Quantifying anisotropy in stratified and rotating turbulence using orthogonal wavelets

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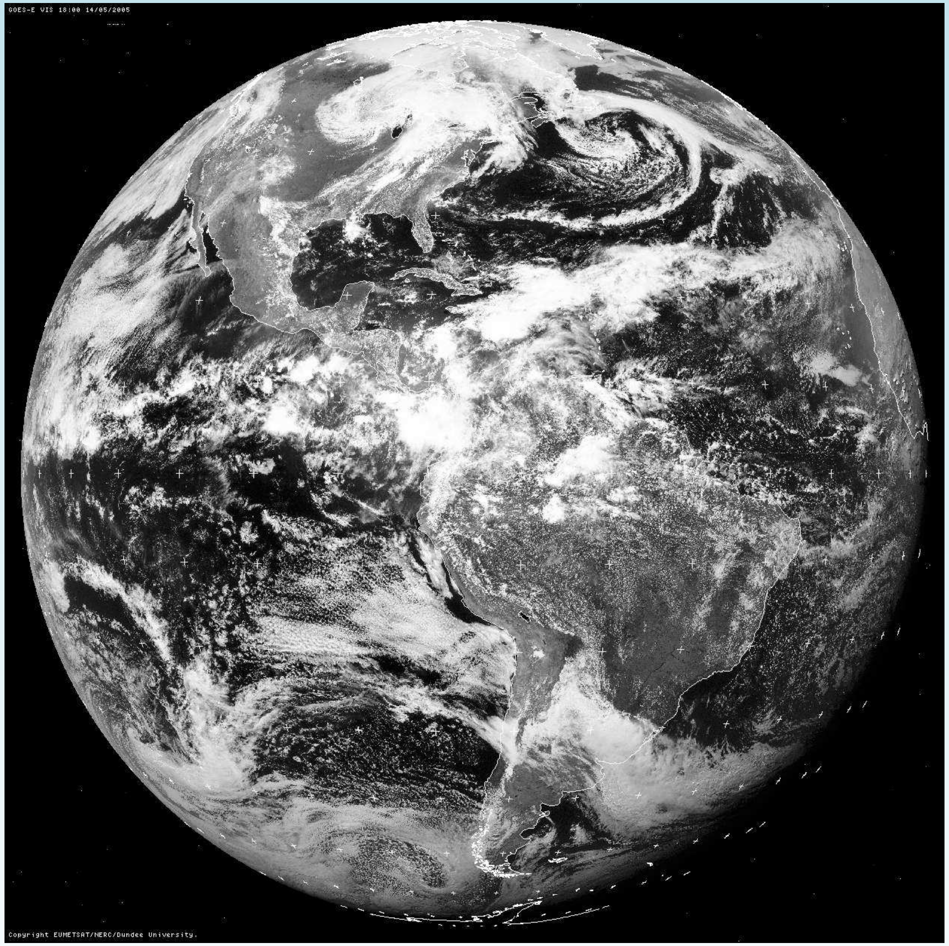
First IMS Turbulence Workshop 'Interscale energy transfers in various turbulent flows'
March 26 – 28, 2007
Institute of Mathematical Sciences, Imperial College London,United Kingdom

Outline

- Introduction
- Rotating and Stratified Turbulence
- Coherent Vortex Extraction
- Anisotropy analysed by Wavelets
- Conclusions and Perspectives

Geophysical turbulence



Geophysical turbulence

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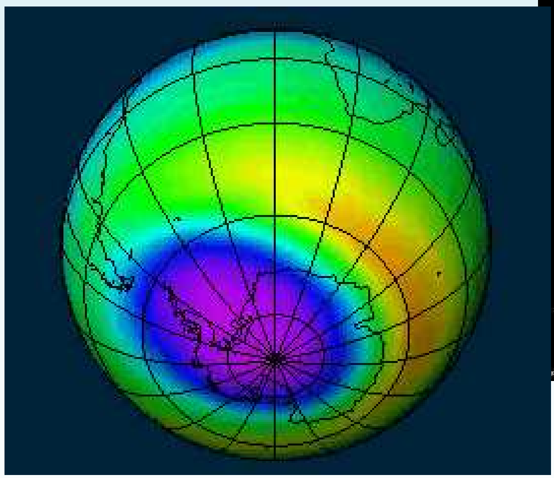
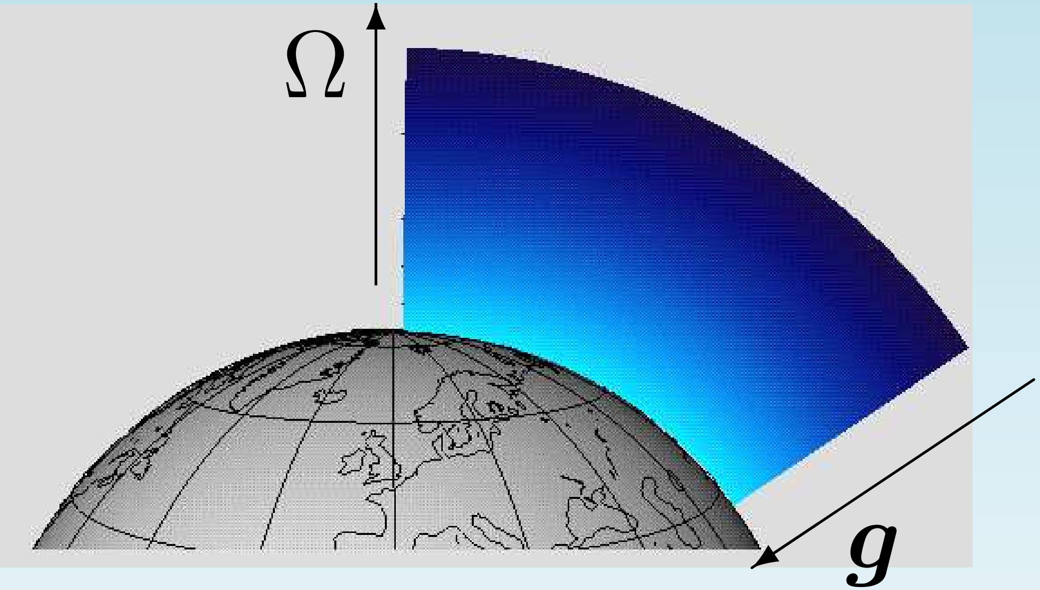


Anisotropic turbulence



Coriolis force with parameter

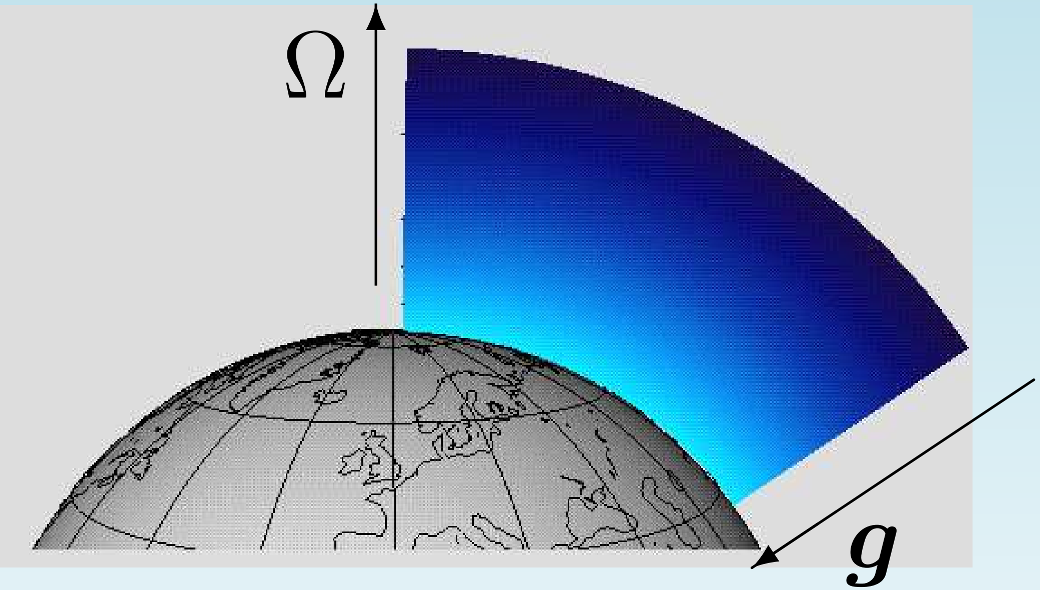
$$f = 2\Omega \sin\phi$$

Buoyancy b due to density gradient

with N,

the Brunt-Vaisala frequency

Anisotropic turbulence



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Simplifications :

Uniform vertical solid body rotation

Constant vertical density gradient

$\Rightarrow$ **Anisotropic homogeneous turbulence**

Boussinesq approximation

Incompressible Navier-Stokes with buoyancy forcing in a rotating frame of reference

$$\frac{\partial \boldsymbol{u}}{\partial t} - \frac{1}{Re}\nabla^2 \boldsymbol{u} = -\nabla\left(p^* + \frac{1}{2}\boldsymbol{u}^2\right) + \boldsymbol{u} \times \nabla \times \boldsymbol{u} - f\,\boldsymbol{n}_3 \times \boldsymbol{u} + b\,\boldsymbol{n}_3$$

$$\nabla \cdot \boldsymbol{u} = 0$$

Equation for buoyancy

$$\frac{\partial b}{\partial t} - \kappa \nabla^2 b = -(\boldsymbol{u} \cdot \nabla) b - N^2 (\boldsymbol{n}_3 \cdot \boldsymbol{u})$$

Direct numerical simulation

- Standard Pseudo-spectral collocation method with a Fourier base
- Periodic cubic domain
- Fully dealiased using a 2/3-rule
- Adams-Bashforth 3rd-order timestepping

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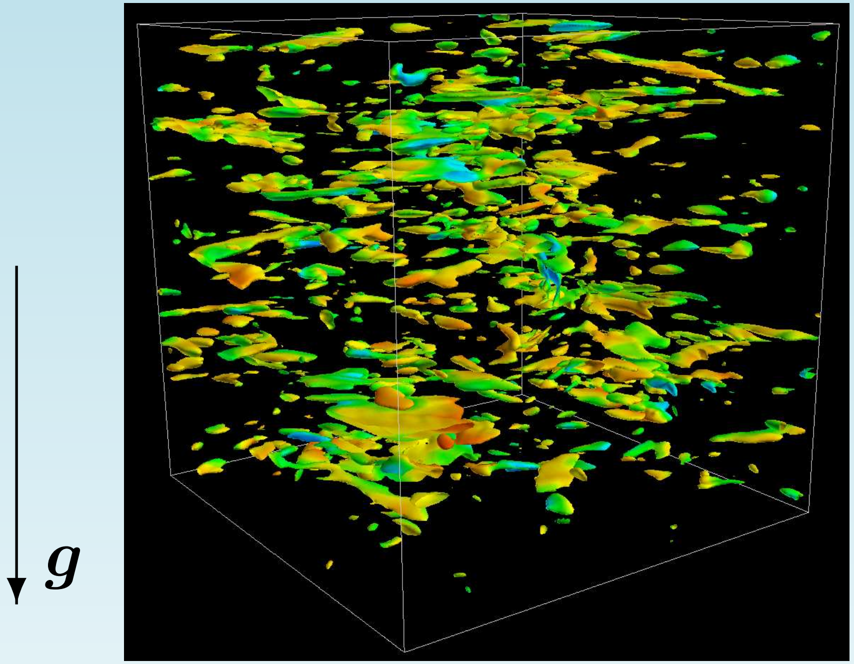
$$R_\lambda \approx 100 \qquad Ro = \frac{u}{fL} \approx 0.01 \qquad Fr = \frac{u}{NL} \approx 0.01$$

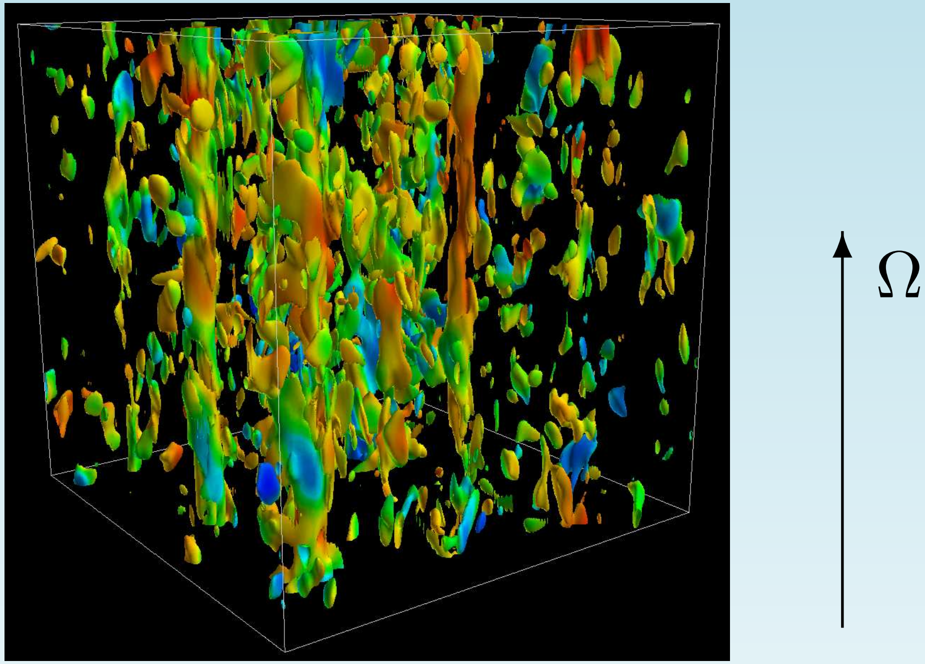
$$\alpha = \frac{f}{N}$$

Homogeneous rotating and stratified turbulence

Isovorticity surfaces at one instant in time. Colours represent vertical velocity.



STRATIFIED
PANCAKES



ROTATING
CIGARS

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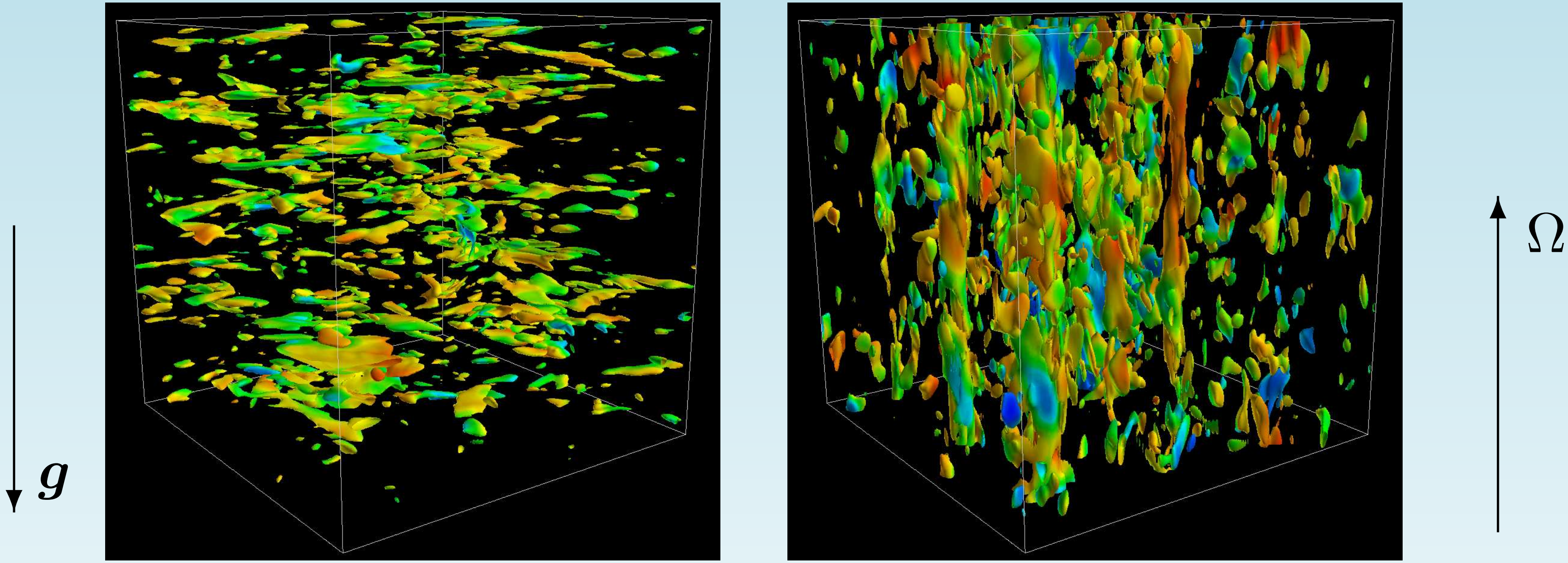

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Typical anisotropic vel. distr. and development of coherent structures with aniso. aspect ratios. Also statistics are anisotropic (see e.g. vel. corr. length scales **JOT 6**, Nr.24).

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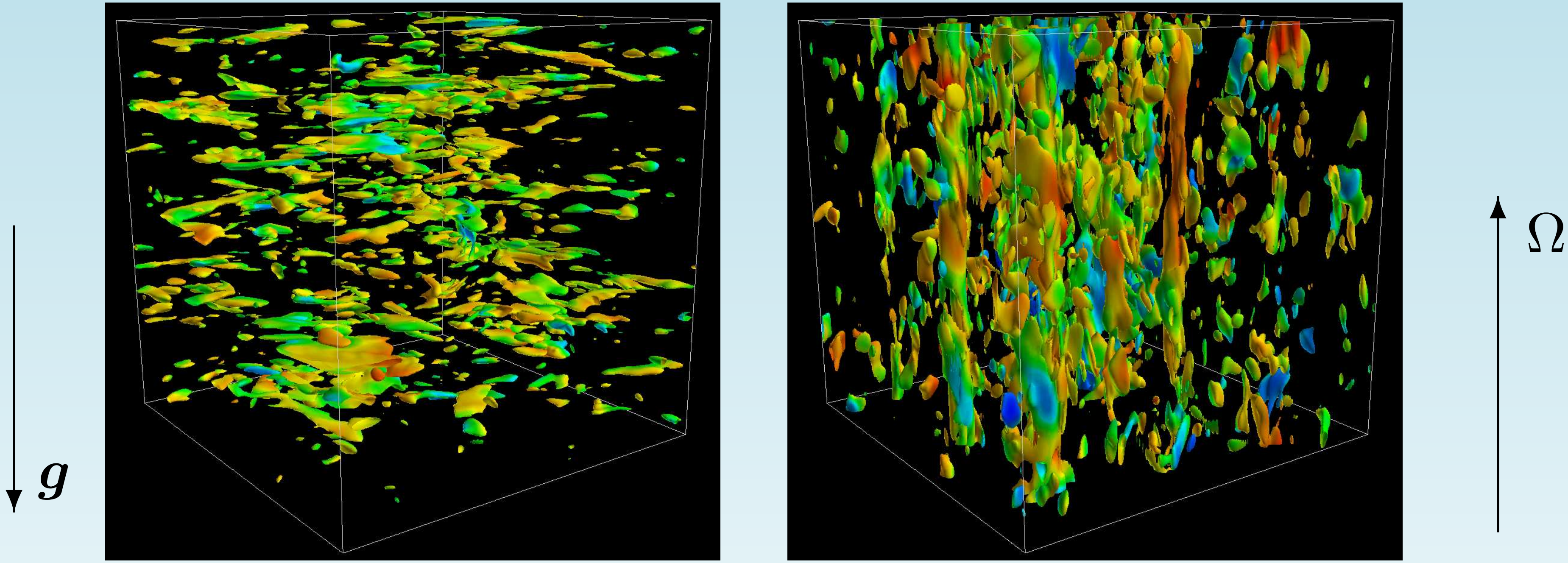

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ROTATING CIGARS

Typical anisotropic vel. distr. and development of coherent structures with aniso. aspect ratios. Also statistics are anisotropic (see e.g. vel. corr. length scales **JOT 6**, Nr.24).

Presence of "vortical" turbulence and internal "waves".

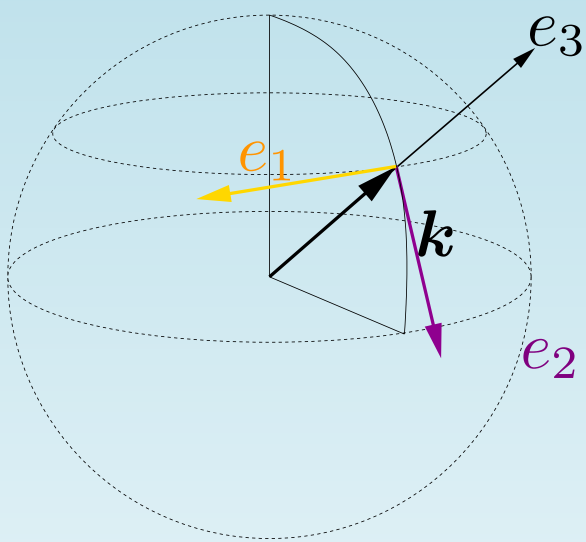
Linear Boussinesq approximation in Fourier space

Transformation of variables :

$$(\hat{u}_1, \hat{u}_2, \hat{u}_3) \longrightarrow (\hat{v}^{(1)}, \hat{v}^{(2)})$$

$$\frac{i}{N} b \longrightarrow \hat{v}^{(3)}$$

Craya-Herring frame of reference



Toroidal

Poloidal

Potential

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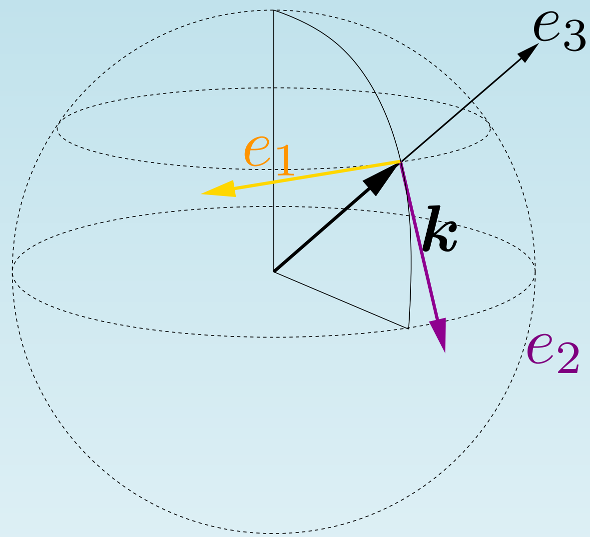

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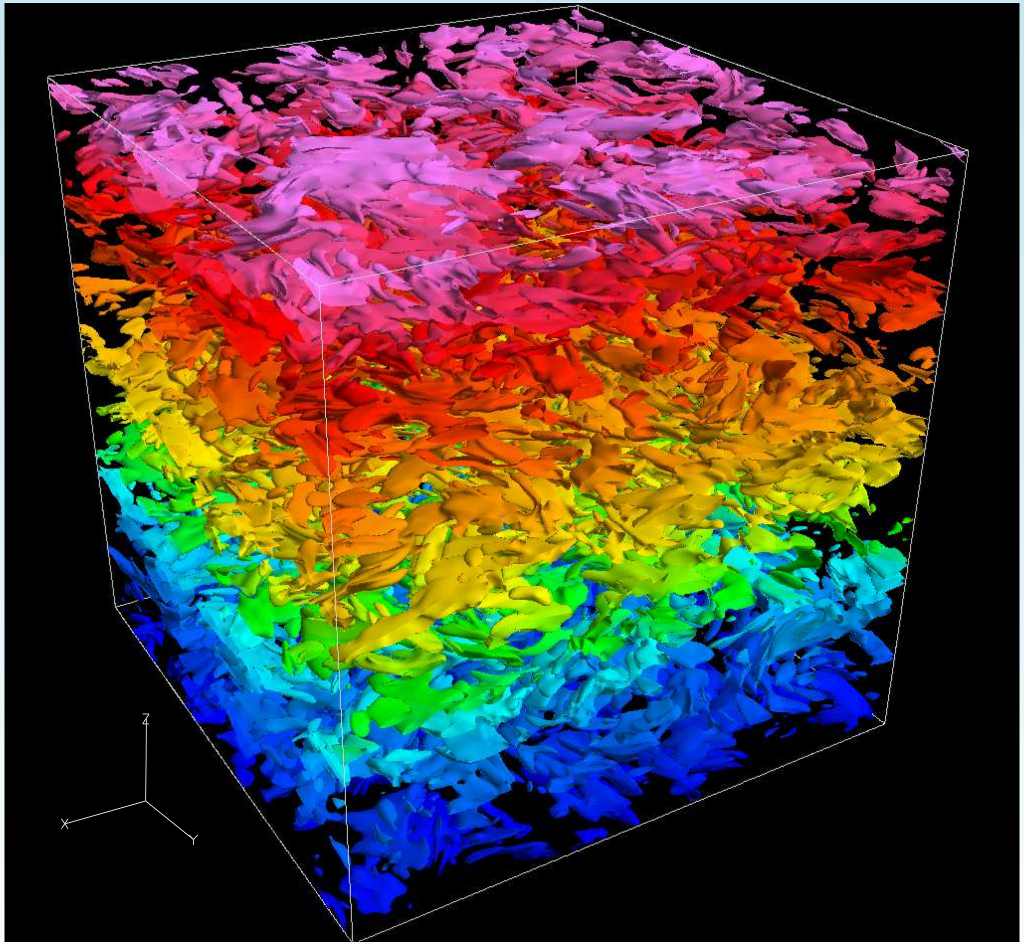
Leading to a linear system of equations, where $\sigma_r = f \cos\theta$ and $\sigma_s = N \sin\theta$:

$$\partial_t \begin{pmatrix} \hat{v}^{(1)} \\ \hat{v}^{(2)} \\ i\hat{v}^{(3)} \end{pmatrix} + \begin{pmatrix} 0 & -\sigma_r & 0 \\ \sigma_r & 0 & -\sigma_s \\ 0 & \sigma_s & 0, \end{pmatrix} \begin{pmatrix} \hat{v}^{(1)} \\ \hat{v}^{(2)} \\ i\hat{v}^{(3)} \end{pmatrix} = 0$$

Iso-vorticity surfaces of rotating and stratified turbulence

$N = 5\pi \quad f = \pi$

Isosurface at $|\omega| = 3 * \sigma$ where σ is the Standard deviation of ω.



One time instant with

Resolution :

$n^3 = 512^3$

Time of analysis :

$T_L = tu/L = 9.2$

which corresponds to

5 inertial turnover times or

25 Brunt-Vaisälä oscillation times

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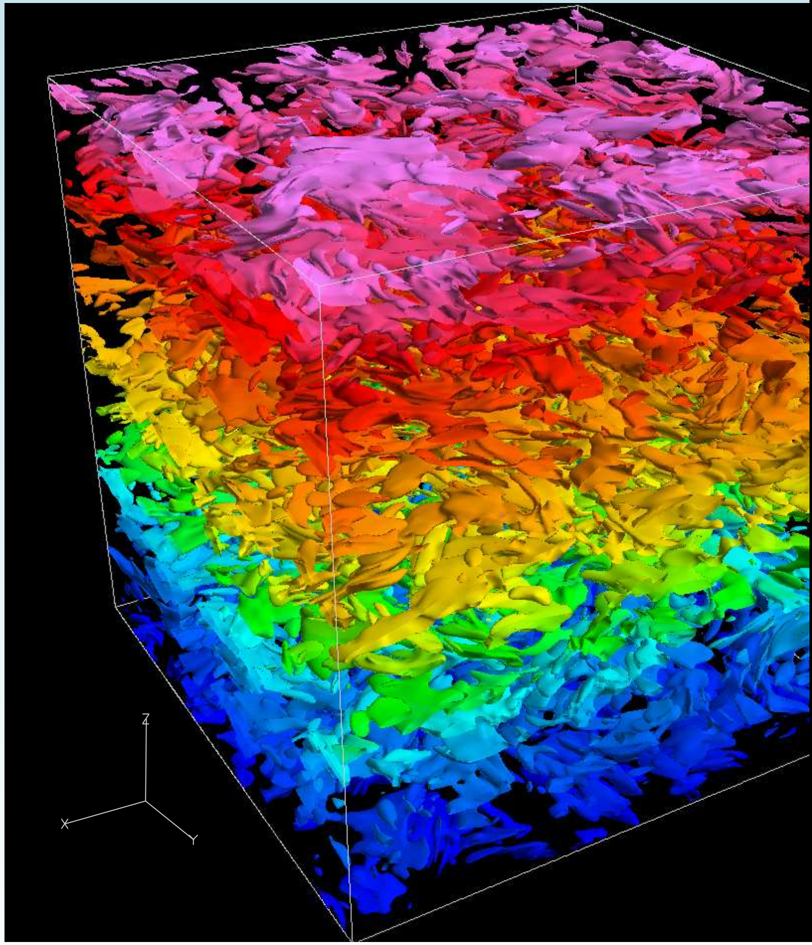
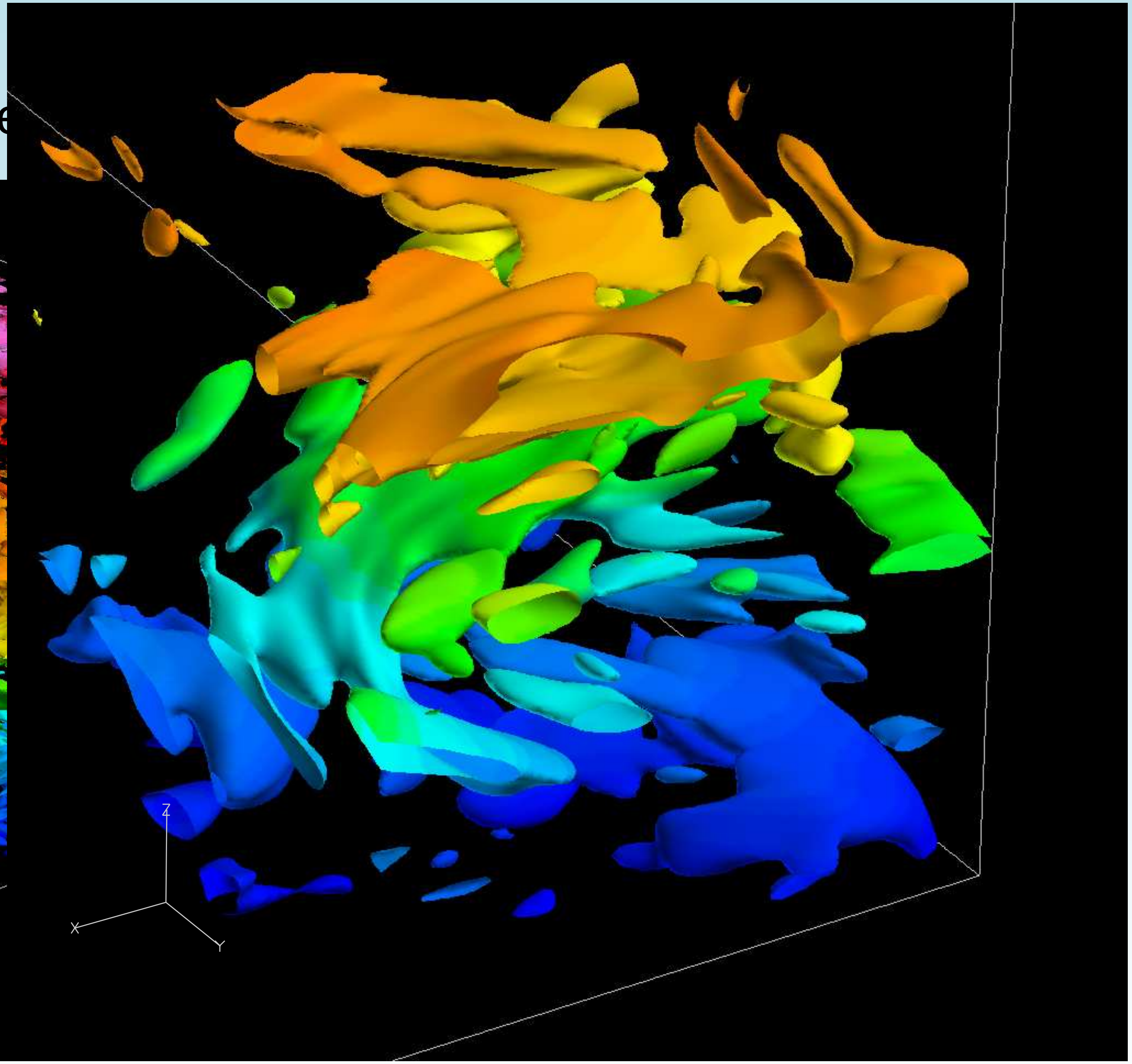
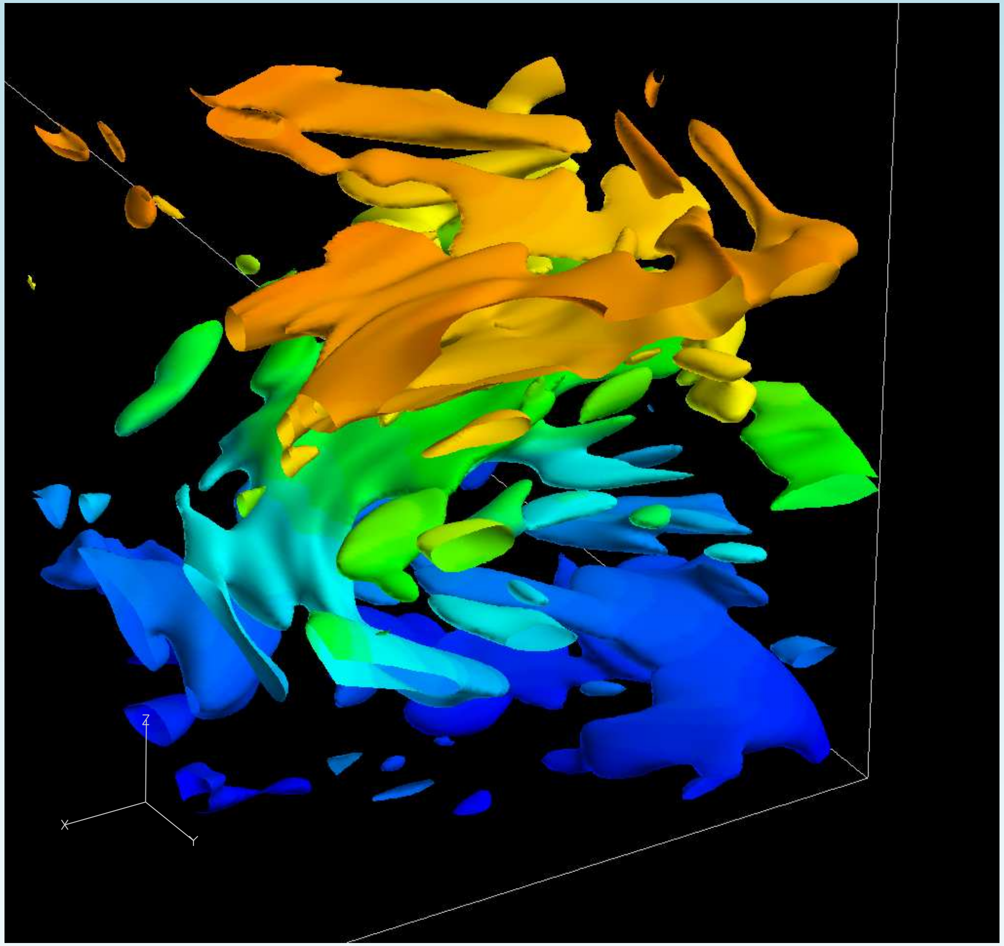




Comparison to isotropic turbulence

Kinetic Energy : $E = 0.0139$
at $T_L = 9.2$



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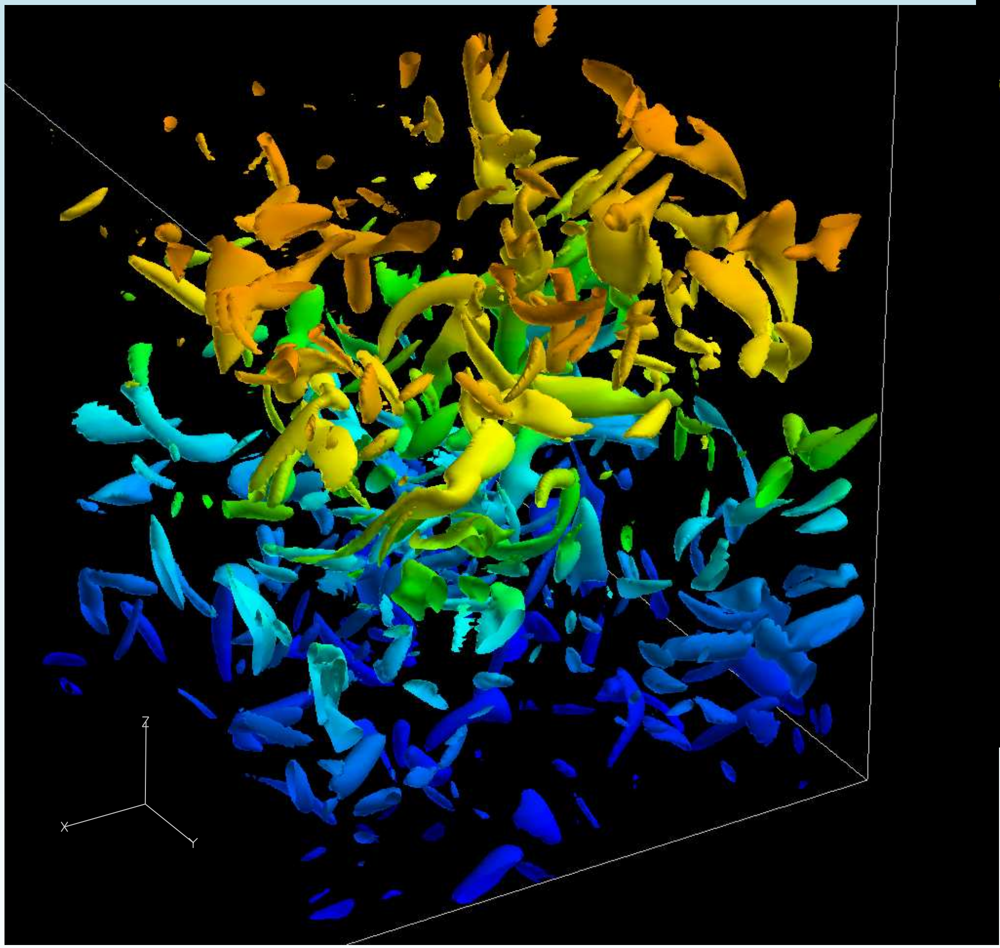
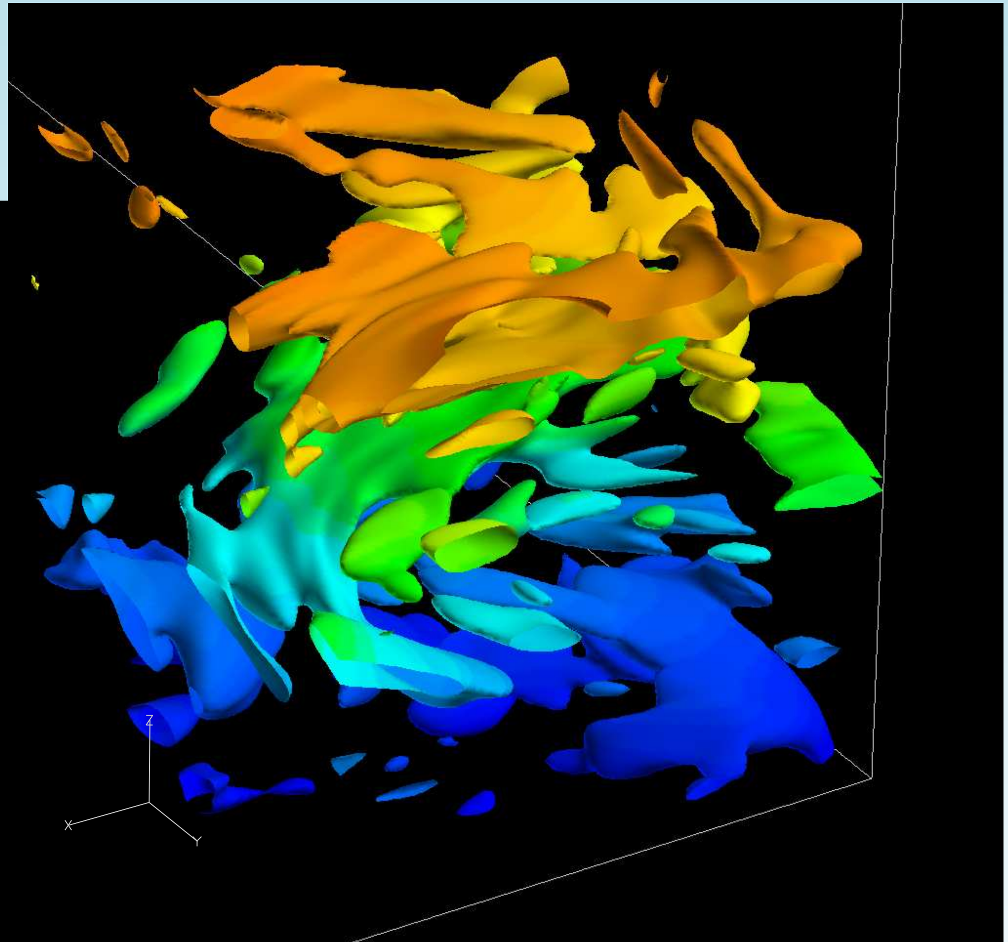




$E = 0.0142$ at $T_L = 2.8$

Coherent Vortex Extraction

Wavelets : A multi-resolution analysis

"position and scale"

Farge, Schneider, Kevlahan, 1999
Farge, Pellegrino, Schneider, 2001

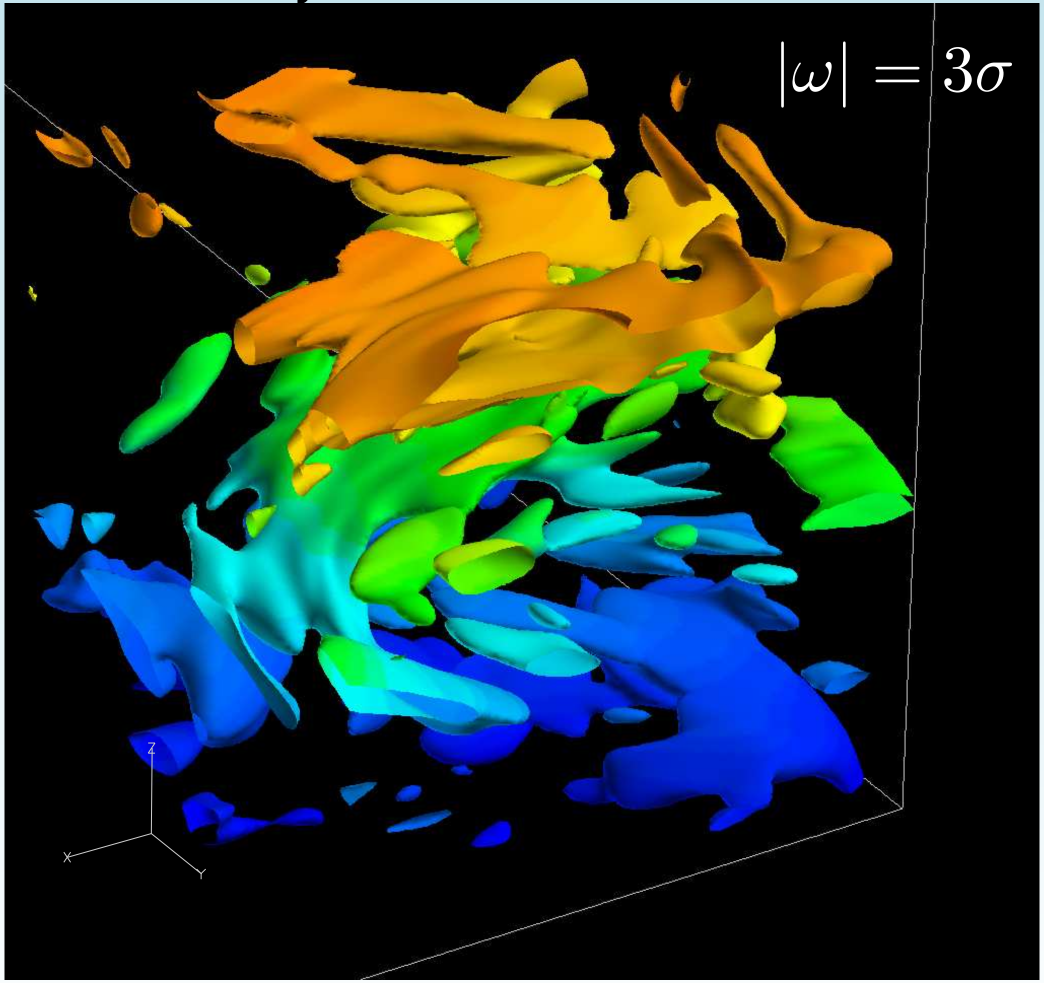
- Vorticity $\boldsymbol{\omega} = \nabla \times \boldsymbol{v}$ at resolution $n = 2^{3J}$
- Wavelet transform $\tilde{\boldsymbol{\omega}} = \langle \boldsymbol{\omega}, \psi_\lambda \rangle$
- Thresholding : $T = (4/3Z \ln n)^{1/2}$ where Z is the Enstrophy

$$\tilde{\boldsymbol{\omega}}_C = \begin{cases} \tilde{\boldsymbol{\omega}} & \text{for} \quad |\tilde{\boldsymbol{\omega}}| \geq T, \\ 0 & \text{for} \quad |\tilde{\boldsymbol{\omega}}| < T \end{cases} \qquad \tilde{\boldsymbol{\omega}}_I = \begin{cases} \tilde{\boldsymbol{\omega}} & \text{for} \quad |\tilde{\vec{\omega}}| < T, \\ 0 & \text{for} \quad |\tilde{\vec{\omega}}| \geq T \end{cases} \tag{1}$$

- Inverse wavelet transform to reconstruct $\boldsymbol{\omega}_C + \boldsymbol{\omega}_I = \boldsymbol{\omega}$
- Apply Biot-Savart operator to reconstruct $\boldsymbol{v}_C + \boldsymbol{v}_I = \boldsymbol{v}$ with $\boldsymbol{v} = \nabla \times \nabla^{-2} \boldsymbol{\omega}$
- Remark : $Z = Z_C + Z_I$ (orth. dec.) and $E \approx E_C + E_I$

Iso-vorticity surfaces of coherent and incoherent parts

Total vorticity : $100\% n^3$ contain $100\% Z$



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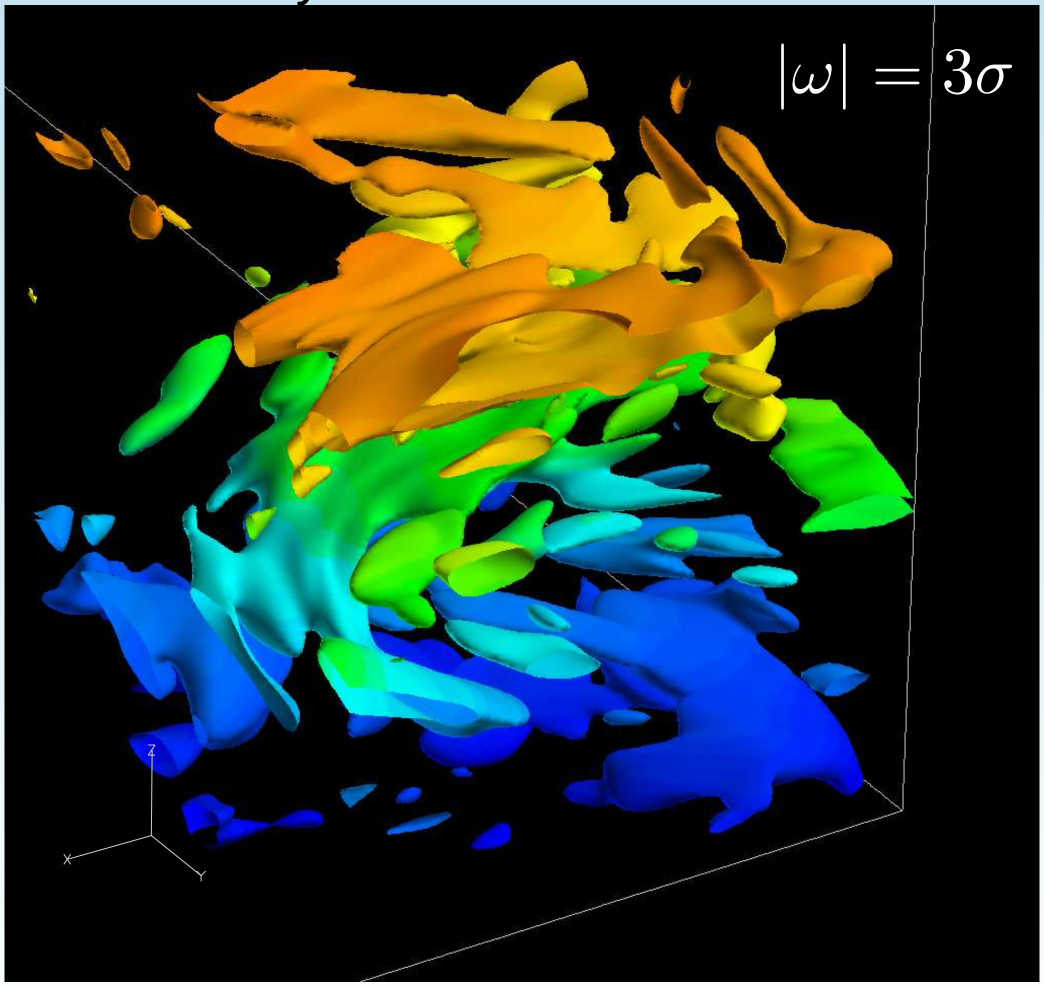
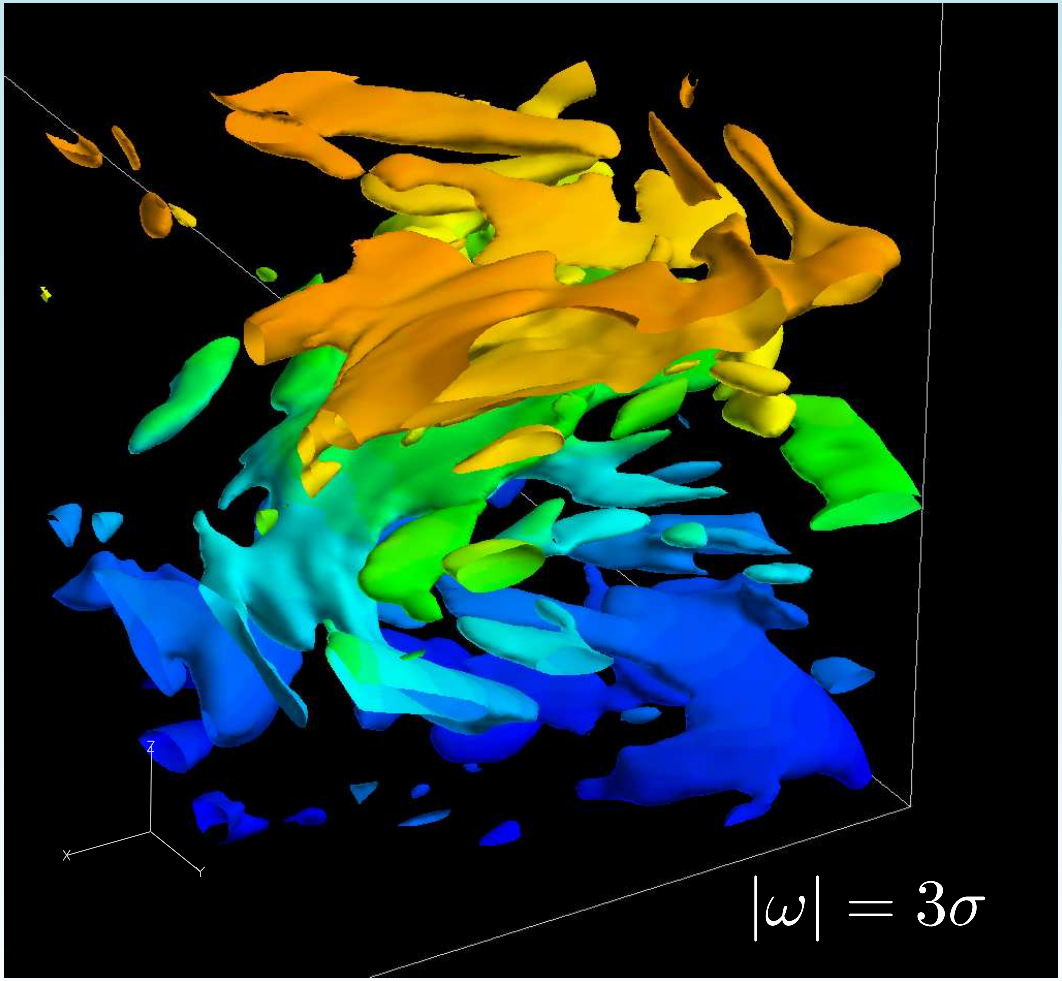
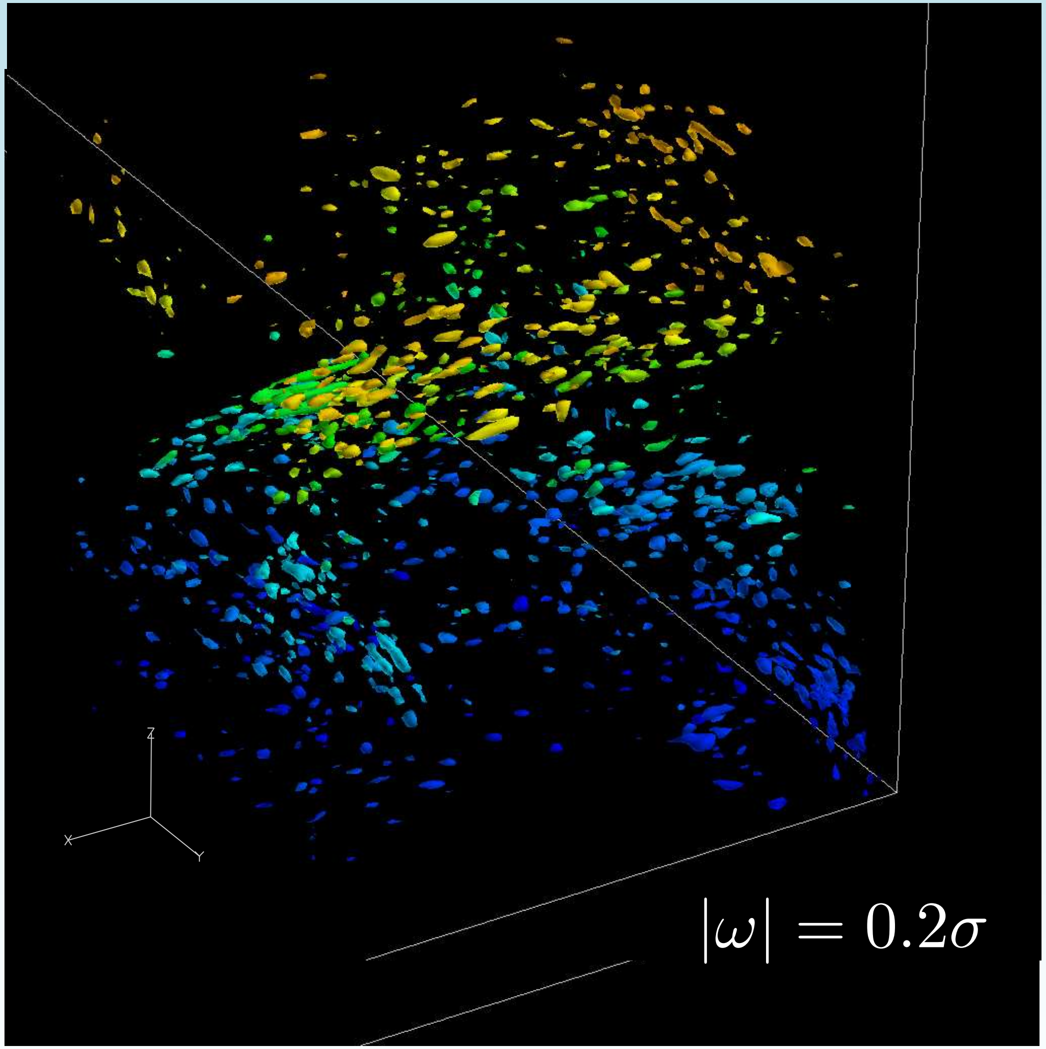


Coherent vort. : $1\% n^3$ contain $99.9\% Z$



Iso-vorticity surfaces of coherent and incoherent parts

Incoherent vort : $99\% n^3$ contain $0.1\% Z$



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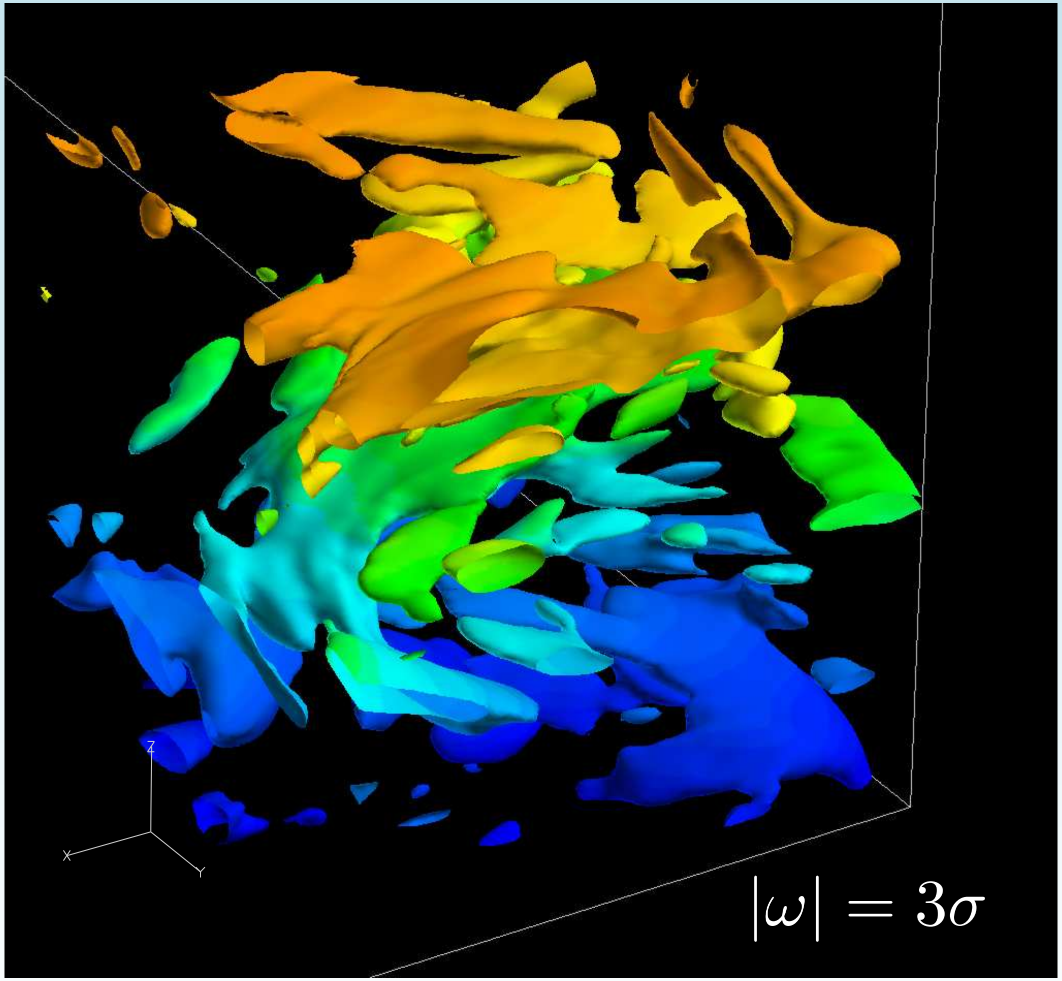

Half-time Conclusions

Anisotropic turbulence :

- Flow field develops anisotropic structures and anisotropic statistics.
- Rotating and stratified turbulence is a “two-mode” flow, composed of a vortical motion and internal waves.
- Modelisation is difficult, due to combined linear/nonlinear mechanisms and anisotropy in the flow (large linear oscillations with a slow nonlinear evolution)

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Coherent vortex extraction of anisotropic turbulence :

– Coherent part contains all of the enstrophy with only 1% of wavelet coefficients.

– Compression rate is much better for anisotropic turbulence than for isotropic turbulence ($\approx 5\%$ contain $\approx 90\%$ of enstrophy), despite a higher R_λ in anisotropic turbulence.

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Which way do we continue ? **Directional wavelet analysis.**

Turbulence and spherically averaged spectra

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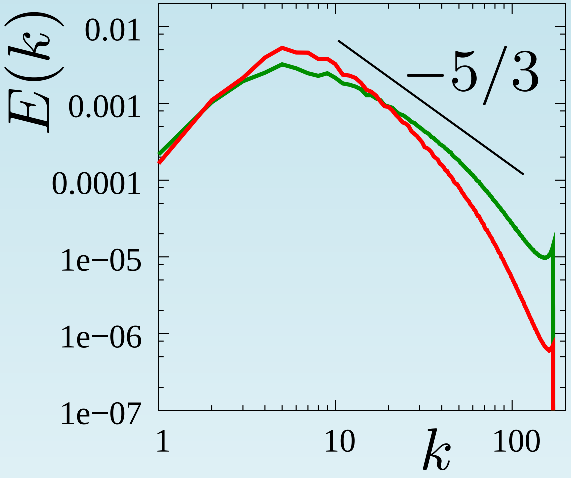
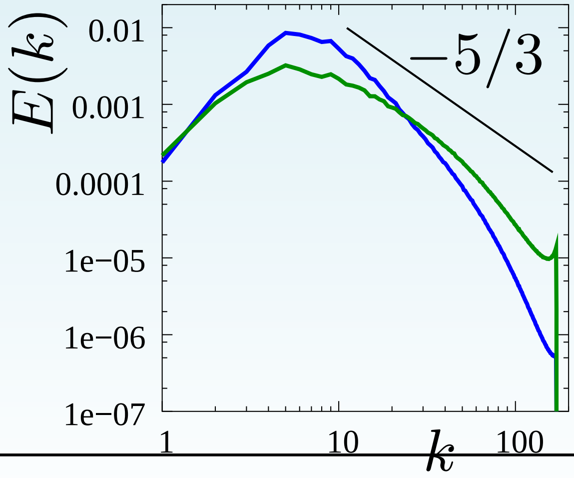
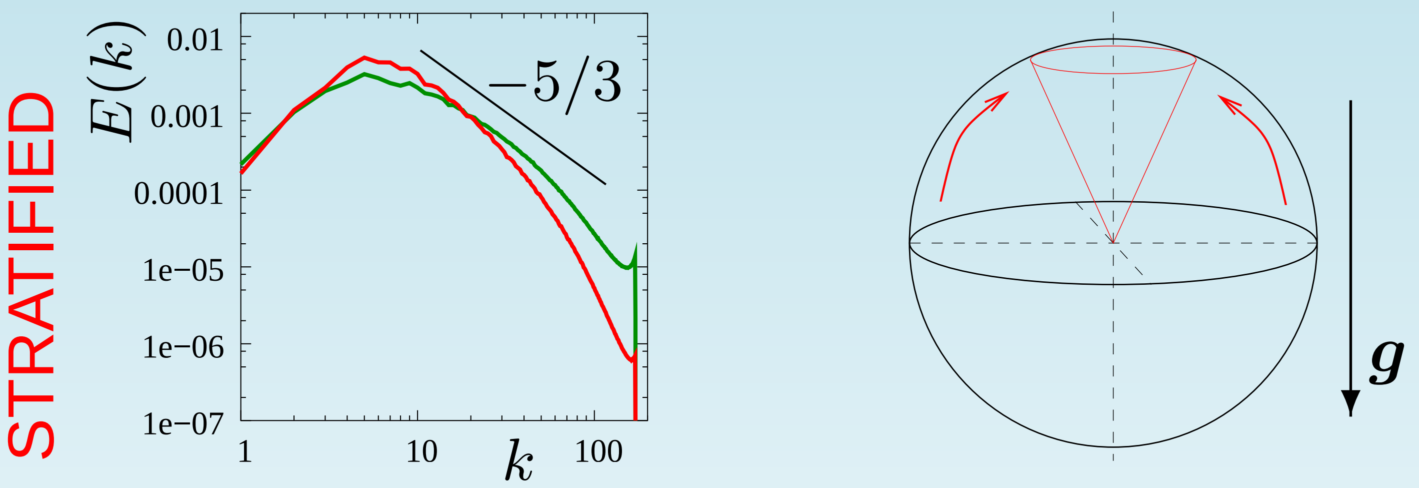
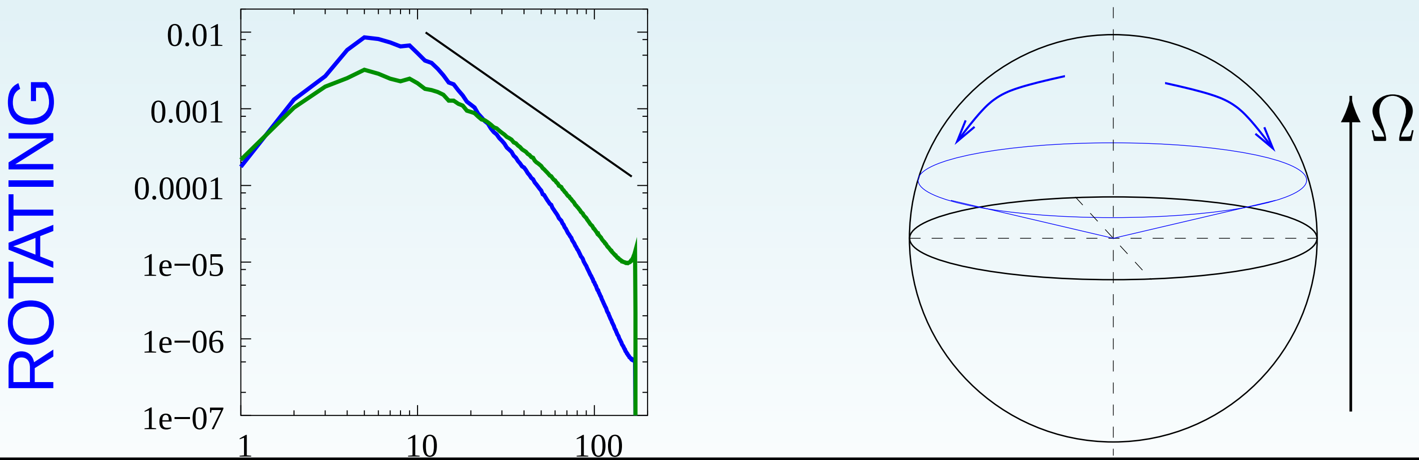


ROTATING



Turbulence and spherically averaged spectra





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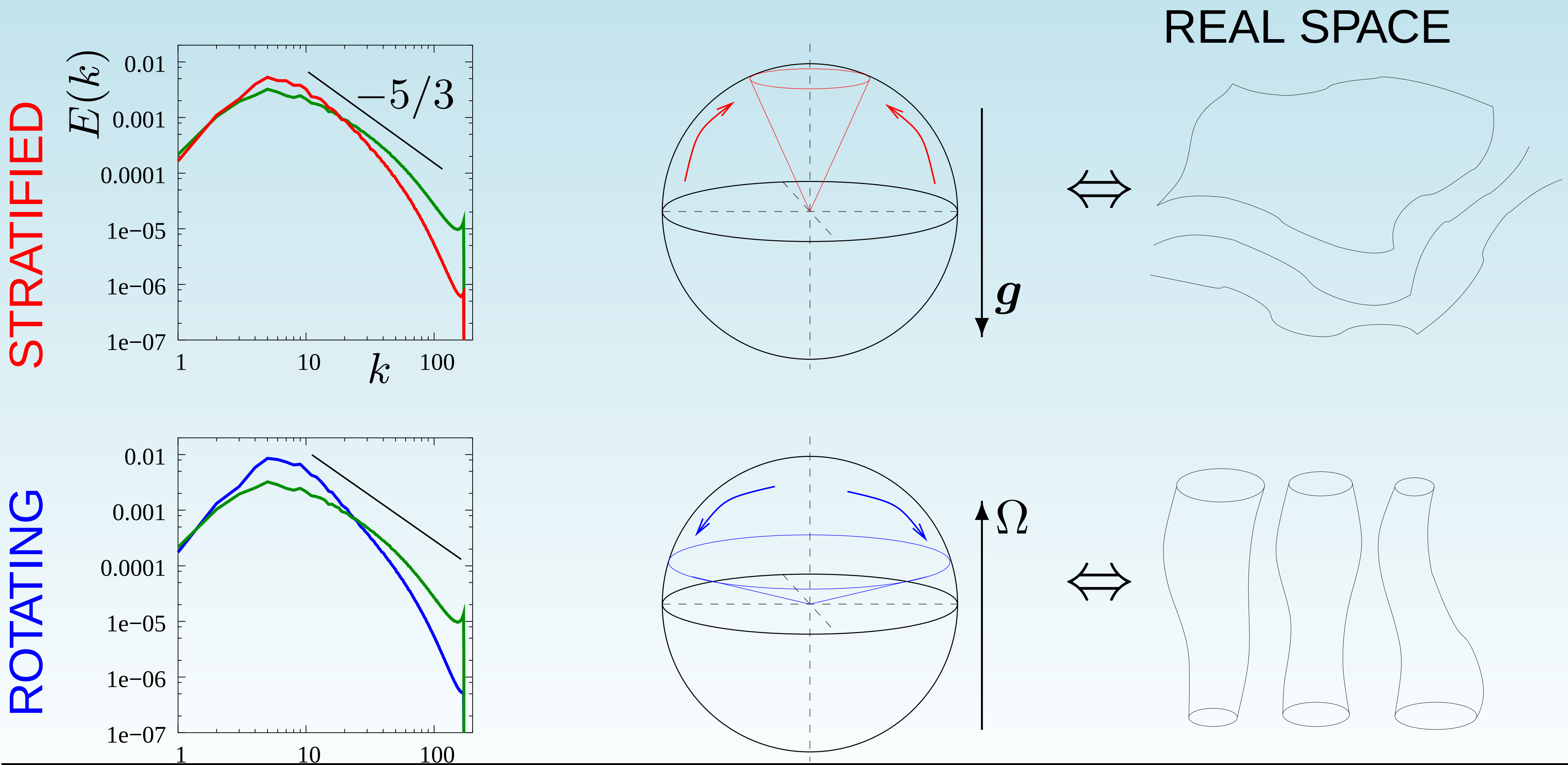
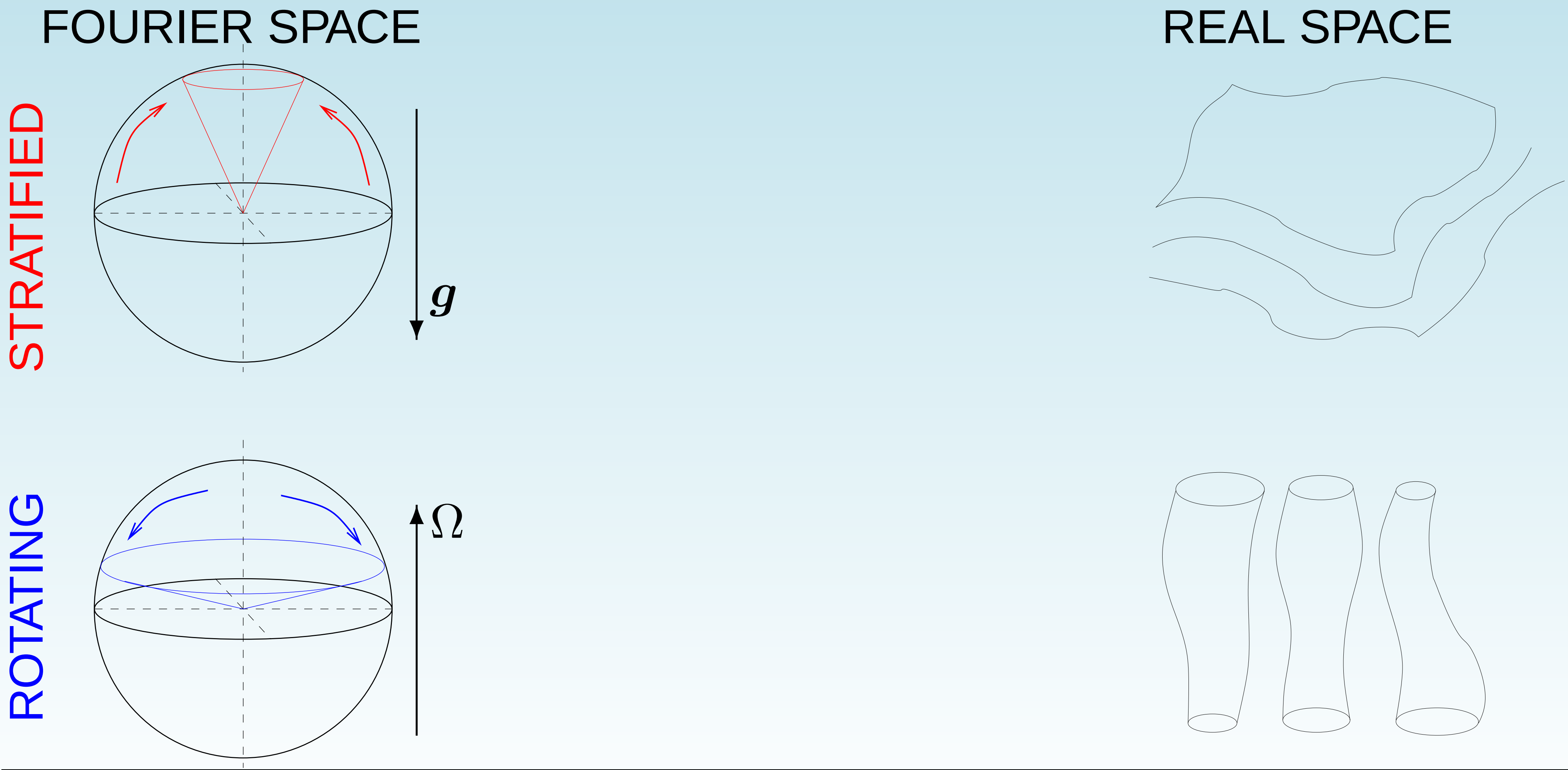


What about wavelet space ?



What about wavelet space ?

FOURIER SPACE | WAVELET SPACE | REAL SPACE

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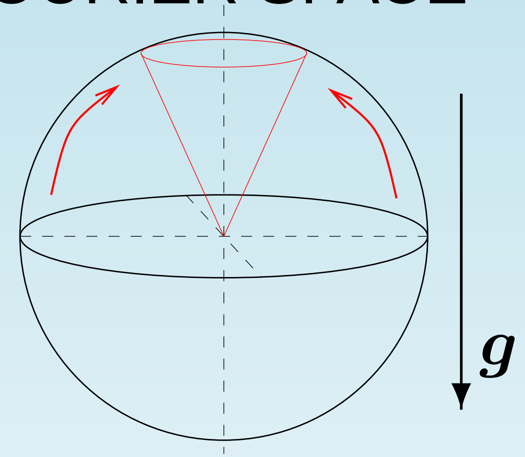
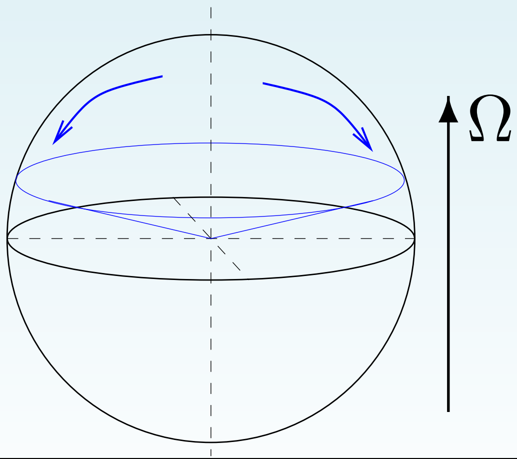


Good localisation in space

Information about scale

Directional information for anisotropy

ROTATING

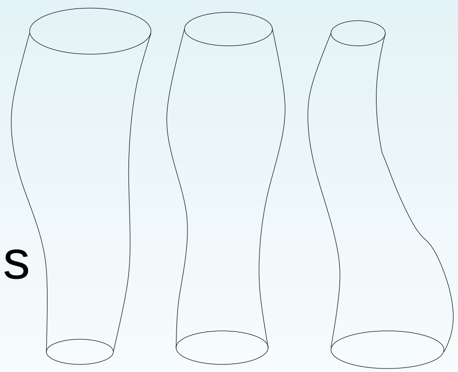


Orthogonal wavelets :

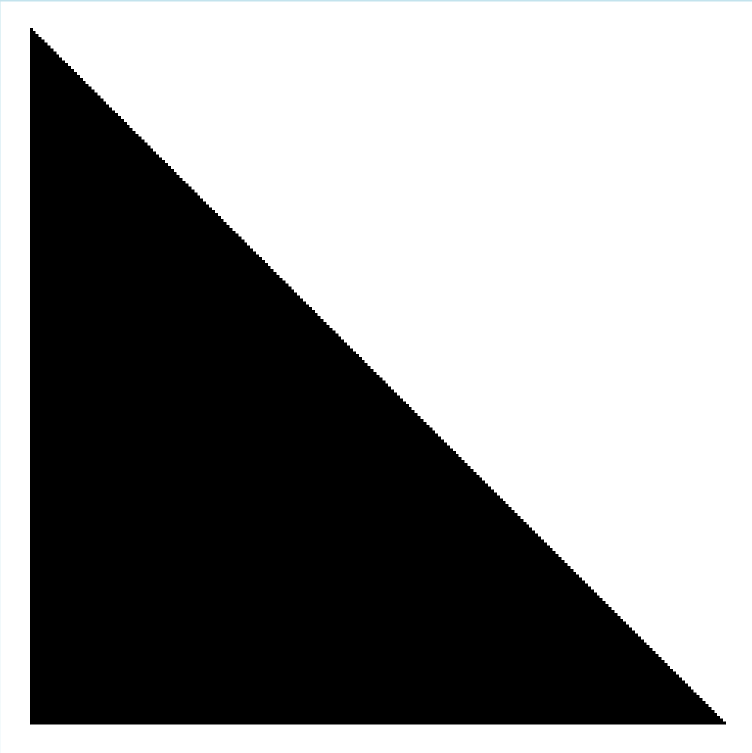
Orthogonal basis functions

Scale dependent spatial resolution

Basis functions in 7 different directions



Directionality in orthogonal wavelet coefficients



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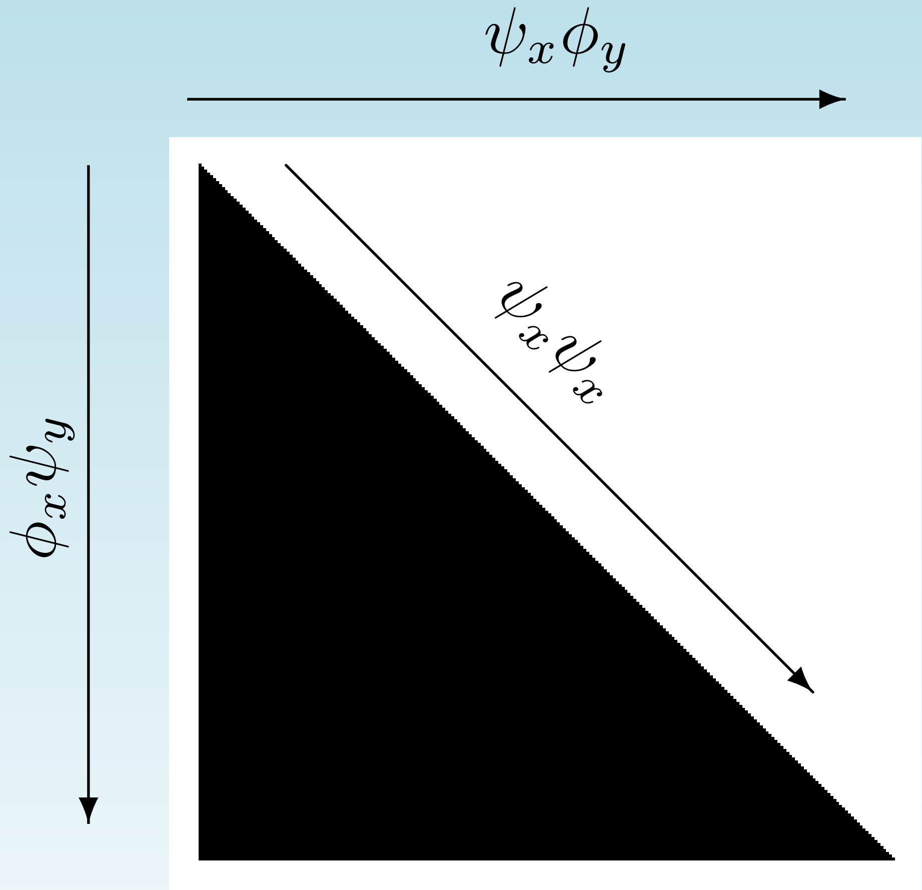
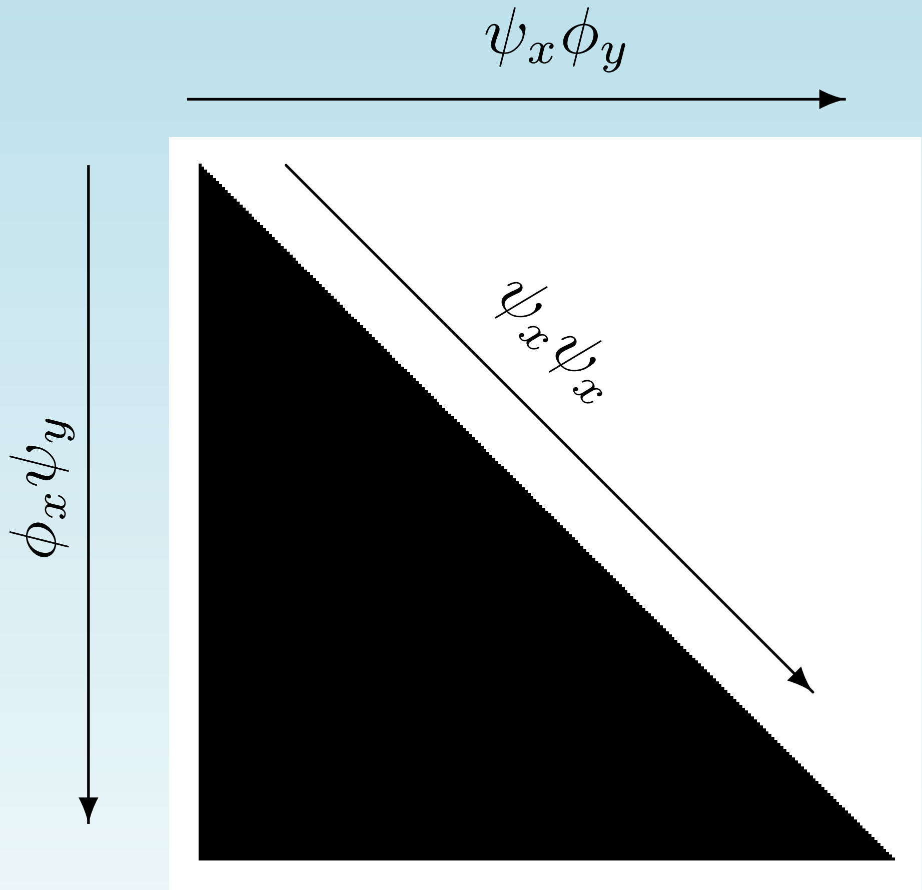


Orthogonal wavelet filtering :
Scaling filter ϕ and wavelet filter ψ

Directionality in orthogonal wavelet coefficients



Mallat-
$\Rightarrow$
Repres.

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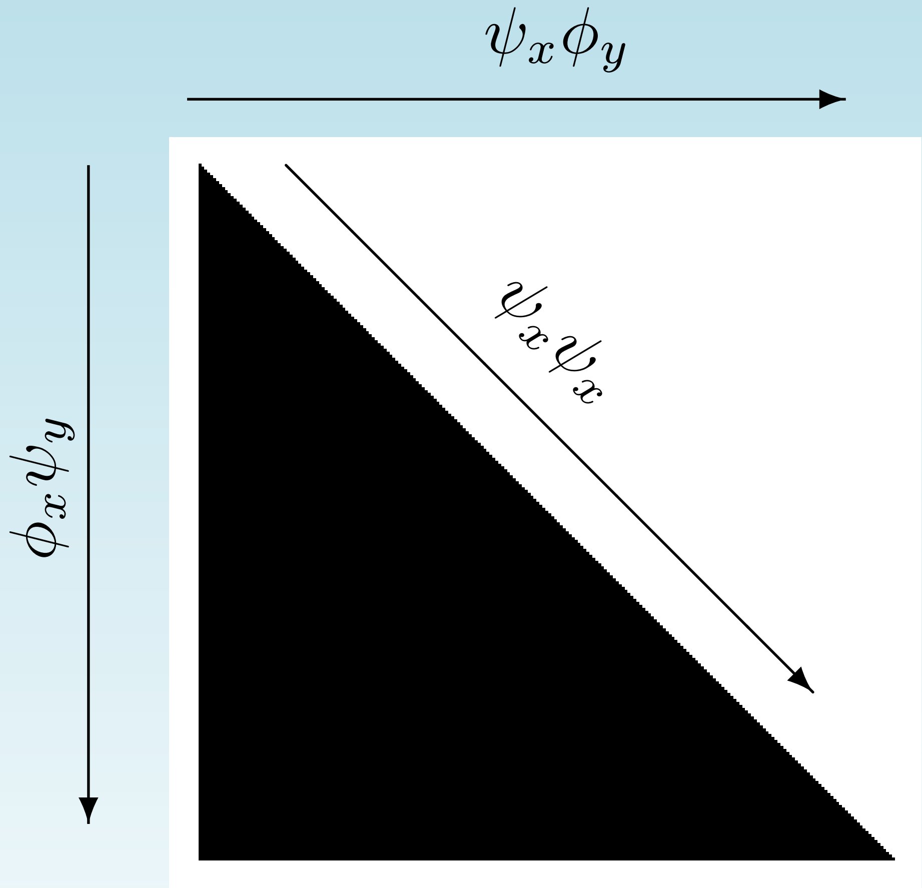

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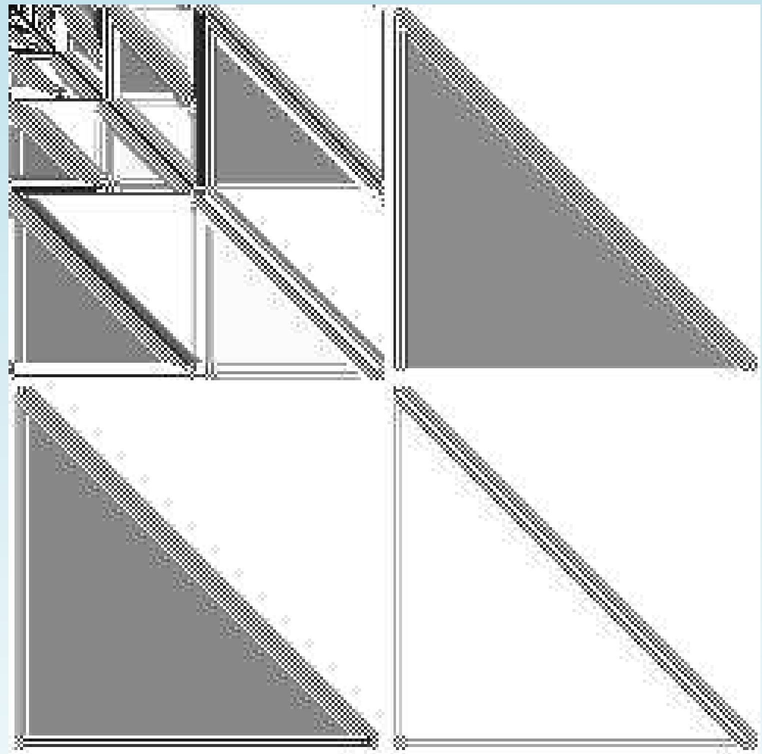

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3 directions for 2-dimensional signals :
Directional information can be extracted.

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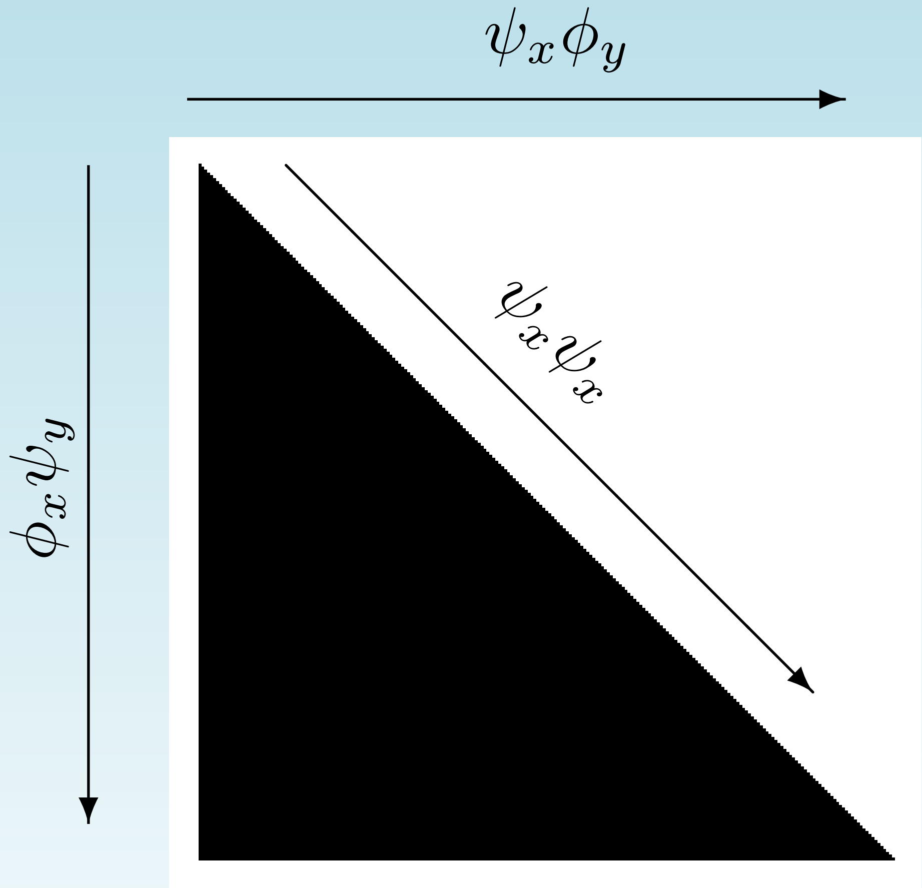

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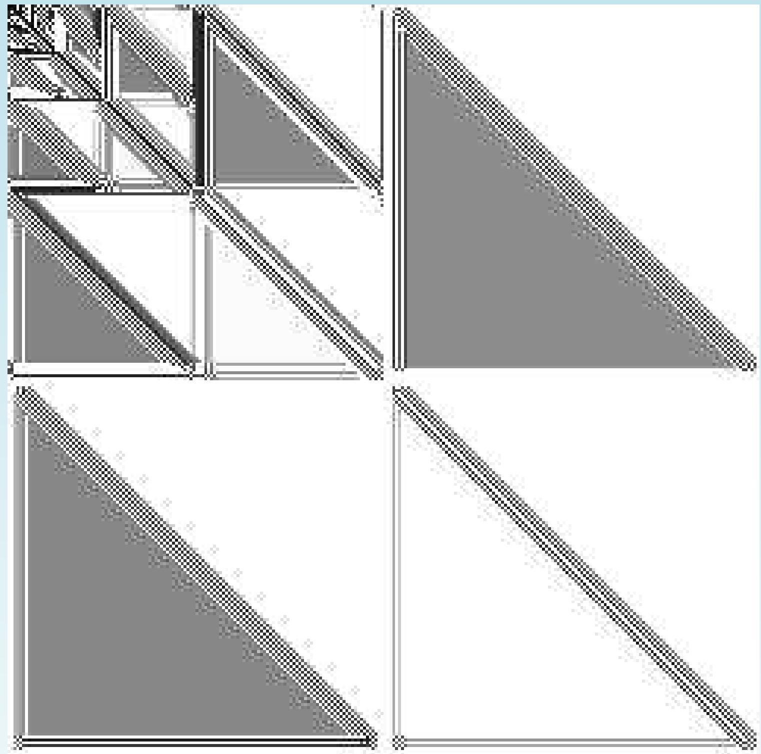

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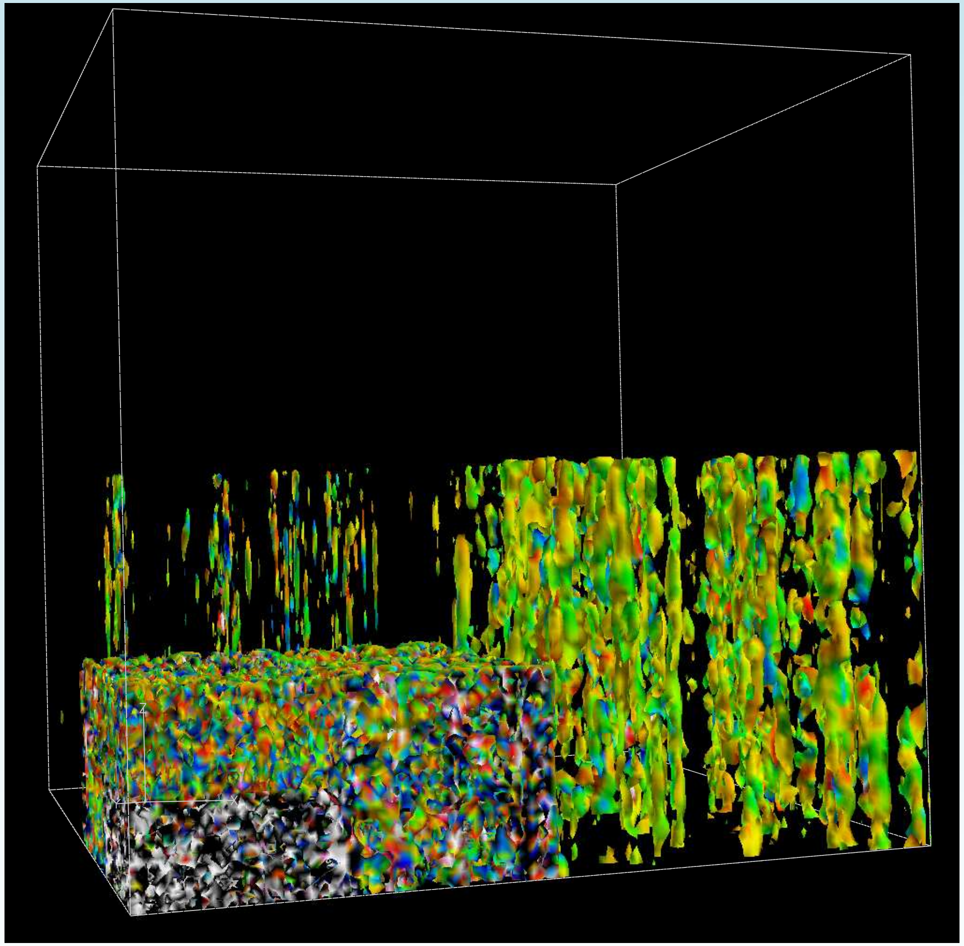
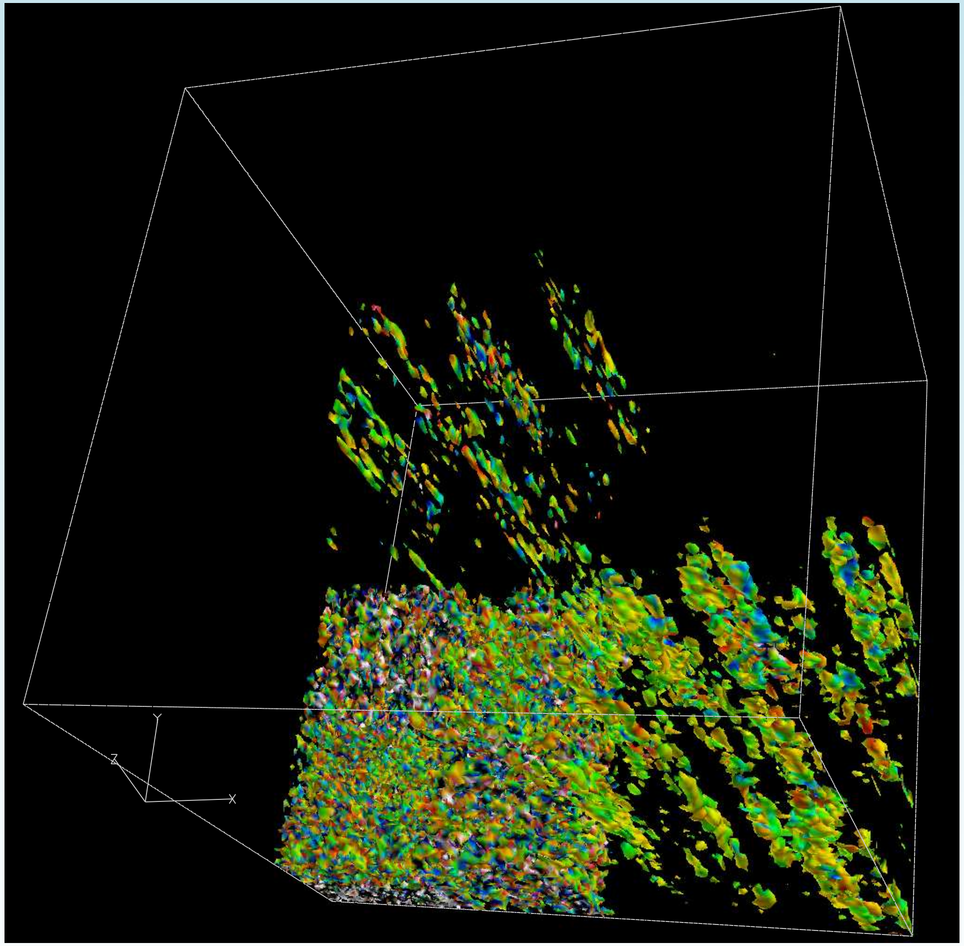
Directional information can be extracted.

Gradient in a direction means strong wavelet coefficients for that direction
(Wavelet coefficients)^2 proportional to energy

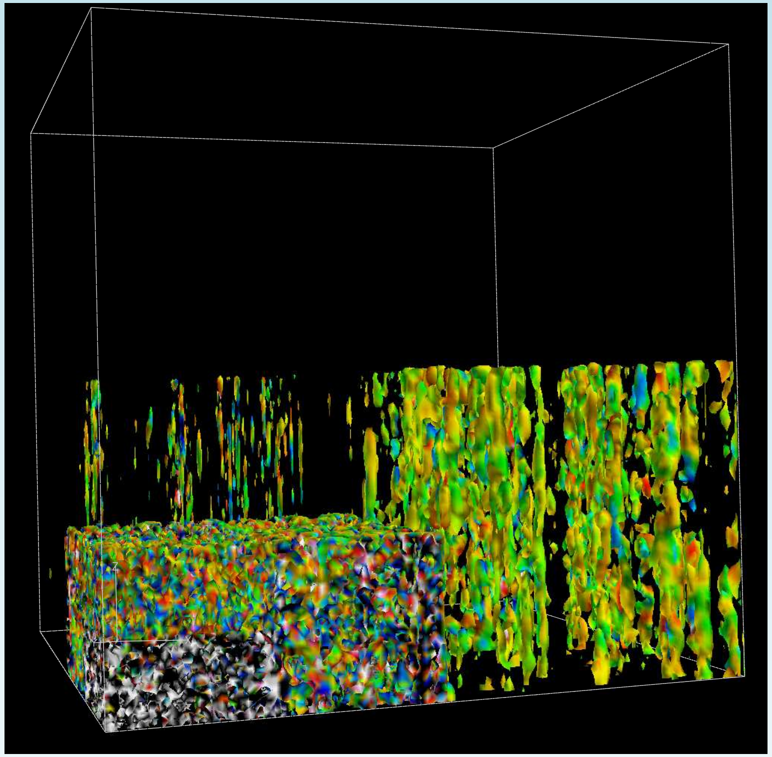
3-D wavelet coefficients of rotating turbulence

Mallat representation of wavelet-coefficients of x-component of vorticity, i.e. $\tilde{\omega}_x$
Wavelet coefficients of z-component $\tilde{\omega}_z$ are superposed by color.





Rotating turbulence in wavelet space



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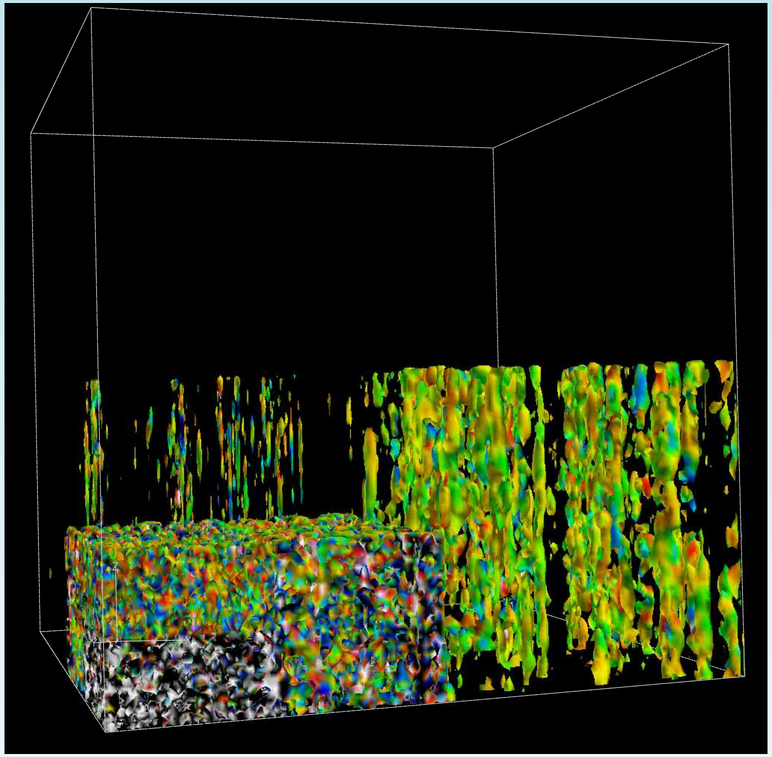
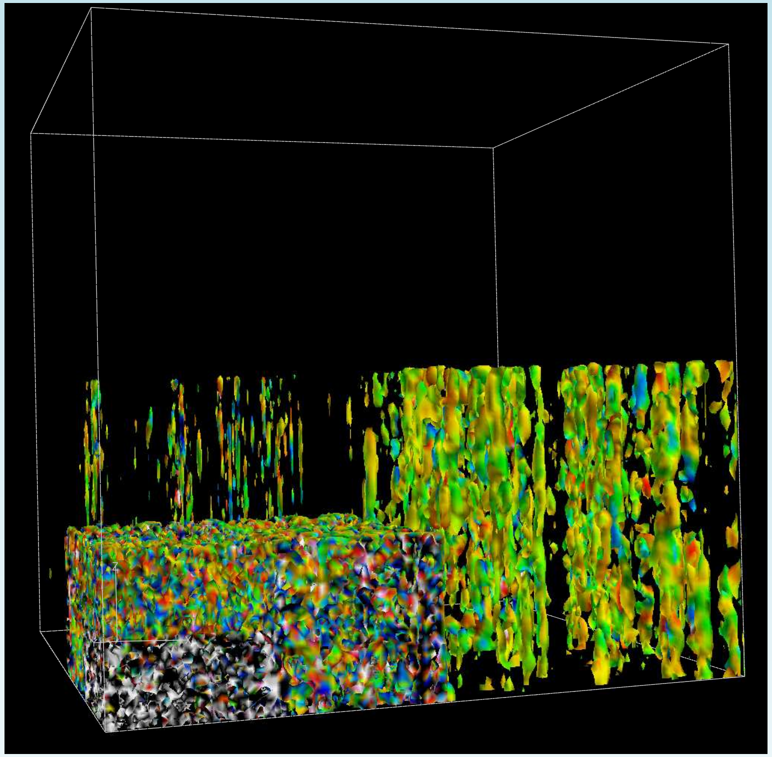


Nearly $3/4$ of wavelet space is empty
(vertical direction ψ_z)
Density in 'transversal direction'
is higher (e.g. ψ_y for $\tilde{\omega}_x$)
Large scales more isotropic than small scales

$\Rightarrow$ Statistics in wavelet space

Rotating turbulence in wavelet space



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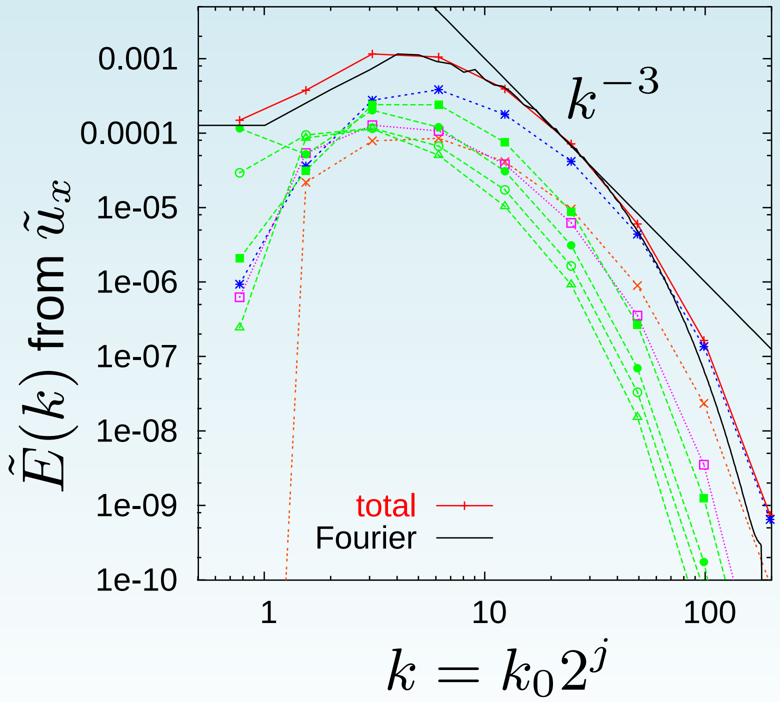
Coeffi cient density $\propto$ Energy density $\Rightarrow$ Directional Spectra

Directional wavelet spectra of rotating turbulence



Directional energies for seven wavelet directions for u, v and w

$\psi_x \phi_y \phi_z$

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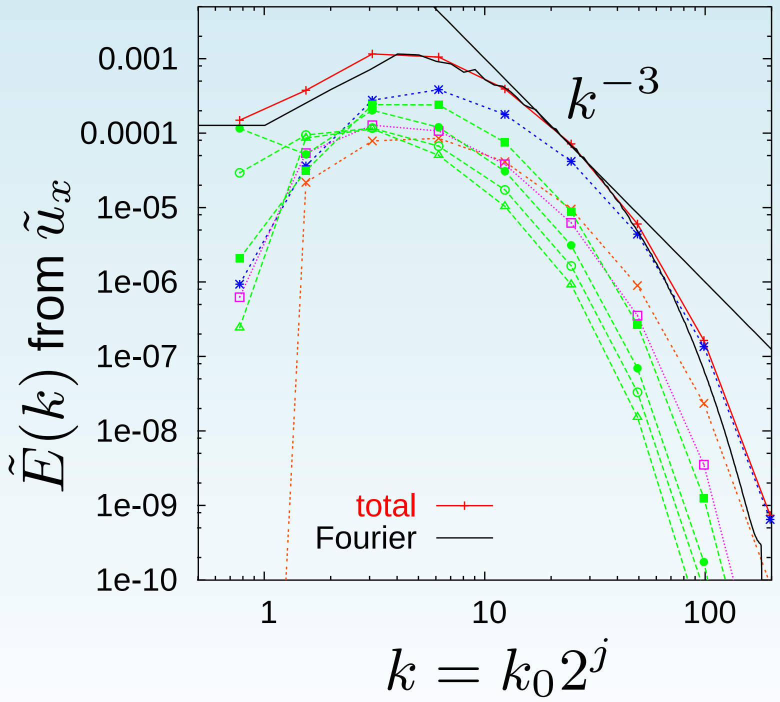
Directional wavelet spectra of rotating turbulence

A good agreement between Fourier spectra and wavelet spectra

Spectra with a vertical wavelet (ψ_z) are smaller

Horizontal 'transversal' spectrum is larger than longitudinal

Isotropy decreases with k



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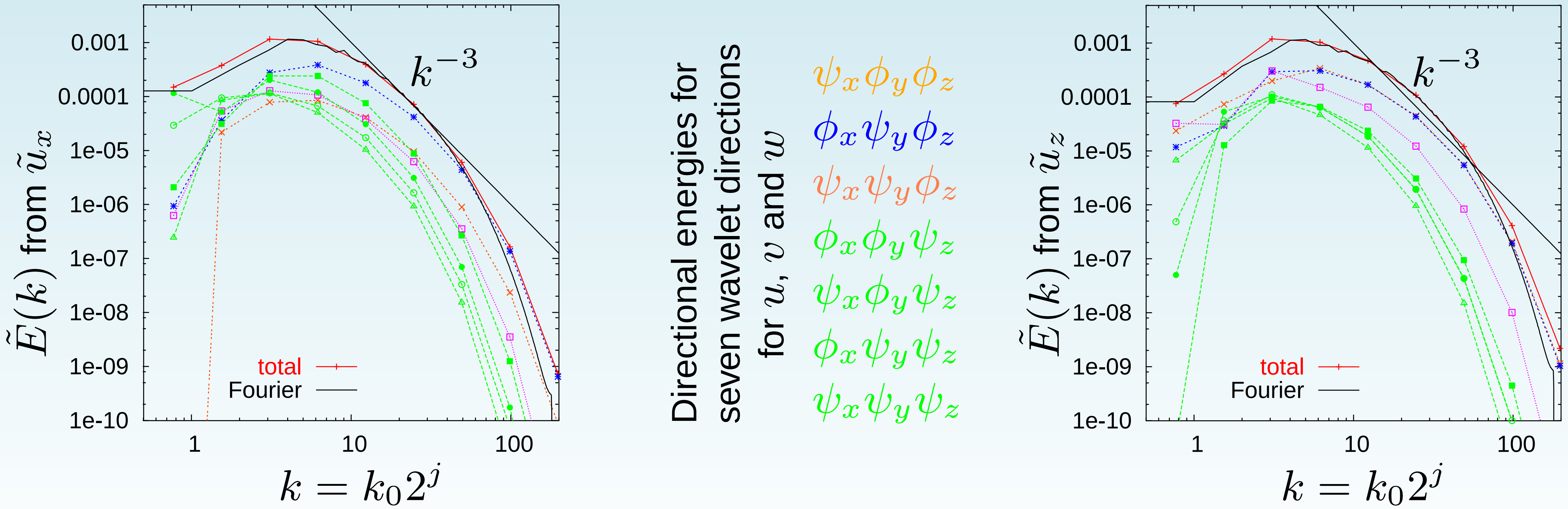
Directional wavelet spectra of rotating turbulence

A good agreement between Fourier spectra and wavelet spectra

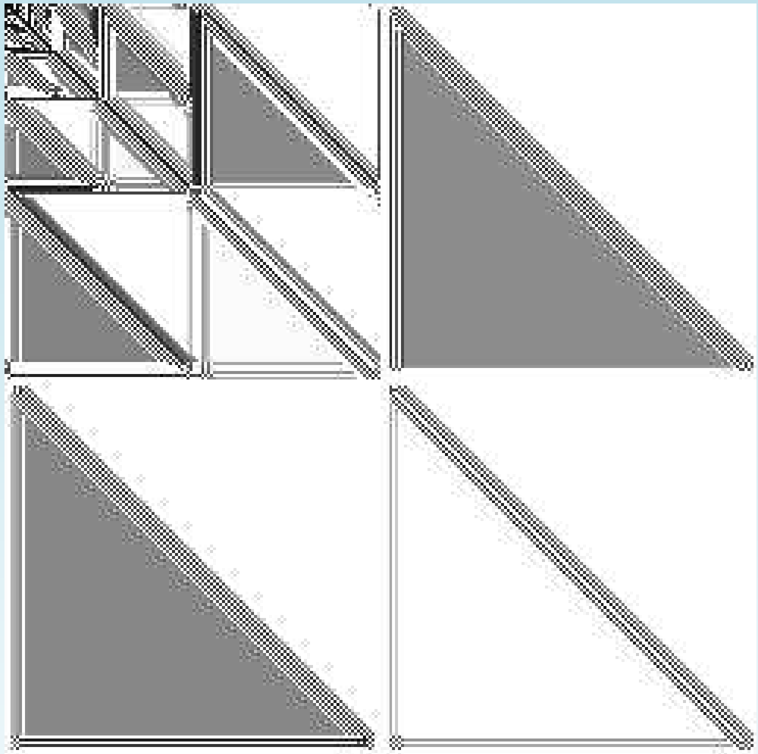
Spectra with a vertical wavelet (ψ_z) are smaller

Horizontal 'transversal' spectrum is larger than longitudinal

Isotropy decreases with k



What is directional energy ?



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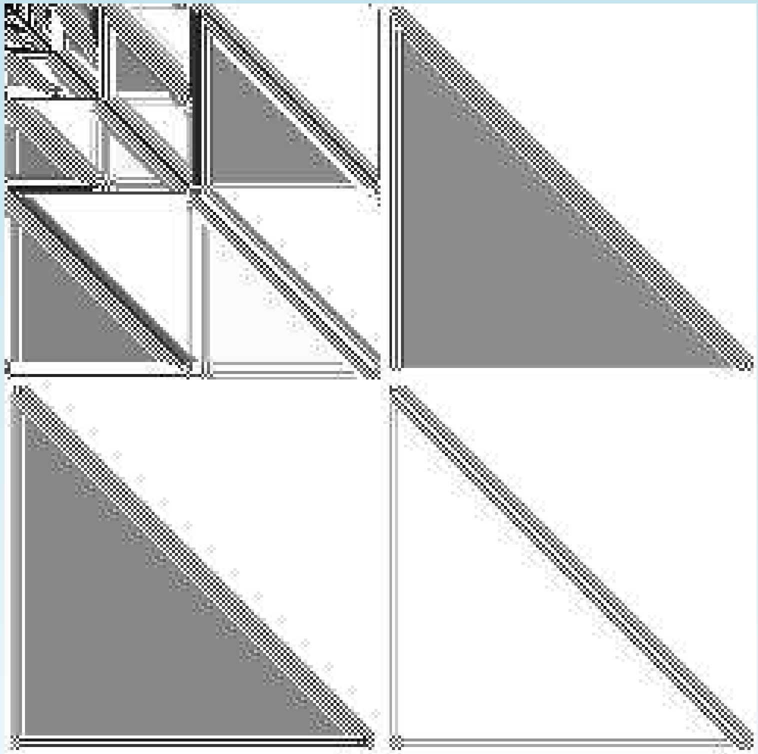

Strong coefficients are found where strong "changes" are present in a direction

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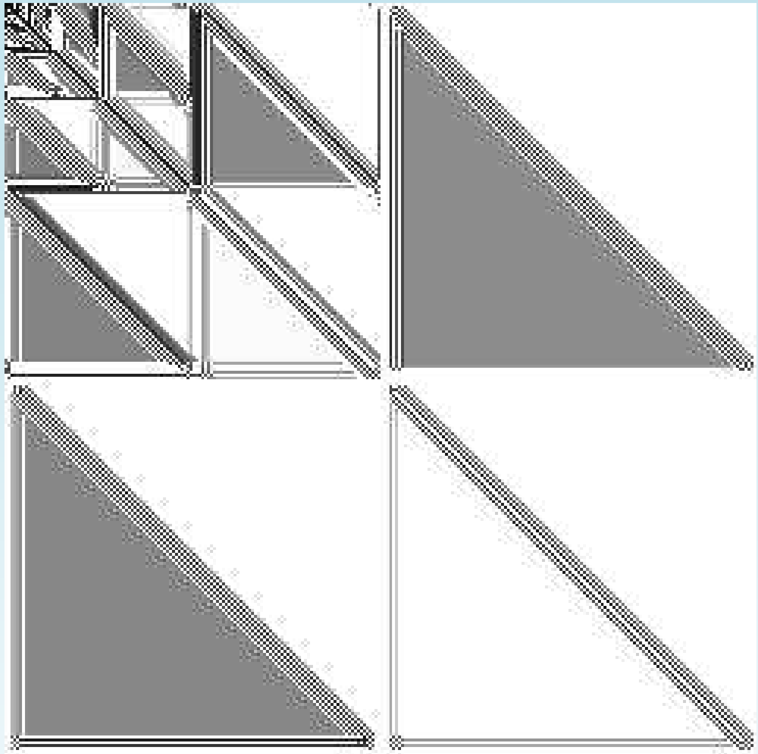

Direct.
$\Rightarrow$
sum

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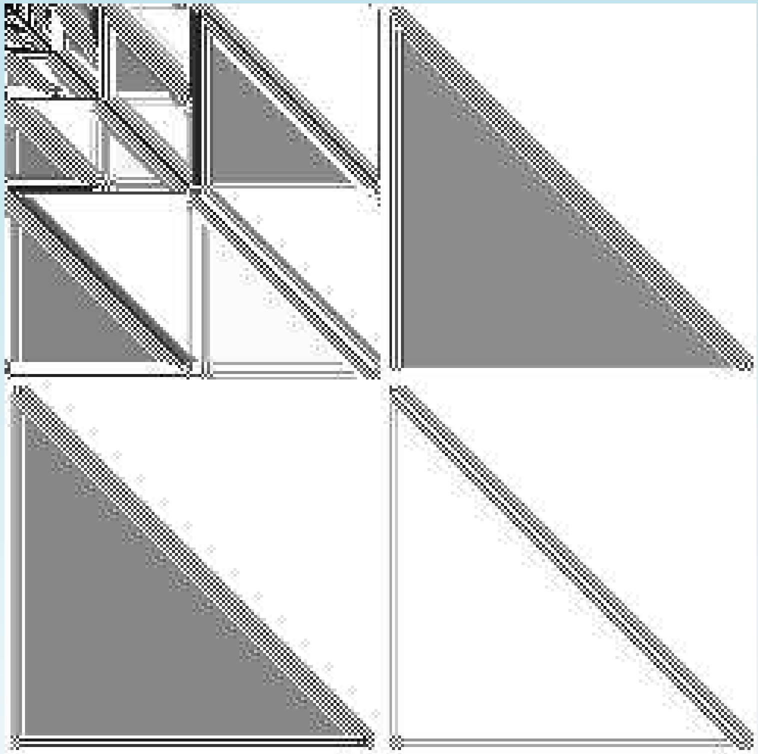

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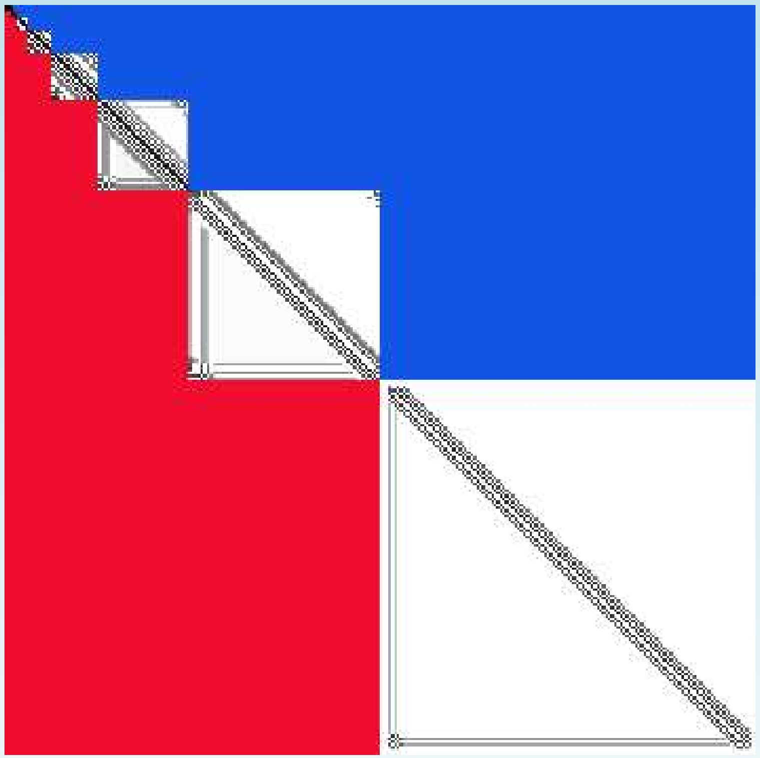
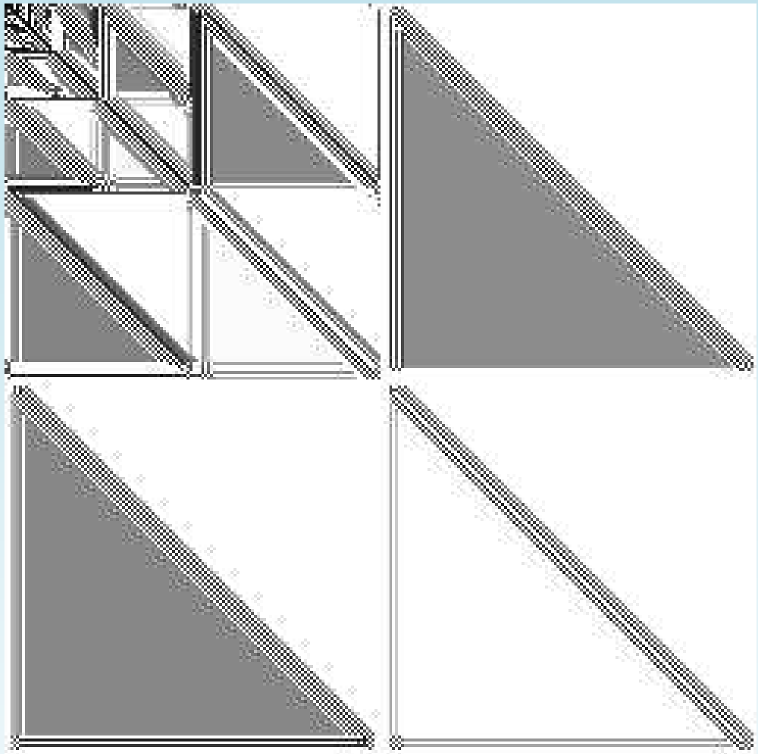


Strong coefficients are found where strong "changes" are present in a direction

Σ (coeffs)^2 for one direction measures the energy of the signal in that direction.

What is directional energy ?



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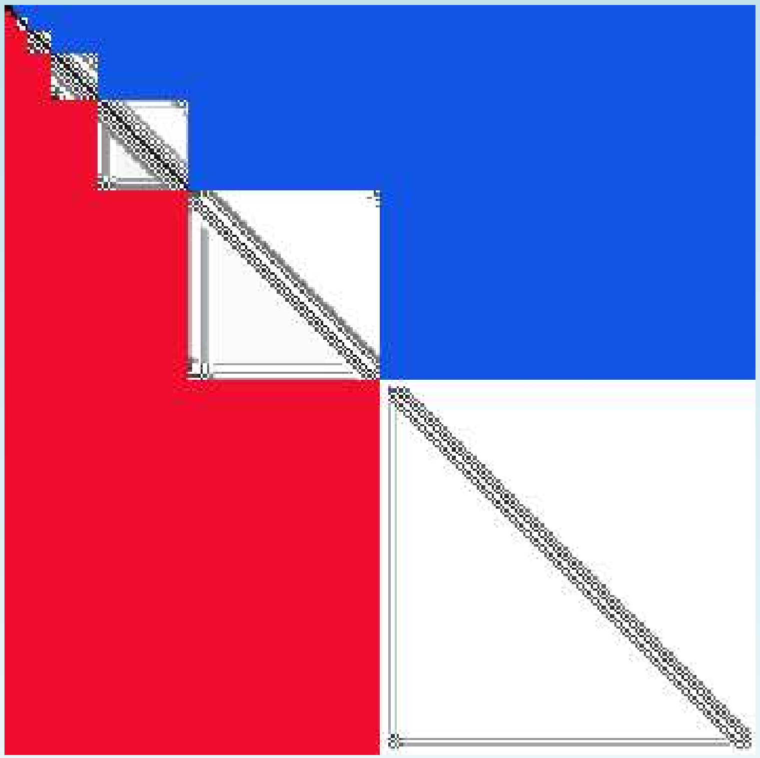

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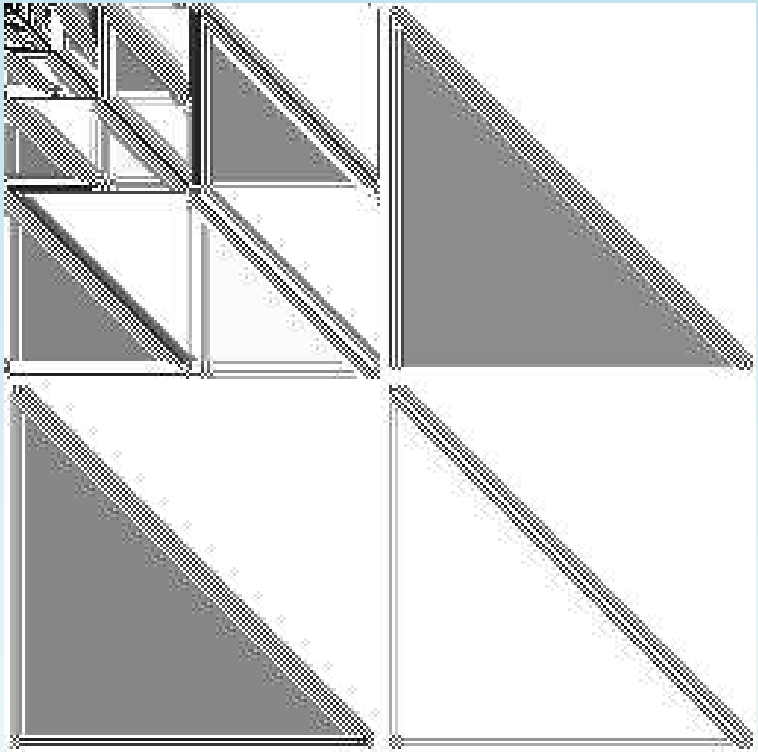
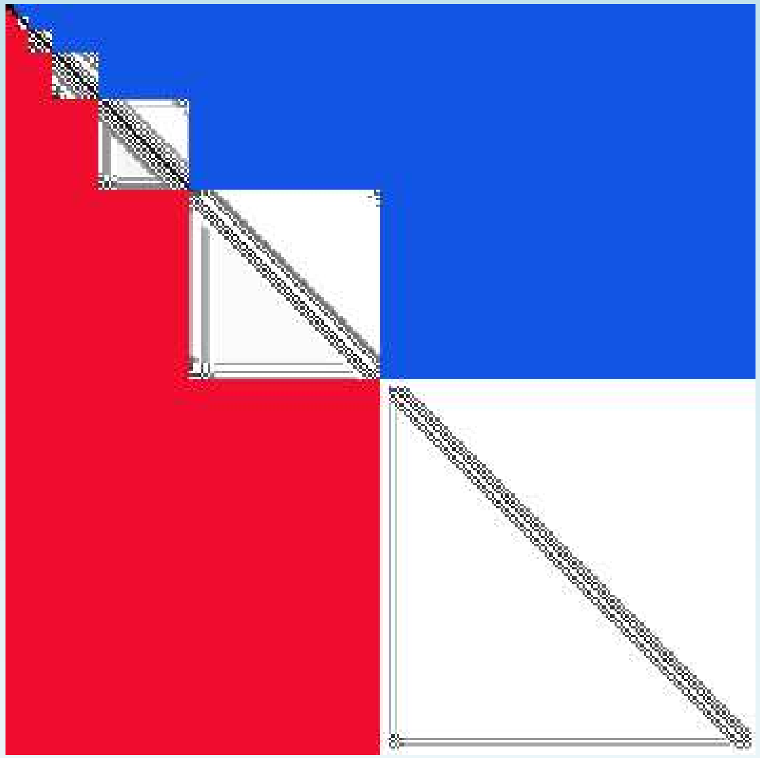
Directional energy compares the importance of different directions.

What is directional energy ?



Direct. $\Rightarrow$ sum



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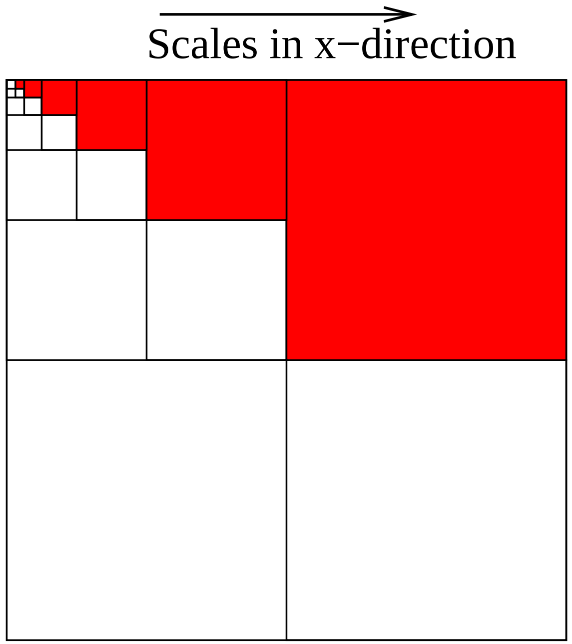
Directional energy compares the importance of different directions.

For a vector field, longitudinal and transversal energies can be defined.

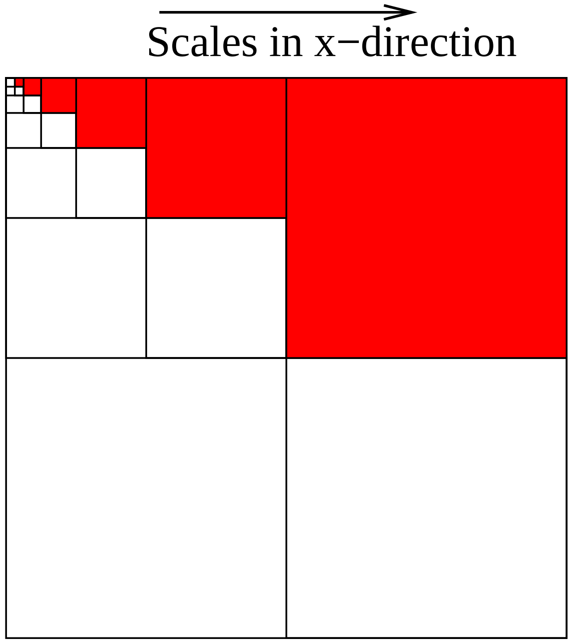
Mallat representation of wavelet space

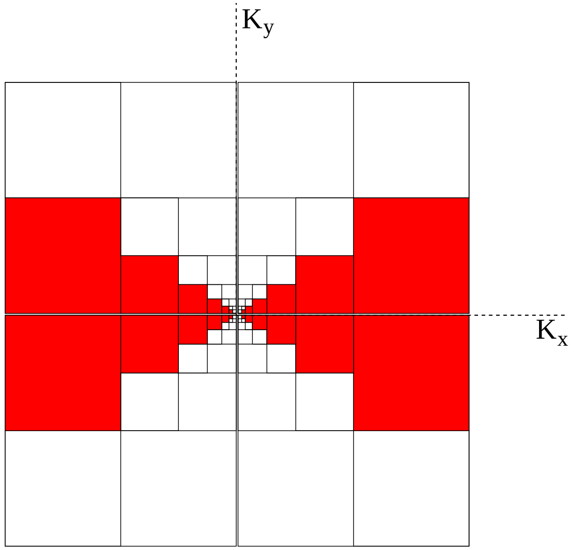


Each box corresponds to a certain scale j and direction.

Each box contains $2^{2(j-1)}$ values, giving information on localization in physical space.

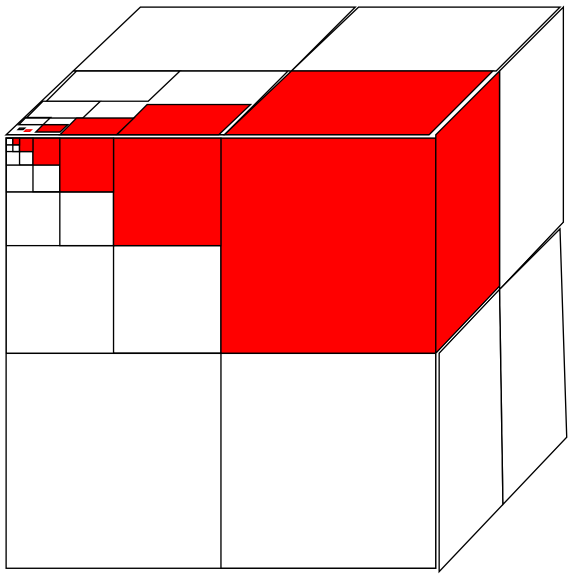
Wavelet space vs. Fourier space 2D

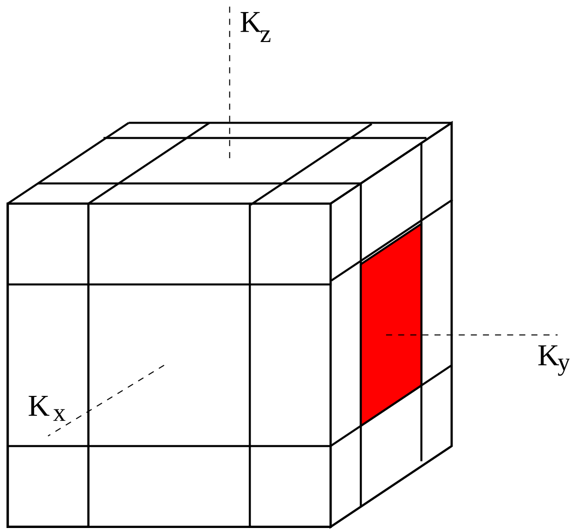




Hereby we can relate a scale j to a wavenumber K^j : $K^j = K_0 2^j$

Wavelet space vs. Fourier space 3D

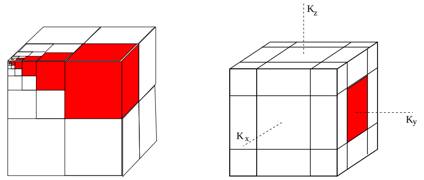


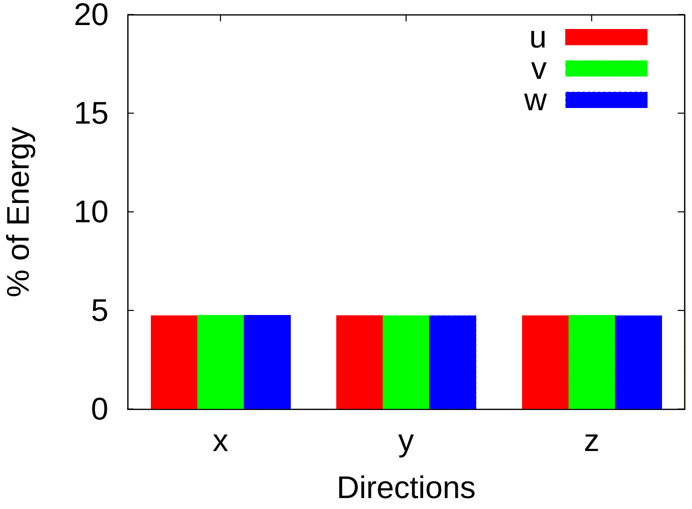


Example : velocity field.

The energy in the red domain corresponds to horizontal velocity fluctuations

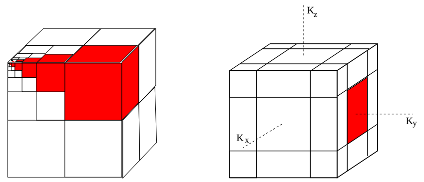
Directional Energy GWN

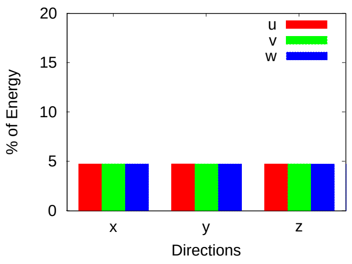


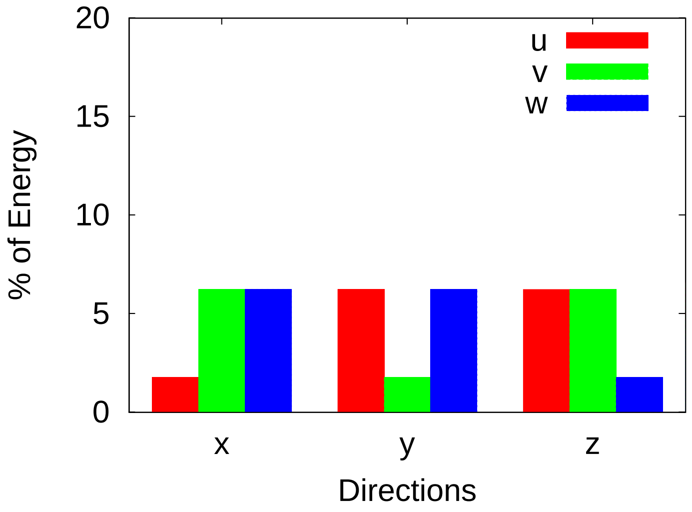


Gaussian White Noise (GWN)

Directional Energy DFGWN

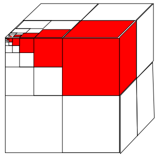


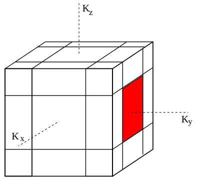


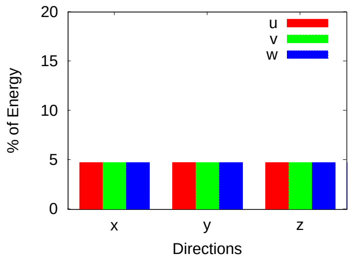


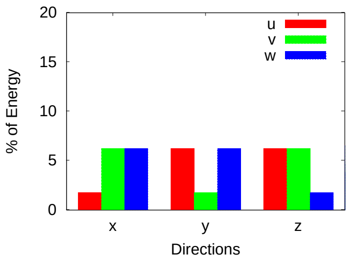
Gaussian White Noise (GWN), Divergence Free GWN

Directional Energy Isotropic Turbulence

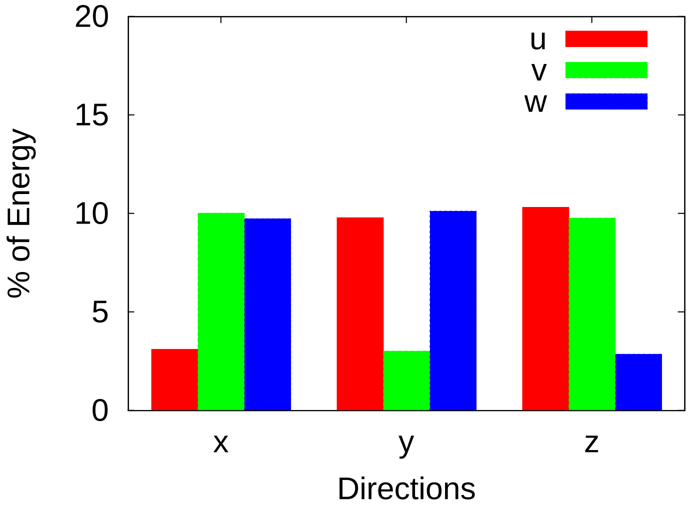






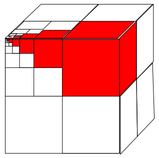


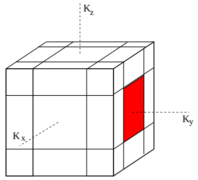
Gaussian White Noise (GWN), Divergence Free GWN

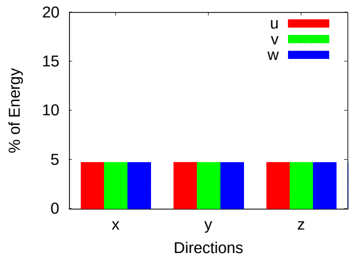


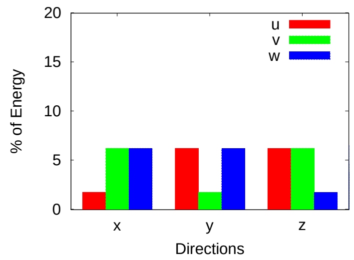
Turbulence : Isotropic

Directional Energy Rotating Turbulence









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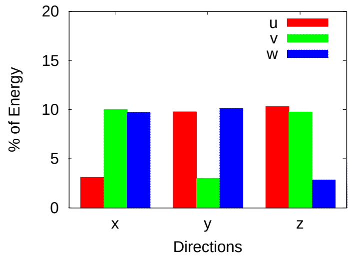
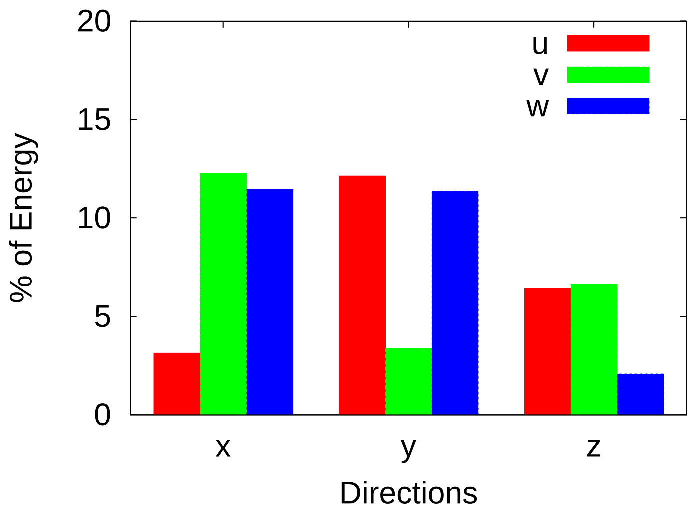
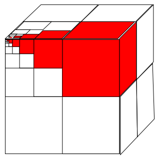
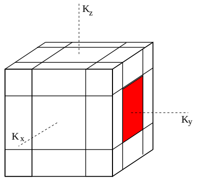


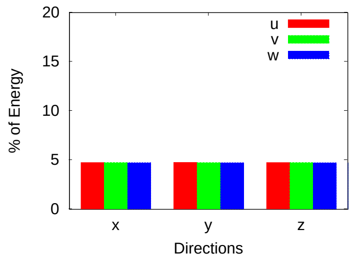


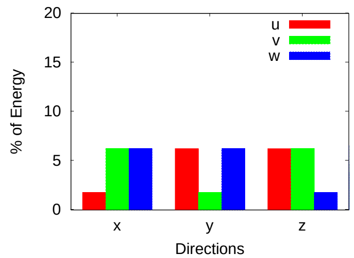
Turbulence : Isotropic, Rotating

Directional Energy Stratified Turbulence

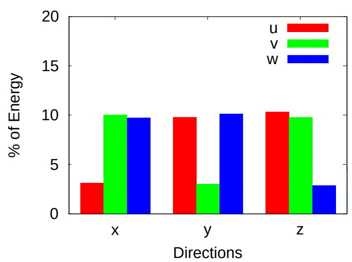


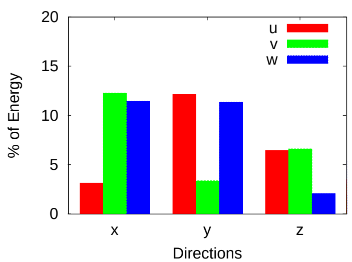


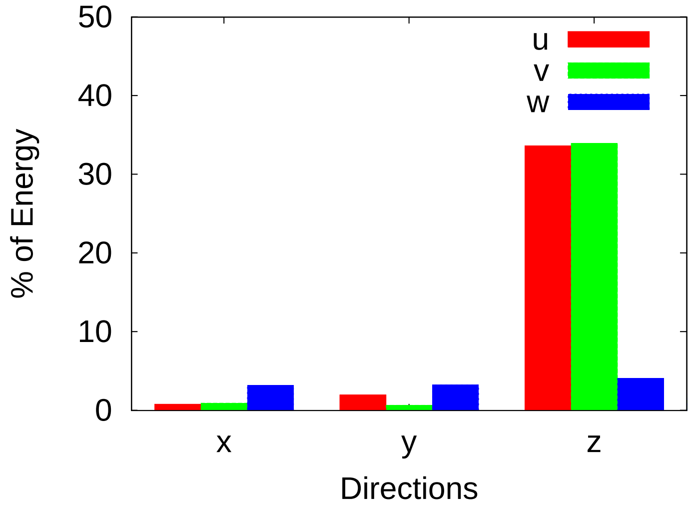




Gaussian White Noise (GWN), Divergence Free GWN

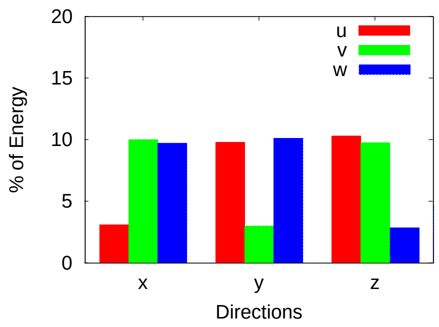


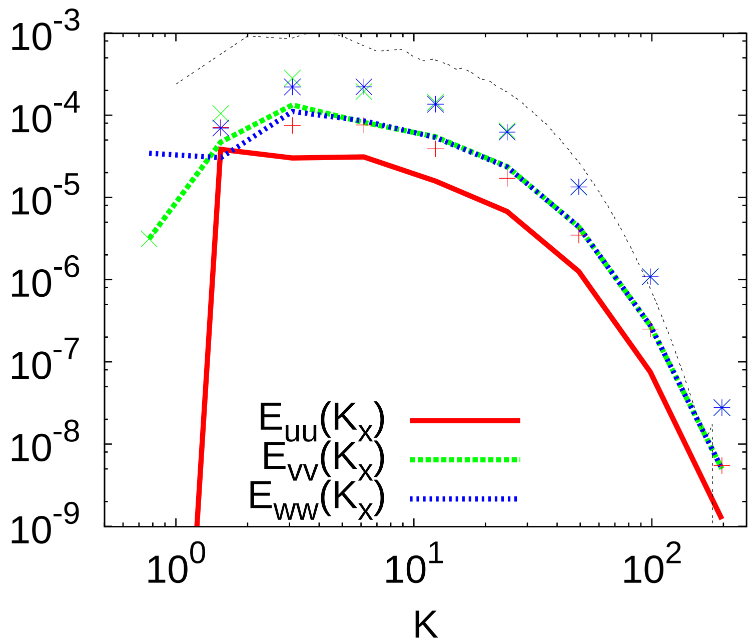




Turbulence : Isotropic, Rotating, Stratified

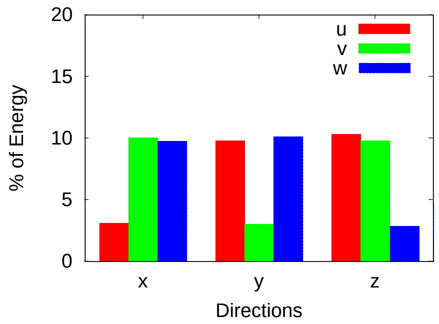
Directional energy distributed over scales : spectra ..

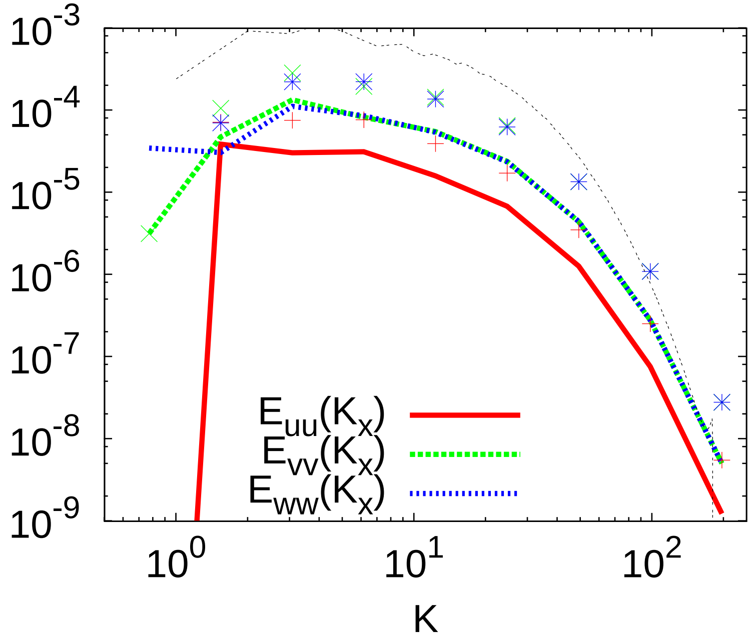




One looses some spectral resolution...

Directional energy distributed over scales : spectra





One looses some spectral resolution...

.. but obtains standard deviation of the spectral distribution

$$\sigma_E(K^j) \sim \left[\overline{e(K^j)^2} - \overline{e(K^j)}^2\right]^{1/2} \quad \text{with} \quad e(K^j) = \overline{\tilde{u}(x,j)^2}$$

Relation standard deviation and Flatness

Flatness :

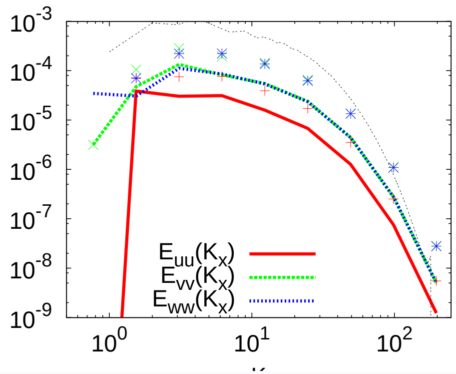
$$F_u(K^j) = \frac{\overline{\tilde{u}(K^j)^4}}{\overline{\tilde{u}(K^j)^2}^{\,2}}$$

and standard deviation of the spectral distribution

$$\sigma_e(K^j) = \left[\overline{e(K^j)^2} - \overline{e(K^j)}^{\,2}\right]^{1/2}$$

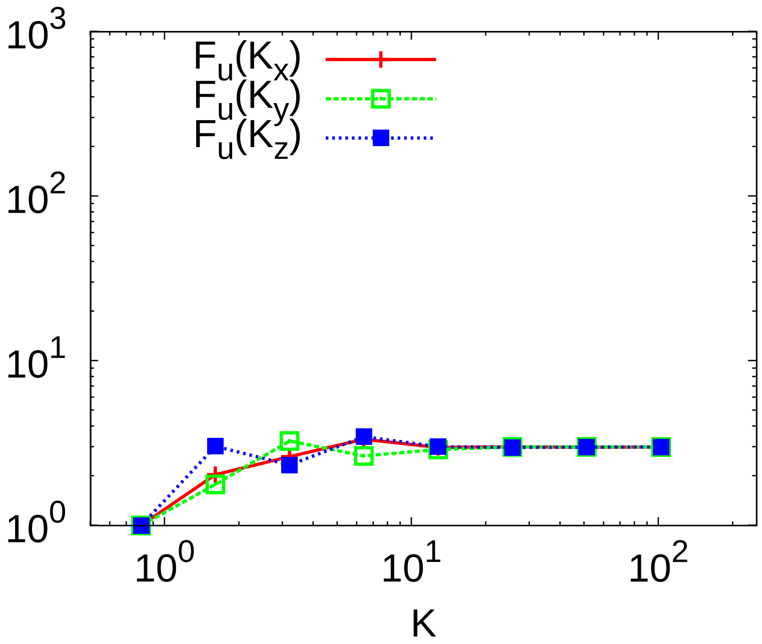
$$\sigma_e(K^j) = \left[(F_u(K^j) - 1)\right]^{1/2} e(K^j)$$

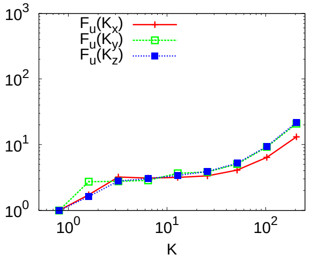
$$\rightarrow F_u(K^j) = \left(\frac{\sigma_E(K^j)}{e(K^j)}\right)^2 + 1$$

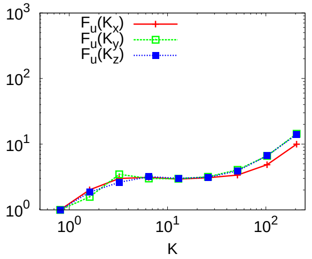


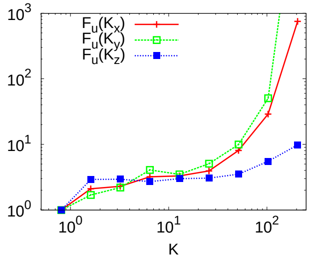
Flatness is related to relative variance of the spectra

Directional Flatness



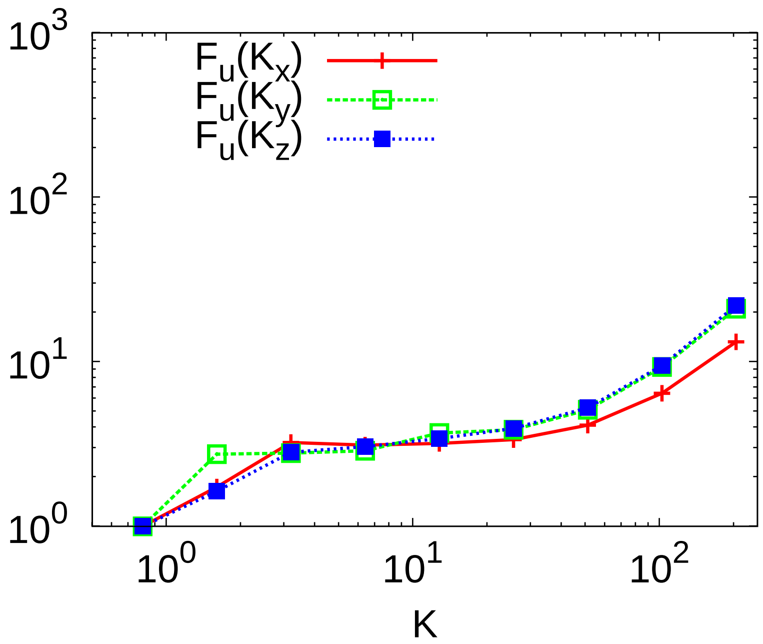


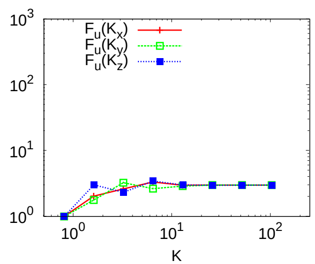


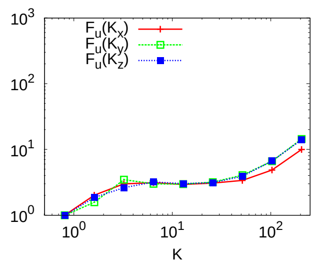


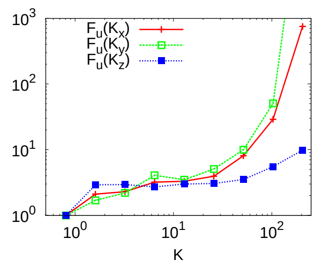
Divergence Free GWN

Directional Flatness



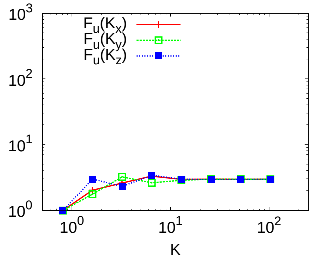


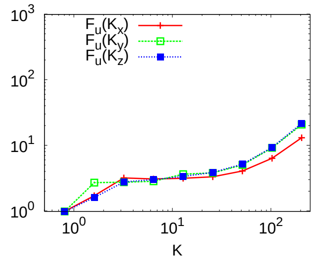


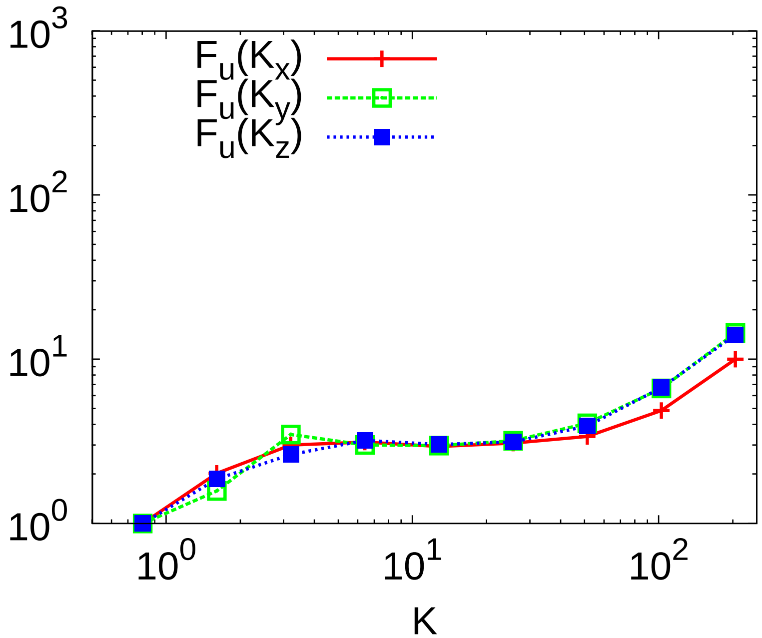


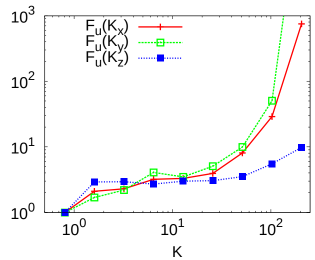
Isotropic turbulence

Directional Flatness



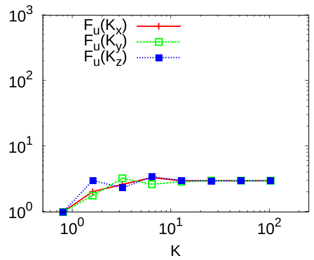


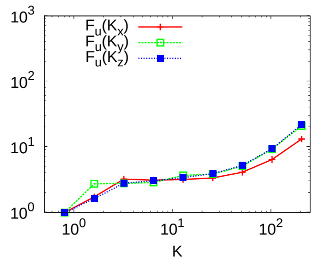


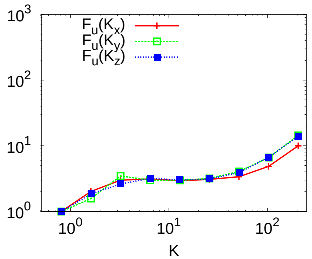


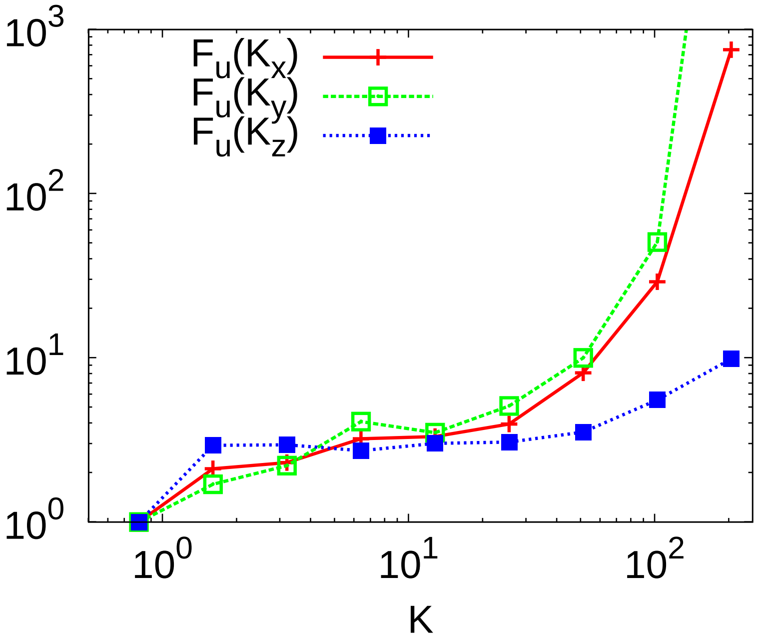
Rotating turbulence

Directional Flatness









Stratified turbulence

Physics ?

– The K_z direction corresponds to the vortex mode in stratified turbulence

– It seems that this mode behaves “isotropically” when considering flatness

– The wavemode yields a dramatical increase of flatness in the small scales

Conjecture : the wavemode is responsible for intermittency and rare events in stratified turbulence

Conclusions

– Compared to non-structured energy distributions, turbulence prefers 'transversal' energy (vortical structure). Average direction of vortices can be determined by directional energies.

– Anisotropic turbulence has non-zero coefficients in real space and Fourier space. However, the majority of the coefficients are very small in wavelet space, suggesting possibilities for models.

Perspectives

– Parametrisation of anisotropy (N and f) in wavelet space.

– Extenstion to Magneto-hydrodynamics.

– Coherent-vortex-simulation of anisotropic turbulence using 1% of the wavelet coefficients..

Ref.: L. Liechtenstein, W. Bos and K. Schneider.
Anisotropy and spatial intermittency in rotating and stratified turbulence.
Preprint, 03/2007.

http://www.l3m.univ-mrs.fr/site/schneider.htm