features of the brane web.

3.4.3 The limit of zero coupling

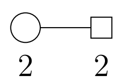
Let us now consider a situation where we take one of the gauge couplings to zero. Let us consider specifically the limit $g_1 \to 0$. What does this do to the brane web? Recall the length of the horizontal D5-branes bounding the right hand face is given by $\frac{1}{g_1^2} + \frac{1}{2}\phi_1$. As we tune g_1 to zero, $\frac{1}{g_1^2}$ will go to infinity. This pushes the edge of the right hand face off to infinity, and the effective picture is a brane web with one face and two extra infinite external

D5-branes on the right hand side of the web. This theory looks like an $SU(2)$ theory with two extra external legs, shown in 3.8. From the field theory point of view, as we take the coupling to zero the mass of the monopole associated to the right face will become infinitely massive and hence won't be excited in any physical process and can be ignored. The W-boson mass remains finite at ϕ_1 however at zero gauge coupling will no longer interact with the rest of the spectrum. Since changing ϕ_1 now moves external legs, it now corresponds to a global deformation and should correspond to a field theory parameter. It is a natural guess that ϕ_1 now represents a bare mass. Indeed the hypermultiplet states $\frac{1}{2}\phi_1 \pm \frac{1}{2}\phi_2$ remain in the spectrum as we take the coupling to zero. This looks just like the mass we would expect for matter quarks with mass $m = \frac{1}{2}\phi_1$. Indeed the contribution to the prepotential from the bifundamental hypermultiplet subbing in m for $\frac{1}{2}\phi_1$ and ϕ_2 for ϕ $\frac{1}{3}\left(|m+\frac{\phi}{2}|^3 + |m-\frac{\phi}{2}|^3\right)$ which one can check is precisely the contribution we would get for two flavours in the fundamental representation of $SU(2)$ with bare mass m. The right gauge factor $SU(2)$ has become a global flavour symmetry group, with the vev of its scalar becoming the hypermultiplet bare mass. We can visualise this easily using quivers. The theory with non-zero gauge coupling has quiver:



(3.4)

Taking the limit of zero coupling we obtain the quiver:



(3.5)

Looking at the brane web, we have learned *adding external D5-branes to a brane web adds matter flavours to the field theory.* The additional external legs add more degrees of freedom the brane web which encdoe the bare masses. In our case the bare mass is equal to half the distance between the external D5-branes. The electric charge n_{e_1} is now a flavour charge. Interestingly the instanton state associated to a D string in the left hand face which had $n_{e_1} = \frac{1}{2}$ at finite coupling is now charged under the flavour group with mass $\frac{1}{g_2^2} + \phi_2 - \frac{1}{2}m$. The fact that instanton masses depend on the masses of quarks in the theory is a non-trivial result from the field theory perspective but completely obvious in the brane web picture. It is possible to take the gauge coupling to zero in two different ways, corresponding to two different

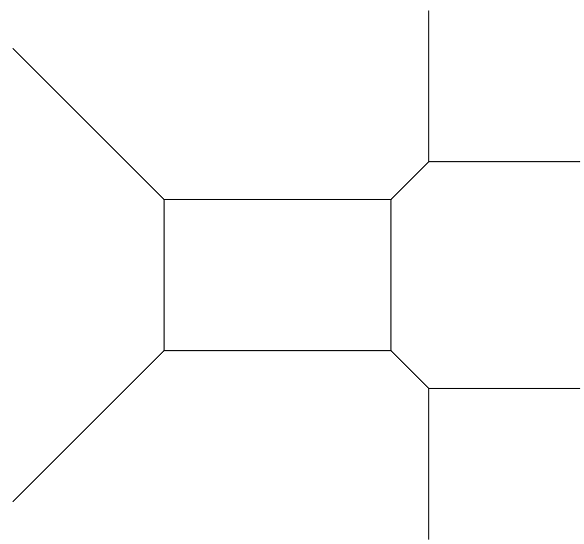


Figure 3.8: Caption

phases of the new $SU(2)$ theory. The first which we have discussed involves taking the coupling of the gauge factor with the larger vev to zero, corresponding in the new theory to the phase $\frac{1}{2}\phi < m$, depicted in 3.8. The other option, depicted in 3.9 is to take the coupling to zero for the gauge factor with the smaller vev, corresponding to the phase $\frac{1}{2}\phi > m$. In this second phase, the state corresponding to the string web in 3.7 survives and so these two phases have different BPS spectra.

The existance of a phase transition at $\frac{1}{2}\phi = m$ is clear from a glance at the brane web. The internal and external D5-branes coinciding as ϕ is tuned to $2m$. We see this phase transition in the spectrum, the quark state with mass $\frac{1}{2}\phi - m$ becomes massless at this point. Massless states often indicate some form of transition. In addition, due to the term $|m - \frac{\phi}{2}|^3$ in the prepotential we see the metric on the Coulomb branch τ_{ij} becomes singular at this point(recall τij is proportional to second derivatives of the prepotential). This kind of transition is known as *flop transition* [17]. The key point of this section is that an external D5-brane on a brane web represents a hypermultiplet. The bare masses of the flavours are associated to the positions of the extenrnal D5-branes. The precise expression for the masses needs to be worked out from each brane web. We can determine this in various ways, for instance by comparing are expressions for the internal phase areas with the monopole tensions calculated from the prepotential, or by identifying the quark states

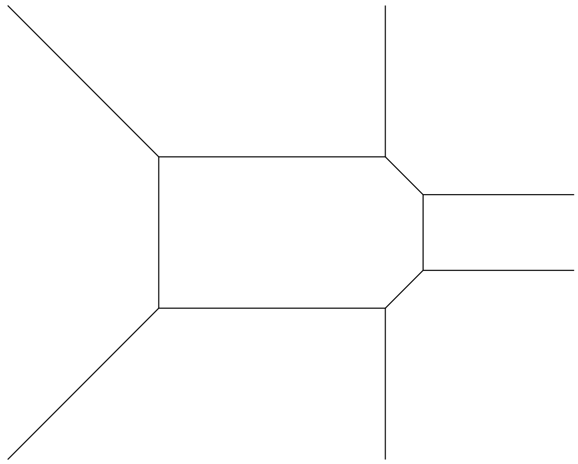


Figure 3.9: Caption

from the flavours within the brane web. For instance, starting with we can detrmine a string instanton inside the (1,1) 5-brane gives a state of mass $\frac{1}{2}\phi - \frac{1}{2}D$, where D is the distance between the external D5-branes. We could then guess that $\frac{1}{2}D = m$. It is interesting to explore the theory at strong coupling. In [13] enhanced global symmetries at strong coupling were found for $SU(2)$ for up to 7 flavours. We have previously examined this enhancement for no flavours for which $U(1)$ is enhanced to $SU(2)$. We saw this could be understood as the W boson and instanton have the same mass at infinite coupling and so can form a doublet under the enhanced $SU(2)$. For the theory with two flavours shown in 3.9 it was shown the global symmetry is enhanced from $SO(4)\times U(1)$ to $SU(3)\times SU(2)$ at infinite coupling (for $SU(2)$ gauge theories the flavour symmetry is $SO(2N_f)$ for N_f flavours). Looking at 3.9 we can see we have three BPS states associated corresponding to the W boson, the instanton arising from a D string stretched across the face and an instanton due to a string web as in 3.7. There are also the two quark states. Taking the coupling to infinity, the hypermultiplet bare masses to zero and taking ϕ to zero we obtain 5 massless states, the correct amount to form a fundamental representation of $SU(3)\times SU(2)$. It is a good guess that the two hypermultiplet states should form a doublet under $SU(2)$ and the W boson and two instanton states a triplet under $SU(3)$. We can see the additional global symmetry emerges due to the requirement of additional diagonal (1,1) and (1-1) branes in order to add the flavours which then allow an additional string web state to be drawn, which in turn provides the enhancement of symmetry. Again a highly non-trivial filed theory phenomenon has a simple geometric explanation.

3.5 $SL(2, Z)$ duality

Type IIB string theory posseses a fundamental $SL(2, Z)$ duality. This duality must also be valid on our brane webs. Recall for an element $A \in SL(2, Z)$, a (p,q) brane transforms as $\begin{pmatrix} p \\ q \end{pmatrix} \to A \begin{pmatrix} p \\ q \end{pmatrix}$ We see $SL(2, Z)$ acts directly on the brane web. Of particular interest to us will be the element $S \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

which rotates the brane web by 90 degrees. This duality is also known as S duality. Crucially this symmetry swaps D5-branes and NS5-branes. Recalling that the gauge symmetry of a theory is determined by the parallel D5-branes which become coincident under some local deformation, we can see that this symmetry transformation can change the gauge group of the theory, giving us a duality between different theories. We note that since a 90 degree rotation does not change the number of internal faces, the rank of the gauge group is preserved under this transformation. Let us see how this acts on the theories we have considered. On the pure $SU(2)$ theory 3.1, rotating the brane web by 90 degrees does not change theory. We still have two parallel D5-branes. S-duality here exchanges the dimensions $\frac{1}{g^2} + \phi$ and ϕ. Interpreting the length of the internal face again as ϕ. S-duality here corresponds to the transformation $\phi \to \frac{1}{g^2} + \phi$, $\frac{1}{g^2} \to -\frac{1}{g^2}$. In five dimensions masses are real and may take on negative values. Since $\frac{1}{g^2}$ is essentially the bare mass of the instanton particle, it being negative does not pose any difficulty. Let us see what happens for the $SU(2) \times SU(2)$ theory 3.4. The rotated theory is depicted in FIG. We can see now there are three parallel D5-branes. This means we are now dealing with a theory with gauge group $SU(3)$. The dimensions of the brane web here have not changed, we are simply looking at it from a different angle. However the mapping from the brane web parameters to the field theory parameters will now be different

shown in 3.10. This theory again has two internal faces corresponding to a rank 2 theory, however this now represents an $SU(3)$ theory. The two external D5-branes represent matter flavours. We conclude we have an $SU(3)$ theory with two flavours.

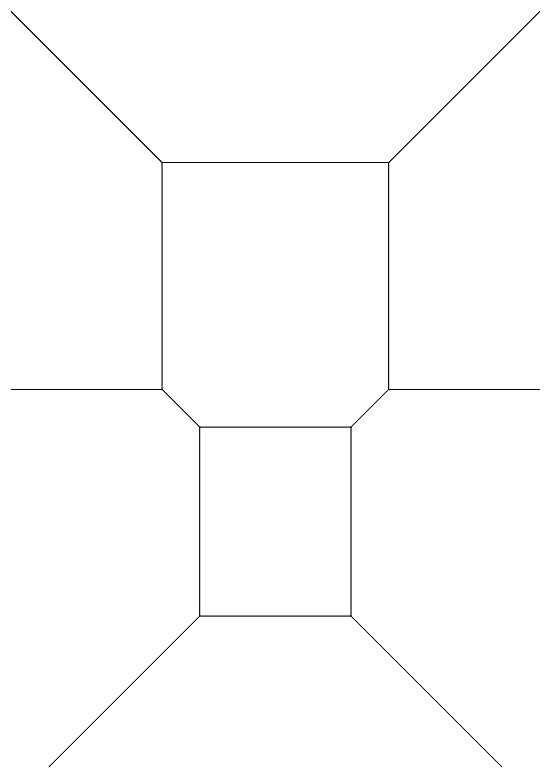


Figure 3.10: S-dual of the $SU(2) \times SU(2)$ which is $SU(3)$ with two flavours.

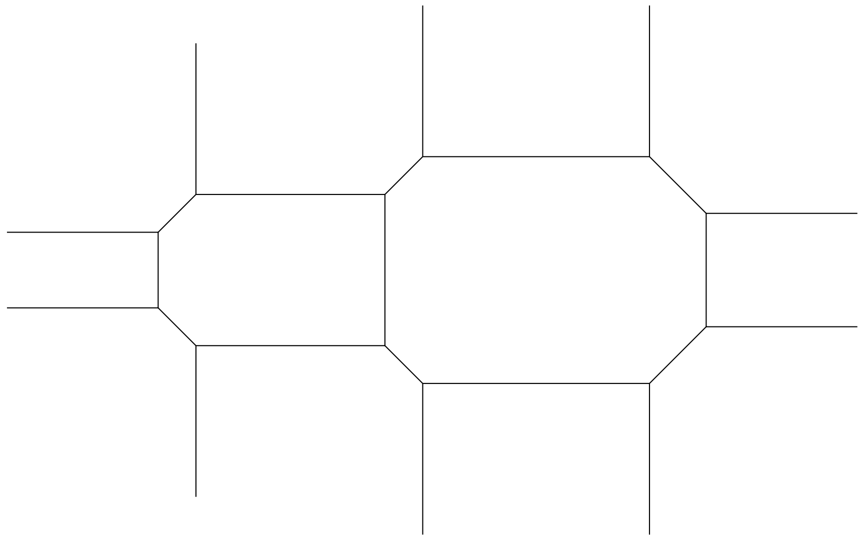


Figure 3.11: Brane web for an $SU(2) \times SU(2)$ theory with two fundamental hypermultiplets charged under each gauge group, as well as a a bifundamental hypermultiplet charged under both. The mass of the bifundmental is set to zero and the masses of each pair fundamentals kept equal.

3.6 Example: $SU(2) \times SU(2)$ with four flavours

As a final example let us consider the theory described by the quiver FIG. This is like the $SU(2) \times SU(2)$ theory we considered previously now with two flavour hypermultiplets charged under each gauge group. The brane web for this theory is shown in 3.11, in the phase $\phi_1 > \phi_2$ and $m_i < \frac{1}{2}\phi_i$, where ϕ_i m_1, m_2 are the vevs of the hypermultiplets charged under ϕ_1 and ϕ_2 respectively. This is two copies of the pure $SU(2)$ theory with two flavours glued together along a central NS5-brane. For simplicity in3.11 we have kept the masses of the fundamental hypermultiplets charged under the same gauge $SU(2)$ equal and the mass of the bifundamental zero. This means the brane web is symmetric under reflection through the horizontal axis. The masses of these flavours is then given by half the distance between the external D5-branes.

The local deformations associated to changing ϕ_1 and ϕ_2 are shown in 3.13 and 3.12. Now the local deformation associated to ϕ_1 does not the change the length of the D5-branes bounding the right hand face and this is just given by $\frac{1}{g_1^2}$. For the left hand face we can see from 3.12 its length is

$\frac{1}{g_2^2} + \frac{1}{2}\phi_2 - \frac{1}{2}\phi_1$. $\frac{1}{g_2^2}$ is simply given by the distance between the leftmost and central external NS5-branes. We calculate the areas of the two faces giving us the monopole tensions:

$$T_{M_1} = \frac{\phi_1}{g_1^2} + \frac{1}{2}\phi_1^2 - \frac{1}{4}\phi_2^2 - m_1^2 \tag{3.6}$$

$$T_{M_2} = \frac{\phi_2}{g_2^2} + \frac{3}{4}\phi_2^2 - \frac{1}{2}\phi_1\phi_2 - m_2^2 \tag{3.7}$$

We can then deduce the prepotential as before:

$$\frac{\phi_1^2}{2g_1^2} + \frac{\phi_2^2}{2g_2^2} + \frac{1}{6}\phi_1^3 + \frac{1}{4}\phi_2^3 - \frac{1}{4}\phi_1\phi_2^2 - \phi_1 m_1^2 - \phi_2 m_2^2 \tag{3.8}$$

One can check we get the same answer using (2.2). As before the theory has a number of different phases. The transitions can be see on the brane web when D5-branes become parallel on the brane web, or equivalently in the prepotential when terms like $|\frac{1}{2}\phi_1 - m_1|$ become zero. Each of these different phases lead to a different prepotential and brane web. Naively there are 2^5 many phases associated as there are 5 of these transitions associated with four bare masses and which of ϕ_1, ϕ_2 is greater. Some of these however will of course be qualitatively the same up to relabelling indices. We will see the importance of these phaes later. The BPS spectrum is like two copies of the BPS spectrum for $SU(2)$ with two flavours, however there is an additional string web state available in the right hand face so long as $\phi_1 \neq \phi_2$.

Upon rotating the brane web by 90 degrees, we see the S-dual theory is an $SU(3)$ gauge theory with 6 flavours.

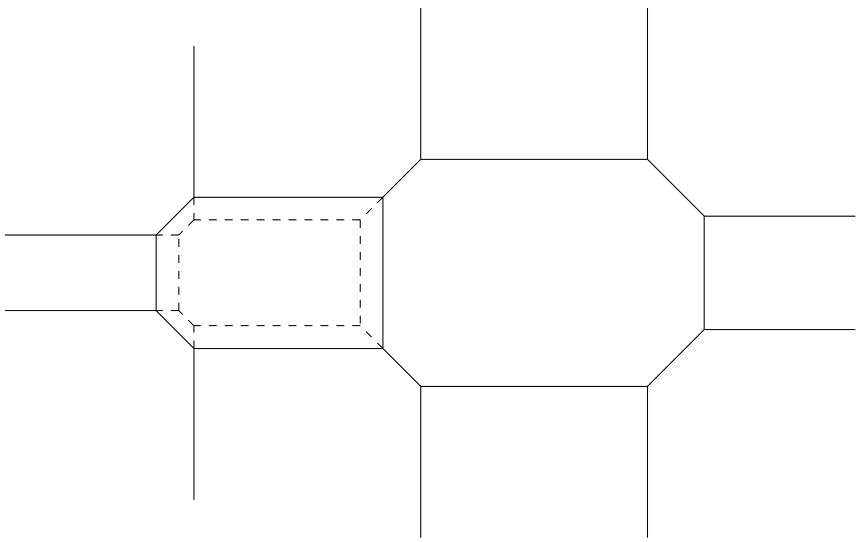


Figure 3.12: Caption

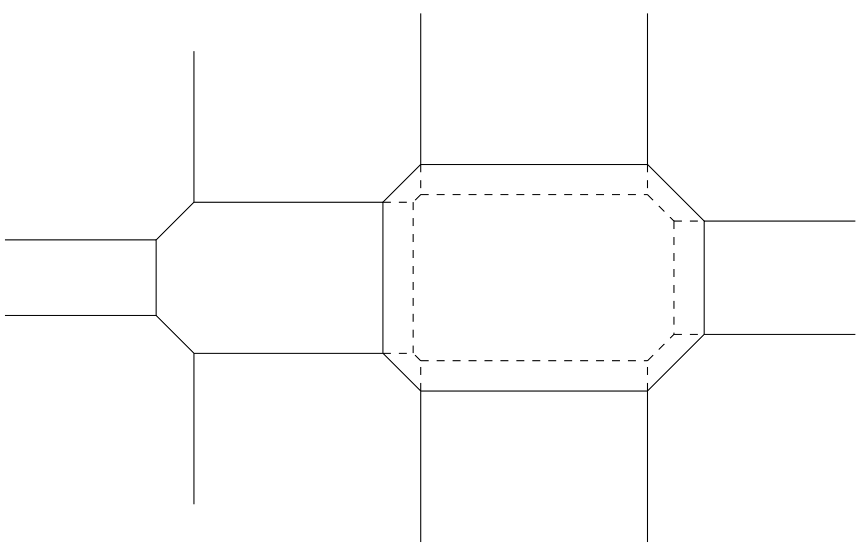


Figure 3.13: Caption

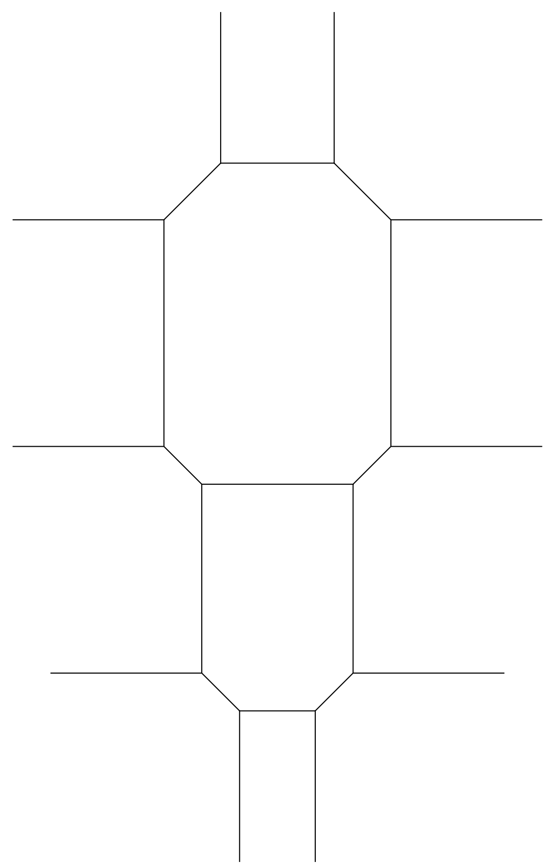


Figure 3.14: S-dual of $SU(2) \times SU(2)$ with 4 flavours which is $SU(3)$ with 6 flavours.

Chapter 4

[p,q] 7-branes, Magnetic Quivers, Tropical Geometry and Higgs branches

We shall now introduce a new tool from string theory in order to aid our study of supersymmetric field theories. We can enhance our brane webs with the use of "[p,q] 7-branes". The application of these objects to brane webs was introduced in [18], from which we will reproduce the basic facts. Again we will not derive the results and simply provide the necessary tools to do computations. In particular we will omit dicussion of monodromy and the Hanany-Witten effect, a more thorough discussion is contained in [18]. Similarly to (p,q) 5-branes and (p,q) strings, the ordinary D7-brane corresponds to a [1,0] 7-brane. (p,q) 7-branes transfrom under $SL(2,Z)$ identically to (p,q) 5-branes and strings and may be introduced into brane webs without breaking any additional supersymmetry, leaving us 8 supercharges. The key fact is that a (p,q) 5-brane can end on a [p,q] 7-brane. The advantage of this is we can now dispense with infinite external legs, letting external (p,q) 5-branes end on [p,q] 7-branes instead of extending to infinity. To see how this works let us consider again the pure $SU(2)$ theory now with 7-branes depicted in 4.1. We will take the 7-branes to occupy dimensions $x^0 \ldots x^4$ and $x^7, \ldots x^9$. They then occupy a point in the x^5, x^6 plane. In our brane webs we will represent the position of a 7-brane by a circle, if necessary writing its [p,q] charge adjacent, as in 4.1. In 4.1 we have two [1,1] and two [1,-1] branes on which the external (1,1) and (1,-1) 5-branes end. The basic argument for replacing external legs with 7-branes as laid out in [18], is that moving the

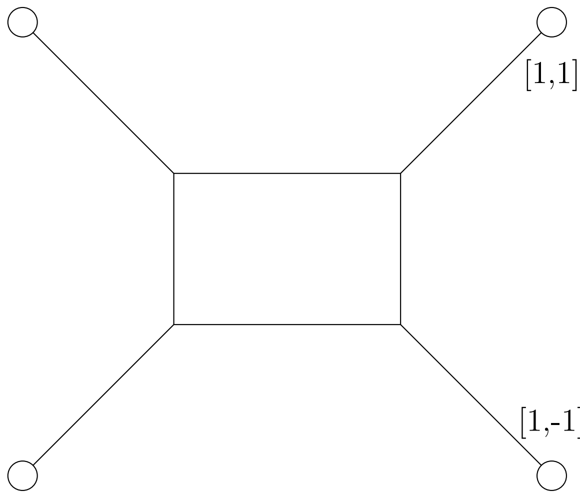


Figure 4.1: Caption

[p,q] 7-branes along the direction of the (p,q) 5-branes should not produce any change in the field theory. Accepting this, we could imagine moving the 7-branes very far away from the brane web, so that the theory is indistinguishable from the theory with infinite external legs. Since the theories differ by moving 7-branes, we conclude they are the same. This allows us to end any external leg on a 7-brane. This is useful for a number of reasons, firstly it removes difficulties that were conjectured to arise due to parallel external legs [23]. Secondly, there is a rich theory of 7-brane symmetry algebras - see for example [24, 25] - which can be used to directly calculate the global symmetry of the theory, for example this is done in [18]. This is particularly useful for strong coupling fixed points, where the global symmetry may be enhanced [13]. This is not something that is readily computable without 7-branes. Finally, 7-branes enable us to understand how we may separate a brane web into subwebs. As we will see, this allows to understand the form of the Higgs branch of the theory. It is this last application we will focus on in this chapter.

4.1 The Higgs Branch

We review here the basic features and results regarding the Higgs branch. Recall the Higgs branch is the component to the moduli space where the vevs of the vector multiplet scalars are zero and the the vevs of matter hypermultiplet scalars are non zero. Recall for the Coulomb branch of a theory with a lagrangian description the vacuum equations restricted the vevs to lie in the cartan subalgebra of the gauge group, and gauge invariance restricted the vevs to lie in a single Weyl chamber. The geometric form of the Coulomb branch is then just $\mathbb{R}^r/W$, where W is the Weyl group of the gauge group and r the rank. So far our analysis of brane webs has been for theories on their Coulomb branch. For the Higgs branch, the story is different. Firstly let us recall that each hypermultiplet has four real scalars, so the space of allowed vevs for the flavour hypermultiplets is $\mathbb{R}^{4N_f}$, where N_f is the number of flavour hypermultiplets. We must then reduce this space down by first restricting to solutions of the vacauum equations - this ensures a configuration of vevs describes a proper supersymmetric vacuum state. Secondly we must factor out the gauge symmetry, ensuring we are not artificially distinguishing between vacua related by a gauge transformation. The first step involves solving algebraic equations, the second stape involves a mathematical construction known as a Hyper-Kähler quotient [26], where one divides out the gauge redundancy of the description. After this computation the Higgs branch is generally found to be a Hyper-Kähler cone, or the union of a number of cones, sometimes with interesting intersections [12]. An important point is that the Higgs branch does not receive quantum corrections due to a supersymmetric non-renormalisation theorem [27]. As a consequence it is determined by an algebraic computation which does not depend on the spacetime dimension, so long as we are dealing with 8 supercharges. Therefore the Higgs branches for given gauge group and matter content is the same for 3d $\mathcal{N}=4$, 4d $\mathcal{N}=2$ and 5d $\mathcal{N}=1$, all theories with 8 supercharges [11]. Although the Higgs branch does not generally receive quantum corrections at finite coupling, in 5d theories at infinite coupling there is a phenomenon where the Higgs branch can be altered, often generating an extra cone. This can be attributed to the appearance of massless instantons [8] as the coupling goes to infinity. Recall we saw briefly in chapter 1 the appearance of enhanced global symmetry at infinite coupling, for $SU(2)$ with zero and two falvours, which could also be understood as being due to the instanton mass going to zero. It was understood in [9] that the Higgs branch at infinite coupling could be computed by calcuating the Coulomb branch of a different theory in three dimensions, a problem that had previously been tackled [6]. In [10] a method was developed for finding the Higgs branch using (p,q) 5 and 7-brane webs. In this method which we will describe, the Higgs branch can be found by seeing how a brane web decomposes into subwebs and considering the intersection of these subwebs. In particular it is just as easy to find the Higgs branch at infinite coupling as it is at finite coupling(indeed in many cases it is easiest at infinite coupling). In what follows, we will provide a pedagogical introduction to this method, and then apply this method to $SU(2) \times SU(2)$ theory with four flavours.

4.2 Decomposing brane webs

We will describe how brane webs can be decomposed into subwebs. The methods and results we will describe here were introduced in [10] and further developed in [11]. Let us recall in our brane webs, a (p,q) 5-brane spans dimensions $x^0, \ldots x^4$, occupies a line in the (x^5, x^6) plane, and a point in the remaining dimensions x^7, x^8, x^9. The (p,q) 7-branes occupy a point in the (x^5, x^6) plane and span the remaining 8 dimensions. Our brane webs so far have just shown the (x^5, x^6) plane taking the branes to be at the same point in (x^7, x^8, x^9). If we have a brane web which can be decomposed into a number of subwebs - where a subweb is a subset of the brane web which itself forms a legitimate brane web, satisfying charge conservation at vertices and having the correct slope - then we have the option to move these subwebs apart from each other in (x^7, x^8, x^9) space. In the same way as the brane webs we have seen so far, which all descrived the Coulomb branch, moving branes apart gives masses to open strings, which in field theory terms means giving mass to gauge bosons by turning on scalar vevs. When we move apart branes in the (x^7, x^8, x^9) dimensions however, we describe the theory on the Higgs branch rather than the Coulomb branch. The simplest example of this decompoisition can be seen for the pure $SU(2)$ theory 4.1. This theory has no matter and so classically does not have a Higgs branch. At infinite coupling however massless instantons generate a moduli space which acts as a Higgs branch. We can see all this from the brane web. 4.2 shows the origin of the Coulomb branch for the pure $SU(2)$ theory at both finite and infinte coupling. At finite coupling there is no way to decompose the brane web. The presence of the two central D5-branes require the (1,1) and (1,-1) branes emerging at either side in order to satisfy charge conservation. We have drawn these D5-branes with a small separation to make the brane web more readable, they should be thought of as coincident at the origin of the Coulomb branch. At infinite coupling however there are no central D5-branes

Figure 4.2: The origin of the Coulomb branch for pure SU(2). The left hand brane web shows the origin at finite coupling, the right hand shows infinite coupling. We have drawn the finite coupling web with a small separation between the finite D5-branes to make the diagram clearer, at the origin these branes become coincident.

at the origin of the Coulomb branch, and the brane web is composed of a (1,1) and (1,-1) 5-brane which intersect. Each of these 5-branes taken individually form valid brane webs, and so the brane web can be decomposed into two subwebs each consisting of a single 5-brane. One can then move these two 5-branes apart in (x^7, x^8, x^9).

Let us now examine the Higgs branch a more standard example of a theory with matter which have seen before, namely $SU(2)$ in two flavours. This theory has two phases shown in 3.9 and 3.9 which correspond respectively to the phases $m < \frac{1}{2}\phi$ and $m > \frac{1}{2}\phi$ where m is the bare mass of the hypermultiplets which we take to be equal. The origin of the Coulomb branch is shown in 4.3 for both phases. Since at the origin ϕ is zero, if the hypermultiplet mass is non-zero it will necessarily be the theory the large mass phase shown in 3.8 at the origin. For both hypermultiplet masses finite, shown on the left hand side of 4.3, we can see the structure is essentially the same as the pure $SU(2)$ theory, with some extra 7-branes attached at the end which don't affect the decomposition of the web. This is what we would expect from field theory, supersymmetric vacuum states must have zero energy [19] and so massive scalars cannot have a non-zero vev. Thus the hypermultiplets may as well not be there as far as the Higgs branch is concerned if they are massive, which is reflected clearly in the brane web. On the right hand side where the hypermultiplet masses are zero, the situation is different. We clearly have an NS5-brane ending on two (0,1) 7-branes which can form a subweb. One might ask if the brane web can be decomposed any further. Naively one could imagine the rightmost D5-brane which is suspended between two (1,0) 7-branes can also form its own brane web. However this decompistion violates something known as the *s-rule*. This was first discussed in [16] for

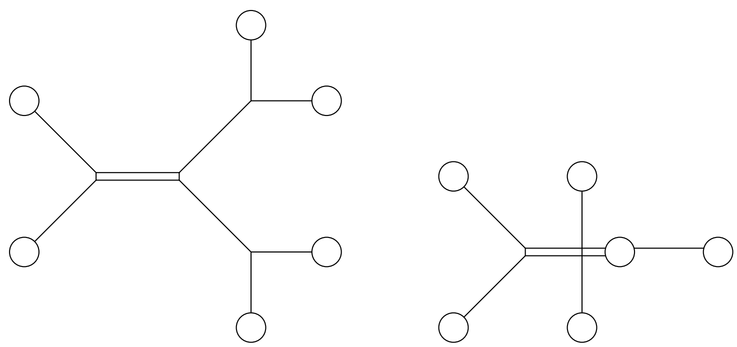


Figure 4.3: The origin of the Coulomb branch for $SU(2)$ with two flavours. The left hand brane web shows the theory for finite hypermultiplet masses, the right hand brane web shows the theory for zero hypermultiplet masses.

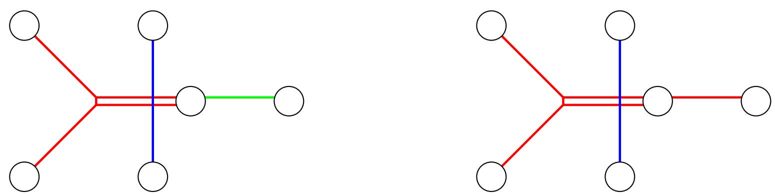


Figure 4.4: Demonstration of the s-rule. Different colours signify a different subweb which can be at different positions for (x^7, x^8, x^9). The left hand brane web violates the s-rule as the red subweb has two D5-branes ending on the same NS5-brane and the same [1,0] 7-brane

brane configurations for 3d theories. In our case the s-rule states that we cannot while preserving supersymmetry have two D5-branes which end on the same NS5-brane and the same 7-brane in a single subweb [11]. We can see an example of this in 4.4. In these figures we use a different colour for each subweb. In the leftmost web, the red subweb has two D5-branes which end on the same NS5-brane and the same 7-brane. Moving apart this web in the (x^7, x^8, x^9) directions would break supersymmetry. The right hand brane web shows a valid decomposition. In this decomposition the two D5-branes end on the same NS5-brane however they end on different 7-branes. One of them ends on the first [1,0] brane and the other continues to end on the rightmost [1,0] brane. Note we are depicting the 7-branes as circles however they are in reality pointlike in the (x^5, x^6) plane.

Applying the s-rule, we find the brane web for $SU(2)$ with two flavours with zero bare mass, the brane web decpomposes into two subwebs at the

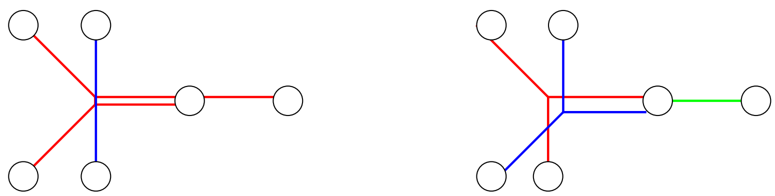


Figure 4.5: The two maximal decompistions for $SU(2)$ with two flavours. In the right hand web we have separated the webs slightly so they are visible, they should however be considered coincident as in the left hand side.

origin of the Coulomb branch. We can immediately calculate the dimension of the Higgs branch from this. Firstly note since hypermultiplets contain four real scalars, the degrees of freedom on the Higgs branch only come in groups of four. Clearly each subweb adds a new degree of freedom however there is one degree of freedom assoicated with translating the entire web in (x^7, x^8, x^9) space. So then if n is the number of subwebs, the dimension is given by $4(n-1)$ [11]. We also call $n-1$ the quaternionic dimension of the Higgs branch. We see the quaternionic dimension of the Higgs branch of $SU(2)$ with two massless hypermultiplets is 1. The right hand side of 4.4 shows the maximal decomposition at finite coupling. Let us see if any thing happens at infinte coupling. In 4.4 this correspond to moving the vertical blue NS5-brane to the point where the the (1,1) and (1,-1) 5-branes on the left hand side intersect, shown on the left hand side of 4.5. One can see this by taking the rectangular part of the face in 3.9 to the left of the external NS5-brane to be a square, and then shrinking the face and the hypermultiplet mass to zero.

At infinite coupling, the finite coupling decompisition is still perfectly valid, however we gain another option, showon on the right hand side of 4.5. We can split the vertical NS5-brane into two separate branes, the top half forms a vertex with the (1,1) 5-brane coming form the bottom left, the bottom half forms a vertex with the (1,-1) 5-brane coming from the top left. These two vertices satsify charge conservation and may both end on the [1,0] 7-brane. Since the two D5-branes are now part of different subwebs, the s-rule does not apply. The right most D5-brane shown in green can now also separate into its own subweb. We have found at infinite coupling two distinct decompositions are possible. One with quaternionic dimension 1 and the other with quaternionic dimension 2. Since brane web admits two inequivalent decompositions we interpret this as corresponding to two different components of the Higgs branch.

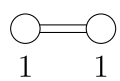
4.3 Tropical Geometry, Magnetic Quivers and the Higgs Branch

With the techniques detailed so far we have been able find the number of components of the Higgs branch and the dimension of these components. There is more to the structure of the Higgs branch than this. The different components are geometric objects called symplectic singularities which have their natural mathematical setting in algebraic geometry [5]. It turns out the form of these spaces can be found by considering how the different subwebs in the decomposition intersect. The process for computing these spaces involves looking at how the different subwebs in a given decomposition intersect. From these intersections we can compute a quiver diagram known as a *magnetic quiver*. This is a quiver diagram like those discussed previously. The component of the Higgs branch associated to that decomposition is then completely specified by the magnetic quiver, except possibly for certain nilpotent elements [11] which we will not discuss here. A magnetic quiver is just a quiver diagram and defines a gauge theory in any dimension. The fundamental result first introudced in [10] is that if we considerr the 3d $\mathcal{N} = 4$ gauge theory specified by the magnetic quiver, and find the corresponding Coulomb branch, this gives us the Higgs branch of the original 5d theory. Finding the Coulomb branch of a 3d $\mathcal{N} = 4$ is a straightforward algorithmic procedure which was introduced in [6]. In this work the Coulomb branch is found by enumerating the different monopole operators in the theory. Using a formula known as the monopole formula, one can compute a generating function known as the *Hilbert series* [28]. The Hilbert series counts gauge invariant operators on the moduli space graded by their charges under abelian global symmetries. If we know the Hilbert series we know all the gauge invariant operators, since only gauge invariant objects have physical meaning this constitutes a full description of the moduli space.

Let us now describe how to obtain the magnetic quiver from the brane webs. The idea is that each subweb froms a circular gauge node, the number of links between each pair of nodes is determined by an integer called an intersection number, also known as the *stable intersection*. Let us consider two subwebs A and B. The contributions to the intersection number I_{AB} due to intersection are as follows. For each intersection $I_i$between a (p,q) 5-brane from subweb A and a (s,t) 5-brane from subweb B, we get a contribution $I_i = |pt - qs|$. The contribution due to intersetions is then $\sum_i I_i$, where the sum is over the mutual intersections of 5-branes belonging to the two subwebs. Let X denote the number of times a (p,q) 5-brane from subweb A and a (p,q) 5-brane from subweb B end on the same [p,q] 7-brane from the *opposite* side, and Y denote the number of times a (p,q) 5-brane from subweb A and a (p,q) 5-brane from subweb B end on the same [p,q] 7-brane from the *same* side. Then the full intersection number between A and B is

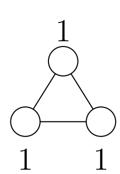
$$I_{AB} = \sum_i I_i + X - Y \tag{4.1}$$

In the magnetic quiver, the number of links between node A and node B is given by I_{AB}. Let us compute this for the brane webs we have seen so far. For the $SU(2)$ theory at infinite coupling 4.5 we have two subwebs comprising a (1,1) and (1,-1) 5-brane respectively. They intersect once with intersection number 2. The corresponding magnetic quiver is:



(4.2)

Now consider $SU(2)$ with two flavours at finite coupling shown in the right hand side of 4.4. There are two subwebs and two intersections between them, both between (1,0) and (0,1) 5-branes. These interesections both have intersection number 1 and so the total intersection is 2, giving the same quiver as $SU(2)$ at infinite coupling. At infite coupling we have the previous quiver associated to one decomposition and a second one shown on the right hand side of 4.5. This has three subwebs. The blue and red subwebs on the left have two intersections with intersection number 1 as drawn, however they also end on the same 7-brane once, leading to an overall intersection number of 1. The both end on the same 7-brane on the opposite side as the green subweb on the right which giving them both an intersection number of 1 with the green subweb. This leads to a magnetic quiver:

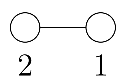


(4.3)

If we have n indentical subwebs which are coincident on the brane web, we treat them as one node of rank n. To find the intersection number between nodes on the magnetic quiver, we only consider the intersection of one of the identical subewebs. For example consider the brane web on the left hand side of 4.6.

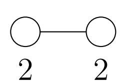
Figure 4.6: Examples of brane webs with identical coincident subwebs.

This decomposes into three subwebs, two of which are identical and coincident(recall we leave a small separation between branes in our diagrams so they can be distinguished, at the origin of the Coulomb branch there is no separation). The two red D5-branes contribute one node in the magnetic quiver of rank 2. To find the intersection with the blue NS5-brane we consider how one of the red D5-branes intersects with the blue subweb, which has intersection number 1. The corresponding magnetic quiver is:



(4.4)

For the brane web on the right hand side of 4.6 we have four subwebs which split into two groups of identical subwebs contributing two nodes of rank two. To find the intersection we consider only one subweb from each group of identical subwebs. So the intersection number is given by the intersection of one D5-brane and one NS5-brane as previously yielding the magnetic quiver:



(4.5)

4.4 Higgs branch of $SU(2)\times SU(2)$ with 4 flavours

In this section we will compute the magnetic quivers for the Higgs branch of the theory given by the quiver 2.2, the brane web for this theory is shown in 3.11. This an $SU(2) \times SU(2)$ theory with 4 flavours. We will study how the Higgs branch of this theory changes in different parts of the parameter space. We saw previously 4.3 the origin of the Coulomb branch looks different depepnding on the the values of the gauge coupling and the hypermultiplet masses. We will keep the bifundamental hypermultiplet at zero mass throughout. The first parameters to consider are the four hypermultiplet bare masses. As we saw previously for the $SU(2)$ theory with two flavours, there are only two cases for the bare mass relevant to the form of the Higgs branch, zero or non-zero. There is no distinction on the Higgs branch as seen from the brane web between a hypermultiplet having non-zero or infinite mass or indeed being removed from the theory. We saw this in 4.3. To find the structure of the Higgs branch for zero mass we can draw the brane web for the case $\frac{1}{2}\phi = m$ and then go to the origin of the Coulomb branch by contracting the internal faces to zero area. This usually corresponds to moving the external D5-branes so they coincide horizontally with the internal finite D5-branes, such as in 4.7.

To find the Higgs branch for a finite bare mass we consider the brane web in the phase $\frac{1}{2}\phi < m$ and go to the origin of the Coulomb branch in that phase. In this case, one external NS5-brane and one external D5-brane effectively gets replaced by an external (1,1) or (1,-1) 5-brane, see for example 4.3. Once we are at the origin of the Higgs branch for a given set of mass parameters, we can then tune the coupling. We will consider two cases of the gauge coupling, finite and infinite. Having deterimined the strucutre of the origin of the Coulomb branch, a gauge coupling can be tuned to infinity by moving NS5-branes together. We saw this in 4.5. The left hand side shows the brane web at finite coupling, in the right hand side we have moved the external NS5-brane so that it coincides with the the internal NS5-brane which correspond to infinite coupling. At this point the Higgs branch changes, an additional decomposition is possible representing the generation of an additional component in the Higgs branch at infinite coupling. We will see many examples of this in the following calculations. Naively there are 4 mass parameters and two gauge couplings which have two meaningful values: zero or non-zero for the masses, finite or infinite for the gauge couplings. This gives 2^6 possible different Coulomb branch origins and corresponding Higgs branches. However unsurprisingly many of these phases are equivalent up to relabelling of indices (in other words up to 90 degree or 180 degree rotations of the brane web) and so in reality the number of phases is less. In what follows we will organise the phases by first stating which masses are zero or non-zero, and then computing the Higgs branch for each possible combination of finite/infinite couplings. In some cases there will three different Higgs branch phases for a given set of mass parameters, corresponding to all couplings finite, one infinite one finite, and all couplings infinite. In other cases there will be four different phases when it matters which gauge coupling is finite. For example, in the case where one hypermultiplet is massive and one coupling is infinite, we get a different Higgs branch depending on which coupling is infinite corresponding to whether or not the massive hypermultiplet is charged under the gauge group factor with infinite coupling. This is all made manifestly clear in the brane web picture. For simplicity in what follows, we will only indicate the rank of a gauge node if it is greater than 1, a gauge node with no indicated rank should be read as a rank 1 node.

4.4.1 Case 1: All masses zero

The brane web for the theory in the phase $\frac{1}{2}\phi = m$ at a point on the Coulomb branch is shown in 4.7. Moving to the origin of the Coulomb branch we obtain the brane web and associated magnetic quiver depicted in 4.8. As we can see there are 7 subwebs giving a Higgs branch of quaternionic dimension 6. The red subwebs are identical giving a rode of rank 2, and the rest intersect once with the red subweb giving the quiver on the right hand side of 4.8. In this and what follows, we have coloured the nodes in the magnetic quiver the same colour as the subweb they correspond to. For simplicity we use the same colour for distinct subwebs when it is clear they are separte subwebs, for instance the blue subwebs in 4.8 are all separate.

The next case is to take one of the gauge couplings to infinity depicted in 4.9. It is clear it does not matter which coupling is taken to infinity in this case. Here we have 9 subwebs giving a quaternionic dimension of 8. The third case corresponding to taking both couplings to infinity is obtained by moving the final NS5-brane to coincide with the others as depicted in 4.10. The corresponding Higgs branch has quaternionic dimension 12. We can clearly see in the previous examples the enhancement of the Higgs branch as each coupling is tuned to infinity, with the dimension increasing from 6 to 9 and then to 12 as the couplings are tuned to infinity.

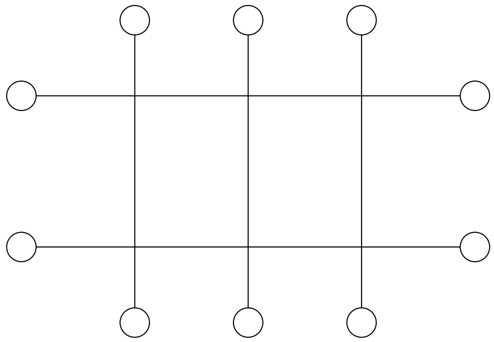
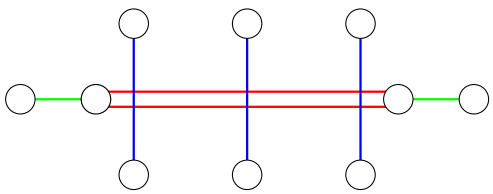


Figure 4.7: The phase $\frac{1}{2}\phi_i = m_i$ on the Coulomb branch.



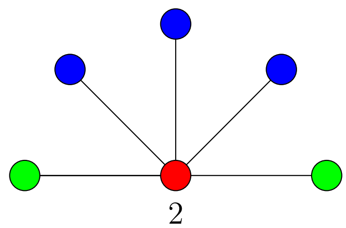
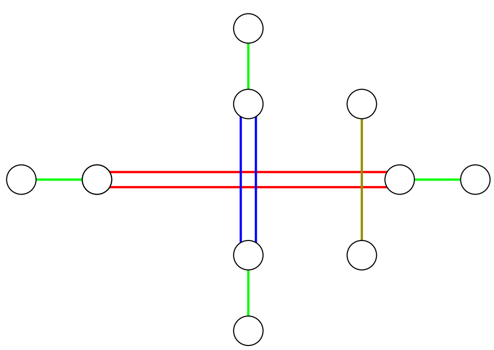


Figure 4.8: The left hand side shows the brane web at the origin of the Coulomb branch for all masse zero and all couplings finite, the different subwebs are coloured differently. We have used the same colours for subwebs which are clearly distinct. The right hand side shows the corresponding magnetic quiver. The nodes are coloured according to their correpsonding subwebs. Nodes without an indicated rank are of rank 1.



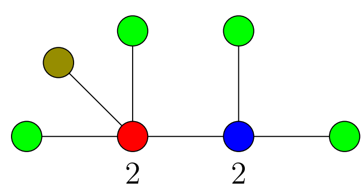


Figure 4.9: All masses zero, one coupling infinite.

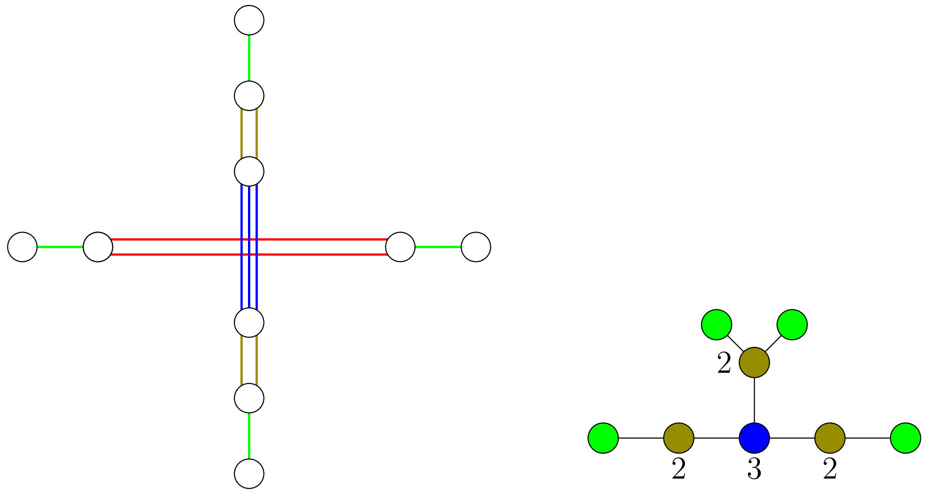


Figure 4.10: All masses zero, all couplings infinite.

4.4.2 Case 2: One mass finite

Now we examine the case where one hypermultiplet is given a finite mass. The relevant phase is depicted on the Coulomb branch on in 4.11, where $\frac{1}{2}\phi_1 < m_1$. The origin of the Coulomb branch and the magnetic quiver with both couplings finite is shown 4.12. In this case there are four phases and four corresponding magnetic quivers. The first phase is shown in 4.12 and has quaternionic dimension 4. There are two cases corresponding to one infinite coupling due to the single hypermultiplet breaking the symmetry of the brane web. The first case depicted in 4.13 corresponds to taking the coupling to infinity for the gauge group factor with no massive hypermultiplets, which has dimension 5. The second case is depicted in 4.14 which corresponds to taking the other coupling to inifnity, which also has dimension 5 but has a different quiver. The final case for both couplings infinite is shown in 4.15 which has dimension 8.

4.4.3 Case 3: Two massive hypermultiplets charged under the same gauge group

The case is when there are two massive hypermultiplets charged under the same gauge group. This is characterised by having two external diagnonal branes on the same side of the brane web. Like Case 2 this gives an asym-

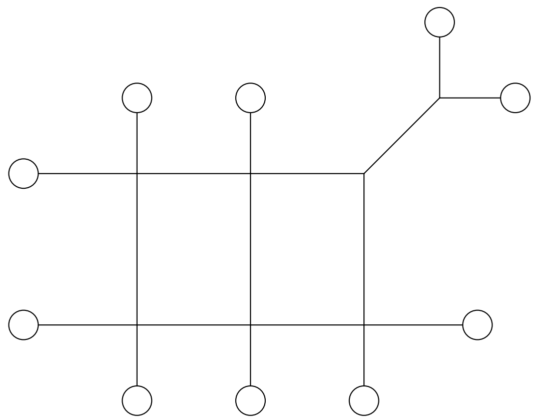


Figure 4.11: Coulomb branch for one mass finite.

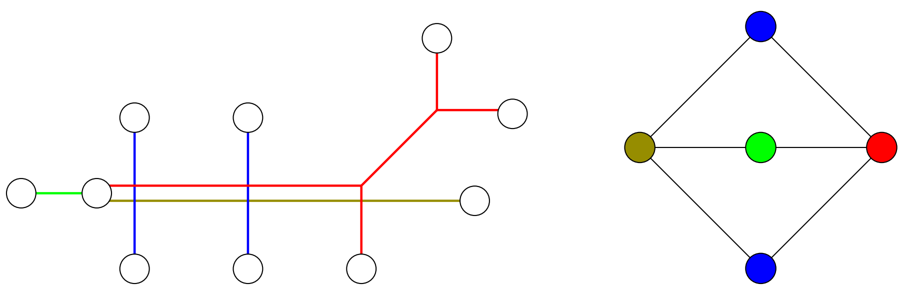


Figure 4.12: One mass finite, all couplings finite

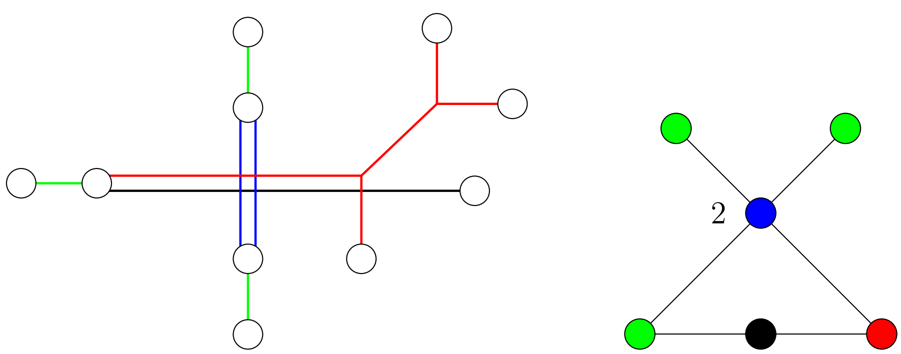


Figure 4.13: One mass finite, 1 coupling finite.

metric brane web yielding four phases in total. The finite coupling phase is depicted in 4.16. In this case the s-rule is essential for determining the decomposition since we have two D5-branes ending on the same NS5-brane.

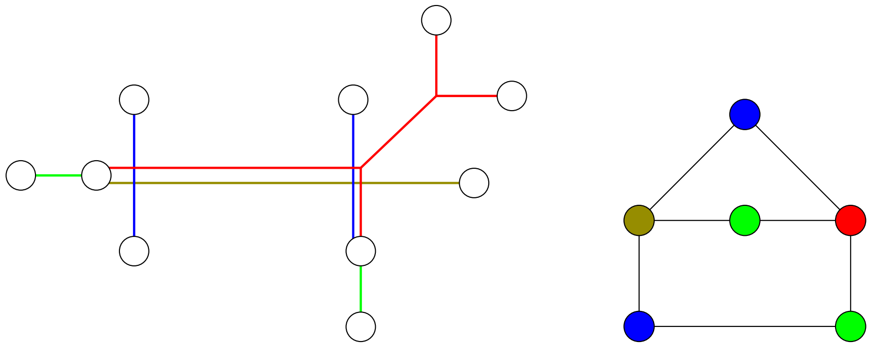


Figure 4.14: 1 mass finite, 1 coupling finite.

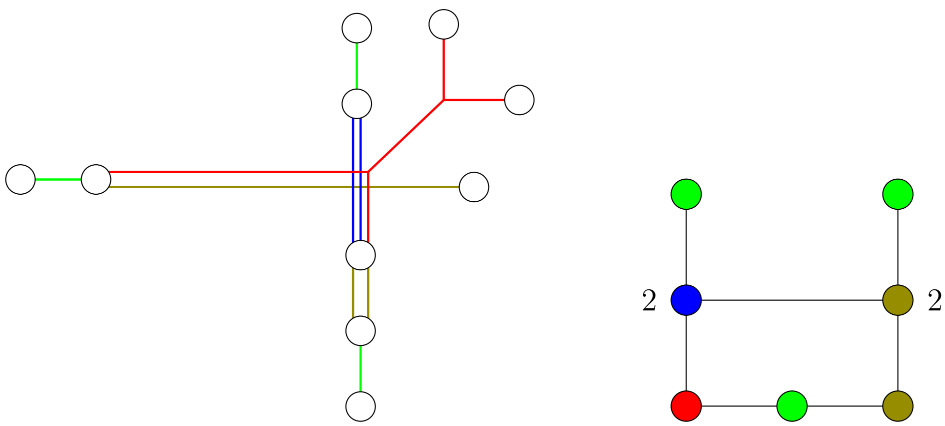


Figure 4.15: One mass finite, all couplings infinite.

We also see our first example of nodes with an intersection number greater than 1. This phase has dimension 2. The first with one infinite coupling is shown in 4.17 which has dimension 3. For the other with one infinite coupling we see our first example of a Higgs branch with mutiple components. We obtain this phase by moving the central NS5-brane to the short NS5-brane on the right where the the two diagonal branes intersect. At this point the decomposition in 4.16 is still perfectly valid however we can now perform a new decomposition shown in 4.18. For this decomposition we split the NS5-brane in two and use it to form new vertices which allows us to separate the original red subweb in two. The leftmost brane (coloured green) may now form its own subweb as the s-rule no longer applies. This new component has dimension 3. The resulting Higgs branch for these parameters then consists of two components specified by the magnetic quivers seen in 4.16 and 4.18. These two components have a trivial intersection as there is no non-maximal decompositon from which they can both be obtained. The final phase with two infinite couplings is depicted in 4.19 which has dimension 5.

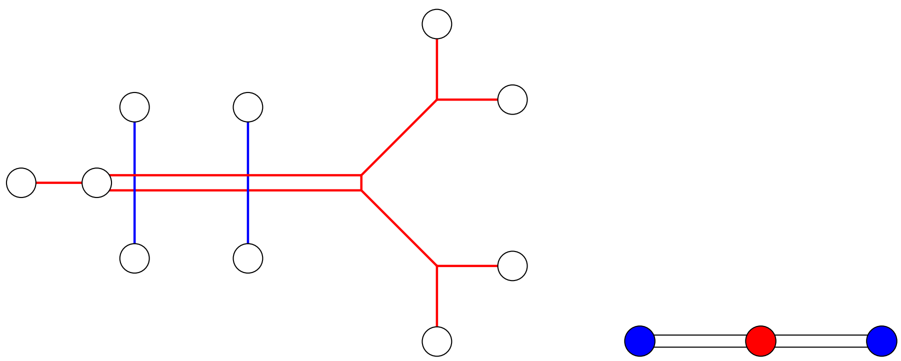


Figure 4.16: 2 masses finite on the same side, all couplings finite.

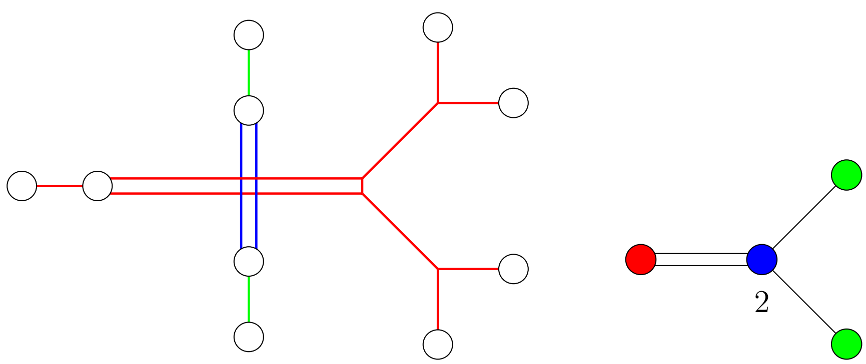


Figure 4.17: 2 masses finite on the same side, one coupling finite.

4.4.4 Case 4: Two masses finite for different gauge groups

In this case we have two massive hypermultiplets but charged under opposite sides. The corresponding brane web is symmetric under reflection and so we only have one phase with one coupling infinite. The finite coupling phase is depicted in 4.20, and with one coupling infinite in 4.21 having dimensions

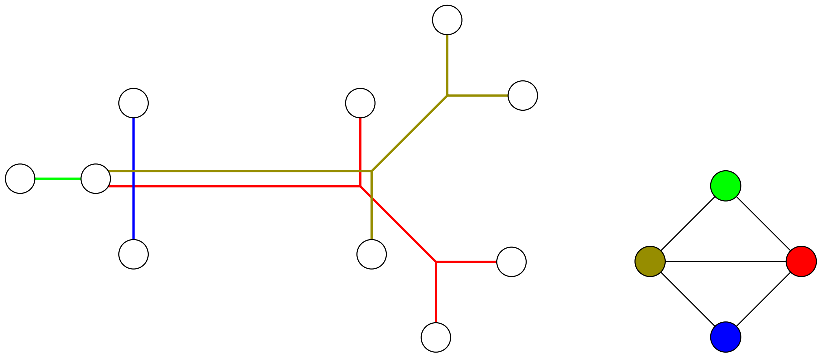


Figure 4.18: 2 masses finite on the same side, one coupling finite.

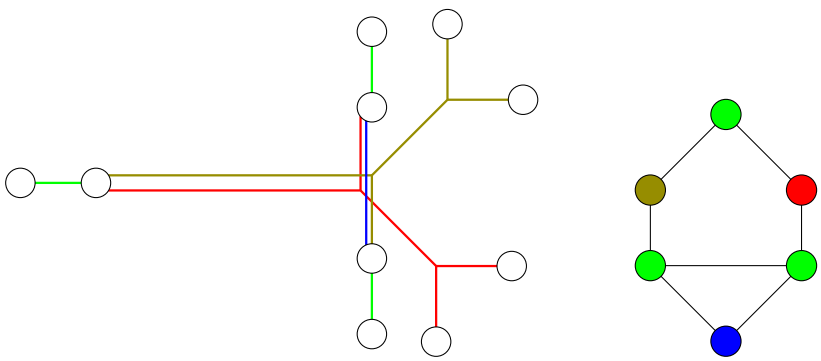


Figure 4.19: 2 masses finite on the same side, all couplings infinite.

2 and 3 respectively. The phase with both couplings infinite again has two components. The first is shown in 4.22 and the second in 4.23 which have dimensions 4 and 5 respectively. Again these have no intersection.

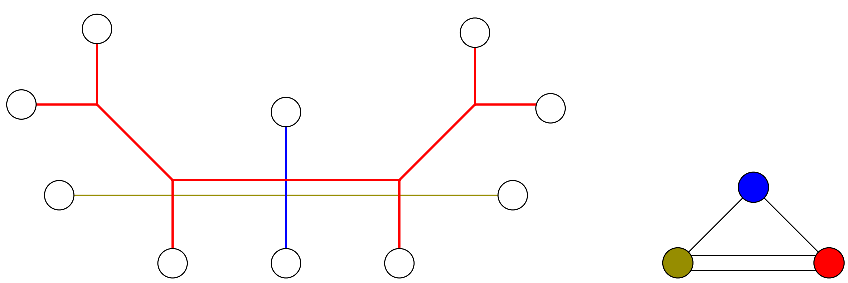


Figure 4.20: 2 masses finite on opposite sides, all couplings finite.

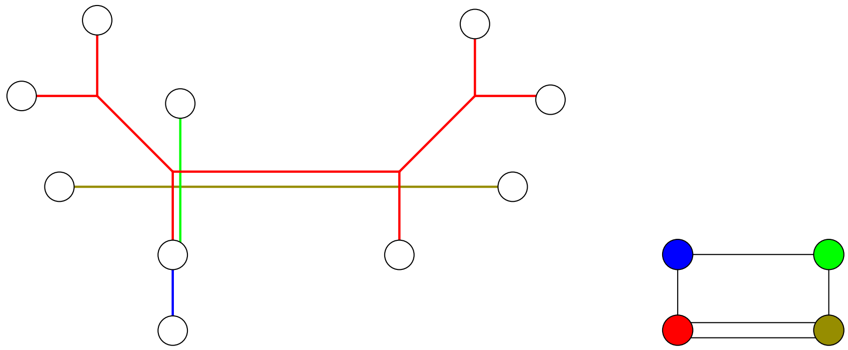


Figure 4.21: 2 masses finite on opposite sides, one coupling finite.

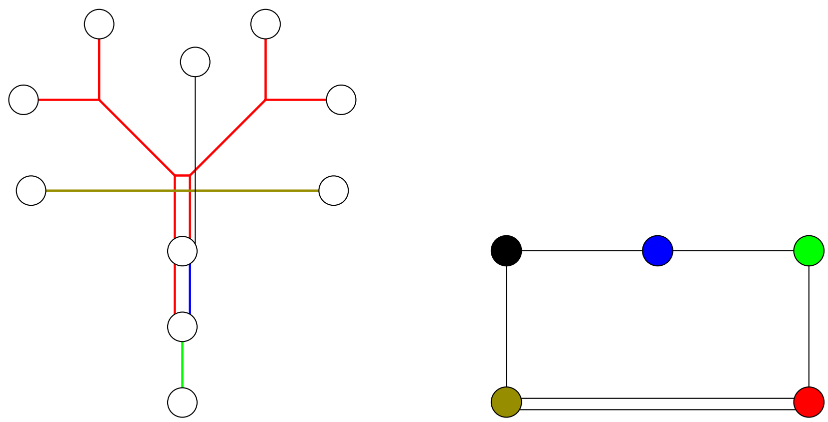


Figure 4.22: 2 masses finite on opposite sides, all couplings infinite, first component.

4.4.5 Case 5: 3 masses finite

In this case there are 3 finite masses. This brane web is asymmetric leading to two phases with one coupling infinite. The finite coupling phase is depicted in 4.24 which has dimension 1. The first case of one coupling finite corresponds to moving the central NS5-brane so that it coincides with the finite NS5-brane on the left hand side. There are two decompositions, the finite coupling one shown in 4.24 and a new one showin in 4.25. This is the same decomposition we saw in Case 3. This phase then has two components of dimension 1. The other phase with one infinite coupling is depicted in 4.26 which has dimension 2. The final case is for both couplings infinite. There are two components, the decomposition from 4.25 and a new decomposition 4.27 with dimension

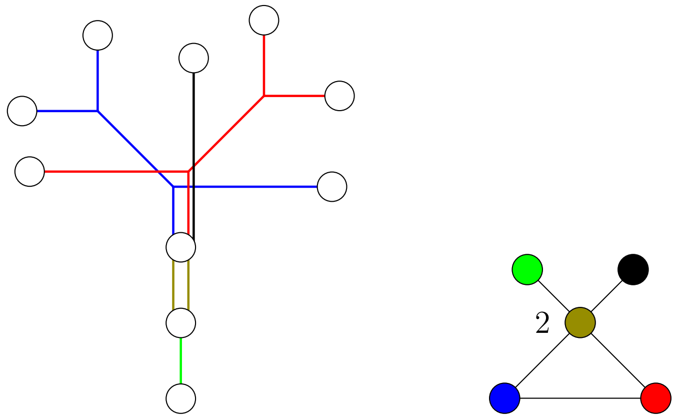


Figure 4.23: 2 masses finite on opposite sides, all couplings infinite, second component.

2. Note the s-rule prevents any further decomposition in 4.25 as the second coupling is brought to infinity. The decomposition in 4.26 is still possible in this case however it is not maximal. the red subweb in 4.26 splits into two subwebs in 4.27.

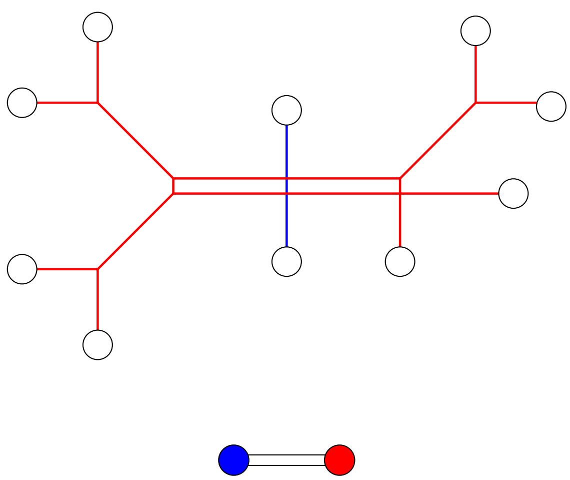


Figure 4.24: 3 masses finite, all couplings finite.

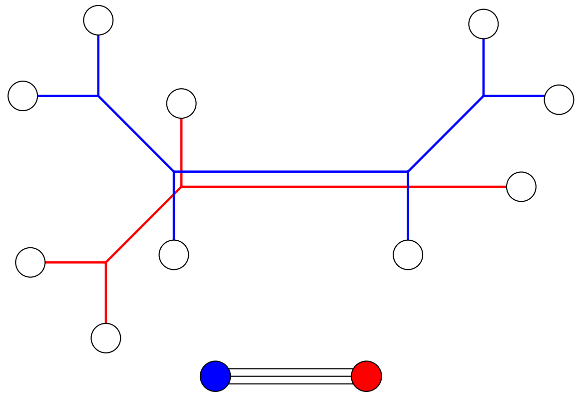


Figure 4.25: 3 masses finite, one coupling finite.

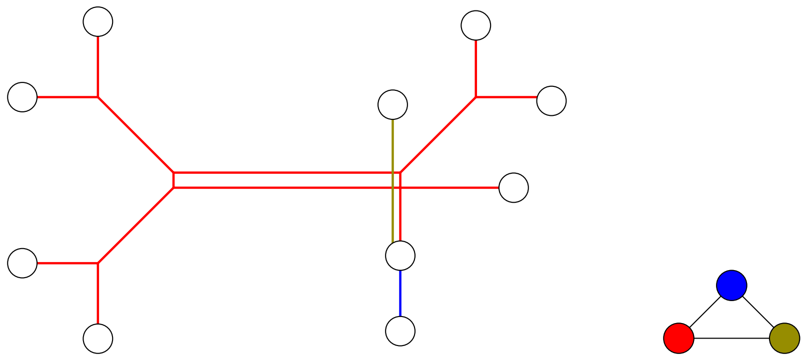


Figure 4.26: 3 masses finite, one coupling finite.

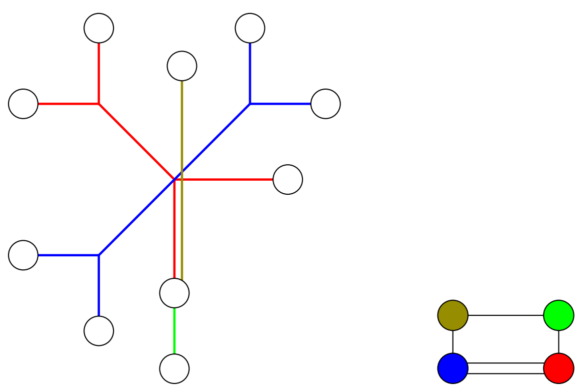


Figure 4.27: 3 masses finite, all couplings infinite.

4.4.6 Case 6: All masses finite

In this case all the hypermultiplets are massive. This will give the Higgs branch of the $SU(2) \times SU(2)$ theory with a bifundamental charged under both gauge group factors we have seen previously. There are 3 phases. The finite coupling phase is shown in 4.28. With one coupling infinite we may decompose the brane web as in 4.19 however this produces a single web corresponding to dimension zero and so does not give us a new Higgs branch. The phase for both couplings infinite is depicted in 4.29 which has dimension 2.

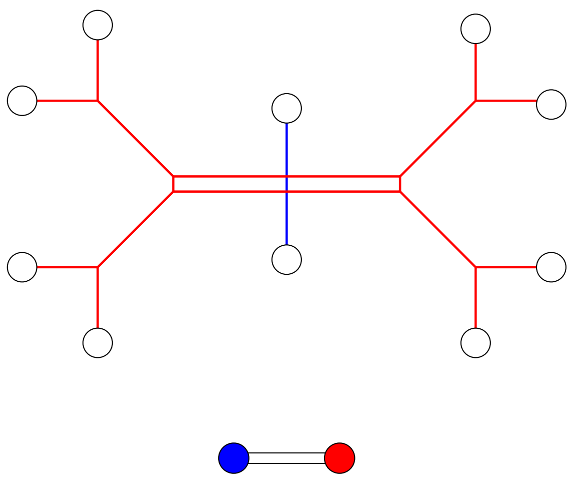


Figure 4.28: All masses finite, all couplings finite.

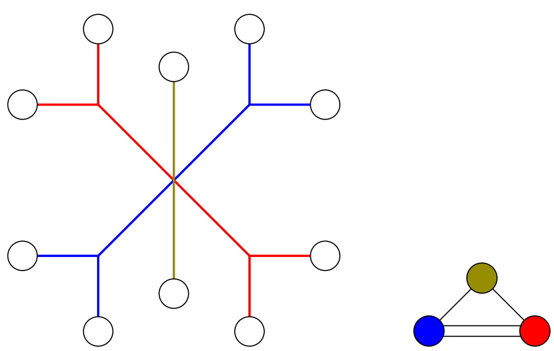


Figure 4.29: All masses finite, all couplings infinite.

4.4.7 Summary

We have seen a single theory has a rich variety of phases in its Higgs branch for different parameter values. The enhancement of the Higgs branch at infinite coupling is clear from the brane web. The dimension always increases due to additional massless instanton states and sometimes additional components are generated.

4.5 Global Symmetry

It is straightforward to use the monopole formula [6] to calculate the Hilbert series and thus determine the structure of the Higgs branch from the magnetic quiver. This is calculationally intensive however and we can use a simpler method [29, 30] This method, referred to as the Balance Global Symmetry in [30] allows us to find the global symmetry of the moduli space associated to a magnetic quiver. For this we need to introduce the concept of balance. In a magnetic quiver, we define the excess e_i of a node to be

$$e_i = f_i - 2r_i \tag{4.6}$$

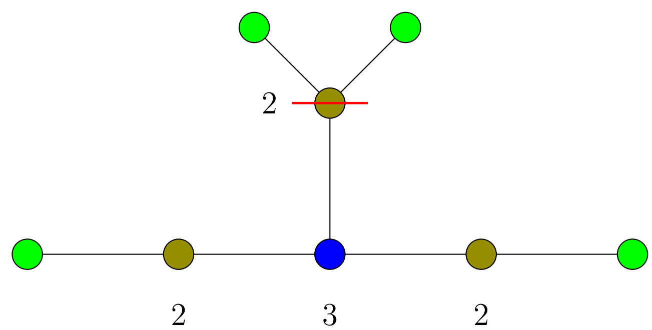
Where f_i is the number of flavours this node sees and r_i is the rank of the node. The number of flavours a node sees is given by $\sum_{adjacent} r_i l_i$ where the sum is over all adjacent nodes, r_i is the rank of an adjacent node and l_i the number of links. For instance an adjacent node of rank 2 linked by one line contributes 2 to the excess, an adjacent node of rank 2 linked by 2 lines contributes 4. A quiver is said to be *balanced* if its excess is zero. There is also a distinction between *framed* quivers and *unframed* quivers. An unframed quiver is one which contains no flavour nodes(no square nodes). A framed quiver is one that contains flavour nodes, for example 2.2. All the magnetic quivers we obtain will be unframed so will state the rules in this case.

- Step 1: Find the excess of each node and identify all unbalanced nodes.
- Step 2: *Ungauge* one unbalanced node of rank 1, this means replace a circular node with a square node. If there are no nodes of rank 1 ungauge a higher rank node.
- Step 3: Considering the remaining balanced gauge nodes, identify the set of connected balanced quivers g_i and the number k of unbalanced

gauge nodes. If we ungauged a node of rank greater than 1, then we subtract 1 from k.

- Step 4: The global symmetry is then given by $\prod_i G_i \times U(1)^k$, where G_i is the group obtained from the quiver g_i, interpreting it as the Dynkin diagram of a lie algebra.

For example let us compute the global symmetry of Higgs branch at infinite coupling with all flavours massless, shown in 4.10. The quiver is:



(4.7)

All nodes are balanced in this quiver except the node of rank 2, indicated with a red line in (4.7). Following the method, we ungauge this node. The remaining connected quivers are the bottom line link of 5 quivers, which is the same is the A_5 Dynkin diagram, and the two isolated green nodes at the top, which is the A_1 Dynkin diagram. We recall that A_n gives the lie algebra of the group $SU(n+1)$. In conclusion we have a global symmetry group $SU(6) \times SU(2) \times SU(2)$. The method we have used above does not always work [30], for certainty it is necessary to compute the Hilbert series. We can however check this result against what we know from the brane web. The brane web has 10 external legs meaning the global symmetry has rank 7. Indeed the rank of $SU(6) \times SU(2)^2$ is 7. Let us compute the global symmtetry for the other phases of the Higgs branch with massless flavours. With one coupling infinite 4.9 we get $SU(4) \times SU(2)^3$ which has rank 6, and for no couplings infinite 4.8 we get $SU(2)^5$ which has rank 5. What is happening here? The number of external legs is not changing and so the rank of the global symmetry should be 7 regardless of coupling. This is precisely the phenomenon we discussed previously of massless instantons enhancing the Higgs branch. Recall the topological $U(1)_{I_i}$ symmetry associated to the instanton charges I_i and instanton masses $\frac{1}{g_i^2}$. Each $SU(2)$ factor in our gauge group contributes an instanon charge to the global symmetry. When we take a coupling to infinity, the instantons become massless and form part of the Higgs branch, and the rank from the $U(1)_I$ associated to its topological symmetry becomes manifest in the symmetry of the Higgs branch. The topological symmetry is still present on the Higgs branch, however the method used above does not detect it. The correct global symmetry should include the $U(1)_I$ factors that do not contribute to symmetry enhancement. For instance the full global symmetries for one and no couplings infinite contain respectively an additional 1 and 2 $U(1)$ factors in their global symmetry. In 4.1 we display a summary of the calculations of the $SU(2) \times SU(2)$ theory above, including the global symmetry calculated using the above method. Examining 4.1 we see the same phenomenon as above when we give the flavours a finite bare mass. The flavours are no longer massless and so do not form part of the Higgs branch. The calculated global symmetries obey the formula $rank = \#infinitecouplings + \#masslessflavours$. Since we kept the bifundamental hypermultiplet massless throughout the Higgs branch is never trivial. Again the real global symmetry must have rank 7. We can conjecture the groups calculated using the balance global symmetry method can be completed to the correct groups by adding $U(1)$ factors to raise the rank to 7.

# Massive Flavours	# Infinite Couplings	# Components	Quaternionic Dimension	Global Symmetry
0	0	1	6	$SU(2)^5$
0	1	1	8	$SU(4) \times SU(2)^3$
0	2	1	12	$SU(6) \times SU(2)^2$
1	0	1	4	$SU(2)^3 \times U(1)$
1	1	1	6	$SU(4) \times SU(2) \times U(1)$
1	1	1	5	$SU(3) \times SU(2)^2 \times U(1)$
1	2	1	8	$SU(5) \times SU(2) \times U(1)$
2 (opposite sides)	0	1	2	$SU(2)^2$
2 ””	1	1	4	$SU(4)$
2 ””	1	2	3,2	$SU(2)^2 \times U(1)$, $SU(2)^2$
2 ””	2	2	4,5	$SU(4)$, $SU(4) \times SU(2) \times U(1)$
2 ”” (same side)	0	1	2	$SU(2) \times U(1)$
2 ””	1	1	3	$SU(3) \times U(1)^2$
2 ””	2	2	4,5	$SU(3) \times U(1)^2$, $SU(4) \times U(1)$
3	0	1	1	$SU(2)$
3	1	2	1,1	$SU(2)$ $SU(2)$
3	1	1	2	$SU(3)$
3	2	2	1,2	$SU(2)$, $SU(3) \times U(1)$
4	0	1	1	$SU(2)$
4	1	2	0, 1	$SU(2)$
4	2	1	2	$SU(2) \times U(1)$

Table 4.1: The structure of the Higgs branch for different parameters for the $SU(2) \times SU(2)$ quiver theory depicted in 2.2. Multiple entries in the dimension and global symmetry columns refer respectively to the different components. The global symmetry listed is the global symmetry calculated from the balance global symmetry method and in most cases does not reflect the full global symmetry.

Chapter 5

Magnetic quivers for some families of quivers

We present here some calculations of the Higgs branch at infinite coupling for some families of quivers, using the methods we have developed. We see it generally requires little extra effort to compute magnetic quivers for an entire series of quivers once we have computed the first one or two cases. There is a nice correspondence between the series of quivers defining 5d supersymmetric theories and the series of magnetic quivers giving the Higgs branch at infinite coupling.

5.1 A series

Let us consider the series of quivers corresponding to the A-series of Dynking diagrams:

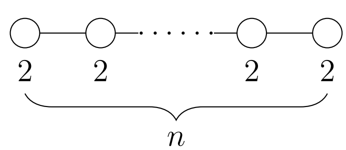
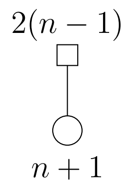


Figure 5.1: The A series of quivers with rank 2 nodes.

Included in this are the pure $SU(2)$ theory A_1 and the $SU(2) \times SU(2)$ theory with one bifundamental hypermultiplet A_2. The theory given by the quiver A_n has gauge group $SU(2)^n$ with $n-1$ hypermultiplets in the bifundmental representation of each pair of $SU(2)$ factors. If we consider the brane web corresponding to this theory it has 4 external diagnonal (1,1)/(1,-1) branes and $2(n-1)$ external vertical NS5-branes. Under S-duality, rotating the brane web 90 degrees the NS5-branes become D5-branes and we get an $SU(n+1)$ gauge theory with $2(n-1)$ flavours. Hence by S-duality, this series of quivers equivalently describes the theories:



(5.1)

In other words the A series describes SQCD theories where $N_f = 2(N_c - 2)$.

The brane webs at infinite coupling for the first 3 quivers in the seires are:

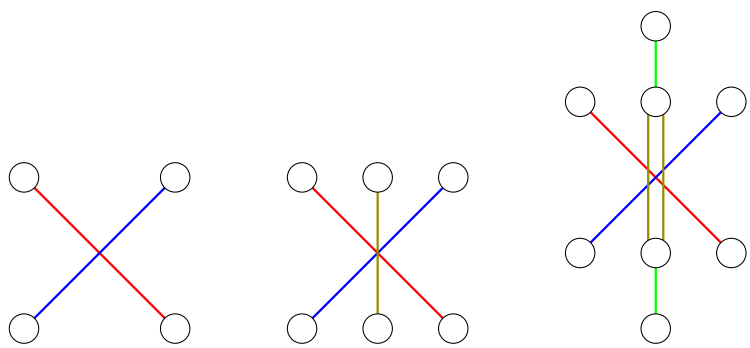


Figure 5.2

These have corresponding magnetic quivers:

Using the balance global symmetry method we can compute these have global symmetries: $SU(2)$, $SU(2) \times U(1)$ and $SU(4) \times U(1)$ respectively.

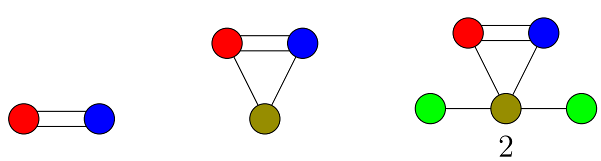
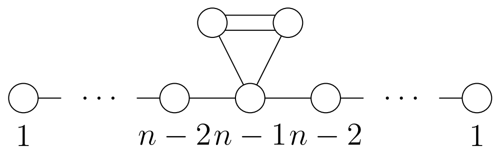


Figure 5.3: Caption

It is easy to infer the general magnetic quiver for A_n:



(5.2)

We can try to calculate the global symmetry using the balance global symmetry method. This gives $SU(2(n-1)) \times U(1)$. This has rank $2n-2$). Comparing with the brane web we have $2(n+1)$ 7-branes for A_n which gives a rank of $2n-1$. We are missing one rank from our computation indicating it has not worked. It is common for the method to not work for quivers with more than one link between two nodes, which are called non-simply laced quivers. A guess for the correct symmetry would be $SU(2(n-1)) \times U(1)^2$. We would need to calculate the Hilbert series using the monopole formula to verify this.

5.2 Framed A series

Next let us consider what we will call the Framed A series which is the A series but with flavour nodes attached to either end:

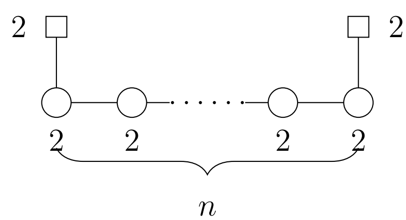
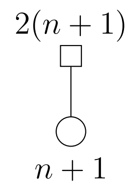


Figure 5.4: The Framed A series of quivers with rank 2 nodes.

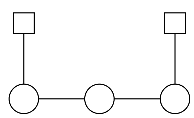
A_n^F describes a theory with gauge group $SU(2)^n$ with $n-1$ bifundamental hypermultiplets as well as two fundamental hypermultiplets charged under the first and last gauge group. The A_n^F brane web contains 4 external D5-branes and $2(n+1)$ external NS5-branes. Rotating by 90 degrees we then get a theory with gauge group $SU(n+1)$ with $2(n+1)$ flavours. Hence we find the framed A series is S-dual to the series:



(5.3)

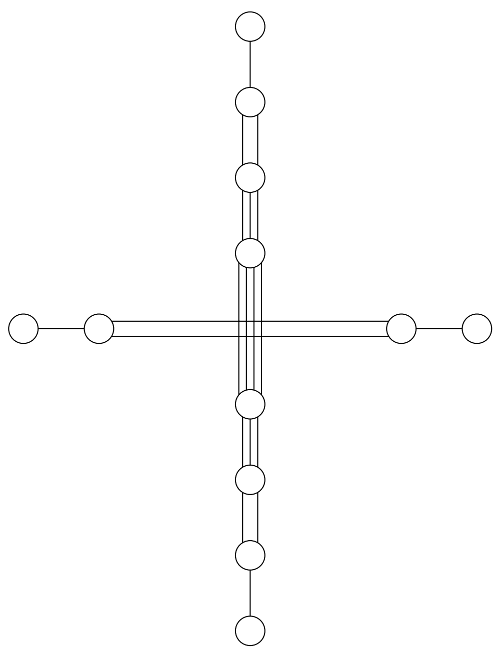
In other words the framed A series describes SQCD theories with $N_f = 2N_c$.

We have examined the first quiver A_2^F extensively in the previous chapter. Its infinite coupling Higgs branch is shown in 4.10. The higher members of the series generalise in a simple way. Let us examine A_3^F to get an intuition. Its quiver is given by:



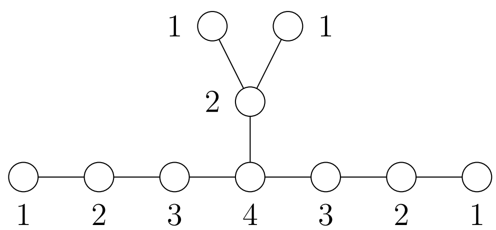
(5.4)

Where all nodes have rank 2. The corresponding brane web is:



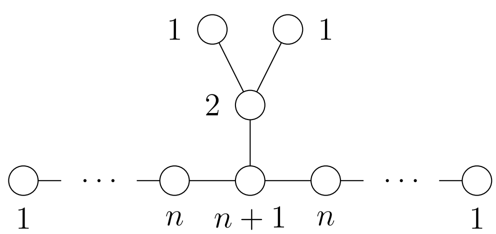
(5.5)

This gives a magnetic quiver:



(5.6)

It is straightforward to infer the magnetic quiver for A_n^F. Like the A series the vertical branes will continue to stack, while the horizontal branes remain unchanged. The general quiver for A_n^F is:



(5.7)

Applying the balance global symmetry method to calculate the global symmetry gives $SU(2(n+1)) \times SU(2)^2$. This has rank $2n+3$ which agrees with the rank of the global symmetry as calculated from the brane web. There are $2(n+3)$ 7-branes for A_n^F (corresponding to $2(n+3)$ external legs. This indicates we could trust this computation. Again one would need to compute the Hilbert series using the monopole formula to verify this.

Chapter 6

Conclusion and Outlook

In this work, we have seen the power of brane webs in understanding 5d $\mathcal{N} = 1$ field theories. We have seen how rather complicated field theory calculations can be replaced by drawing simple pictures, and calculating distances and areas. Phase transitions in the field theory have elegant and simple realisations on the brane web. S-duality has a remarkably simple form for brane webs, if we tilt our head to the side we can immediately discern the S-dual theory. We also saw how computations for infinite coupling phases, which do not have Lagrangian, descriptions present no extra difficulty using brane webs, and indeed are the easiest case to tackle in the case of the Higgs branch. Interesting areas of further study would be to use the monopole formula to calculate exactly the global symmetry of the families of theories discussed in 5. Furthermore we have only discussed in this work unitary quivers, guage theories with orthongonal and symplectic gauge groups can also be studied using brane webs with the addition of orientifold planes [31, 12]. An interesting area of further study would to examine if there is a useful intersection between the brane web tools we have discussed here and the other methods currently being used to study 5d SCFTs. 5D SCFTs can also be approached from the point of view of M-theory and Calabi-Yau geometry [14, 15], and from the point of view of superconformal indices and partition functions [32, 33]. Given these approaches are studying the same field theories, there is a good possibility of finding fruitful connections between different areas in modern theoretical physics by pursuing this direction.

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