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Eleven-Dimensional Supergravity and M2-Brane

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Abstract

This dissertation reviews the formulation of supergravity and supermembrane in eleven dimensions and discusses the dimensional reduction to ten-dimensional superstring theory. We start by giving the actions of eleven-dimensional supergravity and the ten-dimensional type II supergravity. Following this, we describe the classical M2-brane solution to the 11d supergravity. The M2-brane action is introduced as the source term of the 11d supergravity. Additionally, we discuss the κ symmetry and demonstrate that the constraints of preserving κ symmetry are equivalent to the equations of motion of 11d supergravity. Finally, we present how the 10d type IIA superstring action can be obtained from the M2-brane action through double-dimensional reduction.

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Contents

1 Introduction

M-theory remains a conjecture of a consistent interacting quantum theory in eleven dimensions that gives rise to the five different types of superstring theories via various compactifications. These five superstring theories are related via different dualities. The low-energy limits of these superstring theories give rise to five kinds of supergravity theories. This M-theory is postulated to be the ultraviolet (UV) completion of the eleven-dimensional supergravity, a theory that contains the massless graviton, gravitino, and gauge fields, meanwhile preserving supersymmetry. Despite the massless fields, one encounters extended, non-perturbative objects known as "membranes" in M-theory. Notably, these encompass the M2-brane and the M5-brane, coupling to the gauge field, either electrically or magnetically.

Supergravity is a locally supersymmetric theory that contains massless bosons and fermions with spin less than or equals to two [1]. Supergravity in eleven dimensions is special because there is only one possible consistent 11d supergravity theory [2]. Such uniqueness can be attributed to the constraints that the space-time dimensions impose upon the maximum number of supercharges allowed for a supergravity theory.

In ten dimensions, there are five types of supergravities that satisfied the constraints of supersymmetry. Of these, the type II supergravities are the most interesting since they are the maximally supersymmetric gravity theory in ten dimensions. The construction of a maximally supersymmetric supergravity needs two Majorana-Weyl supercharges. The one that has opposite chirality is categorised as type IIA supergravity, whilst its counterpart, having the same chirality, is classified as type IIB supergravity [3]. The other three consistent supergravity theories fall beyond the primary scope of this essay.

Due to the non-renormalisation theorem, a gravity theory in dimension greater or equal to four is not UV-complete, meaning that such theory remains a low-energy effective theory of some underlined UV complete theory at the quantum level. The full theory of superstring consists not only of the massless modes presented in the low-energy effective theory but also of higher dimensional extended objects, strings and branes, and the massive modes of extended objects that decouple from the theory as we approach the low-energy limit. Since string theory is believed to be UV finite, we consider superstrings as the UV completion of the ten-dimensional supergravities.

It was shown that type IIA supergravity can be obtained via dimensional reduction from the 11d supergravity [4]. This relation between type IIA superstring theories and the ten-dimensional supergravities has encouraged us to consider the eleven-dimensional M-theory as the UV completion of the 11d supergravity. We postulated that M-theory plays a role as the origin of all the superstring theories and make it a potential candidate for quantum gravity.

Analogous to the essential role that the one-dimensional string plays in superstring theories, the two-dimensional extended object, the M2-brane, is considered to behave similarly within M-theory [5, 6]. This proposition gained considerable support upon the discovery that the M2-brane action, when wrapping on a circle, could indeed result in the superstring action [4]. It is now believed that 11d supergravity admits a consistent UV completion where the stable M2-brane plays a critical role [7–9]. However, it is still unclear how the UV completion of the eleven-dimensional supergravity is formulated.

This dissertation will review supergravities and the M2-brane. We will start in section 2 by giving the action for supergravity theories in eleven and ten dimensions. The local supersymmetric transformations and classical field equations will be presented. The supergravity theories are then consistently truncated to the bosonic sector, which has a pure bosonic solution. The Killing spinor equation should be satisfied for the solution to preserve the supersymmetry. In section 3, we will solve for the classical solution to the equations of motion of eleven-dimensional supergravity. Under the effect of the M2-brane source, the space-time will be deformed from the flat space. We will present the general p-brane ansatz with worldvolume $\mathrm{ISO}(1, p)$ isometry and isotropic transverse space. Then, we will solve specifically for the eleven-dimensional supergravity. The solution will be obtained by utilising the Killing spinor condition and substituting the ansatz into the equations of motion.

Section 4 will demonstrate the M2-brane action as the source term of the eleven-dimensional supergravity theory. We review the bosonic membrane action in flat space and its generalisation in curved space. Then, the superspace formulation of the supermembrane will be introduced. We will introduce the κ symmetry to preserve supersymmetry of the supermembrane action. It plays a crucial role in relating supermemrbane and supergravity. For example, we will see that the κ symmetry demands the M2-brane action to have an eleven-dimensional supergravity background. The origin of κ symmetry from the Green-Schwarz superstring will also be discussed. Finally, we will give the M2-brane action in a flat space background and present various classical solutions in different supergravity backgrounds.

In the last section, we will review the Kazula-Klein dimensional reduction from eleven-dimensional to ten-dimensional supergravity. Then, we will show how we obtain the superstring action from the M2-brane action through double-dimensional reduction, where we simultaneously compact one dimension in the worldvolume and the target space. We will also introduce how the Weyl invariance of string theory emerges from the three-dimensional diffeomorphism of the M2-brane worldvolume. Finally, we briefly discuss the dilaton term in string sigma model, which arise from the dimensional reduction of the metric.

2 Supergravity

Supergravity theories contain supersymmetries and gravity but without particles with spin higher than two. Hence, there are only limited types of supergravity in a given dimension. In ten dimensions, there are five different types of supergravity theories. Each of them is the low-energy effective theory of a ten-dimensional superstring theory. Those superstring theories can transform into each other via duality. The supergravity theory in eleven dimensions is more special since there is only one consistent supergravity theory, which is believed to be the low-energy effective theory of the so-called M-theory [1].

We will focus on the eleven-dimensional supergravity and the ten-dimensional type II supergravities. In this section, we will roughly review the construction of supergravity theories. The supersymmetric transformation and the equations of motion of supergravity theory in eleven dimensions will be discussed. Then, we will introduce the type II supergravity in ten dimensions as the low-energy effective theories of type II superstrings. We will also determine the bosonic sector of the supergravity theories by consistent truncation and present their residual global supersymmetry by introducing the Killing spinors.

2.1 Construction of Supergravity Theory

There are four ways to construct supergravity theories. The first one is the Noether method, the most often-used method. It was first used in 1970s to construct the N=1 D=4 supergravity in its on-shell [10, 11] and off-shell [12, 13] formulations. Later on, this method also played an essential role in constructing D=11 supergravity [1].

The second method is constructing the supergravity theory via the superspace [14]. Supergravity in the superspace description is similar to the usual general relativity. Using the supervielbein and the spin connection, one can build a supergravity theory invariant under super-transformations. As usual, one constructs the covariant derivatives and then defines standard quantities of torsion and curvature. The key problem of supergravity's superspace formalism is finding which part of the torsion and curvature should be set to zero under the constraint of the Lorentz tangent space group. This method was first carried out in 1970s on N=1 D=4 supergravity [15, 16].

The third method of finding supergravity theories is by gauging certain space-time groups. For example, by gauging the super Poincaré group, one can find the N=1 D=4 supergravity theory [17, 18]. However, theories that involve gravity are not necessarily gauge theories. Therefore, it is necessary to introduce constraints to address this aspect. This method provides the first algebraic proof of the invariance of supergravity theories [17] and is also used in constructing the conformal supergravity theory [19].

Finally, with a given higher-dimensional supergravity theory, one could obtain a lower-dimensional supergravity theory through dimensional reduction. Dimensional reduction involves compactifying some dimensions on circles and suppressing the dependence of fields on those dimensions. Since supergravity theories in higher dimensions are generally simpler and easier to study, we can construct low-dimensional supergravity theories from high-dimensional ones, which can be easily constructed using the methods above.

2.2 D=11 Supergravity

The D=11 Supergravity was initially constructed by E. Cremmer, B. Julia and J. Scherk to study the extended $\mathrm{O}(n), n = 1, \ldots, 8$ supergravity [1]. It gives the lower dimensional supergravity by dimensional reduction.

The eleven dimension is the highest dimension for the supergravity theory, which means a theory with spin two or less [2]. This follows from the irreducible representations of supersymmetry, and there is no consistent interaction for particles with spin higher than 2 in four dimensions.

In a four-dimensional theory, one can construct superspace with N Majorana supercharges, each having four real degrees of freedom, making it $4N$ supercharges. The anti-commutation relations of these supercharges form a Clifford algebra with N pairs of operators that raise and lower helicity by $\frac{1}{2}$. Conventionally, one chooses the raising operators to annihilate the vacuum. Hence, starting with the vacuum of helicity two and acting with lowering operators on it, one obtains the physical states with helicity ranging from 2 down to $2 - (N - 2)$. With the constraint that we cannot have particles with spin higher than 2, the maximum number of supercharges allowed for a four-dimensional supergravity theory is 8, which corresponds to $4 \times 8 = 32$ spinor degrees of freedom.

When considering supergravity theory with $D \geq 4$, one can examine the four-dimensional theory obtained through dimensional reduction, which satisfies the constraint we just mentioned. Dimensional reduction involves compactifying the extra dimensions on a manifold and decoupling the fields from those dimensions. The simplest approach is compactifying on a torus, the most basic compact manifold. This process preserves supersymmetry, thereby keeping the number of supercharges unchanged. Hence, for a given dimension to potentially admit supergravity theory, that dimension must have a spin representation of dimension 32 or less.

The spinor representations for a given dimension include Dirac spinor, Weyl spinor, and Majorana spinor. In even dimensions $D = 2k$, each of these representations has the following real degrees of freedom: 2^{k+1} for Dirac spinor, 2^k for Weyl spinor, and 2^k for Majorana spinor. For odd dimensions $D = 2k+1$, one can incorporate the chirality matrix $\Gamma_* = (-i)^{k-1}\Gamma^0\Gamma^1 \ldots \Gamma^{D-1}$ into the set of gamma matrices of $D = 2k$ to form the odd-dimensional Clifford algebra, thereby maintaining the dimensions of the spinor representations. However, it is not possible to have a Weyl spinor in odd dimensions. Therefore, the highest dimensional supergravity theory is in $D = 11$ with one 32-component Majorana spinor, an $D = 11$, $\mathcal{N} = 1$ supergravity.

The field content can be fixed via on-shell dimensional analysis. One could assume there exists a massless spin-2 gravitational field which is the traceless symmetric tensor representation of SO$(11-2)$, and one Majorana spinor gravitino. The degrees of freedom of graviton is thus $(11-2)\times(11-1)-1 = 44$. The dimension of Majorana gravitino is $\frac{1}{2}(11-3)\times 2^5 = 2^7 = 128$. For a supersymmetric theory, the bosonic degrees of freedom must match the fermionic degrees of freedom. A three-form gauge field is the right we need, as it has degrees of freedom of $\frac{1}{3!}(11-2)\times(11-3)\times(11-4) = 84$. Hence, the field content of D=11 supergravity comprises a graviton g_{mn}, a Majorana gravitino ψ_m, and a 3-form gauge field $A_{m_1m_2m_3}$, with 44, 128 and 84 degrees of freedom respectively. The graviton (metric) is usually written in terms of vielbeins ${e_m}^a$ as $g_{mn} = \eta_{ab}{e_m}^a{e_n}^b$.

The Lagrangian of the D=11 dimensional supergravity is given by,

$$\begin{aligned}S_{11} =& \frac{1}{2\kappa_{11}^2}\int d^{11}x\left(eR(\Omega(e,\psi)) - \frac{e}{48}F_{m_1\cdots m_4}F^{m_1\cdots m_4} - \frac{e}{2}\bar{\psi}_m\Gamma^{mnp}D_n\left(\frac{1}{2}(\Omega+\hat{\Omega})\right)\psi_p\right.\\ &-\frac{1}{384}e\left(\bar{\psi}_{m_1}\Gamma^{m_1\cdots m_6}\psi_{m_2} + 12\bar{\psi}^{m_3}\Gamma^{m_4m_5}\psi^{m_6}\right)\left(F_{m_3\cdots m_6} + \hat{F}_{m_3\cdots m_6}\right)\\ &\left.+\frac{1}{(12)^4}\epsilon^{m_1\cdots m_{11}}F_{m_1\cdots m_4}F_{m_5\cdots m_8}A_{m_9m_{10}m_{11}}\right)\end{aligned} \tag{2.1}$$

where F is the field strength of gauge field A,

$$F_{m_1\ldots m_4} = 4\partial_{[m_1}A_{m_2m_3m_4]} \tag{2.2}$$

and $\hat{F}$ is defined by,

$$\hat{F}_{m_1\dots m_4} = F_{m_1\dots m_4} + 3\bar{\psi}_{[m_1}\Gamma_{m_2 m_3}\psi_{m_4]} \tag{2.3}$$

The spin connections are given by,

$$\Omega_{mab} = \hat{\Omega}_{mab} - \frac{1}{4}\bar{\psi}_n\Gamma_{mab}{}^{np}\psi_p \tag{2.4}$$

$$\hat{\Omega}_{mab} = \hat{\Omega}^{(0)}_{mab}(e) + \frac{1}{2}(\bar{\psi}_m\Gamma_b\psi_a - \bar{\psi}_m\Gamma_a\psi_b + \bar{\psi}_b\Gamma_m\psi_a) \tag{2.5}$$

where $\hat{\Omega}^{(0)}_{mab}$ is the torsion-free spin connection expression in terms of the vielbein,

$$\begin{aligned}\hat{\Omega}^{(0)}_{mab} =& \frac{1}{2}e_a{}^n\left(\partial_m e_{bn} - \partial_n e_{bm}\right) - \frac{1}{2}e_b{}^n\left(\partial_m e_{an} - \partial_n e_{am}\right)\\ &- \frac{1}{2}e_a{}^p e_b{}^q\left(\partial_p e_{qc} - \partial_q e_{pc}\right)e_m{}^c\end{aligned} \tag{2.6}$$

And the covariant derivative is defined by,

$$D_m(\Omega)\psi_b = (\partial_m + \frac{1}{4}\Omega_{mab}\Gamma^{ab})\psi_b \tag{2.7}$$

and,

$$[D_m, D_n] = -\frac{1}{4}R_{mna}{}^b\Gamma^a{}_b \tag{2.8}$$

The hatted objects are corrected with a higher order of κ_{11} so that their variations under supersymmetry do not depend on the derivative of the supersymmetry parameter $\partial_M\epsilon$. This property is referred to as being "super-covariant".

The action of 11D supergravity is invariant under the local supersymmetric transformation:

$$\begin{aligned}\delta e_m{}^a &= \frac{1}{2}\bar{\epsilon}\Gamma^a\psi_m\\ \delta\psi_m &= \hat{D}_m\epsilon\\ \delta A_{m_1 m_2 m_3} &= -\frac{3}{4}\bar{\epsilon}\Gamma_{[m_1 m_2}\psi_{m_3]}\end{aligned} \tag{2.9}$$

Here we defined the supercovariant derivative $\hat{D}_m$ as,

$$\hat{D}_m = \partial_m + \frac{1}{4}\Omega_{mab}\Gamma^{ab} + \frac{1}{288}(\Gamma_m{}^{n_1\dots n_4} - 8\delta^{n_1}_m\Gamma^{n_2n_3n_4})\hat{F}_{n_1\dots n_4} \tag{2.10}$$

where $\Gamma^a = \Gamma^m e_m{}^a$ is the gamma matrix in vielbein basis, and the gamma matrix with n multiple indices represents n gamma matrices anti-symmetrised $\Gamma^{m_1\dots m_n} = \Gamma^{[m_1}\dots\Gamma^{m_n]}$. There is also the usual local gauge symmetry for gauge field,

$$\delta A_{m_1 m_2 m_3} = 3\partial_{[m_1}A_{m_2 m_3]} \tag{2.11}$$

where the gauge parameter $A_{m_2 m_3}$ is a 2-form.

The last term in action, which is the pure gauge field coupling term, is the Chern-Simmons term. The Chern-Simmons term is a topological term and contains many interesting properties which have not been discussed here yet.

We also noticed only one coupling constant in the D=11 supergravity action, the gravitational constant κ_{11}, which has a dimension of mass$^{-9/2}$. Thus we can define Planck's mass m_p by $\kappa_{11} = m_p^{-9/2}$. One could scale the field with the coupling constant, which also scale the action by the correspondent factor. With proper scaling, we can set the equation of motion at classical level to be independent of κ_{11}, meaning that gravitational constant κ_{11} has no classical effect.

2.3 D=10 Supergravity

In this section, we discuss the supergravity theory in ten dimensions, where we have Majorana-Weyl spinors, which possess only 16 degrees of freedom. As mentioned, a supergravity theory can only be constructed based on a supersymmetry algebra with fewer than 32 supercharges. Consequently, there are two types of supergravity theories: $\mathcal{N} = 2$ and $\mathcal{N} = 1$, corresponding to two Majorana-Weyl spinors or one, respectively. The former is type II supergravity, which has 32 supercharges, while the latter is type I supergravity, with 16 supercharges.

We are interested in the dimensional reduction of the eleven-dimensional supergravity theory. All the supersymmetries are preserved when we perform the dimensional reduction on the 11-dimensional supergravity theory by compactifying the eleventh dimension on a circle. Hence, we result in a 10-dimensional supergravity with 32 supercharges, which should be a type II supergravity. Hence, our main focus would be on the type II supergravity theories.

The type II supergravity theories are the low-energy effective theories of the type II superstring theories. The spectrum of superstring theory includes an infinite number of massive modes, which generally decouple from the theory in the low-energy limit. Consequently, as we flow down the energy scale, only the massless modes of the string theory remain, giving rise to the spectrum of the type II supergravity theory.

Before moving on to the effective theories of superstring theory, let us first review the effective action of string theory. The σ-model for bosonic string moving in a background with its massless modes that contains a general metric g_{mn}, an anti-symmetric B_{mn} field, and a dilaton ϕ is given by,

$$I = \frac{1}{4\pi\alpha'}\int d^2\xi\sqrt{\gamma}\left[\gamma^{ij}\partial_i x^m \partial_j x^n g_{mn}(x) + \epsilon^{ij}\partial_i x^m \partial_j x^n B_{mn}(x) + \alpha' R(\gamma)\phi(x)\right], \tag{2.12}$$

where γ_{ij} is the metric on the world sheet of string. The condition for this action to be quantum Weyl anomaly free is that all beta functions vanish. Such condition happens to be the equations of motion of the string effective action given by,

$$I_{\text{eff}} = \int d^D x\sqrt{G}e^{-2\phi}\left[(D-26) - \frac{3}{2}\alpha'\left(R - \frac{1}{12}H_{mnl}H^{mnl} + 4\nabla^2\phi - 4(\nabla\phi)^2\right) + \mathcal{O}\left(\alpha'^2\right)\right] \tag{2.13}$$

where $H_{mmm} = \partial_m B_{nl} + \partial_n B_{lm} + \partial_l B_{mn}$. For flat space to solve this action, the constant term $D-26$ must vanish. This reflects that the critical dimension of the bosonic string is 26. In the superstring theory, a similar NS-NS sector arises with the constant term replaced by $D-10$,

$$I_{\text{eff}} = \int d^{10}x\sqrt{G}e^{-2\phi}\left[R - \frac{1}{12}H_{mnl}H^{mnl} + 4\nabla_m\phi\nabla^m\phi\right] \tag{2.14}$$

indicating that the critical dimension of superstring theory is ten.

From the chirality of the two Majorana-Weyl superalgebras, type II supergravity can be classified into two theories: type IIA and type IIB. These theories possess opposite chirality and the same chirality, respectively. Among them, the supercharges for type IIA supergravity can be combined to form a Majorana spinor with 32 components. Therefore, one might naturally assume that by dimensional reduction from 11-dimensional supergravity, type IIA supergravity is obtained. In fact, this is how type IIA supergravity was initially discovered [20–22].

The field content of type IIA supergravity contains a graviton g_{mn} written in vielbein ${e_m}^a$, a dilaton ϕ, a 2-form NS-NS gauge field B_{mn}, a 1-form R-R gauge field A_m, and a 3-form R-R field C_{mnp} and fermionic fields gravitino ψ_m^α and dilatino λ^α. The index m here represents the world index of the ten dimensions.

Starting from the 11-dimensional supergravity action, we can perform the dimensional reduction to obtain the action for type IIA supergravity. The detailed discussion of dimensional reduction is given in Section 5. The resultant ten-dimensional action can be separated into bosonic sector and fermionic sector,

$$S_{IIA} = S_{IIA}^B + S_{IIA}^F \tag{2.15}$$

With proper field redefinition, the bosonic sector is given by,

$$
\begin{aligned}
S^B_{IIA} =& \frac{1}{2\kappa^2_{10}} \int d^{10}x \left\{ e \left(R - \frac{1}{2}\partial_m\phi\partial^m\phi - \frac{1}{12} e^{\phi} H_{m_1m_2m_3} H^{m_1m_2m_3} \right.\right. \\
& \left. - \frac{1}{48} e^{-\frac{\phi}{2}} \hat{F}_{m_1m_2m_3m_4} \hat{F}^{m_1m_2m_3m_4} - \frac{1}{4} e^{-\frac{3}{2}\phi} F_{m_1m_2} F^{m_1m_2} \right) \\
& \left. + \frac{3}{4\times(12)^3} \epsilon^{m_1\cdots m_{10}} F_{m_1m_2m_3m_4} F_{m_5m_6m_7m_8} B_{m_9m_{10}} \right\},
\end{aligned} \tag{2.16}
$$

where

$$
\begin{aligned}
F_{m_1m_2} &= 2\partial_{[m_1} A_{m_2]}, \\
H_{m_1m_2m_3} &= 3\partial_{[m_1} B_{m_2m_3]}, \\
F_{m_1m_2m_3m_4} &= 4\partial_{[m_1} C_{m_2m_3m_4]}, \\
\hat{F}_{m_1m_2m_3m_4} &= F_{m_1m_2m_3m_4} + 4A_{[m_1} H_{m_2m_3m_4]}.
\end{aligned} \tag{2.17}
$$

Notice that the dilaton coupling is not uniform in this frame, called the Einstein frame. We can rewrite the bosonic action via the Weyl-rescaling on the metric,

$$
g^{(e)}_{mn} = e^{-\frac{\phi}{2}} g^{(s)}_{mn} \tag{2.18}
$$

to put it into the string frame where the dilaton coupling is uniformly $e^{-2\phi}$,

$$
\begin{aligned}
S^B_{IIA} =& \frac{1}{2\kappa^2_{10}} \int d^{10}x \left\{ e \left[e^{-2\phi} \left(R(g^{(s)}) - 4\partial_m\phi\partial^m\phi - \frac{1}{12} H_{m_1m_2m_3} H^{m_1m_2m_3} \right) \right.\right. \\
& \left. + \left(- \frac{1}{48} \hat{F}_{m_1m_2m_3m_4} \hat{F}^{m_1m_2m_3m_4} - \frac{1}{4} F_{m_1m_2} F^{m_1m_2} \right) \right] \\
& \left. + \frac{3}{4\times(12)^3} \epsilon^{m_1\cdots m_{10}} F_{m_1m_2m_3m_4} F_{m_5m_6m_7m_8} B_{m_9m_{10}} \right\},
\end{aligned} \tag{2.19}
$$

We can see that the NS-NS sector, which is multiplied by the factor $e^{-2\phi}$, matches with the effective action of string.

The fermionic part is given by,

$$
\begin{aligned}
S^F_{IIA} =& \frac{1}{\kappa^2_{10}} \int d^{10}x e \left\{ -\frac{1}{2} \bar{\psi}_{m_1} \Gamma^{m_1m_2m_3} D_{m_2} \psi_{m_3} - \frac{1}{2} \bar{\lambda} \Gamma^m D_m \lambda + \frac{\sqrt{2}}{4} \bar{\lambda} \Gamma^{11} \Gamma^{m_1} \Gamma^{m_2} \psi_{m_1} \partial_{m_2}\phi \right. \\
& + \frac{1}{96} e^{\frac{\phi}{4}} \left(-\bar{\psi}_{m_1} \Gamma^{m_1\cdots m_6} \psi_{m_2} - 12 \bar{\psi}^{m_3} \Gamma^{m_4m_5} \psi^{m_6} + \frac{1}{\sqrt{2}} \bar{\lambda} \Gamma^{11} \Gamma^{m_1} \Gamma^{m_3\cdots m_6} \psi_{m_1} \right. \\
& \left. + \frac{3}{4} \bar{\lambda} \Gamma^{11} \Gamma^{m_3\cdots m_6} \lambda \right) F_{m_3\ldots m_6} - \frac{1}{24} e^{-\frac{\phi}{2}} \left(\bar{\psi}_{m_1} \Gamma^{11} \Gamma^{m_1\cdots m_5} \psi_{m_2} - 6 \bar{\psi}^{m_3} \Gamma^{11} \Gamma^{m_4} \psi^{m_5} \right. \\
& \left. - \sqrt{2} \bar{\lambda} \Gamma^{m_1} \Gamma^{m_3m_4m_5} \psi_{m_1} \right) H_{m_3m_4m_5} - \frac{1}{16} e^{\frac{3\phi}{4}} \left(\bar{\psi}_{m_1} \Gamma^{11} \Gamma^{m_1\cdots m_4} \psi_{m_2} + 2 \bar{\psi}^{m_3} \Gamma^{11} \psi^{m_4} \right. \\
& \left. \left. + \frac{3}{\sqrt{2}} \bar{\lambda} \Gamma^{m_1} \Gamma^{m_3m_4} \psi_{m_1} - \frac{5}{4} \bar{\lambda} \Gamma^{11} \Gamma^{m_3m_4} \lambda \right) F_{m_3m_4} + \text{ quartic terms in fermions} \right\},
\end{aligned} \tag{2.20}
$$

where the covariant derivative is defined with the usual spin connection. However, in the upcoming section, we will focus on the bosonic part when solving the classical equation of motion.

We achieve this by setting all fermions to zero and imposing the Killing spinor conditions. Therefore, there is no need to be concerned about the fermionic sector.

Type IIB supergravity possesses two Majorana-Weyl spinors with the same chirality as the superalgebra. This theory includes a 4-form R-R gauge field C_{mmpq}, which has a self-dual 5-form field strength $F_{m_1 \dots m_5} = *F_{m_1 \dots m_5}$. However, such a field does not admit an action [23], thereby preventing the construction of a supersymmetric action for IIB supergravity. In addition to the 4-form gauge field, the bosonic fields of type IIB supergravity encompass a graviton g_{mn} expressed in terms of the vielbein ${e_m}^a$, a dilaton ϕ, a 2-form NS-NS gauge field H_{mn}, a 2-form R-R gauge field C_{mn}, and an 0-form R-R gauge field χ. Within the fermionic sector are two left-handed Majorana-Weyl gravitinos ψ_m and two left-handed Majorana-Weyl dilatinos λ.

Thus, in the superstring theory, we encounter field strengths ranging from rank 1 to rank 5. By utilising the ϵ symbol, we can determine the dual field strengths with ranks 6 to 9, taking into account that the rank 5 field strength is self-dual.

2.4 Classical Equations of Supergravity

The classical solutions always have vanishing fermionic fields. Therefore, when discussing classical solutions, one may focus only on the bosonic part of the action by setting the gravitino to zero. We will discuss how to obtain such a simpler action through consistent truncation in later section.

Supergravity theories are, in general, graviton coupling to scalars and different ranks of gauge fields with additional fermions. The bosonic part of a general supergravity action in D-dimension is:

$$S_{D_B} = \frac{1}{2{\kappa_D}^2} \int d^D x e \left(R - \frac{1}{2} \partial^m \phi \partial_m \phi - \sum_i \frac{1}{2 \times n_i!} e^{a_i \phi} F_{m_1 \dots m_{n_i}} F^{m_1 \dots m_{n_i}} \right), \tag{2.21}$$

where $F_{m_1 \dots m_{n_i}} = n_i \partial_{[m_1} A_{m_2 \dots m_{n_i}]}$ and e is the determinant of vielbein e^m_M. The constant κ_D is related to the gravitational constant G_D via $2\kappa_D^2 = 16\pi G_D$. The constant a_i, which controls the coupling of a scalar field with gauge fields, is negative by convention.

The equation of motion are obtained by varying with respect to g_{mn}, the gauge fields A, and the scalar fields called dilaton ϕ,

$$\begin{aligned} R_{mn} =& \frac{1}{2}\partial_m\phi\partial_n\phi + \sum_i \frac{1}{2\cdot n_i!}e^{a_i\phi} \times \big(n_i F_m{}^{m_2\cdots m_{n_i}} F_{nm_2\cdots m_{n_i}} \\ & - \frac{n_i - 1}{D-2} g_{mn} F_{m_1\cdots m_{n_i}} F^{m_1\cdots m_{n_i}}\big), \end{aligned} \tag{2.22}$$

$$\partial_m \left(\sqrt{-g} e^{a_i\phi} F^{mm_2\cdots m_{n_i}}\right) = 0, \tag{2.23}$$

$$\frac{1}{\sqrt{-g}}\partial_m\left(\sqrt{-g}\partial^m\phi\right) - \sum_i \frac{a_i}{2\cdot n_i!} e^{a_i\phi} F_{m_1m_2\cdots m_{n_i}} F^{m_1m_2\cdots m_{n_i}} = 0. \tag{2.24}$$

For the supergravity theory in eleven dimensions, the bosonic part of the action is the graviton couple to a 3-form gauge field, with a Chern–Simmons term,

$$S_{11_B} = \frac{1}{2\kappa_{11}^2}\int d^{11}xe\left(R - \frac{1}{48}F_{m_1\cdots m_4}F^{m_1\cdots m_4}\right) + \frac{1}{(12)^4}\epsilon^{m_1\cdots m_{11}}F_{m_1\cdots m_4}F_{m_5\cdots m_8}A_{m_9m_{10}m_{11}} \tag{2.25}$$

or written concisely in,

$$S_{11_B} = \frac{1}{2\kappa_{11}^2}\int d^{11}xe\left(R - \frac{1}{48}F^2\right) + \frac{1}{6}F\wedge F\wedge A \tag{2.26}$$

Adding the Chern–Simmons term only modify the equation of motion of the gauge field by $\frac{3!}{12^4}\epsilon^{m_1\cdots m_{11}}F_{m_1\cdots m_4}F_{m_5\cdots m_8}$. Then, the equation of motion of D=11 supergravity are,

$$R_{mn} = \frac{1}{12}\left(F_m{}^{m_2m_3m_4}F_{nm_2m_3m_4} - \frac{1}{12}g_{mn}F_{m_1\cdots m_4}F^{m_1\cdots m_4}\right), \tag{2.27}$$

$$\partial_m\left(eF^{mm_1m_2m_3}\right) + \frac{1}{1152}\epsilon^{m_1\cdots m_{11}}F_{m_4\cdots m_7}F_{m_8\cdots m_{11}} = 0 \tag{2.28}$$

We also have the Bianchi identity for gauge field strength F,

$$dF = 0 \tag{2.29}$$

2.5 Residual Supersymmetry

Supergravity action is invariant under any supersymmetric transformation parameterised by the spinor ϵ. The general form of the local supersymmetric transformation is given by,

$$\begin{aligned} \delta_\epsilon B &\sim \epsilon F, \\ \delta_\epsilon F &\sim \partial\epsilon + B\epsilon, \end{aligned} \tag{2.30}$$

where B and F stands for the bosonic part and the fermionic part, separately [24]. We will only interested in the pure bosonic solution in the later section when we solve for the classical solution, since the only macroscopic fields we observed in nature are bosonic fields.

Killing spinors are related with special classical solution of the equation of motion of supergravity. Specifically, there are only a subset of spinor ϵ that leave the solution unchanged under the supersymmetric transformation. And these subset of spinors are known as Killing spinors.

The Killing spinor is analogous to the Killing vector. Killing vectors are vectors that parameterise the diffeomorphisms that kept the metric unchanged. It describes the preserved global symmetry of the solution. Similarly, Killing spinors are spinors that characterised the preserved global supersymmetry of a solution.

In order for the action to respect the local supersymmetric transformation, all supersymmetric variations should vanish after we substitute in the classical solution of equation of motion. As we mentioned previously, the classical solution, also referred to as background, usually has vanishing fermionic part. While the supersymmetric transformation of bosonic vectors is typically proportional to some fermionic fields that vanish in the classical solution, the requirement for a Killing spinor is primarily related to the supersymmetric transformation of the spinor fields that is independent of spinors in the theory.

In our case for D=11 supergravity, we take the bosonic solution of the field equations. The supersymmetric variation of the gravitino does not vanish unless $\hat{D}_m\epsilon = 0$. Hence, setting the fermionic parts to zero breaks the local supersymmetry of the solution. However, we can still preserve the global supersymmetry by demanding the supersymmetric variation of the gravitino to be vanished, which defines Killing spinors. Therefore, the Killing spinors in 11-dimensional supergravity are given by:

$$\hat{D}_m\epsilon = \left[\partial_m + \frac{1}{4}\Omega_{mab}\Gamma^{ab} + \frac{1}{288}({\Gamma_m}^{n_1\dots n_4} - 8\delta_m^{n_1}\Gamma^{n_2n_3n_4})F_{n_1\dots n_4}\right]\epsilon = 0 \tag{2.31}$$

which ensures the vanishing of all supersymmetric variations. Given the Killing spinor, the rest of the bosonic solution will be constrained and are parameterised by certainn constants.

3 M2-Brane Solution of Supergravity

In this section, we solve the equations of motion of the supergravity theory. We start by making the brane ansatz with a d-dimensional worldvolume and a spherical symmetric transverse space, which captures the effect of a M2-brane on the supergravity space-time structure. Then, using the Killing spinor condition and field equation of the gauge field, we can determine the metric and the gauge field as the solution to the equations of motion of supergravity.

3.1 Brane Ansatz

To solve the field equations, we make an ansatz that preserves a certain amount of unbroken supersymmetry. For simplicity, we demand that the ansatz admits $\mathrm{ISO}(1,p) \times \mathrm{SO}(D-p-1)$ symmetry, where we assume the isotropy of the $(D-p-1)$ dimensions that are transverse to the $(p+1)$ dimensions [25]. This can be viewed as a $(p+1)$-dimensional hyperplane embedded in D-dimensional space-time. Additionally, one can think of it as the worldvolume of a p-dimensional extended object.

Therefore, we make an $(p+1)$ - $(D-p-1)$ split of the coordinate $x^m = (x^i, y^s)$, where $m = (0, 1, \ldots, D-1)$ indices split into two ranges: $i = (0, \ldots, p)$ representing the worldvolume that admits $\mathrm{ISO}(1,p)$ isometries, and $s = (p+1, \ldots, D-1)$ representing the isotropic transverse direction to the worldvolume. The ansatz of metric that respects the above symmetry is,

$$ds^2 = e^{2A(r)} dx^i dx^j \eta_{ij} + e^{2B(r)} dy^s dy^t \delta_{st} \tag{3.1}$$

where $r = \sqrt{y^s y_s}$ is the radial coordinate in the transverse space.

The ansatz for scalar dilaton ϕ is not used in the $D = 11$ supergravity solution, but we also gives it here:

$$\phi = \phi(r). \tag{3.2}$$

The ansatz for the gauge field is more complicated as we have to consider two different cases related to each other by duality. The first case is the $(n-1)$-form gauge field A. The Maxwell one-form vector potential couples to the worldline of a charged particle, which is a 0-dimensional extended object. Analogous to that, the $(n-1)$-form gauge field should couple to a worldvolume of a $(p = n-2)$-dimensional extended charged object. The charge arises from the analogue of applying Gauss' law to the field strength F of A. Furthermore, we can relate the worldvolume dimension $p+1$ to the $(n-1)$-form gauge field it couples to via $p+1 = n-1$. This ansatz is referred to as the "electric" ansatz for the gauge field:

$$A_{i_1 \ldots i_{n-1}} = \epsilon_{i_1 \ldots i_{n-1}} e^{C(r)}, \tag{3.3}$$

with all other components set to zero. The term "electric" refers to the analogy between this ansatz and the electrical coupling of the Maxwell gauge field to a charged particle. Equivalently, given this ansatz in terms of the field strength,

$$F^{\text{el}}_{s i_1 \ldots i_{n-1}} = n \epsilon_{i_1 \ldots i_{n-1}} \partial_s e^{C(r)} \tag{3.4}$$

with all other components set to zero.

The other way of coupling the d-dimensional worldvolume to gauge field is by considering the dual field strength $*F$. The Hodge dual of an n-form F is a $(D-n)$-form. By the same analogy, the gauge field corresponding to this field strength $*F$ will couple to the worldvolume of a $(p = D - n - 1)$-dimensional extended charged object. In principle, one could invert $*F$ to obtain the gauge field, but it turns out the solution would be non-local. Hence, we express this ansatz using the related n-form field strength F:

$$F^{\text{mag}}_{s_1 \ldots s_n} = \lambda \epsilon_{s_1 \ldots s_n t} \frac{y^t}{r^{n+1}}, \tag{3.5}$$

where the magnetic charge λ is an undetermined integration constant. This ansatz is referred to as the "magnetic" ansatz in contrast to the previous "electric" ansatz. In eleven-dimensional supergravity, this ansatz corresponds to the M5-brane, which is not our main focus here.

3.2 M2-Branes Solution

The general solution to the Mp-brane solution for D-dimensional general supergravity theory is given by [26],

$$ds^2 = H^{\frac{-4\tilde{d}}{(D-2)\Delta}} dx^i dx^j \eta_{ij} + H^{\frac{4d}{(D-2)\Delta}} dy^s dy^t \delta_{st} \tag{3.6a}$$

$$e^{\phi} = H^{\frac{2a}{\varsigma\Delta}} \quad \varsigma = \begin{cases} +1, & \text{electric} \\ -1, & \text{magnetic} \end{cases} \tag{3.6b}$$

$$H(y) = 1 + \frac{k}{r^{\tilde{d}}}, \tag{3.6c}$$

where $d = p + 1$ and $\tilde{d} = D - d - 2$ with $C(r)$ satisfied the relation given in (B.23),

$$e^C = \frac{2}{\sqrt{\Delta H}}. \tag{3.7}$$

The integration constant k is related to the magnetic charge λ via,

$$k = \frac{\sqrt{\Delta}}{2\tilde{d}}\lambda. \tag{3.8}$$

where this relation comes from the analysis of the magnetic brane. The detailed discussion of general solution to supergravity is given in appendix B.

After considering the general case, we now focus on a specific solution involving a 3-dimensional extended object corresponding to the 2-brane solution in $D = 11$ supergravity. The eleven-dimensional supergravity theory is widely recognised as the low-energy effective theory of the "M-theory," which gives rise to the concept of "M-branes." Hence, this particular 2-brane solution is commonly referred to as "M2-branes."

By employing the ansatz of $\mathrm{ISO}(1, d-1) \times \mathrm{SO}(D-d)$, specifically $\mathrm{ISO}(1, p) \times \mathrm{SO}(D-p-1)$, we can substitute $D = 11$, $d = 3$, $p = 2$, and $n = 4$ into the ansatz for the gauge field and the equations of motion. It is worthwhile to restate the ansatz for clarity:

$$ds^2 = e^{2A(r)} dx^i dx^j \eta_{ij} + e^{2B(r)} dy^s dy^t \delta_{st} \tag{3.9}$$

where $i = (0, 1, 2)$, and $M = (3, \ldots, 10)$. Following from this ansatz, the 3-form gauge field, which electrically couples to the M2-Brane worldvolume, is given by,

$$A_{i_1 i_2 i_3} = \epsilon_{i_1 i_2 i_3} e^{C(r)}, \tag{3.10}$$

and in terms of the field strength, it takes the form,

$$F_{M i_1 i_2 i_3} = 4\epsilon_{i_1 i_2 i_3} \partial_M e^{C(r)}. \tag{3.11}$$

Using the equations of motion obtained earlier, we can derive the equations of motion with the variable $S = e^{-3A+C} C'$. The Einstein equation (2.27) now takes the form,

$$\begin{aligned} A'' + 3(A')^2 + 6A'B' + \frac{7}{r}A' &= \frac{1}{3}\left(C'^2\right) e^{-6A+2C} \\ B'' + 3A'B' + 6(B')^2 + \frac{13}{r}B' + \frac{3}{r}A' &= -\frac{1}{6}\left(C'^2\right) e^{-6A+2C} \\ 6B'' + 3A'' - 6A'B' + 3(A')^2 - 6(B')^2 - \frac{6}{r}B' - \frac{3}{r}A' &= \frac{1}{2}\left(C'^2\right) e^{-6A+2C} \end{aligned} \tag{3.12}$$

Notice that the scalar dilaton vanishes in this case, resulting in the equation of motion for the dilaton (B.9) being trivially satisfied. And, the equation of motion of the gauge field becomes,

$$\nabla^2 C + (C' - 3A' + 6B')C' = 0. \tag{3.13}$$

Since the scalar coupling constant a is zero, Δ is determined to be 4. Consequently, according to Equation (B.23), the relationship between A, B, and C is given by:

$$3A = -6B = C \tag{3.14}$$

Additionally, we will arrive at the same relationship by requiring the solution to be supersymmetric, which will be demonstrated by solving for the Killing spinor in the upcoming section.

Substitute all the field contents' ansatz into the field equation of gauge field, we obtain,

$$\nabla^2 e^{-C} = 0 \tag{3.15}$$

And the general solution to this Laplacian equation is given by,

$$e^{-C} = 1 + \frac{k}{r^6} \tag{3.16}$$

Then, the M2-brane solution is given by,

$$\begin{aligned} ds^2 &= \left(1 + \frac{k}{r^6}\right)^{-2/3} dx^i dx^j \eta_{ij} + \left(1 + \frac{k}{r^6}\right)^{1/3} dy^s dy^t \delta_{st} \\ A_{ijk} &= \epsilon_{ijk} \left(1 + \frac{k}{r^6}\right)^{-1}, \end{aligned} \tag{3.17}$$

where all other components of the 3-form gauge field and other field contents are zero. Since the transverse part of space-time is isometric, it is convenient to write them in the spherical symmetric form,

$$ds^2 = \left(1 + \frac{k}{r^6}\right)^{-2/3} dx^i dx^j \eta_{ij} + \left(1 + \frac{k}{r^6}\right)^{1/3} \left(dr^2 + r^2 d\Omega_7^2\right) \tag{3.18}$$

By taking the $r \to 0$ limit and discarding the constant 1 in the brackets, we obtained the near horizon metric,

$$ds^2 = k^{1/3} \left(\frac{r^4}{k} dx^i dx^j \eta_{ij} + \frac{1}{r^2} dr^2 + d\Omega_7^2\right). \tag{3.19}$$

By substitute $\rho = k^{1/2}/2r^2$, we can see immediately that the geometry near $r = 0$ has an $\mathrm{AdS}_4 \times \mathcal{S}^7$ geometry,

$$ds^2 \to \frac{1}{4} k^{1/3} \left[\frac{1}{\rho^2} \left(dx^i dx^j \eta_{ij} + d\rho^2\right) + 4 d\Omega_7^2\right] \tag{3.20}$$

A more rigorous analysis can give rise to the full picture of the entire space-time. Like the Schwarzschild metric, the metric of M2-brane solution has a singularity at $r = 0$. However, one can calculate the space-time curvature from the curvature components to show that this is actually a coordinate singularity. This coordinate singularity corresponds to a horizon. Hence we can extend through the coordinate singularity into other region of the space-time. By changing the parameter r to R,

$$r = \frac{k^{\frac{1}{6}} R^{\frac{1}{2}}}{(1 - R^3)^{\frac{1}{6}}} \tag{3.21}$$

we rewrite the solution into the form,

$$\begin{aligned} ds^2 =& \left[R^2 \left(dx^i dx^j \eta_{ij} \right) + \frac{1}{4} k^{1/3} R^{-2} dR^2 \right] + k^{1/3} d\Omega_7^2 \\ &+ \frac{1}{4} k^{1/3} \left[\left(1 - R^3\right)^{-7/3} - 1 \right] R^{-2} dR^2 + k^{1/3} \left[\left(1 - R^3\right)^{-1/3} - 1 \right] d\Omega_7^2 \\ A_{ijk} =& R^3 \epsilon_{ijk}. \end{aligned} \tag{3.22}$$

As we approach the $r = 0$ horizon, which is now the $R = 0$ horizon, the metric tends to,

$$ds^2 \to \left[R^2 \left(dx^i dx^j \eta_{ij} \right) + \frac{1}{4} k^{1/3} R^{-2} dR^2 \right] + k^{1/3} d\Omega_7^2, \tag{3.23}$$

where Ω_7 is the volume of the unit sphere in 7 dimensions. We can further write the metric in the bracket into the standard AdS metric form by letting $\rho = -\frac{k^{1/6}}{2R}$

$$ds^2 \to \frac{1}{4} k^{1/3} \left[\frac{1}{\rho^2} \left(dx^i dx^j \eta_{ij} + d\rho^2 \right) + 4 d\Omega_7^2 \right] \tag{3.24}$$

once again we obtain the standard form of $\mathrm{AdS}_4 \times \mathcal{S}^7$.

Thus we conclude that the near horizon $M2$-brane space-time geometry is $\mathrm{AdS}_4 \times \mathcal{S}^7$ as $R \to 0$, while become flat space as $R \to 1$. This metric only covers part of the entire space-time. We can proceed through the horizon and approach a true singularity as $R \to -\infty$. However, unlike the Schwarzschild metric, this $R \to -\infty$ singularity is timelike [26].

One may notice that the solution to the Laplacian equation (3.15) is a linear differential equation. Hence, any linear combination of solutions can solve the equation, and e^{-C} takes the form

$$e^{-C} = 1 + \sum_i \frac{k}{(r - r_i)^6} \tag{3.25}$$

where r_i specifies the position of each membrane. Thus, the single membrane solution can be generalised into the multi-membrane solution to the supergravity equation of motion, which is given by,

$$\begin{aligned} ds^2 &= \left(1 + \sum_i \frac{k}{(r - r_i)^6} \right)^{-2/3} dx^i dx^j \eta_{ij} + \left(1 + \sum_i \frac{k}{(r - r_i)^6} \right)^{1/3} dy^s dy^t \delta_{st} \\ A_{ijk} &= \epsilon_{ijk} \left(1 + \sum_i \frac{k}{(r - r_i)^6} \right)^{-1}. \end{aligned} \tag{3.26}$$

The reason for that we can perform linear superposition on membrane solutions is due to the no-force condition [25]. It states that the gravitational attraction between two widely separated membrane is exactly cancelled by an equal and opposite repulsive form from the contribution of the anti-symmetric tensors, which is to say that the gravitational force is cancelled by the electric force between two separated membranes, allowing us to linearly stack them together.

3.3 Killing Spinor

As mentioned earlier in Section 2.4, we discussed the pure bosonic solution to supergravity, truncating all the fermionic terms. In order to determine the number of preserved supersymmetries in this truncation, we need to analyse the Killing spinor given in Equation (2.31). Since Equation (2.31) is linear in the Grassmannian parameter ϵ, the anti-commuting identity of ϵ is not relevant for solving the equation, so we can treat it as an ordinary commuting object [26].

To solve the Killing spinor in the p-brane background, we need to choose a basis for the Γ matrices. Considering the Γ matrix in (2.31), we can define them in the vielbein basis,

$$\{\Gamma_a, \Gamma_b\} = \eta_{ab}. \tag{3.27}$$

where η_{ab} is the Minkowski metric. We can express the usual Γ matrices in terms of the world coordinates by using the vielbein as $\Gamma_m = \Gamma_b e_m{}^b$. Additionally, we can perform a $3-8$ split on the Γ matrices, separating them into components for $D = 3$ and $D = 8$:

$$\Gamma_m = \left(\gamma_i \otimes \Sigma_9, \mathbf{1}_{(2)} \otimes \Sigma_M\right) \tag{3.28}$$

where γ_i and Σ_M are the Dirac matrices for $D = 3$ and $D = 8$, respectively. Here, $\Sigma_9 \equiv \Sigma_3 \cdots \Sigma_{10}$ forms the chiral projection operators $\frac{1}{2}(1 \pm \Sigma_9)$.

The spinor parameter can also be split as:

$$\epsilon = \varepsilon \otimes \eta(r) \tag{3.29}$$

where ε is a constant SO(2, 1) spinor, and $\eta(r)$ is an SO(8) spinor that is isotropic. The SO(8) spinor $\eta(r)$ can be further decomposed into chiral spinors using the projection operators.

Plugging in the gauge field solution, spin connection, and the Γ matrices, the supercovariant derivative can be written as:

$$\begin{aligned}
\hat{D}_i &= \partial_i - \frac{1}{2}\gamma_i e^{-A}\Sigma^M \partial_M e^A \Sigma_9 - \frac{1}{6}\gamma_i e^{-3A}\Sigma^M \partial_M e^C, \\
\hat{D}_M &= \partial_M + \frac{1}{4}\mathrm{e}^{-B}\left(\Sigma_M \Sigma^N - \Sigma^N \Sigma_M\right)\partial_N \mathrm{e}^B - \frac{1}{24}\mathrm{e}^{-3A}\left(\Sigma_M \Sigma^N - \Sigma^N \Sigma_M\right)\partial_N \mathrm{e}^C \Gamma_9 \\
&\quad - \frac{1}{6}\mathrm{e}^{-3A}\partial_M \mathrm{e}^C \Gamma_9.
\end{aligned} \tag{3.30}$$

We substitute the supercovariant derivatives into Equation (2.31) and solve for the Killing spinor conditions, obtaining:

$$3A = -6B = C \tag{3.31}$$

which is the same condition for determining the M2-brane solution. Additionally, we find:

$$\eta(r) = \eta_0 e^{-\frac{1}{6}C} \tag{3.32}$$

where η_0 is a constant SO(8) spinor, which is a chiral spinor given by,

$$\frac{1}{2}(1 + \Sigma_9)\eta_0 = 0 \tag{3.33}$$

This reduces the number of supersymmetries by half, indicating that half of the rigid supersymmetry is preserved in the M2-brane background.

4 M2-Brane Action

We have seen that the supergravity in ten dimensions is the low-energy effective theory to the ten-dimensional superstring theory. Analogue to that, the eleven-dimensional supergravity might be related to some supermembrane theory. Bergshoeff *et al.* [5] constructed such action for the supersymmetric M2-brane that known as the supermembrane action, which is given by,

$$S = T_2 \int d^2\xi \left[-\frac{1}{2}\sqrt{-\gamma}\gamma^{ij} E_i{}^a E_j{}^b \eta_{ab} + \frac{1}{2}\sqrt{-\gamma} + \frac{1}{6}\epsilon^{i_1 i_2 i_3} E_{i_1}{}^{A_1} E_{i_2}{}^{A_2} E_{i_3}{}^{A_3} A_{A_3 A_2 A_1} \right]. \tag{4.1}$$

The action contains the Polyakov version of the 2-brane action terms and a Wess-Zumino term involving the super 3-form. We will explain this action in the following sections. For such action of M2-Brane in eleven dimension background to preserve supersymmetry, the space-time background has to satisfy certain constraints, which turns out to be the equation of motion of the eleven-dimensional supergravity [4,5]. This constrain coming from demanding supersymmetry is inherited from the kappa symmetry in the Green-Schwarz superstring action, which was inspired by the kappa symmetry preserved in the action of Brink–Schwarz superparticle [27]. In this section, we will demonstrate the formulation of the supermembrane action, and demonstrate the close relation between the super-M2-brane action and the eleven-dimensional supergravity.

4.1 M2-Brane as a Source of Eleven Dimensional Supergravity

We can view the supermembrane action as the consistent source term of the supergravity. In the section 3.2, we have found the M2-brane solution for 11-dimensional supergravity, and we have also seen that the solution does not cover the entire space. To make this solution valid within the entire space-time, we have to introduce the source term [25]. For example, we can add a delta source at $r = 0$, therefore modify the key equation of solving for the M2-brane solution (3.15) into,

$$\nabla^2 e^{-C} = 6k\Omega_7 \delta^8(y) \tag{4.2}$$

where Ω_7 is the volume of a unit seven-sphere S^7. The similar effect also applies to the Einstein equation.

Therefore we give the combination of pure supergravity (2.1) and the action for supermembrane (4.1),

$$
\begin{aligned}
S =& \frac{1}{2\kappa_{11}^2}\int d^{11}x\left[e\left(R-\frac{1}{48}F_{m_1\cdots m_4}F^{m_1\cdots m_4}\right)+\frac{1}{(12)^4}\epsilon^{m_1\cdots m_{11}}F_{m_1\cdots m_4}F_{m_5\cdots m_8}A_{m_9m_{10}m_{11}}\right]\\
&+T_2\int d^3\xi\left[-\frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\partial_iX^a\partial_jX^b\eta_{ab}+\frac{1}{2}\sqrt{-\gamma}+\frac{1}{6}\epsilon^{i_1i_2i_3}\partial_{i_1}X^{m_1}\partial_{i_2}X^{m_2}\partial_{i_3}X^{m_3}A_{m_3m_2m_1}\right]
\end{aligned}
\tag{4.3}
$$

Once again, we consider only the bosonic terms by setting all the fermionic fields to zero and discard the Grassmann odd coordinate dependence. The formulation of the supermembrane action will be discussed in the following subsections.

With the source term included, the equations of motion get modified. The Einstein equation are enhanced by the matter tensor from the source term,

$$
R_{mn}-\frac{1}{2}g_{mn}R=\kappa_{11}^2T_{mn}, \tag{4.4}
$$

where T_{MN} now being,

$$
\begin{aligned}
\kappa_{11}^2T_{mn} =& \frac{1}{12}\left(F_m{}^{m_2m_3m_4}F_{nm_2m_3m_4}-\frac{1}{12}g_{mn}F_{m_1\cdots m_4}F^{m_1\cdots m_4}\right)\\
&-\kappa_{11}^2T_2\int d^3\xi\frac{1}{e}\sqrt{-\gamma}\gamma^{ij}\partial_iX_m\partial_jX_n\delta^{11}(x-X)
\end{aligned}
\tag{4.5}
$$

The field equation for the 3-form gauge field is given by,

$$
\begin{aligned}
\partial_m\left(eF^{mm_1m_2m_3}\right)&+\frac{1}{1152}\epsilon^{m_1\cdots m_{11}}F_{m_4\cdots m_7}F_{m_8\cdots m_{11}}\\
&=2\kappa_{11}^2T_2\int d^3\xi\epsilon^{i_1i_2i_3}\partial_{i_1}X^{m_1}\partial_{i_2}X^{m_2}\partial_{i_3}X^{m_3}\delta^{11}(x-X).
\end{aligned}
\tag{4.6}
$$

And the membrane field equation is given by varying X^m,

$$
\begin{aligned}
\partial_i\left(\sqrt{-\gamma}\gamma^{ij}\partial_jX^ng_{mn}\right)&+\frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\partial_iX^n\partial_jX^p\partial_mg_{np}\\
&+\frac{1}{6}\epsilon^{i_1i_2i_3}\partial_{i_1}X^n\partial_{i_2}X^p\partial_{i_3}X^qF_{mnpq}=0
\end{aligned}
\tag{4.7}
$$

Then, the solution to the X^m field equations is given by,

$$
\begin{aligned}
X^i&=\xi_i, \quad i=0,1,2\\
X^s&=0, \quad s=3,\ldots,10.
\end{aligned}
\tag{4.8}
$$

Substituting this classical solution into the field equation of gauge field, one obtained the source-enhanced version of (3.15), which is given by,

$$
\nabla^2e^{-C}=2\kappa_{11}^2T_2\delta^8(y). \tag{4.9}
$$

Comparing this with (4.2), the relation of tension in supermembrane action and the gravitational constant is given by,

$$
k=\frac{\kappa_{11}^2T_2}{3\Omega_7}. \tag{4.10}
$$

This solution can also be generalised to N-membrane solution as mentioned in (3.26). Then the constant k and tension T_2 of 2-brane will scale with the factor N as

$$\begin{aligned} k &\to \frac{k}{N}, \\ T_2 &\to \frac{T_2}{N}. \end{aligned} \tag{4.11}$$

4.2 Bosonic p-Brane Action

In order to construct the super-p-brane action, we should start with the simplest bosonic brane action. The bosonic p-brane action in D-dimensional space-time is given by the Nambu-Goto action,

$$S = -T_p \int d^{p+1}\xi \sqrt{-\det h_{ij}} \tag{4.12}$$

where ξ^i are the coordinates on the worldvolume, and h_{ij} is the induced metric on the worldvolume defined through the metric on the flat target space η_{mn},

$$h_{ij} = \partial_i X^m \partial_j X^n \eta_{mn}. \tag{4.13}$$

We can also deduce the same equation of motion via the Polyakov action by introducing an auxiliary metric $\gamma_{ij}(\xi)$,

$$S = T_p \int d^{p+1}\xi \left(-\frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\partial_i X^m \partial_j X^n \eta_{mn} + \frac{1}{2}(p-1)\sqrt{-\gamma} \right). \tag{4.14}$$

The special conformal case $p = 1$, where the cosmological term vanishes, is the string theory case that we are familiar with.

We can generalise this action to formulate the bosonic part of p-brane action in the curved space-time by changing $\eta_{mn} \to g_{mn}$. This is the Howe-Tucker form of p-brane coupled to gravity,

$$S = T_p \int d^{p+1}\xi \left(-\frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\partial_i X^m \partial_j X^n g_{mn} + \frac{1}{2}(p-1)\sqrt{-\gamma} \right). \tag{4.15}$$

The equation of motion of X^m is given by,

$$\gamma^{ij}\left(\partial_i\partial_j X^m - \Omega^k_{ij}(\gamma)\partial_k X^m + \Omega^m_{np}(g)\partial_i X^n \partial_j X^p\right) = 0 \tag{4.16}$$

where $\Omega^k_{ij}(\gamma)$ and $\Omega^m_{np}(g)$ are the Christoffel connections of γ_{ij} and g_{mn}, respectively.

The field equation of γ_{ij} is give by,

$$\gamma_{ij} = \partial_i X^m \partial_j X^n g_{mn}. \tag{4.17}$$

We can substitute the field equation of γ into the equation of motion of x^m to obtain the same equation of motion for X^m that we determined from the Nambu-Goto form action generalised into curved space-time,

$$S = -T_p \int d^{p+1}\xi \sqrt{-\det\left(\partial_i X^m \partial_j X^n g_{mn}\right)} \tag{4.18}$$

demonstrating that (4.18) is equivalent to (4.15) at classical level.

4.3 Supermembrane Action

With the bosonic membrane action in hand, we can extend to superspace to construct the supermembrane action. We start with the formulation of the coordinate Z^M for the superspace:

$$Z^M = (X^m, \theta^\mu) \tag{4.19}$$

with the Grassmann even coordinate X^m and Grassmann odd coordinate θ^μ. We also define the supervielbein $E_M{}^A(Z)$ analogue to the vielbein. Indices M are the world indices with $M = (m, \mu)$, and indices A are the tangent space vielbein indices $A = (a, \alpha)$. We also define the pull-back of the supervielbein from the superspace to the worldvolume,

$$E_i{}^A = \left(\partial_i Z^M\right) E_M{}^A \tag{4.20}$$

The gauge field that the super-p-brane couple to must also be the super-$(p+1)$-form gauge field $A_{A_{p+1}\ldots A_1}$.

Superbranes can be classified into various types. Those which dynamics are fully determined by the worldvolume of the superbrane itself are called the simple superbrane. There are other types of superbrane that have gauge fields in its worldvolume. For example, the D-brane requires a vector field A_μ in its worldvolume in addition to the superspace coordinates. However, the later is beyond the scope of this dissertation.

The M2-brane is a simple superbrane, making it easy to formulate. The general simple superbrane action can be written in two parts: the kinetic term and the gauge coupling term. The kinetic term is the supersymmetric generalisation of the bosonic p-Brane action,

$$S_K = T_p \int d^{p+1}\xi \left[-\frac{1}{2}\sqrt{-\gamma}\gamma^{ij} E_i{}^a E_j{}^b \eta_{ab} + \frac{1}{2}(p-1)\sqrt{-\gamma}\right] \tag{4.21}$$

We notice that the summation of the target space metric is only over the Grassmann even indices, a feature of the superspace formalism [16]. The second part, namely the gauge coupling term, contains the coupling to the background fields,

$$S_G = q \int d^{p+1}\xi \; \epsilon^{i_1 \dots i_{p+1}} E_{i_1}{}^{A_1} \cdots E_{i_{p+1}}{}^{A_{p+1}} A_{A_{p+1} \dots A_1} \tag{4.22}$$

These two parts are related by the κ symmetry, which matches the number of bosonic and fermionic degrees of freedom. This κ symmetry is inspired by the κ symmetry in the superparticle action [28]. The κ symmetry also constraints the number of p one can have in a given D dimensional space-time. We will discuss these in detail later. For now, we will adapt the result that the value of p must satisfy [29],

$$D - p - 1 = \frac{1}{4} n_{min} N, \quad \text{for } p \geq 2, \tag{4.23}$$

where n_{min} is the minimum degrees of freedom of spinor in D dimensions, and N is label for supersymmetry. Apparently, the M2-brane has $D = 11$, $p = 2$, $n_{min} = 32$ and $N = 1$, which satisfied the constraint.

Combining all together, we obtain the general action for the super-p-brane [5]:

$$\begin{aligned} S = T_p \int d^{p+1}\xi \Big[&-\frac{1}{2}\sqrt{-\gamma}\gamma^{ij} E_i{}^a E_j{}^b \eta_{ab} + \frac{1}{2}(p-1)\sqrt{-\gamma} \\ &+ \frac{1}{(p+1)!}\epsilon^{i_1 \dots i_{p+1}} E_{i_1}{}^{A_1} \cdots E_{i_{p+1}}{}^{A_{p+1}} A_{A_{p+1} \dots A_1} \Big]. \end{aligned} \tag{4.24}$$

which is the generalised supersymmetric action for (4.14) plus super-gauge field coupling term (Wess-Zumino term.) At $p = 1$, this supermembrane action reduces to the Green-Schwarz superstring action, as seen in the later section.

The equation of motion of the worldvolume metric γ_{ij} gives rise to the so-called embedding equation,

$$\gamma_{ij} = E_i{}^a E_j{}^b \eta_{ab} = \partial_i X^m \partial_j X^n g_{mn} \tag{4.25}$$

The supermembrane action is manifestly target-space supersymmetric. Therefore, the bosonic and fermionic degrees of freedom should match to satisfy the supersymmetric condition. However, this is not the case as we considered initially. For example, in the case of $D = 11$, $p = 2$ supermembrane, the bosonic target-space coordinates take 11 values, while the Majorana spinor in eleven-dimension has 32 spinor degrees of freedom. We should also consider the gauge symmetry in the worldvolume.

The three-dimensional worldvolume diffeomorphism invariance is given by,

$$\begin{aligned} \delta_{u^{(3)}} Z^M &= u^i \partial_i Z^M, \\ \delta_{u^{(3)}} \gamma_{ij} &= u^k \partial_k \gamma_{ij} + 2\partial_{[i} u^k \gamma_{j]k} \end{aligned} \tag{4.26}$$

which should be subtracted from the number of bosonic degrees of freedom, leaving only $11-3=8$ bosonic degrees of freedom. On the other hand, the Majorana spinor satisfies the first-order equation of motion on the worldvolume, rather than the second-order equation of motion satisfied by the bosons. Hence, we expect that the spinor degrees of freedom will match twice the bosonic degrees of freedom. Even though we considered all the bosonic gauge symmetries and the order of field equations, there is still a discrepancy of a factor of 2. This difference can only be resolved by introducing the fermionic gauge symmetry, also known as the κ symmetry.

The κ symmetry is a fermionic gauge symmetry generated by a Majorana spinor κ^α, which is both a tangent space spinor and a worldvolume scalar [30]. The transformation parameterised by κ^α has a projector that reduces the number of degrees of freedom by half [31].

The transformation rules of the κ symmetry for general super-p-brane action are given by,

$$\begin{aligned} \delta E^a &= 0, \\ \delta E^\alpha &= \kappa^\beta (1+\Gamma)^\alpha{}_\beta. \end{aligned} \tag{4.27}$$

where $\delta E^A = \delta Z^M E_M{}^A$, $A=(a,\alpha)$ and $\Gamma^\alpha{}_\beta$ is given by,

$$\Gamma^\alpha{}_\beta = \frac{(-1)^{\frac{d(d-3)}{4}}}{d!\sqrt{-\gamma}} \epsilon^{i_1 \dots i_d} E_{i_1}{}^{a_1} \cdots E_{i_d}{}^{a_d} (\Gamma_{a_1 \dots a_d})^\alpha{}_\beta \tag{4.28}$$

where $d=p+1$ is the worldvolume dimension, and Γ_a are the Dirac matrices in the vielbein basis. The choice of $\Gamma^\alpha{}_\beta$ is made such that it satisfies the equation

$$\Gamma^\alpha{}_\gamma \Gamma^\gamma{}_\beta = \delta^\alpha_\beta, \tag{4.29}$$

provided that the equations of motion of the action are satisfied. Consequently, the operators $\frac{1}{2}(1 \pm \Gamma)^\alpha{}_\beta$ become projection operators. As a result, kappa symmetry implies that we can gauge away half of the fermionic degrees of freedom E^α. This is why the term "fermionic gauge symmetry" is used. With such fermionic gauge symmetric, we can match the physical bosonic and fermionic degrees of freedom in eleven-dimension, and therefore demonstrates the manifest supersymmetric.

Several constraints must be satisfied for the super-p-brane action to admit κ symmetry [31]. The field strength $F=dA$ should satisfies,

$$\begin{aligned} F_{\alpha a_{p+1} \dots a_1} &= 0 \\ F_{\alpha\beta\gamma a_{p-1} \dots a_1} &= 0 \\ F_{\alpha\beta a_p \dots a_1} &= \frac{(-1)^{p+\frac{1}{4}(p+1)(p-2)}}{2p!} \left(\Gamma_{a_1 \dots a_p}\right)_{\alpha\beta} \end{aligned} \tag{4.30}$$

The superspace torsion is given by,

$$T^A = dE^A + E^B \Omega(g)_B{}^A = \frac{1}{2} E^B E^C T_{BC}{}^A \tag{4.31}$$

where $\Omega(g)_B{}^A$ is the connection one-form on the curved target space. The constraints on the torsion are then given by,

$$\begin{aligned} \eta_{ca}(T^c_b)_\alpha &= 0, \\ T^a_{\alpha\beta} &= \Gamma^a{}_{\alpha\beta}. \end{aligned} \tag{4.32}$$

We noticed a remarkable consequence that these constraints imply the equations of motion of a maximally supersymmetric supergravity theory, *e.g.* $D = 11$ supergravity or $D = 10$ type II supergravity. Therefore, the background is demanded to be the corresponding maximally supersymmetric supergravity for the supermembrane action to be supersymmetric.

4.4 Green-Schwarz Superstring in Flat Superspace

Before we move on to solve the supermembrane action, it is convenient for us to look at the superstring action. We will provide a covariant description of type IIA superstrings in the flat space background by combining the global super-Poincare invariance and local κ symmetry [27].

The Green-Schwarz superstring action is generalised from the Brink–Schwarz superparticle action. The covariant action for massless superparticle is known as,

$$S = \frac{1}{2} \int d\tau e^{-1} E^m E^n \eta_{mn} \tag{4.33}$$

where $E^m = \partial_\tau X^m - i\bar{\theta}\gamma^m \partial_\tau \theta$, and e is the square root of the one-dimensional metric. The dotted variable stands for taking the derivative with respect to the worldline parameter τ. The superparticle action is invariant under the global supersymmetry transformations parameterised by a constant Majorana spinor ϵ [32],

$$\delta X^m = i\bar{\epsilon}\Gamma^m \theta, \quad \delta\theta = \epsilon, \tag{4.34}$$

and a hidden local κ symmetry given by [28],

$$\delta X^m = i\bar{\theta}\Gamma^m \delta\theta, \quad \delta\theta^\mu = i(\Gamma_m)^\mu_\nu E^m \kappa^\nu, \quad \delta e = -4e\partial_\tau \bar{\theta}^\mu \kappa_\mu \tag{4.35}$$

where κ_μ is an arbitrary function of τ.

Given the Brink–Schwarz superparticle action, it is natural that one may think of generalising it to the superstring action, which is given as,

$$S = T_1 \int d^2\xi \sqrt{-\gamma}\gamma^{ij} E_i{}^m E_j{}^n \eta_{mn}. \tag{4.36}$$

Considering that the type IIA superstring has two Majorana-Weyl fermions θ^1, θ^2 of opposite chirality and each with 16 components, the pull-back of supervielbein $E_i{}^m$ is given by,

$$E_i{}^m = \partial_i X^a - i\left(\theta^{1\mu}(\Gamma^a)_{\mu\nu}\partial_i\theta^{1\nu} + \theta^{2\mu}(\Gamma^a)_{\mu\nu}\partial_i\theta^{2\nu}\right). \tag{4.37}$$

as we only interested in the bosonic part. However, this action does not admit the κ symmetry. We restored this symmetry by introducing an extra term in the action,

$$\begin{aligned} S = T_1 \int d^2\xi \Big(& -\frac{1}{2}\sqrt{-\gamma}\gamma^{ij} E_i{}^m E_j{}^n \eta_{mn} \\ & - i\epsilon^{ij}\partial_i X^m \left(\bar{\theta}^1\Gamma_m\partial_j\theta^1 - \bar{\theta}^2\Gamma_m\partial_j\theta^2\right) + \epsilon^{ij}\bar{\theta}^1\Gamma^m\partial_i\theta^1\bar{\theta}^2\Gamma_m\partial_j\theta^2\Big). \end{aligned} \tag{4.38}$$

This action is invariant under both the global supersymmetric transformation and local κ symmetry transformation.

4.5 The M2-brane in D=11 Flat Superspace

Section 4.2 shows that the eleven-dimensional supergravity is a consistent background of the supermembrane in the eleven-dimensional target space. The flat space is a trivial solution to the field equations of eleven-dimensional supergravity and thus can be chosen as the background where the supermembrane moves in [31].

The flat space supervielbein in $D = 11$ is given by,

$$E_M{}^a = (\delta_m{}^a, -i(\Gamma^a)_{\mu\nu}\theta^\nu), \quad E_M{}^\alpha = (0, \delta_\mu{}^\alpha). \tag{4.39}$$

As stated in section 4.4, we will not distinguish contracting between a and m. Then, $E_i{}^A$ is given by,

$$E_i{}^A = \left(\partial_i X^a - i\bar{\theta}^\rho(\Gamma^a)_{\rho\sigma}\partial_i\theta^\sigma, \partial_i\theta^\alpha\right) \tag{4.40}$$

We eliminate the 3-form gauge field and obtain the super-2-brane action in flat background [33, 34],

$$\begin{aligned} S = & T_2 \int \xi^3 \Big[-\frac{1}{2}\sqrt{-\gamma}\gamma^{ij}E_i{}^a E_j{}^b\eta_{ab} + \frac{1}{2}\sqrt{-\gamma} \\ & -i\epsilon^{ijk}\bar{\theta}\Gamma_{ab}\partial_i\theta\left(E_j{}^a E_k{}^b + iE_j{}^a\bar{\theta}\Gamma^b\partial_k\theta - \frac{1}{3}\bar{\theta}\Gamma^a\partial_j\theta\bar{\theta}\Gamma^b\partial_k\theta\right)\Big]. \end{aligned} \tag{4.41}$$

Notice that this supersymmetric action is similar to (4.33) and (4.38). So, it is natural that one consider this action as a further generalisation of the Green-Schwarz Superstring action that admits the local ϵ supersymmetry and local κ symmetry.

The equations of motion, after integrating out the 3-form gauge field A, with respect to γ^{ij}, X^m, and θ are given by,

$$\gamma_{ij} = E_i{}^m E_{im}, \tag{4.42}$$

$$\partial_i \left(\sqrt{-\gamma}\gamma^{ij} E_j{}^m\right) + \epsilon^{ijk} E_i{}^n \partial_j \bar{\theta}\Gamma^m{}_n \partial_k \theta = 0, \tag{4.43}$$

$$(1-\Gamma)\gamma^{ij} E_i{}^m \Gamma_m \partial_j \theta = 0, \tag{4.44}$$

respectively. The Γ is given in (4.28) taking $p = 2$.

The flat space admit the super-Poincare symmetry, which generators are momentum P^m, angular momentum M^{mn}, and super charges Q. The canonical momentum $K^m = \frac{\partial \mathcal{L}}{\partial \dot{X}_m}$ is given by,

$$K^m = -\frac{T_2}{2}\left[\sqrt{-\gamma}\gamma^{0j} E_j{}^m + \epsilon^{0jk}\left(E_j{}^n + \frac{i}{2}\bar{\theta}\Gamma^n \partial\theta\right)\bar{\theta}\Gamma_k{}^m \partial_j \theta\right] \tag{4.45}$$

We can further integrating over $\xi^1\xi^2$ to obtain the conserved momentum P^m, which is given by,

$$P^m = -\frac{T_2}{2}\int d\xi^1 d\xi^2 \left[\sqrt{-\gamma}\gamma^{0j} E_j{}^m + \epsilon^{0jk}\left(E_j{}^n + \frac{i}{2}\bar{\theta}\Gamma^n \partial\theta\right)\bar{\theta}\Gamma_k{}^m \partial_j \theta\right] \tag{4.46}$$

Similarly, we can also obtain the conserved angular momentum M^{mn}, which is given by,

$$\begin{aligned} M^{mn} = \int d\xi^1 d\xi^2 \Big[& X^m K^n - X^n K^m - \frac{iT_2}{4}\sqrt{-\gamma}\gamma^{0i} E_i{}^p \bar{\theta}\Gamma_p \Gamma^{mn}\theta + \frac{T_2}{8}\epsilon^{0ij} E_i{}^p E_j{}^q \bar{\theta}\Gamma_{pq}\Gamma_{mn}\theta \\ & + \frac{iT_2}{8}\epsilon^{0ij}\left(E_i^p + \frac{i}{3}\bar{\theta}\Gamma^p \partial_i \theta\right)\left(\bar{\theta}\Gamma^q \Gamma^{mn}\theta\bar{\theta}\Gamma_{pq}\partial_j\theta + \bar{\theta}\Gamma_{pq}\Gamma^{mn}\theta\bar{\theta}\Gamma^q \partial_j \theta\right)\Big] \end{aligned} \tag{4.47}$$

The rest of the supercharges Q^μ are determined to be,

$$\begin{aligned} Q^\mu = -\frac{T_2}{2}\int d\xi^1 d\xi^2 \Big[& 2i\sqrt{-\gamma}\gamma^{0i} E_i{}^m (\Gamma_m)^\mu{}_\nu \theta^\nu - \epsilon^{0ij} E_i{}^m E_j{}^n (\Gamma_{mn})^\mu{}_\nu \theta^\nu \\ & - \frac{4i}{3}\epsilon^{0ij}\left(E_i{}^m + \frac{2i}{5}\bar{\theta}\Gamma^m \partial_i \theta\right)\left((\Gamma_{mn})^\mu{}_\nu \theta^\nu \bar{\theta}\Gamma^n \partial_j \theta + (\Gamma^n)^\mu{}_\nu \theta^\nu \bar{\theta}\Gamma_{mn}\partial_j \theta\right)\Big]. \end{aligned} \tag{4.48}$$

4.6 Classical Solutions of M2-Brane Action in Flat Superspace

Unlike string theory, we cannot choose a particular gauge to put the equations of motion of the M2-brane into linear form, making it impossible to find the general solution. However, we can find specific classical solutions in a particular background [31]. The fermionic solution is always solved by $\theta = 0$; therefore, the equation of motion is reduced to that of the bosonic membranes. For a classical solution to the bosonic membrane with fermionic terms set to zero, one can verify the condition $\delta S = 0$ to check if the solution is supersymmetric, like we did in section 3 for the supergravity action. We present here some stable classical solution in flat space:

(1) *String Solution.* We can perform a double dimensional reduction to compactify one dimension on a circle; the ansatz is given by,

$$X^{10} = \xi^2, \quad \partial_{\xi^2} X^m = 0, \quad m = 0, \ldots, 9, \tag{4.49}$$

resulting in a flat ten-dimensional background $(\text{Mink})_{10} \times T^1$. We will discuss in detail the double dimensional reduction in section 5. Then, the theory reduces to a ten-dimensional superstring theory, and all the string solutions will solve the equations of motion of the M2-brane.

(2) *Static Toroidal Membrane.* We can further perform a double dimensional reduction, namely wrapping around a torus, to reduce to a nine-dimensional superparticle theory. The ansatz are give by,

$$X^{10} = \xi^2, \quad X^9 = \xi^1, \quad \partial_{\xi^2} X^m = 0, \quad m = 0, \ldots, 8, \tag{4.50}$$

The extended object has reduced to a massive point particle, whose mass is given by the energy on the toroidal membrane. The equation of motion of the particle is given by,

$$X^m = X^m_0 + p^m \xi^0, \quad p^2 = m^2 \tag{4.51}$$

(3) *Spherical Solutions.* A spherically symmetric solution of M2-brane was first founded by Dirac when solving for the static model of membrane carrying electric charges [35]. There is no static solution without the presence of electric charge; however, a periodically pulsating membrane solution is valid, similar to the pulsating string solution of string

theory [36]. The solution is given by,

$$X^0 = \xi^0, \quad X^1 = \rho(\xi^0)\sin\xi^1\cos\xi^2, \qquad X^2 = \rho(\xi^0)\sin\xi^1\sin\xi^2,$$
$$X^m = 0, \quad m = 3, \ldots, 10. \tag{4.52}$$

where $\rho(\xi^0)$ is solved from the equation,

$$\dot{\rho} + \frac{1}{\rho_0^2}\sqrt{(\rho_0^4 - \rho^4)} = 0, \quad \rho_0 > 0 \tag{4.53}$$

(4) *Pancake Membrane.* Kikkawa and Yamasaki initially propose this solution as a solution of an open membrane propagating in a Minkowski space-time with dimension greater or equal to 5 [37]. The solution for the M2-brane in the eleven-dimensional background can be written as,

$$X^0 = \xi^1,$$
$$X^1 + iX^2 = \xi^1 e^{i\omega\xi^0}, \quad X^3 + iX^4 = \xi^2 e^{i\omega\xi^0}, \tag{4.54}$$
$$X^m = 0, \quad m = 5, \ldots, 10.$$

This solution describes a disk spinning in the five-dimensional space-time.

The spinning disc solution is an open membrane solution that breaks the supersymmetry. Hence, we considered only the closed membrane solution to avoid the issue [31]. For consistency, the open membrane solution was generalised to the closed membrane solution by identifying the boundaries of two copies of the disc-like membrane [38].

One can perform the semi-classical quantisation by fluctuating around these classical solutions [6].

4.7 p-Branes as Solitons

We have introduced the supermembrane action as the source term of the 11d supergravity; nevertheless, there is another interpretation of supermembranes as solitons. The solitons are the topological defects in quantum field theory [39], leading to the non-perturbative effects of the theory. The p-brane can be viewed as the solitons that solve the equations of motion of the corresponding quantum field theory [40].

The soliton approach of supermembranes was through the effective action for the extended solitons of supersymmetric field theories. It was first noticed that the effective action of the

$D = 6$, $N = 1$ abelian supersymmetric gauge theory is the Green-Schwarz covariant superstring action in four-dimensional space-time [30]. Such a solution to the gauge theory is referred to as the soliton. Later on, the generalisation of this superstring theory inspired the construction of the supermembrane action. It was argued that certain types of membranes correspond to solitons of some supergravity background [41]. The types of supermembranes are listed as the $\mathcal{H}, \mathcal{C}, \mathcal{R}$ sequences, which are sequences of p-branes in D-dimensional target space-time [29]. It was also postulated in the same paper that the $\mathcal{O}$ sequence, which contains the string in 10d supergravity and 2-brane in 11d supergravity, would present the same property. Later studies have shown that these extended objects include the σ-model source and hence are not solitons [25, 42]. However, the super-five-brane was proved to be the soliton solution to the 11d supergravity, but it is not our main focus here [43].

5 Dimensional Reduction of Supergravity and M2-Brane

It was shown that the D=10 type IIA supergravity can be obtained from the D=11 supergravity by dimensional reduction [20]. It is also known that we can deduce the action of type IIA superstring from the action of supermembrane in $D = 11$ supergravity background. However, we have seen that the extended object in D=11 supergravity is a two-dimensional membrane, not the one-dimensional string. Hence, to match the dimensions, one must reduce the worldvolume dimension from three to two and the target space dimension from eleven to ten. In this section, we will show how we derive the D=10 type IIA supergravity from the D=11 supergravity via the Kazula-Klein dimensional reduction, and how one can obtain the superstring action from the supermembrane action via the double dimensional reduction.

The string theory has thought to be special due to the Weyl symmetry that only presents when $p = 1$. It is not obvious that the Weyl symmetry van emerge from the double dimensional reduction from the M2-brane theory that does not admit Weyl symmetry itself. We will see that the Weyl symmetry does arise from the residual symmetry of the three-dimensional worldvolume diffeomorphism after dimensional reduction.

5.1 From $D = 11$ Supergravity to $D = 10$ Type IIA Supergravity

In this section, we rewrite the index convention for clarity: the hatted indices run over all eleven-dimensional coordinates, and non-hatted indices run over all ten-dimensional coordinates.

To carry out the dimensional reduction, naturally, we can separate ten space-time coordinates from the original eleven dimensions,

$$x^{\hat{m}} = (x^m, y), \quad m = 0, \dots, 9. \tag{5.1}$$

The dimensional reduction is to compactify the eleventh dimension on a circle $\mathcal{S}^1$, which means that the eleventh coordinate is identified as $y = y + 2\pi R_c$. Then, we can represent the periodic dependence of the eleventh coordinate by its Fourier modes. For example, for a field $\phi(x^{\hat{m}})$ in eleven dimension, it can be expressed as,

$$\phi(x^{\hat{m}}) = \phi(x^m, y) = \phi(x^m) + \sum_{n \neq 0} e^{in\theta} \phi_n(x^m) \tag{5.2}$$

where $\theta = \frac{y}{R}$. Thus, one field in eleven dimensions will correspond to an infinite number of fields in ten dimensions, as n can take any possible integers. However, the non-zero modes, where $n \neq 0$, will be massive in ten-dimensional space-time due to their momentum in the compactifying eleventh dimension. Such massive particles are called Kaluza–Klein particles.

Such massive particle looks troublesome. However, one could see that the momentum of such massive modes is inversely proportional to the radius of the compactifying s=dimension R_c. Hence, as we take the $R_c \to 0$ limit, such modes become infinitely massive and entirely decoupled from the theory, with only the massless zero modes left.

The general form of metric that is reduced by one dimension is given by,

$$ds^2_{d+1} = e^{2\alpha_d \phi} ds^2_d + e^{-2(d-2)\alpha_d \phi} \left(dy + A_m dx^m\right)^2 \tag{5.3}$$

where ϕ is a scalar field and

$$\alpha_d = \sqrt{\frac{1}{2(d-1)(d-2)}}. \tag{5.4}$$

The choice of the constant α_d ensures that the action after dimensional reduction takes the canonical form without the extra exponential of ϕ. Substituting $d = 10$ into this general expression, we find the decomposition of the eleven-dimensional metric,

$$ds^2_{11} = e^{\frac{1}{6}\phi} ds^2_{10} + e^{-\frac{4}{3}\phi}(dy + A_m dx^m)^2 \tag{5.5}$$

And the metric is given by,

$$g^{(11)}_{\hat{m}\hat{n}} = \begin{pmatrix} e^{\frac{1}{6}\phi} g_{mn} + e^{-\frac{4}{3}\phi} A_m A_n & A_m e^{-\frac{4}{3}\phi} \\ A_n e^{-\frac{4}{3}\phi} & e^{-\frac{4}{3}\phi} \end{pmatrix} \tag{5.6}$$

Or, we can write in terms of the vielbein,

$$e^{(11)}{}_{\hat{m}}{}^{\hat{a}} = \begin{pmatrix} e^{\frac{1}{12}\phi} e_m{}^a & e^{-\frac{2}{3}\phi} A_a \\ 0 & e^{-\frac{2}{3}\phi} \end{pmatrix}, \quad e^{(11)}{}_{\hat{a}}{}^{\hat{m}} = \begin{pmatrix} e^{-\frac{1}{12}\phi} e_a{}^m & -A_n e^{-\frac{1}{12}\phi} \left(e^{-1}\right)_a{}^n \\ 0 & e^{\frac{2}{3}\phi} \end{pmatrix} \tag{5.7}$$

where $\hat{a}$ runs through 0 to 10 and a runs through 0 to 9. We use the local Lorentz symmetry to set $e_y{}^a = 0$. To obtained the pre-factor factor from compactifying on the circle, we integrate out the eleventh dimension,

$$\int_0^{2\pi R_c} dy \sqrt{g^{11}_{yy}} = 2\pi R_c e^{-\frac{2}{3}\phi} \tag{5.8}$$

With the vielbein given above, we can deduce the dimensional reduction of the Einstein term in the action,

$$\int d^{11}x e^{(11)} R(g^{(11)}) = 2\pi R_c \int d^{10}x e^{(10)} \left(R(g^{(10)}) - \frac{1}{2}\partial_m \phi \partial^m \phi - \frac{1}{4} e^{-\frac{3}{2}\phi} F_{mn} F^{mn}\right), \tag{5.9}$$

where $F_{mn} = 2\partial_{[m}A_{n]}$ is the field strength of A_m. Notice that the e on the LFS is of the eleven-dimensional vielbein while the e on the RHS is the ten-dimensional. One can check that these terms match those in (2.16).

Then, we consider the dimensional reduction of the gauge field term $\frac{e}{48}F^{(11)}_{\hat{m}_1\ldots\hat{m}_4}F^{(11)\hat{m}_1\ldots\hat{m}_4}$. The 3-form gauge field $A_{\hat{m}\hat{n}\hat{p}}$ can be divided into two ten-dimensional objects, one with the circle direction y and one without,

$$A^{(11)}_{\hat{m}\hat{n}\hat{p}} = (C_{mnp}, B_{mn}) \tag{5.10}$$

where we take $A^{(11)}_{mn10}$ to be B_{mn} and $A^{(11)}_{mnp}$ to be C_{mnp} to agree with the convention in type IIA supergravity. Therefore, the 4-form field strength $F_{\hat{m}\hat{n}\hat{p}\hat{q}}$ decomposes into,

$$\begin{aligned} F^{(11)}_{mnp10} &= 3\partial_{[m}B_{np]} = H_{mnp} \\ F^{(11)}_{mnpq} &= 4\partial_{[m}C_{npq]} = F_{mnpq} \end{aligned} \tag{5.11}$$

and written on vielbein basis,

$$\begin{aligned} F^{(11)}_{a_1a_2a_310} &= e^{-\frac{5}{12}\phi}H_{a_1a_2a_3} \\ F^{(11)}_{a_1a_2a_3a_4} &= e^{-\frac{1}{3}\phi}\left(F_{a_1\ldots a_4} + 4A_{[a_1}H_{a_2a_3a_4]}\right) = \hat{F}_{a_1a_2a_3a_4} \end{aligned} \tag{5.12}$$

Then we got,

$$\frac{1}{48}F^{(11)}_{\hat{a}_1\ldots\hat{a}_4}F^{(11)\hat{a}_1\ldots\hat{a}_4} = \frac{1}{48}e^{-\frac{2}{3}\phi}\hat{F}_{a_1a_2a_3a_4}\hat{F}^{a_1a_2a_3a_4} + \frac{4}{48}e^{-\frac{5}{6}\phi}H_{a_1a_2a_3}H^{a_1a_2a_3} \tag{5.13}$$

Substituting back the vielbein, we obtain the dimensional reduction of the gauge field term,

$$\begin{aligned} &\int d^{11}x e^{(11)}\frac{1}{48}F^{(11)}_{\hat{m}_1\ldots\hat{m}_4}F^{(11)\hat{m}_1\ldots\hat{m}_4} \\ &\qquad = 2\pi R_c\int d^{10}x e^{(10)}\left(\frac{1}{48}e^{-\frac{1}{2}\phi}\hat{F}_{m_1\ldots m_4}\hat{F}^{m_1\ldots m_4} + \frac{1}{12}e^{\phi}H_{m_1\ldots m_3}H^{m_1\ldots m_3}\right) \end{aligned} \tag{5.14}$$

Finally, the Chern-Simons term can be transformed easily considering that the antisymmetry of the wedge products,

$$F^{(11)}_4 \wedge F^{(11)}_4 \wedge A^{(11)}_3 = F^{(10)}_4 \wedge F^{(10)}_4 \wedge B^{(10)}_2 \tag{5.15}$$

Putting everything together, we derived the dimensional reduction of the $D = 11$ supergravity action on a circle,

$$\begin{aligned} S_{10} = \frac{2\pi R_c}{2\kappa^2_{11}}\int d^{10}x e^{(10)}\Big(&R(g^{(10)}) - \frac{1}{2}\partial_m\phi\partial^m\phi - \frac{1}{4}e^{-\frac{3}{2}\phi}F_{m_1m_2}F^{m_1m_2} \\ &-\frac{1}{48}e^{-\frac{1}{2}\phi}\hat{F}_{m_1\ldots m_4}\hat{F}^{m_1\ldots m_4} - \frac{1}{12}e^{\phi}H_{m_1\ldots m_3}H^{m_1\ldots m_3}\Big) \\ &+\int F_4\wedge F_4\wedge B_2 \end{aligned} \tag{5.16}$$

which is precisely the bosonic part of the action for type IIA supergravity. Thus, we have obtained the $D = 10$ type IIA supergravity from the $D = 11$ supergravity.

5.2 Double Dimensional Reduction

One can obtain an action of the type IIA superstring coupled to an $N = 2$, $D = 10$ supergravity background from the action of supermembrane coupled to an $D = 11$ supergravity background via double dimensional reduction [4]. The notion of "double" comes from the fact that we simultaneously compact the worldvolume and target space-time on a circle to reduce dimension, which reduces the target space and worldvolume dimension by one. We can also see that the conformal symmetry of string theory arises from the dimensional reduction process as the residual symmetry of the three-dimensional diffeomorphism [44].

Following the dimensional reduction on the supergravity action, we move to the dimensional reduction on the action of the supermembrane coupled to the supergravity background. The action is given in (4.1). We will first work on the pure bosonic sector given in (4.3).

Also, we can do the two-one splitting for the worldvolume coordinates,

$$\xi^{\hat{i}} = (\xi^i, \rho), \quad i = 0, 1. \tag{5.17}$$

and also for the target space coordinates,

$$X^{(11)\hat{m}} = \left(X^{(10)m}, Y\right). \tag{5.18}$$

Then, we can choose the partial static gauge choice such that $\rho = Y$. Several ansatz were made for the double dimensional reduction to be carried out. We demand that,

$$\partial_\rho X^m = 0 \tag{5.19a}$$

$$\partial_\rho \gamma_{\hat{i}\hat{j}} = 0 \tag{5.19b}$$

$$\partial_Y g_{\hat{m}\hat{n}} = 0 \tag{5.19c}$$

$$\partial_Y A_{\hat{m}\hat{n}\hat{p}} = 0 \tag{5.19d}$$

Then, a suitable choice of ten-dimensional components from the eleven-dimensional fields is given by,

$$g^{(11)}_{\hat{m}\hat{n}} = e^{-\frac{2}{3}\phi} \begin{pmatrix} g_{mn} + e^{2\phi} A_m A_n & A_m e^{2\phi} \\ A_n e^{2\phi} & e^{2\phi} \end{pmatrix} \tag{5.20}$$

Using the embedding equation (4.25), we determine the induced metric on the worldvolume after dimensional reduction,

$$\gamma^{(3)}_{\hat{i}\hat{j}} = g^{(3)}_{\hat{i}\hat{j}} = e^{-\frac{2}{3}\phi} \begin{pmatrix} \gamma_{ij} + e^{2\phi} A_i A_j & A_i e^{2\phi} \\ A_j e^{2\phi} & e^{2\phi} \end{pmatrix} \tag{5.21}$$

where $\gamma_{ij} = g_{ij} = \partial_i X^m \partial_j X^n g_{mn}$ is the induced metric ion the world-sheet in ten-dimensional target space, and $A_i = \partial_i X^m A_m$. We determine that,

$$\det\left(\gamma^{(3)}_{\hat{i}\hat{j}}\right) = \det\left(\gamma^{(2)}_{ij}\right) \tag{5.22}$$

since $\gamma^{(3)}$ is a three-by-three matrix.

Recall that the membrane field equation is given by (4.7), by substituting the ansatz above, we reduce the dimension to ten,

$$\begin{aligned} \partial_i \left(\sqrt{-\gamma}\gamma^{ij}\partial_j X^n g_{mn}\right) + \frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\partial_i X^n \partial_j X^p \partial_m g_{np} \\ + \frac{3}{6}\epsilon^{i_1 i_2 i_3}\partial_{i_1} X^n \partial_{i_2} X^p \partial_{i_3} Y F_{mnp10} = 0 \end{aligned} \tag{5.23}$$

Using (5.19), we have only $\partial_\rho y \neq 0$. Therefore, the membrane field equation becomes,

$$\begin{aligned} \partial_i \left(\sqrt{-\gamma}\gamma^{ij}\partial_j X^n g_{mn}\right) + \frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\partial_i X^n \partial_j X^p \partial_m g_{np} \\ + \frac{1}{2}\epsilon^{ij}\partial_i X^n \partial_j X^p H_{mnp} = 0 \end{aligned} \tag{5.24}$$

where $H_{mnp} = F_{mnp10}$ is the 3-form field strength from dimensional reduction of $F_{\hat{m}\hat{n}\hat{p}\hat{q}}$. We notice that (5.24) is just the equation of motion of the action,

$$S = \int d^2\xi \left(\frac{1}{2}\sqrt{-\gamma}\gamma^{ij}\partial_i X^m \partial_j X^n g_{mn} + \frac{1}{2}\epsilon^{ij}\partial_i X^m \partial_j X^n B_{mn}\right). \tag{5.25}$$

with respect to X^m. This action describes a one-dimensional string coupled to the type IIA supergravity background.

Consequently, dimensional reduction reduces the eleven-dimensional membrane to a ten-dimensional string, eliminating the cosmological term $\sqrt{-\gamma}$. It also changes the Wess-Zumino term from being coupled to a three-form into being coupled to a two-form. The other field contents in the type IIA supergravity, C_{mnp}, A_m, and ϕ are decoupled from the bosonic sector but will still couple to the theory in the fermionic sector.

The double dimensional reduction differs from conventional Kaluza-Klein dimensional reduction by making the partial gauge choice $\rho = Y$, which means we compactify the target space and worldvolume on the same circle. The ansatz (5.19b) indicates that we discard the massive modes on the worldvolume in the reduction. Such truncation of massive modes is proved to be consistent with the membrane and the background supergravity [45].

5.3 From Supermembrane to Superstring

We have discussed the dimensional reduction of the bosonic part of the eleven-dimensional membrane action coupled to supergravity. Likewise, we can perform dimensional reduction on the full supermembrane action in superspace [4]. For simplicity, we eliminate the auxiliary metric by writing in the Nambu-Goto form of action,

$$S^{(11)} = T_3 \int d^3\xi \left[-\sqrt{-\det\left(E_{\hat{i}}{}^{\hat{a}} E_{\hat{j}}{}^{\hat{b}} \eta_{\hat{a}\hat{b}} \right)} + \frac{1}{6} \epsilon^{\hat{i}_1\hat{i}_2\hat{i}_3} E_{\hat{i}_1}{}^{\hat{A}_1} E_{\hat{i}_2}{}^{\hat{A}_2} E_{\hat{i}_3}{}^{\hat{A}_3} A_{\hat{A}_3\hat{A}_2\hat{A}_1} \right]. \tag{5.26}$$

Similarly, we separate the eleventh dimension of the bosonic sector,

$$Z^{\hat{M}} = (X^m, Y, \theta^\mu) = (Z^M, Y), \tag{5.27}$$

All 32 supercharges are preserved under the dimensional reduction from $D = 11$ to $D = 10$, hence all the Grassmann odd coordinates are preserved. We imposed the same ansatz,

$$\partial_\rho Z^M = 0 \tag{5.28}$$

and identify $\rho = y$ as in the bosonic case.

Then, the Kaluza-Klein ansatz of the supervielbein is given by,

$$\begin{aligned} E^{(11)}{}_{\hat{M}}{}^{\hat{A}} &= \begin{pmatrix} E_M{}^a & E_M{}^\alpha & E_M{}^{10} \\ E_y{}^a & E_y{}^\alpha & E_y{}^{10} \end{pmatrix} \\ &= \begin{pmatrix} E_M{}^a & E_M{}^\alpha + A_M \chi^\alpha & \Phi A_M \\ 0 & \chi^\alpha & \Phi \end{pmatrix}, \end{aligned} \tag{5.29}$$

where $E_M{}^a$ and $E_M{}^\alpha$ are the components of the ten-dimensional supervielbein $E_M{}^A$, A_M is the superspace U(1) gauge field. Φ and χ^α are superfields. The leading orders of them are dilaton ϕ and dilatino λ^α, respectively. As we did for the vielbein, we imposed local Lorentz symmetry to set $E^{(11)}{}_y{}^a = 0$.

The gauge fields $A_{\hat{M}\hat{N}\hat{P}}$ decomposed into,

$$A^{(11)}_{\hat{M}\hat{N}\hat{P}} = (C_{MNP}, B_{MN}) \tag{5.30}$$

where we have identified $A^{(11)}_{MNP} = C_{MNP}$ and $A^{(11)}_{MNy} = B_{MN}$.

Substituting all the reduced ten-dimensional fields into the supermembrane action (5.26) and setting $\theta^\mu = 0$, we obtained the action of superstring that coupled to a supergravity background,

$$S^{(10)} = T_1 \int \mathrm{d}^2\xi \left(-\sqrt{-\det E_i{}^a E_j{}^b \eta_{ab}} + \frac{1}{2}\epsilon^{ij}\partial_i Z^M \partial_j Z^N B_{NM} \right). \tag{5.31}$$

where all constants are absorbed into $T_1 = 2\pi R_c T_2$.

We can determine the κ symmetry after dimensional reduction. Substituting the K-K ansatz, the transformation law in ten dimensions is determined to be,

$$\begin{aligned} \delta Z^m &= 0, \\ \delta Z^\alpha &= \kappa^\beta (1 + \Gamma^{(10)})^\alpha{}_\beta. \end{aligned} \tag{5.32}$$

where $\Gamma^{(10)}$ is determined from $\Gamma^{(11)}$ via identifying $a_3 = 11$,

$$\begin{aligned} (\Gamma^{(10)})^\alpha{}_\beta &= \frac{1}{6\sqrt{-\gamma}}\epsilon^{ij} E_i{}^{a_1} E_j{}^{a_2} 3 \left(\Gamma^{(11)}_{[a_1 a_2} \Gamma^{(11)}_{11]} \right)^\alpha{}_\beta \\ &= \frac{1}{2\sqrt{-\gamma}}\epsilon^{ij} E_i{}^{a_1} E_j{}^{a_2} \left(\Gamma^{(11)}_{a_1 a_2} \Gamma^{(11)}_{11} \right)^\alpha{}_\beta \end{aligned} \tag{5.33}$$

which is equivalent to the one obtained from the general expression (4.28) since $\Gamma^{(11)}_a = \Gamma^{(10)}_a$ for $a = 1, \ldots, 10$. The compact coordinate y also transformed under the κ symmetry,

$$\delta_y = -\kappa^\beta \left(1 + \Gamma^{(10)}\right)^\alpha{}_\beta A_\alpha \tag{5.34}$$

Hence, to maintain the partial gauge choice $y = \xi^3$, a worldvolume diffeomorphism of

$$u^{(3)} = (0, 0, \kappa^\beta (1 + \Gamma^{(10)})^\alpha{}_\beta A_\alpha) \tag{5.35}$$

must be made.

5.4 Weyl Invariance from the Three-Dimensional Worldvolume Diffeomorphism

Now we consider the symmetry of the theory after dimensional reduction. The superstring theory should have Weyl symmetry on the two-dimensional world sheet. However, with the ansatz given in (5.19), the worldsheet only admits a two-dimensional diffeomorphism with no further residual local symmetry. The diffeomorphism parameter is given by,

$$u^{(3)\hat{i}}(\xi^1, \xi^2, \rho) = \left(u^{(2)i}(\xi^1, \xi^2), 0\right) \tag{5.36}$$

It was shown that the Weyl symmetry can arise from the residual three-dimensional diffeomorphism invariance by imposing the appropriate form of Kaluza-Klein ansatz [44].

The reason we do not retain the Weyl invariance is that we put the theory in the partial gauge $\rho = y$. Instead, we can release the gauge choice by setting $\partial_\rho y = 1$, or equivalently,

$$y = \rho + f(\xi^0, \xi^1) \tag{5.37}$$

where f is an arbitrary function. Now the diffeomorphism parameter $u^{\hat{i}}$ is given by,

$$u^{(3)\hat{i}}(\xi^1, \xi^2, \rho) = \left(u^{(2)i}(\xi^1, \xi^2), u^\rho(\xi^1, \xi^2)\right) \tag{5.38}$$

and the residual symmetry group is the two-dimensional diffeomorphism

$$\begin{aligned} \delta_{u^{(2)}} x^m &= u^i \partial_i x^m \\ \delta_{u^{(2)}} \gamma_{ij} &= u^k \partial_k \gamma ij + (\partial_i u^k)\gamma_{kj} + (\partial_j u^k)\gamma_{ki} \end{aligned} \tag{5.39}$$

with additional residual symmetries given by,

$$\begin{aligned} \delta_{u^\rho} x^m &= u^\rho \partial_\rho x^m = 0 \\ \delta_{u^\rho} y &= u^\rho \partial_\rho y = u^\rho \\ \delta_{u^\rho} \gamma_{\hat{i}\hat{j}} &= (\partial_{\hat{i}} u^\rho)\gamma_{\rho\hat{j}} + (\partial_{\hat{j}} u^\rho)\gamma_{\rho\hat{i}} \end{aligned} \tag{5.40}$$

Those residual symmetries yield the transformation law of f,

$$\delta f(\xi^1, \xi^2) = u^\rho(\xi^1, \xi^2) \tag{5.41}$$

Then, we modify the decomposition of the worldvolume metric to be depended on $f(\xi^0, \xi^1)$,

$$\gamma^{(3)}_{\hat{i}\hat{j}} = e^{-\frac{2}{3}\phi'} \begin{pmatrix} e^{-f(\xi^0,\xi^1)}\gamma_{ij} + e^{2\phi'} G_i G_j & G_i e^{2\phi'} \\ G_j e^{2\phi'} & e^{2\phi'} \end{pmatrix} \tag{5.42}$$

while the decomposition of the target space metric $g_{\hat{m}\hat{n}}$ is kept the same as in (5.20). We can determine that relation of the determinant of the induced metric is given by,

$$\sqrt{-\det \gamma^{(2)}_{ij}} = e^f \sqrt{-\det \gamma^{(3)}_{\hat{i}\hat{j}}} \tag{5.43}$$

considering that γ_{ij} is an 2×2 matrix, from which we can further determined that,

$$\delta_{u^\rho} \gamma_{ij} = u^\rho \gamma_{ij}. \tag{5.44}$$

Thus, u^ρ plays the role of the Weyl symmetry factor e^ϕ under the Weyl transform $\gamma'_{ij} = e^\phi \gamma_{ij}$. Substituting the embedding equation we can also determine γ_{ij}, G_i, and ϕ',

$$\begin{aligned} \phi' &= \phi, \\ G_i &= \partial_i x^m A_m + \partial_i f, \\ \gamma_{ij} &= e^f \left(\partial_i x^m \partial_j x^n g_{mn}\right). \end{aligned} \tag{5.45}$$

5.5 Dilaton Dependence of Type II Superstring Theory

The ten-dimensional type II supergravity theory presents an NS-NS sector multiplied by the tree-level factor $e^{-2\phi}$ while the R-R sector is not. Studying the dilaton coupling to the R-R sector from the view of the worldsheet would provide a clear understanding of the dilaton dependence of higher order terms as we go beyond the effective theory [46].

The conjecture was made that the R-R fields have an extra e^{ϕ} factor that cancelled the tree-level dilaton coupling. Therefore, the dilaton dependence vanishes in the kinetic term in the superstring action while the presence in the Weyl anomaly beta function, which has the standard form of $R + D^2\phi + H^2 + e^{2\phi}(F_{[2]}^2 + F_{[4]}^2) = 0$.

Studying the eleven-dimensional origin of the dilaton will help us understand the nature of dilaton dependence. As mentioned in the previous section, we can obtain the $D = 10$ supergravity from the supermembrane coupled to the $D = 11$ supergravity via double-dimensional reduction. The origin of the dilaton is the (11,11) component of the $D = 11$ metric; the R-R one-form gauge field is the (11,m) vector in the $D = 11$ metric, while the RR three-form gauge field is from the (m,n,p) component of the $D = 11$ three-form gauge field. We consider only RR fields coupling terms obtained from the Wess-Zumino term, expanding in the field contents to the leading order gives the following expression,

$$\begin{aligned}\frac{1}{6}\epsilon^{\hat{i}_1\hat{i}_2\hat{i}_3}E_{\hat{i}_1}{}^{\hat{A}_1}E_{\hat{i}_2}{}^{\hat{A}_2}E_{\hat{i}_3}{}^{\hat{A}_3}A_{\hat{A}_3\hat{A}_2\hat{A}_1} \\ = -\frac{1}{8}\epsilon^{\hat{i}_1\hat{i}_2\hat{i}_3}\partial_{\hat{i}}Z^{\hat{M}}\hat{E}_{\hat{M}}{}^{\hat{a}}\partial_{\hat{j}}Z^{\hat{N}}\hat{E}_{\hat{N}}{}^{\hat{b}}\partial_{\hat{k}}Z^{\hat{P}}\hat{E}_{\hat{P}}{}^{\hat{\alpha}}\left(\bar{\theta}\gamma_{\hat{a}\hat{b}}\right)_{\hat{\alpha}} + \ldots.\end{aligned} \tag{5.46}$$

From the double dimensional reduction we obtained the ten-dimensional Wess-Zumino term, the leading order is an NS-NS gauge coupling term while the $O(\theta^2)$ order term is an R-R coupling term, which is given by,

$$\frac{1}{2}e^{\phi}\epsilon^{ij}\partial_i x^m\partial_j x^n\bar{\theta}\left[\frac{1}{8}F_{mp}\Gamma_n\Gamma^p + \frac{1}{288}F_{klpq}\Gamma_{11}\Gamma_n\left(\Gamma^{klpq}{}_m - 8\Gamma^{klp}\delta_q^s\right)\right]\theta. \tag{5.47}$$

Similarly, the Nambu-Goto term in the supermembrane action after double dimensional reduction and expanding in the higher order of θ of superfields give rise to a parity-even R-R gauge fields coupling with the same e^{ϕ} pre-factor.

The existence of the e^{ϕ} factor of the R-R form gauge fields after double dimensional reduction is consistent with our conjecture of the extra e^{ϕ} factor made previously. One can also determine that for higher kth order coupling terms of R-R gauge fields in type IIA superstring effective action, the extra factor would be $e^{(k+n-2)\phi}$, where n is the genus on the string worldsheet.

Other than the pre-factor of e^{ϕ}, the type II supergravity also admit a dilaton term that takes the form,

$$\int d^2\xi\sqrt{-\gamma}\phi R(\gamma) \tag{5.48}$$

This term does not present in the classical analysis in our previous discussion in double dimensional reduction. In fact, this term can be regarded as arising from quantising the membrane [47]. In the same paper, a conclusion is also presented that this construction of dilaton term only works with critical dimension, which rules out the non-critical supermembrane.

6 Conclusions and Outlook

In this dissertation, we delved into the fundamental concepts of the M2-brane. We reviewed how the M2-brane couples to 11d supergravity and how it reduces to superstrings in ten dimensions. We began by introducing supergravity in eleven dimensions and type II supergravities in ten dimensions. The discussion then proceeded to the Killing spinor condition for the pure bosonic solution that admits global supersymmetry.

Subsequently, we derived the classical solution of supergravities. In particular, we solved the M2-brane solution of the eleven-dimensional supergravity by employing the brane ansatz, which features an isotropic transverse space and a worldvolume recognising a three-dimensional diffeomorphism. For detailed calculation, readers are referred to Appendix B. This section also highlighted that the M2-brane solution possesses a horizon at $r = 0$. The geometry near the horizon is $\mathrm{AdS}_4 \times \mathcal{S}^7$, and we demonstrated that we can extend the metric to the entire spacetime by introducing the M2-brane source term. We also mentioned the possibility of stacking multiple M2-branes.

The next segments of this dissertation presented the action for the M2-brane in eleven dimensions, followed by the derivation of the classical solution to the M2-brane action. Firstly, the bosonic p-brane action in flat and curved backgrounds was given. The generalisation of the bosonic brane action into the supermembrane action was illustrated. Then, the κ symmetry was introduced for supermembrane action to match the bosonic and fermionic degrees of freedom. We noted that the requirement of the supermembrane action admitting κ symmetry is equivalent to the equations of motion of the eleven-dimensional supergravity, demonstrating the close relation between them. We described Brink-Schwarz superparticle and Green-Schwarz superstring action to discuss the κ symmetry further. Classical solutions of the M2-brane action in flat space were also discussed. Finally, we ended this section by discussing the soliton interpretation of supermembrane.

In the last part, we discussed the dimensional reduction of supergravity and the M2-brane. First, we introduced the Kaluza-Klein reduction of the 11D supergravity to 10D type II A supergravity by compacting one spatial degree of freedom on a circle. The decomposition of field components into lower-dimensional fields was illustrated. Then, using a partial gauge choice, we compacted both the target space and worldvolume of the M2-brane action, obtaining the Green-Schwarz superstring coupled to the supergravity background. We verified that the

κ symmetry is preserved in the dimensional-reduced superstring theory. Moving to the Weyl symmetry exclusive to string theory, we established that with an appropriate gauge choice, the Weyl symmetry can be revived from the residual symmetry of the worldvolume diffeomorphism for the M2-brane action. The final point of discussion centred on the dilaton dependence in type II superstring theory and the dilation term emerging from the quantisation of the membrane action.

The exploration of the M2-brane and M-theory remains an active and evolving area. Over the past several decades, various properties of the M2-brane have been discovered, providing us strong confidence that it is the fundamental object in the proposed M-theory. Moreover, the role of the M2-brane is important in formulating a consistent quantum gravity theory in eleven dimensions. Notably, the M2-brane in M-theory is not a straightforward analogue to the fundamental strings in string theory. This complexity arises because the M2-brane is non-perturbative, making it difficult to quantise. Nonetheless, a way to approach a quantum theory is through semi-classical quantisation, which is to study the fluctuation around the classical solutions [6, 48, 49].

Moreover, the advancements in the AdS/CFT correspondence give a new way to study the M-theory. Specifically, the ABJM theory, which is a gauge theory, is postulated to be the dual for M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$ [50, 51]. Delving into these topics will offers insights into properties of M2-branes, extending beyond the scope of this review.

A Notation Convention

In this essay we use the following conventions:

$$g_{mn} = (-, +, \ldots, +), \tag{A.1}$$

$$\epsilon_{012} = \epsilon^{012} = -1, \tag{A.2}$$

$$\{\Gamma^m, \Gamma^n\} = -2\eta^{mn} \tag{A.3}$$

We label the indices with the following rules:

(1) *In the bosonic case:* We split the coordinate in to (p+1,D-p-1) form: $x^m = (x^i, y^s)$. The transverse coordinates labelled by indices s when discussing the membrane ansatz.

(2) *In the supersymmetric case:* The M for the superspace coordinate Z^M, and can be divided into Grassmann even coordinate x^m and Grassmann odd coordinate θ^μ, $M = (m, \mu)$. The supervielbein $E_M{}^A$ with tangent space supervielbein indices $A = (a, \alpha)$.

(3) *In world-volume formalism:* The worldvolume coordinates ξ^i with worldvolume indices i. And target space coordinates are labelled by the corresponding superspace indices.

(4) *In dimensional reduction:* The hatted indices are the indices of higher dimensions, while the lower dimension coordinates are labelled by un-hatted indices.

B Solving for the M2-brane Ansatz

With the ansatz for all the field contents in hand, we can insert them into the field equations to solve for the p-brane (a p-dimensional extended object) solution. Using the vielbein, we can obtain the Ricci tensor, which is needed for later analysis. The differential one-form is given by $e^{\underline{m}} = e_m{}^{\underline{m}} dx^m$. Similar to the space-time indices, the tangent space indices also split as $\underline{m} = (\underline{i}, \underline{s})$. Substituting the ansatz, we obtain the vielbein basis:

$$e^{\underline{i}} = e^{A(r)} dx^i \delta_i^{\underline{i}}, \quad e^{\underline{s}} = e^{B(r)} dy^s \delta_s^{\underline{s}}. \tag{B.1}$$

The spin connection 1-form $\omega^{\underline{ij}} = \omega_i{}^{\underline{ij}} dx^i$ can be obtained from the torsion-free Cartan structure equation,

$$de^{\underline{i}} + \omega^{\underline{i}}{}_{\underline{j}} \wedge e^{\underline{j}} = 0. \tag{B.2}$$

The spin connection is determined as following,

$$\begin{aligned}
&\omega^{\underline{ij}} = 0, \\
&\omega^{\underline{it}} = e^{-B(r)}\delta^{\underline{tt}}\partial_t A(r) e^{\underline{i}} \\
&\omega^{\underline{st}} = e^{-B(r)}\delta^{\underline{tt}}\partial_t B(r) e^{\underline{s}} - e^{-B(r)}\delta^{\underline{ss}}\partial_s B(r) e^{\underline{t}}.
\end{aligned} \tag{B.3}$$

The curvature 2-form $R^{\underline{m}}{}_{\underline{n}}$ can be determined via the equation:

$$R^{\underline{m}}{}_{\underline{n}} = d\omega^{\underline{m}}{}_{\underline{n}} + \omega^{\underline{m}}{}_{\underline{p}} \wedge \omega^{\underline{p}}{}_{\underline{n}} \tag{B.4}$$

From the curvature 2-form, we can obtain the Ricci tensor, with the non-zero terms given by:

$$\begin{aligned}
R_{ij} =& -\eta_{ij} e^{2(A-B)} \left(A'' + d\,(A')^2 + \tilde{d}A'B' + \frac{(\tilde{d}+1)}{r} A' \right) \\
R_{st} =& -\delta_{st} \left(B'' + dA'B' + \tilde{d}\,(B')^2 + \frac{(2\tilde{d}+1)}{r} B' + \frac{d}{r} A' \right) \\
& - \frac{y_s y_t}{r^2} \left(\tilde{d}B'' + dA'' - 2dA'B' + d\,(A')^2 - \tilde{d}\,(B')^2 - \frac{\tilde{d}}{r} B' - \frac{d}{r} A' \right),
\end{aligned} \tag{B.5}$$

where $d = p + 1$ represents the worldvolume dimension, $\tilde{d} = D - d - 2$, and prime denotes the derivative with respect to r.

Then, we can substitute the curvature components, Ricci tensor, and the gauge field ansatz into the Einstein equation:

$$A'' + d\,(A')^2 + \tilde{d}A'B' + \frac{(\tilde{d}+1)}{r} A' = \frac{\tilde{d}}{2(D-2)} S^2 \tag{B.6a}$$

$$B'' + dA'B' + \tilde{d}\,(B')^2 + \frac{(2\tilde{d}+1)}{r} B' + \frac{d}{r} A' = -\frac{d}{2(D-2)} S^2 \tag{B.6b}$$

$$\tilde{d}B'' + dA'' - 2dA'B' + d\,(A')^2 - \tilde{d}\,(B')^2 - \frac{\tilde{d}}{r} B' - \frac{d}{r} A' + \frac{1}{2}\,(\phi')^2 = \frac{1}{2} S^2 \tag{B.6c}$$

where the first equation is derived from R_{ij}, the second is from the δ_{st} sector of R_{st}, and the third one is from the $\frac{y_s y_t}{r^2}$ sector. The S term on the right-hand side (RHS) of the equations is given by,

$$S = \begin{cases} \left(e^{\frac{1}{2}a\phi - dA + C} \right) C' & \text{electric: } d = n-1, \varsigma = +1 \\ \lambda \left(e^{\frac{1}{2}a\phi - \tilde{d}B} \right) r^{-\tilde{d}-1} & \text{magnetic: } d = D-n-1, \varsigma = -1 \end{cases} \tag{B.7}$$

when considering the graviton coupling to the $n-1$-form gauge field. We can also determine the equation of motion for the gauge field:

$$\nabla^2 C + (C' - dA' + \tilde{d}B')C' = 0. \tag{B.8}$$

where ∇^2 is the Laplacian in the transverse space $\{y^m\}$, and for the dilaton:

$$\phi'' + dA'\phi' + \tilde{d}B'\phi' + \frac{(\tilde{d}+1)}{r}\phi' = -\frac{1}{2}\varsigma a S^2 \tag{B.9}$$

where $\varsigma = \pm 1$ for the electric or magnetic ansatz.

One may notice that the left-hand side (LHS) of the dilaton's equation of motion bears a resemblance to the Laplacian of the isotropic field ϕ in the transverse space:

$$\nabla^2\phi = \phi'' + \frac{\tilde{d}+1}{r}\phi'. \tag{B.10}$$

As we will see in the later section, in order to preserve supersymmetry, the solution should satisfy the following condition:

$$dA' + \tilde{d}B' = 0. \tag{B.11}$$

Substituting this constraint into the equations (B.6) and (B.9) and using the Laplacian, we obtain:

$$\nabla^2 A = \frac{\tilde{d}}{2(D-2)}S^2, \tag{B.12a}$$

$$\nabla^2\phi = -\frac{1}{2}\varsigma a S^2, \tag{B.12b}$$

$$d(D-2)\left(A'^2\right) + \frac{1}{2}\tilde{d}\left(\phi'^2\right) = \frac{1}{2}\tilde{d}S^2, \tag{B.12c}$$

where we have eliminated B from the equations [52].

We can deduce from equation (B.12a) and (B.12b) that the ansatz for gauge field and dilaton admit a linear relation:

$$\phi' = \frac{-\varsigma a(D-2)}{\tilde{d}}A'. \tag{B.13}$$

For simplicity, we introduce a new notation Δ:

$$a^2 = \Delta - \frac{2d\tilde{d}}{D-2}. \tag{B.14}$$

Then, we can write (B.12c) in terms of Δ:

$$S^2 = \frac{\Delta\phi'^2}{a^2}, \tag{B.15}$$

hence the equations of motion become one equation of ϕ only:

$$\nabla^2\phi + \frac{\varsigma\Delta}{2a}\phi'^2 = 0 \tag{B.16}$$

and can be further simplified into the form of the Laplace equation:

$$\nabla^2 e^{\frac{\varsigma\Delta}{2a}\phi} = 0. \tag{B.17}$$

This Laplace equation can be easily solved with the assumption that the transverse space is isotropic

$$\frac{1}{r^{D-d-1}}\partial_r\left(r^{D-d-1}\partial_r e^{\frac{\varsigma\Delta}{2a}\phi}\right)=0 \tag{B.18}$$

by the solution

$$e^{\frac{\varsigma\Delta}{2a}\phi}=1+\frac{\kappa}{r^{D-d-2}}=1+\frac{\kappa}{r^{\tilde{d}}}\equiv H(y),\quad \kappa>0 \tag{B.19}$$

where the boundary condition is set to 0, *i.e.* $\phi_{|r\to\infty}=0$, for asymptotically flat solution. The integration constant κ is set to be positive to ensure there is no naked singularity at finite r, like the mass M in the Schwarzschild metric. Apply the same flat boundary condition on A and B, using (B.13), we can obtain the expression for A in terms of H,

$$e^A=H^{\frac{-2\tilde{d}}{(D-2)\Delta}} \tag{B.20}$$

Recall we have the constraint $dA'+\tilde{d}B'=0$, we can also express B in terms of H,

$$e^B=H^{\frac{2d}{(D-2)\Delta}} \tag{B.21}$$

We can obtain the relation between C that determined the gauge field and A that determine the metric via equation (B.7) and (B.15),

$$\partial_r e^C=\frac{-\sqrt{\Delta}}{a}e^{-\frac{1}{2}a\phi+dA}\phi'. \tag{B.22}$$

where we have used that a is negative when taking the square root. One can also check that this solution satisfies the equation of motion of the gauge field (B.8). In electric coupling case where $\varsigma=1$, we can rewrite (B.22) by absorbing proper coefficient into the derivative,

$$\partial_r e^C=\frac{2}{\sqrt{\Delta}}\partial_r e^{dA-\frac{1}{2}a\phi} \tag{B.23}$$

Thus, collecting all the terms, the Mp-brane solution for D-dimensional general supergravity theory is given by,

$$ds^2=H^{\frac{-4\tilde{d}}{(D-2)\Delta}}dx^i dx^j\eta_{ij}+H^{\frac{4d}{(D-2)\Delta}}dy^s dy^t\delta_{st} \tag{B.24a}$$

$$e^\phi=H^{\frac{2a}{\varsigma\Delta}}\quad \varsigma=\begin{cases}+1, & \text{electric}\\ -1, & \text{magnetic}\end{cases} \tag{B.24b}$$

$$H(y)=1+\frac{k}{r^{\tilde{d}}}, \tag{B.24c}$$

with $C(r)$ satisfied the relation given in (B.23),

$$e^C=\frac{2}{\sqrt{\Delta}H} \tag{B.25}$$

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