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Kolmogorov's contributions to the physical and geometrical understanding of small-scale turbulence and recent developments

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This paper reviews how Kolmogorov postulated for the first time the existence of a steady statistical state for small-scale turbulence, and its defining parameters of dissipation rate and kinematic viscosity. Thence he made quantitative predictions of the statistics by extending previous methods of dimensional scaling to multiscale random processes. We present theoretical arguments and experimental evidence to indicate when the small-scale motions might tend to a universal form (paradoxically not necessarily in uniform flows when the large scales are gaussian and isotropic), and discuss the implications for the kinematics and dynamics of the fact that there must be singularities in the velocity field associated with the $-\frac{5}{3}$ inertial range spectrum. These may be particular forms of eddy or 'eigenstructure' such as spiral vortices, which may not be unique to turbulent flows. Also, they tend to lead to the notable spiral contours of scalars in turbulence, whose self-similar structure enables the 'box-counting' technique to be used to measure the 'capacity' D_{K} of the contours themselves or of their intersections with lines, D'_{K}. Although the capacity, a term invented by Kolmogorov (and studied thoroughly by Kolmogorov & Tikhomirov), is like the exponent $2p$ of a spectrum in being a measure of the distribution of length scales (D'_{K} being related to $2p$ in the limit of very high Reynolds numbers), the capacity is also different in that experimentally it can be evaluated at local regions within a flow and at lower values of the Reynolds number. Thus Kolmogorov & Tikhomirov provide the basis for a more widely applicable measure of the self-similar structure of turbulence. Finally, we also review how Kolmogorov's concept of the universal spatial structure of the small scales, together with appropriate additional physical hypotheses, enables other aspects of turbulence to be understood at these scales; in particular the general forms of the temporal statistics such as the high-frequency (inertial range) spectra in eulerian and lagrangian frames of reference, and the perturbations to the small scales caused by non-isotropic, non-gaussian and inhomogeneous large-scale motions.

1. Kolmogorov's papers: review and comments

(a) *Introduction*

In this review we join with the other contributors to this special publication in celebrating some of Kolmogorov's great contributions to fluid mechanics and mathematics, and showing in some small way how his genius has inspired further

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researches along the many lines he pioneered. (For a full account of his life and research see Kendall (1990).) We concentrate here on Kolmogorov's contributions to the study of the statistical structure of the small scales of turbulence, and his mathematical studies of random processes that have provided new methods for analysing these motions. Although Kolmogorov's analyses were framed in the language of mathematical analysis, especially probability theory and set theory, and not in physical or geometrical terms, subsequent mathematical studies have shown how Kolmogorov's results lead to general hypotheses about the dynamical and geometrical structure of turbulence, which can now be tested against the details of actual flow fields and trajectories using recent experiments and computer simulations of turbulence.

(*b*) *Turbulence* (K41 *a*)

The great new approach of Kolmogorov (1941 *a*, *b*; hereinafter referred to as K41 *a*, *b*) was to show how to combine some of the methods and ideas of statistical physics (though Kolmogorov used the term probability theory) with those of dimensional analysis and scaling, and to apply them to fluid mechanics, in particular the study of turbulence. From the earliest studies of turbulence the analogy had been drawn between turbulent eddy motion and the motion of gas molecules, and it was hoped that the kinetic theory of gases might provide a useful model for turbulent motions (see, for example, Prandtl 1925). Kolmogorov in fact relied on these concepts for modelling the Reynolds shear stresses produced by the larger energy containing eddies in turbulent shear flows (Kolmogorov 1942), in a generalization of the earlier ideas of Prandtl (1925) (see Spalding 1991).

But in K41 *a* he introduced the more general idea from statistical physics of a state of statistical equilibrium. Further he made the hypothesis that the structure of the small-scale motions (defined by the velocity difference $\delta u(l)$ over a distance l, see figure 1) are uncorrelated with the large-scale motions U of the flow and therefore their statistics (e.g. $\langle \delta u(l)^2 \rangle$) must be universal, provided these motions are defined in terms of relative velocities (such as $\delta u(l)$). The large-scale motions and the overall dynamics of the flow determine the magnitude of the motions in any given flow.

The first reviews of Kolmogorov's work in the English language scientific literature by Batchelor (1947, 1953, pp. 6, 7) pointed out the importance of Kolmogorov's results and the novelty of the approach in his analysis, for example in the differences between Kolmogorov's hypothesis of local isotropy and G. I. Taylor's hypothesis of the isotropy of the total velocity field; in fact his 1947 paper was translated into Russian and used as the best introduction to K41 in Moscow and elsewhere! Reading the extended footnote in K41 describing the author's conception of the interactions between different scales of eddy motion, one is struck, as was Batchelor in 1947, by the similarity with Richardson's (1922) poetic image of this process. No reference was given to Richardson in K41 *a*, although surprisingly Kolmogorov referred to Prandtl's momentum transfer model to explain the 'cascade'. However, in the first paragraph of his later paper, Kolmogorov (1962, referred to hereinafter as K62) generously made up for this omission. This point is also emphasized by Monin & Yaglom (1971) in the introduction to their comprehensive book on turbulence.

Batchelor (1953, pp. 6, 7) indicated that there had been some previous theoretical work which might have suggested the assumption of statistical equilibrium for the small scales of turbulence, because 'there is a tendency for dynamical systems with

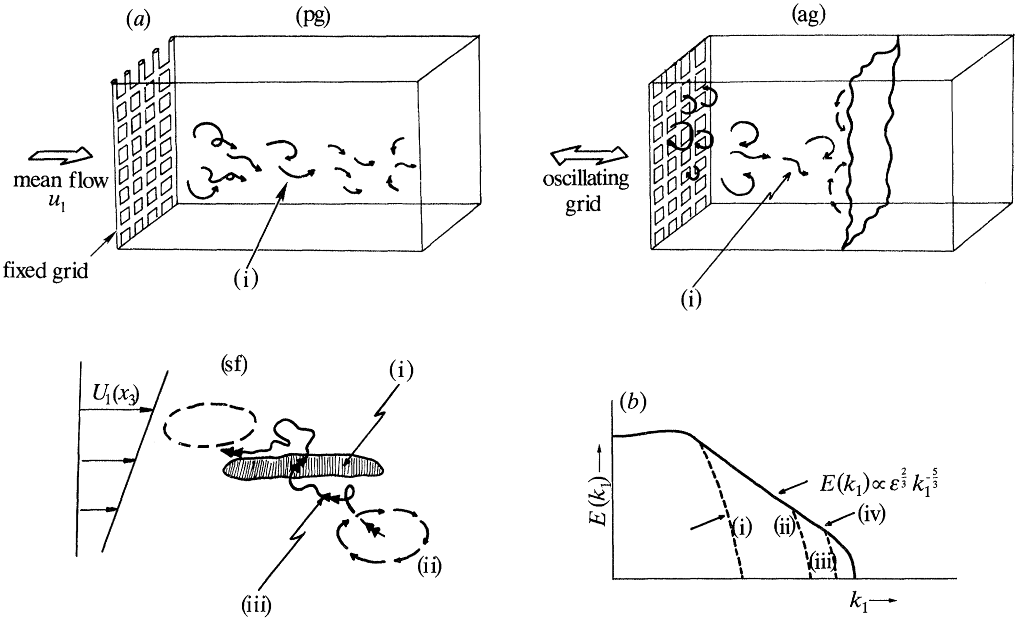

Figure 1. Small-scale turbulence; generation by different large-scale motions and examples of spectra. (*a*) Schematic diagrams of turbulent flows: (pg) passive grid, (i) turbulence transported by u_1 and decays; (ag) active grid, (i) turbulent transport by interaction between eddies; (sf) shear flows (after Rogers & Moin 1987), (i) streak of high speed ($\partial u_1/\partial x_2$ large), (ii) streamwise of 'roller eddy' (in moving frame), (iii) vortex line bent into wiggly horseshoe. (*b*) One-dimensional spectra for laboratory experiments. (i), (ii), (iii) Moderate *Re* ($\approx 10^2$) flows as depicted in (*a*), (iv) high *Re* form of any of these flows.

a large number of degrees of freedom (and not independent), to approach a statistical state which is independent (partially, if not wholly) of the initial conditions' and even, perhaps, 'independent of the precise form of the governing equations'. Although there is no reference to this concept in K41*a*, it was certainly being widely considered by researchers into turbulence in the 1930s, such as Burgers (1940), Tollmien (1933) and others referenced by Dryden (1943).

To apply dimensional or scaling analysis to the small-scale relative motions, it was necessary to introduce a physical quantity representative of the dynamics in this relative frame; it was the brilliant idea of Kolmogorov to define this as ϵ, the average rate of transfer of energy between large and small scales of motion (where viscous stresses are negligible), but which was also equal to the rate of energy dissipation by viscous stresses for the smallest scales of motion. Clearly the kinematic viscosity ν had to be the other dimensional quantity introduced to define these scales $((\nu^3/\epsilon)^{\frac{1}{4}})$.

The main quantitative results of the theory were that, for scales large enough that the viscous stresses are negligible compared with inertial stresses, i.e. $|\delta u(l)|\, l/\nu \gg 1$, the mean square velocity difference over a length l (or structure function) is given by

$$\langle \delta u(l)^2 \rangle \sim \epsilon^{\frac{2}{3}} l^{\frac{2}{3}}. \tag{1.1}$$

For smaller scales where these stresses are of the same order, the functional form of the velocity difference can be stated as

$$\langle \delta u(l)^2 \rangle = F(l/(\nu^3/\epsilon)^{\frac{1}{4}}). \tag{1.2}$$

From the analysis of Karman & Howarth (1938) these results could be expressed in tensor form and in such a way that the statistics of these small-scale motions are invariant under rotations and reflections of the coordinates.

All previous applications of scaling arguments had been applied to solve flow problems where there were only one or two length and velocity scales involved, such as boundary layer flows. Kolmogorov showed how it was possible to apply these concepts to the multiscale phenomena of turbulence. (Further examples of this approach are given in §3.)

Until that time the only practical and well understood techniques that scientists and engineers had available to analyse such phenomena consisted of Fourier methods (as used by Rayleigh (1877) who, like others, used the theorem of Parseval (1806) (but gave no reference) and the recently developed theory of spectra and correlations (Wiener 1933; Khintchine 1933). Taylor (1938) had been the first to apply these methods to turbulence.

For an understanding of recent developments in the dynamics of small-scale motions one should qualify one important aspect of K41. Kolmogorov stated – for no obvious reason – that the small-scale velocity fields $\delta u(l)$ were effectively constant on these local timescales (e.g. the time defined by the local velocity gradient $l/|\delta u(l)|$). His model did not essentially make use of this assumption, but following Batchelor (1947, p. 535) it is now recognized (and established by various models, theories, simulations and experiments) that $l/|\delta u(l)|$ is the characteristic timescale for the eddy motions of length scale l to change with time (see §3*b*).

Obukhov (1941) had similar ideas to those in K41*a*, though he used power spectra, e.g. $E(k)$ for wave number k, to describe the statistical structure of the small-scale motions in statistical equilibrium. The description of turbulence in terms of spectra brings the statistical approach closer to many dynamical models which represent turbulence as a set of interacting modes. Then the given number of modes that are simulated or specified in a representation indicates the range of scales in a spectrum. There is no equivalent direct interpretation between the correlation function and the simulation or calculation of the flow field. In the inertial range of length scales (where (1.1) is valid), either using the Wiener–Khintchine relation between correlations and spectra, or using dimensional arguments, it follows that

$$E(k) \propto \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}. \tag{1.3}$$

Obukhov (1949) later extended the concepts to the statistical distribution of temperature or other scalars, discussed in this issue by Gibson (1991) and Sreenivasan (1991) (see figure 2*a*, *b*).

(*c*) *Turbulence* (K41*b*, K62)

In the second paper Kolmogorov wrote (four months after the first) on the small-scale structure of the turbulence in 1941 (K41*b*), he examined the consequences of applying the statistical hypotheses of his earlier paper to the Navier–Stokes equations governing fluid motion. As Karman & Howarth (1938) had already shown, for homogeneous and isotropic turbulence exact general relations could be derived from these equations between the second-order correlations $B_{rr} = \langle(\delta u(l))^2\rangle$ and the third-order correlations $B_{rrr}\langle(\delta u(l))^3\rangle$ of the relative velocity of two neighbouring fluid particles along the line joining them. In particular, it follows that

$$B_{rrr} = -\tfrac{4}{5}\epsilon l + 6\nu\, \mathrm{d}B_{rr}/\mathrm{d}l. \tag{1.4}$$

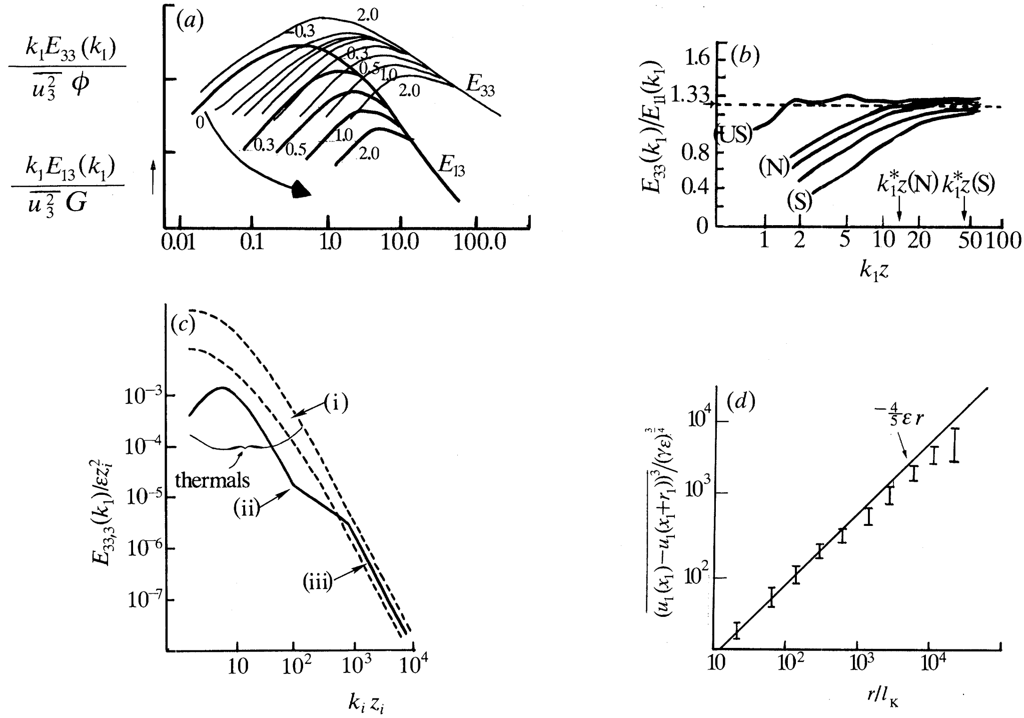

Figure 2. High Reynolds number turbulence in the atmospheric boundary layer to show which ranges of scales and for which statistics the universal structure is and is not observed. (*a*) E_{33} $(\propto \epsilon^{\frac{2}{3}} k_1^{-\frac{5}{3}})$, $E_{13}\,(\propto \epsilon^{\frac{1}{3}} \mathrm{d}u/\mathrm{d}z)\,k_1^{-\frac{7}{3}})$: second-order spectra showing that E_{33} follows the isotropic form while the shear stress co-spectrum is non-zero, but depends on the isotropic spectrum. (Kaimal *et al.* 1973). Arrow indicates increasing shear and stable buoyancy forces. (*b*) E_{33}, E_{11}: second-order spectra of the normal components showing how the shear reduces this ratio from its isotropic form for $k_1 < k_*$ (Kaimal *et al.* 1973). (US) corresponds to unstable buoyancy faces and very weak shear $z/L_{\mathrm{MO}} < -2.0$, where L_{MO} is the Monin–Obukhov length, (N) to no buoyancy forces and moderate shear, (S) to stable buoyancy forces and strong shear $z/L_{\mathrm{MO}} > 0.5$. (*c*) $E_{33,3}(\kappa_1)$, co-spectra of u_3^2 and u_3 in a convective boundary layer. Note how the inertial range contributes to $\overline{u_3^3}$ in this case (Hunt *et al.* 1988). (i) 100–300 m, (ii) 22 m, 50 m; (iii) $E_{33,3} \approx 3\epsilon k_1^{-2}$. (*d*) $B_{ddd}(r) = \langle (u_1(x) - u_1(x+r_1))^3 \rangle (r_1)$; third-order structure function in a near neutral boundary layer (where the large scales are approximately gaussian). Note comparison with the prediction of K41*b* $(= -\frac{4}{5}\epsilon r)$ which has no arbitrary constant. Note the normalization on the velocity and length scales of the Kolmogorov microscale (Van Atta & Chen 1970).

Since for values of $l \gg (\nu^3/\epsilon)^{\frac{1}{4}}$ that lie in the inertial range, the second, viscous term, is negligible, (1.4) shows that

$$B_{rrr}(l) = -\tfrac{4}{5}\epsilon l. \tag{1.4a}$$

Note that there is no arbitrary constant in this expression; it only depends on the hypotheses of the statistical isotropy and homogeneity of the small scales, and the forms of the continuity equation and momentum equations for inviscid flow. (The fundamental importance of this result was emphasized even in the brief chapter on turbulence by Landau & Lifschitz (1960, §33). It is surprising that this result was not discussed in the first post-Kolmogorov book on turbulence by Batchelor (1953), although it was described in his 1947 paper. He informs us that this was because between 1947 and 1953 the results of experimental studies indicated that high Reynolds number for turbulence was unlikely to be measurable. He now admits he was too pessimistic.) An alternative expression for (1.4) involves the co-spectrum of δu^2 and δu, namely

$$E_{u^2,u} \propto \epsilon k^{-2}. \tag{1.4b}$$

Kolmogorov pointed out the important statistical consequence of this dynamical result, which was that the small scales of turbulence are non-gaussian so that the skewness S of the velocity difference ($S = \langle \delta u(l)^3 \rangle / \langle \delta u(l)^2 \rangle^{\frac{3}{2}}$) is a constant, independent of l, for l lying in the inertial range.

At first sight, an experimental measurement of B_{rrr} or $E_{u^2,u}$ to test (1.4) would appear to be a more fundamental and searching test of the hypothesis than verifying the forms for the second-moment correlations and spectra (1.1) and (1.3). But it turns out to be a restrictive test, and cannot be used to indicate whether or not the second moments satisfy the predictions of K41. The reasons are that the experiments are difficult, because reliable measurements of third moments require more extensive data (which means over longer periods when in the environment conditions may change), and because, as explained in §3c, third-order moments can be more strongly influenced than second-order moments by the structure of the large-scale motions. However, measurements in the atmosphere by Van Atta & Chen (1970) and Anselmet *et al.* (1984) are certainly consistent with the predictions of (1.4a), including the value of the coefficient $\frac{4}{5}$ (see also §2b) (see figure 2c, d).

Between 1941 and 1961, when Kolmogorov again published a paper on the small-scale structure (published in 1962, but presented in Marseille in 1961), there had been several experimental studies and theoretical models suggesting that the significant small-scale motions contributing most to the dissipation of energy must be intermittently distributed through the flow. Although intermittency had already been suggested by Taylor in 1938, in his paper K41a Kolmogorov assumed that the rate of dissipation ϵ was constant wherever there was significant small-scale motion. So in the light of the later studies, Kolmogorov proposed a modification to the theory of K41a that allowed for the fluctuations in ϵ. Equations (1.1) and (1.3) respectively were replaced by

$$\langle \delta u(l)^2 \rangle \sim \epsilon^{\frac{2}{3}} l^{\frac{2}{3}+\mu}, \quad E(k) \propto \epsilon k^{-\frac{5}{3}-\mu}, \tag{1.5}$$

where μ is a universal constant relating to the postulated log-normal distribution of the energy dissipation. Also, the property of the constancy of skewness in the inertial range derived in K41b was replaced by $S(l) \sim l^{-3\mu/2}$.

This theory is reviewed in this volume by Gibson (1991), Van Atta (1991) and further developments are discussed by Frisch (1991). However, for the majority of phenomena connected with small-scale turbulent motions these corrections can be neglected in the statistical description of the turbulence (when there is no local sampling of special 'events', such as spots of high strain rate or dissipation), and the average value of ϵ can be used satisfactorily in formulae such as (1.1) to (1.4), as we show in §§2 and 3.

On the other hand the insight from this theory (and its subsequent modifications by others) has stimulated many different kinds of measurement where samples are taken of special 'events'. Although in K61 the statistical analysis to include these 'events' was framed in terms of two point correlations, in many recent studies of these events (Sreenivasan 1991; Mandelbrot 1991; Frisch 1991) other analyses are used, drawn in part from the contribution to 'fractal' analysis by Kolmogorov's papers (Kolmogorov 1958; Kolmogorov & Tikhomirov 1959, hereafter referred to as K58 and KT59 respectively).

It is worth emphasizing that Kolmogorov appealed to a schematic concept of eddies in formulating his model, and did not consider afterwards whether his theoretical predictions could be used to make a more definite model of the spatial distribution and internal structure of turbulent eddies. Some progress in this direction is discussed in §2.

(*d*) *Stochastic processes* (K58, KT59)

Turbulence is not the only field where random intermittent signals occur. In fact, they are of great interest in many dynamical systems and in communication systems (following the pioneering developments of Shannon & Weaver (1963)). It was these fields that stimulated Kolmogorov to develop in 1959, in collaboration with Tikhomirov (KT59), the theory and, of great practical importance, the methods for analysing and characterizing complex multiscale functions that are also highly intermittent. These might be random functions of one variable, such as signals in a communications circuit, or convoluted contour surfaces occurring in local regions of a turbulent flow field.

The mathematical problem was to characterize these signals more efficiently and more revealingly, than the previously available correlation and spectra methods, especially if they were self-similar in some way, e.g. in some local region $\delta u(l) \propto l^n$, or in a statistical sense $\langle \delta u(l)^2 \rangle \propto l^n$, over some interval of l, such as the inertial sub-range of turbulent flows. (Paradoxically, for these and other similar irregular signals it is instructive to assume that the signal has the same irregular and non-differentiable form down to infinitesimal scales, and then analyse this non-smooth function by new techniques. This is similar to representing localized but smooth functions as singularities to simplify the analysis.)

For these kind of signals K58 introduced the measure of 'capacity' D_K (because of its relevance to the capacity of electronic circuits), which is related to the minimum number of elements $N(\delta)$ of size δ that is required to 'cover' the signal. It is found that $N(\delta) \sim \delta^{-D_K}$ as $\delta \rightarrow 0$, and D_K increases as the irregularities of the signal become increasingly space filling over a large range of length scales. The irregularities of a more restricted class of signals had earlier been given a quite different mathematical definition by Hausdorff (1918) who introduced the concept that a 'dimension' could have a fractional value D_H. Mandelbrot (1982) has termed these signals 'fractal'. K58 and KT59, which do not refer to Hausdorff's work, apply to the wide class of self-similar functions that are not everywhere differentiable and have a fine structure with detail on all scales, including those Mandelbrot (1982) termed 'fractals'. In the appendix of his book, Mandelbrot (1982) stated that the 'box-counting' technique discussed in K58 and KT59 and the measure of D_K was essentially equivalent to the concept of dimension introduced earlier by Bouligand (1929).

But it is certain that the paper KT59 not only put this method of analysis on a firm mathematical foundation, but following its discussion by Mandelbrot (1982), it stimulated a new approach to the measurement of intermittent and multiscale phenomena in turbulence and many other fields. These studies (see, for example, Sreenivasan & Menevau 1986) have not only shown many phenomena (e.g. rates of dissipation, contours of concentration) that are self-similar on different scales within a particular flow as quantified by D_K or distributions of D_K, but have also shown that the self-similarity of structure occurs in different kinds of turbulent flows (e.g. jets, wakes, clouds, etc.). These results are clearly consistent with Kolmogorov's hypothesis of a universal small-scale statistical equilibrium; but if the measurements do not also exhibit the universal forms of the two point correlations or spectra (1.1)–(1.4), is there a contradiction or perhaps is the satisfying of these particular two-point statistical forms simply a more stringent condition for the existence of certain universal features of small-scale motion? (A point further discussed in §2*e*.)

2. Observing and interpreting the results and assumptions of Kolmogorov's theory

(*a*) *Formulating questions*

The central fluid mechanical question raised by Kolmogorov's papers is: what are the implications for the kinematics and dynamics of turbulent motion of (*a*) the assumption that the small-scale motions have a universal structure, independent of the large-scale motions, (*b*) the statistical results given by (1.1)–(1.5)?

This general question may be best answered by considering a number of subsidiary problems.

1. What kinds of velocity fields are consistent with the assumptions (*a*) and the results (*b*) of Kolmogorov's models (K41*a*, *b*, K62). For example, what are the characteristics and the defining parameters of the turbulent flow fields, and the range of length scales that are necessary for these statistical conditions to be satisfied, whether defined by spectra, correlations or by the existence of fractal dimensions of certain quantities?

2. Assuming the velocity field satisfies certain constraints, which might simply be that the velocity is finite and satisfies continuity, or might be the more demanding constraint that it satisfy the Navier–Stokes equations, what kinematic or topological features can be expected in the velocity distribution in any one realization, such as the accumulation of high derivatives in local regions of the flow (which can be examined in other ways, for example using KT59)? The answer to this question helps define the appropriate flow conditions considered in the previous question.

3. Do other non-turbulent flows also have the same forms of spectra and other statistics predicted for small-scale turbulence (which would require them to have quite a complex structure with a distribution of scales wide enough to have a $(-\frac{5}{3})$ spectrum)?

4. If the spectrum or other statistical measures predicted by K41*a*, *b* occur in other flows, does it mean that small-scale turbulence contains particular flow patterns or structures that are not unique to turbulence? Or, in other words, is some of the structure of the flow field not necessarily dependent on the assumption (*a*) of the universality of the small-scale structure of the turbulence (Frisch 1991)?

5. The previous question can be taken even further to ask whether it is possible for such a turbulent flow to consist of an ensemble of elements with no internal structure (i.e. represented by a small range of length scales, such as a spherical vortex, as suggested by Synge & Lin (1943))? Or, as our question 3 implies, do the governing equations inevitably lead to a turbulent flow consisting of an ensemble of 'eigenstructures', e.g. spiral vortices or vortex sheets (which may not necessarily be independent of each other, and in each of which the local statistical structure of the velocity field is similar to the statistical structure of the whole field)? Calculations of flame in model 'eddies' reviewed by Bray & Cant (1991) demonstrates the practical importance of the answers to this question.

We cannot answer all these questions yet. In the following sections we review some recent and current attempts to do so.

(b) *Measurements and computations of small-scale turbulence*

(i) *Criteria*

The essential criteria for the existence of a state of statistical equilibrium of the small scales of motion are that: (i) there is a very large number of modes that are at least partly independent of each other; (ii) the ratio r_l of the largest scale L_0 to the smallest Kolmogorov length scale $l_K = (\nu^3/\epsilon)^{\frac{1}{4}}$ is very large, i.e.

$$r_l = L_0/l_K \gg 1; \tag{2.1a}$$

(iii) so that the dissipation be confined to the smallest scales, the ratio r_s of the spectrum of the strain rate $(k^2E(k))$ at the smallest scales to the strain rate spectrum at the large scales must be much larger than 1, i.e.

$$r_s = (\nu\epsilon)^{-\frac{1}{3}}\epsilon/(U^2/L_0) \gg 1. \tag{2.1b}$$

Since ϵ is determined by the large scales of motion its order of magnitude is U^3/L_0. Therefore in terms of the Reynolds number of the turbulence, $Re = UL_0/\nu$, the second criterion (2.1*a*) is satisfied if

$$Re^{\frac{3}{4}} \gg 1, \tag{2.2a}$$

whereas the third criterion requires that

$$Re^{\frac{1}{4}} \gg 1. \tag{2.2b}$$

(ii) *Low to moderate Reynolds numbers*

For a minimum separation of the rates of dissipation between small and large scales the ratio r_s should be about 10, which implies (from 2.2) that $Re \approx 10^4$ and $L_0/l_K \approx 10^{\frac{3}{2}}$.

In most laboratory experiments $Re \approx 200$, although in some shear flows the velocity is great enough that $Re \approx 10^4$ (Anselmet *et al.* 1984). In recent grid turbulence experiments conducted in large wind tunnels the value of Re has reached 10^3 (Gagne 1991).

The present upper limit of the value of Re using direct numerical simulation is about 200 (She *et al.* 1991). In some of the experiments where $Re \leqslant 200$ there is a range of wave numbers k typically varying over a factor of 10 (a 'decade' in the jargon) over which the spectrum $E(k)$ has the same form as predicted by K41 for the inertial range. But in these flows the strain spectra do not generally satisfy the criterion (2.1*b*).

Whether or not in a particular turbulent flow a significant 'inertial range' is found, has been shown (largely empirically) to depend on the 'non-ideal' nature of the flow, especially at these relatively low values of Re. Using local statistical properties of the turbulence, normalized with respect to the velocity and length scales of the turbulence u_0, L_0, the flows can be characterized quantitatively in terms of 'non-ideal' factors N, namely the spatial gradients of the kinetic energy of the turbulence,

$$N_{G(u)} = |\nabla u_0^2|/(u_0^2/L_0); \tag{2.3a}$$

or of the integral scale

$$N_{G(L)} = |\nabla L_0|; \tag{2.3b}$$

or the anisotropy of the large scales (Lumley 1978),

$$N_A = II_A = b_{ij}b_{ji} \quad \text{where} \quad b_{ij} = 1 - \overline{u_i u_j}/\tfrac{1}{3}\overline{u_k u_k}; \tag{2.4}$$

or by random body forces per unit mass, f, defined by their variance

$$N_f = f^2/(u_0^4/L_0^2), \tag{2.5}$$

and by their spectrum

$$N_{E(f)} = E_f(k)/k^3E^2(k); \tag{2.6}$$

or by the magnitude of the straining by the mean flow

$$N_S = |\nabla U|/(u_0/L_0), \tag{2.7}$$

and its form as defined by $II_U = (\partial U_i/\partial x_j)(\partial U_j/\partial x_i)/(u_0/L_0)$, which is equal to the proportion of irrotational to rotational straining. Note that in a pure shear flow $II_U = 0$.

For example, in approximately isotropic turbulence, whether produced in wind tunnel flows past grids or numerically, all these 'non-ideal' factors are small. For moderate $Re \leqslant 100$ it is found that there is a narrow range of eddy scales and that the spectra decay rapidly with k (for example, Champagne *et al.* (1970) and Rogallo (1981) found that $E(k) \propto \exp[-(kL_0)^n]$, where $1 < n < 2$). At higher values of Re, as mentioned above, a small 'inertial' range has been observed in more recent laboratory experiments and numerical simulations (She *et al.* 1991).

But other kinds of approximately isotropic turbulence produced in the laboratory have a significantly different spectrum, as is the case of turbulence below an oscillating grid in a tank of static fluid. There is no mean motion, but the turbulence is significantly inhomogeneous so that $N_G \approx 1$. Unlike the more homogeneous wind tunnel or numerical turbulence, in this case there is a significant 'inertial' range over a 'decade' at a value of $Re \approx 100$ (Hannoun *et al.* 1988). A possible explanation is that since turbulence is advected from the grid by the mutual 'Biot–Savart' induction between eddies, this also leads to significant nonlinear interactions between them and therefore to a broader range of scales than occurs in the decay of turbulence behind a grid. Also, an effect of the gradient of the turbulence energy is to cause a skewed probability distribution with intermittent high-velocity eddies moving away from the grid and producing high-energy small-scale motions.

Turbulence in the presence of a mean shear velocity (e.g. U_i ($i = 1, 2, 3$)) is another kind of 'non-ideal' flow, where $N_S \approx 1$. At moderate values of Re in laboratory and numerical experiments, measurements show that the spectrum decays more slowly with k ('algebraically', as opposed to exponentially), typically as

$$E(k) \propto u_0^2 L_0 (k/L_0)^{-n}, \tag{2.8}$$

where $\frac{5}{3} < n < 3$ (Champagne *et al.* 1970; Rogallo 1981; Ho & Huerre 1984). This great change in the form of the spectrum at a same value of Re is mainly caused by the linear distortion of the eddies by the mean flow providing high shear regions at the edges of the elongated 'streaks' or contour lines of locally high or low velocity parallel to the mean flow (Hunt & Carruthers 1990).

These two 'non-ideal' examples have shown how large-scale motions can force the spectra to have a form with a significant distribution of energy over a wide range of length scales. Therefore the large scales can stimulate nonlinear interactions, and thence the generation of a wide range of length scales, even at moderate values of Re. Thus, particular kinds of 'non-ideal' large-scale motion, can paradoxically stimulate the formation by the small scales of a state of local statistical equilibrium!

By contrast the length scales of turbulence in atmospheric and oceanic flows are so large that the values of Re are high enough to satisfy all the criteria for the existence of the small-scale statistical equilibrium, as defined by (2.1) and (2.2) (e.g. for the atmospheric boundary layer, $L_0 \approx 10$–100 m, $u_0 \approx 0.1$–1 m s^{-1}, $\nu \approx 10^{-5}$, so that $Re \approx 10^5$–10^7, while in the upper ocean where $L_0 \approx 1$ m, $u_0 \approx 0.1$ m s^{-1}, $\nu \approx 10^{-6}$, so that $Re \approx 10^5$). For the large scales the 'non-ideal' factors are usually of order unity. So the question is whether the statistics of the small scale motions become universal and independent of the large scales in any particular flow.

In all (sufficiently detailed) measurements of velocity spectra an 'inertial range' has been found where $E(k) = C_K \epsilon^{\frac{2}{3}} k^{-\frac{5}{3}}$ over a number of 'decades' that appears to increase as the magnitudes of the non-ideal factors decrease, for example reaching as much as about five decades in aircraft measurements of tropospheric turbulence, and as Gibson (1991) reports eight decades for galactic turbulence! The values of ϵ have been measured from the strain rates of the smallest scales $\epsilon = \nu(\partial u_i/\partial x_j)^2$, often by using different instruments for these scales, and thence the coefficient C_K could be evaluated. It is remarkable that this coefficient ('Kolmogorov's constant') has been found to be approximately the same (*ca.* 1.5), whether the turbulence is driven by mean shear ($N_S \approx 1$), or by buoyancy forces ($N_G \approx 1, N_f \approx 1$) (Kaimal *et al.* 1973, 1982).

Measurements of the different spectra $E_{11}(k)$, $E_{22}(k)$, $E_{33}(k)$, for the different velocity components in the three directions in the atmospheric boundary layer have confirmed in many different conditions the K41 *a* hypothesis of isotropy of the small-scale motions in the inertial range (see, for example, Kaimal *et al.* 1973; Monin & Yaglom 1975). Usually this is tested by comparing the ratios of the spectra $E_{22}(k_1)/E_{11}(k_1)$ and $E_{33}(k_1)/E_{11}(k_1)$ for wave numbers in the direction of the mean flow (subscript 1) with the theoretical value for isotropy of $\frac{4}{3}$. In these experiments it is found that this ratio differs from the isotropic ratio for about the same wave numbers that $E(k)$ departs from the inertial range form (1.3) (figure 2). (In many of the laboratory turbulence experiments, although the spectra may have an 'inertial subrange', the small-scale turbulence is not necessarily found to be isotropic.)

When these spectra are inspected very closely, it is found that there is a small systematic discrepancy between the measurements and the form of (1.3) predicted in K41 *a*. As explained in this volume by Gibson (1991) and Frisch (1991) the data agree more closely with the spectra predicted by Kolmogorov's 1961 modification to the 1941 theory which allowed for the variability of the rate of dissipation ϵ.

Although in these very high Re turbulent flows, where the large scales are 'non-ideal', the second-order spectra and correlations are consistent with the hypotheses of K41 *a* of small-scale isotropic statistical equilibrium, the higher order statistics may not be so consistent. Some reasons and data are given in §3 (see also Frisch 1991; Sreenivasan 1991).

So far we have discussed measurements of spectra, although the original results of K41 *a* were expressed in terms of correlations. As explained in §1 *b*, a given value of Re essentially determines the range of independent modes in the turbulence and, if Re is high enough, the ratio L_0/l_K of the highest to the lowest wave numbers in the inertial range. Thence by calculating the correlation or structure function $\langle \delta u(l)^2 \rangle$ from the Fourier transform, it can be shown that the asymptotic result (1.1) for the inertial range is only a good approximation if $(L_0/l_K)^{\frac{1}{3}} \gg 1$. In other words, the structure form of K41 *a* is only likely to be found experimentally at even higher values of Re than the spectral form, as shown by Anselmet *et al.* (1984) and Fung *et al.* (1991). But this also raises the question of whether there is another statistical

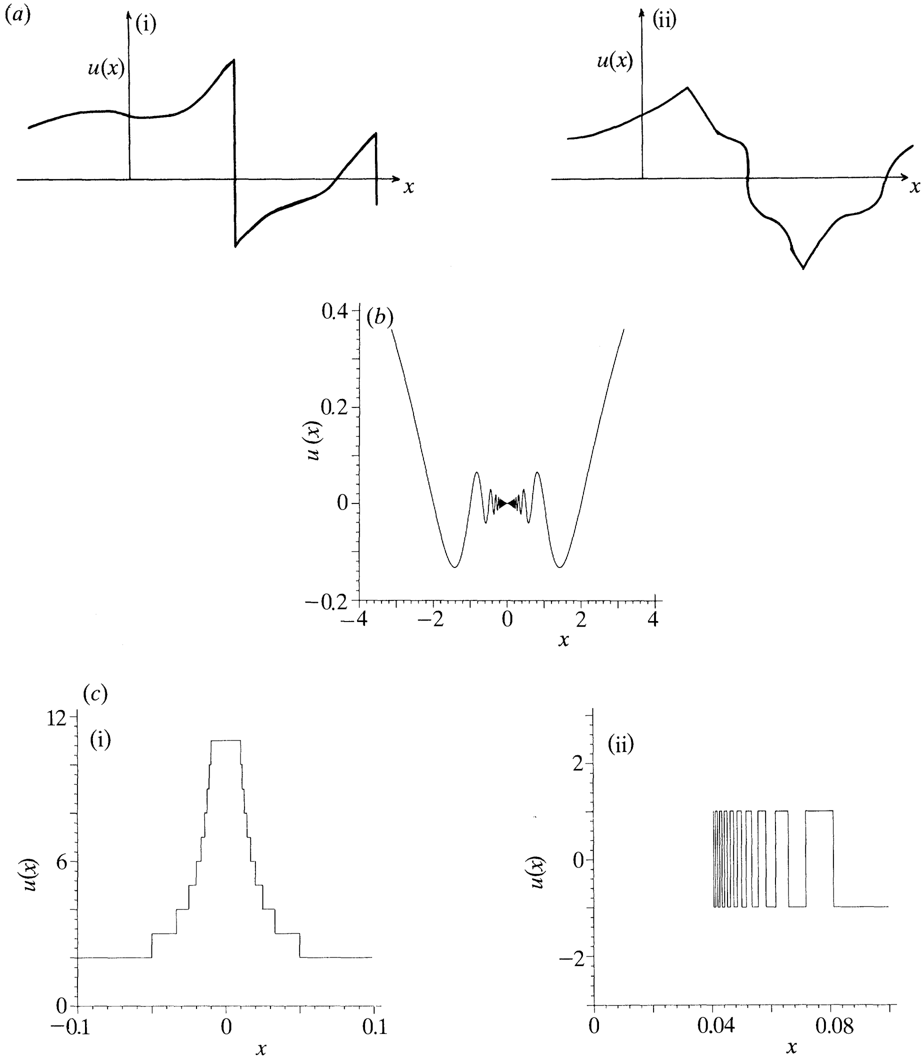

Figure 3. Different kinds of signals whose singularities or accumulations lead to 'power-law' spectra of the form $E(\kappa_1) \propto \kappa_1^{-2p}$ as $\kappa_1 \to \infty$. (*a*) Distinct discontinuities: (i) $p = 1$, $[u(x)] = O(1)$; (ii) $p = 2$, $[u'(x)] = O(1)$. (*b*) Oscillatory accumulation $u(x) = x^{\frac{3}{4}} \cos(1/x)$; ($|x| < 3$); $p = \frac{7}{6}$ (indicative of the velocity of a particle in a vortical flow, where x is the time). (*c*) Singularities and accumulations: (i) $f(x) = f_0 - \Sigma^M H(x - x_m)$; $x_m \propto m^{-\alpha}$. (Note the growth of $f(x)$ towards the point $x = 0$.) (ii) $f(x) = \Sigma^M (-1)^m H(x - x_m)$. (This is analogous to a section of a scalar field.)

measure less demanding than the spectrum which indicates the existence of some universal features of small-scale turbulence?

In this section we have examined only the statistics of the velocity field. Many of the same remarks can be applied to the scalar fields. But the reviews by Gibson (1991) and Sreenivasan (1991) in this volume show that there are experimental situations, even at very high *Re* where, although the velocity spectra follow an inertial form with the constant equal to the general value, the scalar spectra do not follow a general form. Their examples are drawn from laboratory and oceanic experiments, whereas atmospheric boundary layer experiments seem to show a more

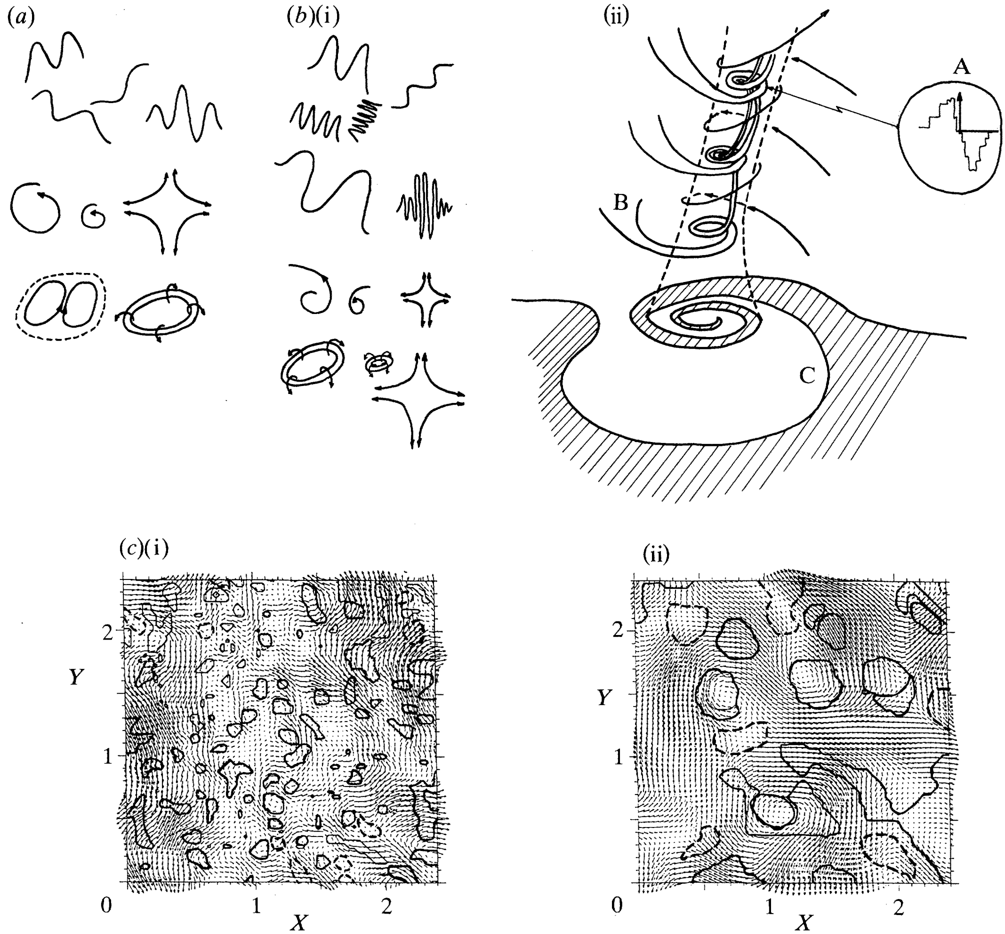

Figure 4. Representations of eddy structure of turbulence and variations with Reynolds number. (*a*) Moderate (*ca.* 10^2) Reynolds number. (*b*) Higher (*ca.* 10^3) Reynolds number: (i) Eddies as sine waves, pulses of sine waves, or simple structures (e.g. vortices, straining flows, vortex rings, which are smaller but remain similar as *Re* increases); (ii) Eddies as complex 'eigen structures' which become *more* complex (i.e. greater range of scales) as *Re* increases. A, complex internal structure; B, sheets of high vorticity entering vortex; C, scalar contours forming spiral shapes (e.g. visualized by light sheet). (*c*) Simulation of three-dimensional flow fields consisting of random uncorrelated Fourier modes, with different spectra: (i) $E(\kappa) \propto \kappa^{-\frac{5}{3}}$; (ii) $E(\kappa) \propto \exp(-\kappa L_0)$, showing how there are different regions where there is intense vorticity (marked with heavy lines) and where there is strong straining. Note how these are smaller and more intermittently distributed for the first case (in a real flow the effect is more marked).

uniform behaviour for the temperature and water vapour spectra, even when the large scales are quite different and the buoyancy forces have different effects (Monin & Yaglom 1975).

(*c*) *Forms of velocity fields consistent with the theory*

It is not always realized by fluid dynamicists that the predictions in K41*a* for the forms of the spectra and correlation functions in the inertial range of length scales necessarily imply that there are mathematical 'singularities' in the velocity field, i.e.

points x_m where the derivatives of the velocity (on the length scales within the inertial range) tend to infinity (i.e. $(u(x)-u(x+h))/h \to \infty$ as $h/L_0 \to 0, h/l_K = O(1)$). In this section we analyse these regions in order to show which kinds of velocity and scalar fields are consistent with Kolmogorov's and Obukhov's theories (which helps answer question (2) of §2*a*).

The spectrum $E(k)$ of a random function $u(x)$, defined in the range $0 < x < X$, is proportional to mean square of the modulus of the complex coefficients, a_n, of the Fourier series of $u(x)$, i.e.

$$E(k) = (X/2\pi)|a_n|^2 \quad \text{where} \quad u(x) = \sum_n a_n \exp(\mathrm{i}k_n x). \tag{2.9}$$

There is a well established, but generally overlooked, property relating the rate of convergence of the coefficients to the degree of singularity of the function. If $u(x)$ and all its derivatives are piecewise continuous, then

$$|a_n| = O(n^{-p}) \quad \text{as} \quad n \to \infty, \tag{2.10a}$$

and

$$E(k) = O(k^{-2p}) \quad \text{as} \quad k/L_0 \to \infty, \tag{2.10b}$$

where $p-1$ is the lowest order derivative that is discontinuous (Thomson & Tait 1879, §77; Courant & Hilbert 1953, p. 74).

Thus if $u(x)$ is discontinuous, as at a vortex sheet, $|a_n| = O(n^{-1})$, and $E(k) \propto k^{-2}$ as $k \to \infty$. (Perhaps first pointed out in the context of turbulence by Batchelor & Townsend (1949)?) A typical function of this kind can be represented as (H is the Heaviside function)

$$u(x) = \sum_m u_m H(x-x_m), \tag{2.11}$$

which is a schematic representation of turbulent velocity in the lateral direction in a class of sheared turbulent flow (Hunt & Carruthers 1990).

The inverse relation is not so straightforward. For example, if $|a_n| = O(n^{-p})$, it does not necessarily mean that $u(x)$ has simple discontinuities of the form of (2.11). In fact $u(x)$ may have a second kind of singularity that is an 'accumulating oscillation'; for example, if

$$u(x) = \sum_m |x-x_m|^s \cos\left(\frac{1}{|x-x_m|^t}\right), \tag{2.12a}$$

then the Fourier coefficients can be evaluated as $n \to \infty$, either using Hardy's formula (see Watson 1958, p. 183), or by the method of stationary phase which leads to

$$p = (2s+t+2)/2(t+1) \tag{2.12b}$$

for $-t < s < 1$ and $t > 0$. In particular, when the exponent of the spectrum is $p = 1$ it does not necessarily follow that the signal is discontinuous (as in a vortex sheet) because (2.12*b*) shows that there may be accumulating oscillations, for example with $t = 1$ and $s = \frac{1}{2}$ (see figure 3). Clearly the function in (2.12*a*) does not satisfy the usual Dirichlet conditions, which as well as requiring piecewise continuity, also require a finite number of maxima and minima; but it does have a Fourier series and a spectrum!

There is a third kind of singularity in functions having these kind of spectra which vary like k^{-2p}, in which the function and its derivatives are a sequence of discontinuities at x_{mr} which 'accumulate' towards the points x_m. For example, if the qth derivative undergoes jumps in proportion to r^{β} at points x_{mr} spaced in a geometrical sequence from x_m defined by $|x_{mr}-x_m| \propto r^{-\alpha}$, then

$$\frac{\mathrm{d}^q u(x)}{\mathrm{d}x^q} = \sum_r \sum_m u_{mr}^{(q)} H(x_{mr}-x_m), \quad \text{where} \quad u_{mr}^{(q)} \propto r^{\beta}. \tag{2.13}$$

This might be the velocity on intersections between a spiral vortex and a line (see figure 4). Deriving the Fourier coefficients for $u(x)$ as $n \to \infty$, again using the method of stationary phase, leads to

$$p = (1+1) - g/\alpha'. \tag{2.14a}$$

If the discontinuities have the same sign

$$\alpha' = \alpha \quad \text{and} \quad g = (1+\beta), \tag{2.14b}$$

but if they are oscillating in sign (i.e. $u_{mr} \propto (-1)^r$),

$$\alpha' = 2(1+\alpha) \quad \text{and} \quad g = (1+2\beta) \tag{2.14c}$$

(see also Moffatt 1984; Gilbert 1988).

Hence if $E(k) \propto k^{-2}$ it is possible for the function to have an accumulation of discontinuities of $u(x)$ that oscillate in sign at points converging harmonically, i.e. $\alpha = 1$ and $q = 0$. Putting $p = 1$ into (2.14a) and using (2.14c) shows that $\beta = -\frac{1}{2}$. This implies that $|u(x)| \propto |x-x_m|^{\frac{1}{2}}$, which is consistent with our calculation for the continuous second kind of singularity. But if the discontinuities of $u(x)$ are of the same sign with the same spacing ($\alpha = 1$), there is an even sharper minimum with $|u(x)| \propto |x-x_m|$.

We now consider the more singular spectra, where $p < 1$, such as predicted by the Kolmogorov–Obukhov theory. In this case $E(k) \propto k^{-2p}$ where $p = \frac{5}{6}$, for $k \gg 1/L_0$, where L_0 is an integral scale. Because p is not an integer, there can be no function $u(x)$ that has this spectrum and has the first (simple) kind of singularity in $u(x)$ or its derivatives. Therefore functions with this spectrum generally have the second or third kind of singularities and accumulation points, unless they are fractal functions, e.g. fractal Weierstrass functions which are also known (see Mandelbrot 1982) to have a spectrum of the type $E(k) \propto k^{-2p}$ with non-integer p. In that case such fractal functions have 'singularities' everywhere, because they are nowhere differentiable even though they are continuous. (If one includes statistics in the definition of the function $u(x)$, it should be mentioned that random gaussian functions were shown by Orey (1970) to have a high wave-number spectrum of that type too; particular realizations of such random functions are similar to Weierstrass functions (see Mandelbrot 1982).)

For the second oscillatory kind, the velocity near the singularity (say at $x = 0$) could have the form

$$u(x) = |x|^s \cos(1/|x|^t), \tag{2.15}$$

where s and t are given by (2.12b). Thus if $p = \frac{5}{6}$, $2s = \frac{2}{3}t - \frac{1}{3}$. So the value of s or t depends on other information about the singularity, either dynamical or kinematical.

The third kind of accumulation of discontinuities have the form given by (2.13). For the velocity spectra on a line where $p = \frac{5}{6}$, it follows that there could be an accumulation of discontinuities (or vortex sheets, $\beta = 0$), of one sign. Substituting into (2.14), with $q = 0$ and $\alpha' = \alpha$, leads to $\alpha = 6$, which means that the sequence of discontinuities converges rapidly onto the singularity. But the velocity increases slowly towards a singular value at the singular point in proportion to $x^{-\frac{1}{6}}$ (we expect viscous processes to dominate very close to the accumulation point itself).

Another dynamical possibility is an accumulation of vorticity (i.e. $\mathrm{d}u/\mathrm{d}x$), which increases by the same amount at each discontinuity, so that $\beta = 0$. Then from (2.14b) $q = 1$, and $\alpha = \frac{6}{7}$. Clearly the latter accumulation is more gradual; from observations it should be possible to detect the difference. Note that in this case also the velocity increases to a singular maximum in proportion to $x^{-\frac{1}{6}}$.

If the motion is essentially two-dimensional near 'spiral' accumulation points, the velocity remains of the same order and has discontinuities that oscillate in sign. Then, if $\beta = 0$, and if $p = \frac{5}{6}$, from (2.14c) it follows that $\alpha = 5$, which is also a rapidly convergent accumulation.

The singularity of a scalar with concentration C does not lead to a maximum or infinite value at $x = 0$. The distortion of C by the velocity field tends to lead to discontinuities of $\mathrm{d}^qC/\mathrm{d}x^q$ (of alternating sign). Hence if $p = \frac{5}{6}$, $q = 1$, and $\alpha' = 2(1+\alpha)$, then $\alpha = 2$. Note that this is a different rate of accumulation to that for the velocity!

This somewhat elaborate analysis has been necessary to emphasize that the one kind of velocity distribution that is not mathematically consistent with a spectrum which decreases with a power law $E(k) \propto k^{-2p}$, is a distribution of isolated 'simple eddies' (i.e. no internal singular structure) with length scale l having velocities $\delta u(l) \propto l^{(p-\frac{1}{2})}$. But it would be possible to have a distribution of such 'eddies', for example with a simple form such as $\exp[-(x/l)^2]$, only if the eddies were spaced in a particular way, so that the total effect of all the eddies would produce the required singularities in the velocity field for consistency with the spectrum. Thus the conventional picture of 'smooth eddies' is only correct if it is suitably qualified in this way, which may be important for the applications of models based on such pictures.

An alternative velocity field that is consistent with these forms of spectra consists of 'eddies' that are not 'smooth' but have some complex internal structure with a local velocity distribution that contains the same order of singularities required by the total velocity field, for example velocity discontinuities for the case of $p = 1$ or oscillatory accumulations if $p < 1$. In this case the velocity may consist of isolated 'eddies', and there is no restriction on their relative spacing (provided there is not an accumulation leading to greater singularities in the velocity field).

These alternative descriptions of the velocity field are sketched in figure 3.

One way to explore these kinematical alternatives is to compare velocity fields constructed in different ways. One method is to superpose a series of random uncorrelated gaussian Fourier solenoidal components having the correct spectrum in the inertial range, for example

$$\boldsymbol{u}(\boldsymbol{x},t) = \sum_{n=1}^{N} \boldsymbol{a}_n \exp(\mathrm{i}\boldsymbol{k}_n \cdot \boldsymbol{x} + \omega_n t), \tag{2.16}$$

where $\boldsymbol{k}_n \cdot \boldsymbol{a}_n = 0$, $|\boldsymbol{a}_n|^2 \propto E(k_n)$, and $\omega_n \propto \epsilon^{\frac{1}{3}} k^{\frac{2}{3}}$ (Kraichnan 1966; Fung *et al.* 1991). A convenient form for the series is that a_n and k_n are proportional to n^{-p} and n^q respectively; that series is a type resembling in form the Weierstrass function (Falconer 1990; see also Richardson (1926) who started his paper by asking 'does the wind possess a velocity' by referring to these functions), which, depending on the phases of a_n and on the relative amplitudes of a_n and k_n, may or may not be differentiable at some or even at all points (in the limit of an infinite inertial range).

In general, for a given value of N there are particular points in the field where the derivatives are very large and points where there are 'accumulations' of maxima and minima.

However, plotting the velocity field shows how the large-scale eddies determines that the overall structure of the flow largely consists of a set of spiral vortices separated by stagnation regions with high local irrotational strain rates (figure 4*c*) (Fung *et al.* 1991). The smaller-scale eddies also produce similar patterns, when the local streamlines are plotted in moving frames of reference (cf. Prandtl 1925). Both large- and small-scale motions are significant in controlling the relative motions of pairs of fluid elements, and thence the movements of interfaces (see §2*e*).

The usual way to explore the velocity structure is to compute solutions of the Navier–Stokes equations (Vincent & Meneguzzi 1991; She *et al.* 1991), which certainly show that the largest derivatives are found in isolated eddies with complex internal structure; in their case they find elongated regions of intense vorticity having longitudinal straining motion, so that the streamlines form spirals. These structures apparently change their form quite slowly.

New experimental methods are also beginning to provide evidence of spiralling streamlines and streaklines within eddies (Hunt *et al.* 1991), and thin layers of high vorticity bunched together (Schwarz 1990), both of which phenomena are consistent with there being accumulation points in the velocity field. The new technique of Wavelet analysis of the fields (see Meyer 1990) may enable the local description of these regions to be made precise.

These are kinematical consequences of the form of the spectra, but the dynamical implications are important, as we show in the next sections.

(*d*) *Dynamical implications of the inertial range spectra*

The theory of K41*a* is based on the premise that the parameter ϵ, which defines the amplitude of the inertial range eddies has the same order of magnitude at least over a region of the flow comparable to L_0 the scale of the large eddies. This leads directly to $E(k) \propto k^{-\frac{5}{3}}$. However, we have just seen in §2*c* that the consequence of this result is that there must be singularities in the derivative of the velocity field (on scales much larger than l_K), where the rate of dissipation ϵ is locally very large (ϵ being defined as $\nu\overline{(\partial u_i/\partial x_j)^2}$.

If the type of non-fractal isolated singularity discussed in §2*c* exists in the solutions of the Navier–Stokes equations, then the new hypothesis of Kolmogorov's later paper (K62) that the dissipation rate is an intermittent process would be implicit in the result of the K41 paper (see Frisch 1991). Perhaps this explains in part why, although considerations of intermittency are not included in K41, the theory describes measured statistics very satisfactorily, as shown in §2*b* and by other papers in this issue.

There have been a number of theoretical and computational studies to investigate whether certain deterministic flows also have the same spectra, because this might explain why some turbulent flows are observed to have certain forms of spectra and correlations that are the same as these in the K41 theory, without the conditions of the theory being satisfied.

One method has been to study the key elements of observed turbulent flow fields in isolation such as vortices in straining flows. Lundgren (1982) derived solutions to the Navier–Stokes equations for a vortex sheet rolling up while being stretched by a large-scale straining motion (figure 4). He found the K41 form of the energy spectrum $E(k) = Ak^{-\frac{5}{3}}$, but the dimensional constant A is not simply related to the average dissipation in the flow.

This is one possible type of accumulation near a singular point, which in this case consists of an accumulation of singularities of the velocity derivative (on a scale greater than l_K). As we showed in §2*c* other kinds of accumulation are also kinematically possible involving discontinuities of higher order, or even none at all, though no dynamical solutions for such cases have yet been found.

Two-dimensional vortices rolling up vortex sheets have been shown by Gilbert (1988) to be solutions of the inviscid Euler equations, which have spectra that are similar to those of two-dimensional turbulence.

There are important physical implications of these kind of solutions.

First, they demonstrate that a distribution of the second, 'complex' kind of eddy is dynamically possible, as well as being kinematically consistent with the spectra for the whole velocity field.

Secondly, they indicate that the regions of high rate of dissipation may well be thin sheets that are convoluted within 'complex' eddies, rather than very small-scale 'simple eddies'. In that case, as the Reynolds number increases, the sheets become thinner so that there can be more turns or convolutions within the eddy, rather than a new generation of smaller eddy being formed (figure 3). This mechanism may be an alternative means for ϵ being proportional to u_0^3/L_0 when $Re \gg 1$.

Thirdly, there are isolated and steady velocity fields with particular external velocity fields; in that sense they are 'eigensolutions' of the equations. The actual velocity field may be a collection of these and other possible 'eigensolutions', but it remains to be seen how closely these correspond to the velocity in actual computations or measurements. It also remains to be understood how these nonlinear 'eigen solutions' evolve and interact with each other.

Fourthly, it is clear that these particular solutions are geometrically similar (e.g. as in a spiral vortex) with respect to the centre of the vortical region. Furthermore, because these are nonlinear solutions and because steady vortices do not exist on rolled up vortex sheets (Moore 1976), other similar 'eddies' could not be superposed or coexist with these 'complex' or 'eigen' eddies. Therefore the streamlines of these particular types of eddy have continuous curvature and do not have a 'fractal' form in Mandelbrot's (1982) sense.

Flow visualization and computer simulation show that most of the eddies in turbulent flows are formed during the interaction between eddies, which are often large vortices. Consequently there have been a number of computational studies of such interactions, for example when line vortices with finite diameter intertwine around each other and then split again into two separate vortices. In this second kind of study of the dynamics of flows with spectra similar to that in the inertial range, it has been found that after such interactions there are several small regions of vorticity, with a wide range of scales, and in some cases the spectrum has been found to approximate the inertial range form, though the dimensional coefficient A has no physical interpretation in this case (see, for example, Kiya 1991; Kida *et al.* 1991). This is another example to show that the inertial range form may arise from particular deterministic flows, that may also be quite frequent in high Reynolds number turbulence, but this point is yet to be proved!

The third kind of dynamical study is to seek for a general theoretical description of the kinds of velocity field that could form such steady or slowly changing 'eigen' eddies. Moffatt (1987), following earlier work by Arnol'd (1974), has shown that there

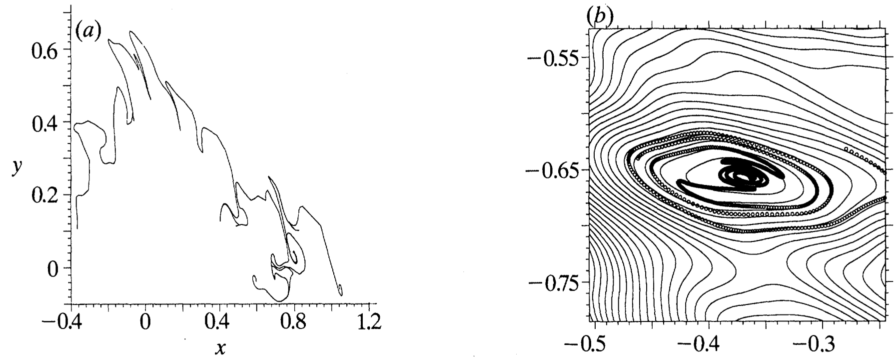

Figure 5. A scalar interface as a two-dimensional velocity field with spectrum $E(k) \propto k^{-2p}$. (*a*) Interface over a large scale, showing how the convolutions are localized. (*b*) Nature of the interface in a vortical or 'eddy' region where $D_k \gtrsim 0.3$. (The computations have run for longer times to expose the detail (Vassilicos 1989).)

are permanent solutions of the Euler equations, and has shown that they may be derived by considering a novel kind of mathematical limiting process. It appears that there is no one type of form that emerges from the limiting process, but a likely velocity field would consist of isolated regions where the velocity is approximately parallel to the vorticity (not unlike strained vortex tubes), but separated by vortex sheets.

(*e*) *Other measures of local eddy structure*

The statistical structure of the small-scale turbulence is determined by local regions where the velocity and scalar quantities have very large derivatives or have rapid variations in their magnitude or that of their derivatives. These are regions surrounding points that are mathematically singular in the different senses described in §2*c*. Recent research techniques, including those deriving from Kolmogorov, are providing new insights into these singular regions, which as was shown in §2*d*, largely determine the small-scale dynamics.

A revealing computational study of a material interface in a two-dimensional solenoidal random field (figure 5) with a spectrum $E(k) \propto k^{-2p}$ shows how the interface rolls up in the vortical regions (where $II_u = (\mathrm{d}u_i/\mathrm{d}x_j)(\mathrm{d}u_j/\mathrm{d}x_i) < 0$), and is stretched out in the straining regions, where $II_u > 0$ (Vassilicos 1989). Clearly figure 5 suggests that the value of p is likely to be determined by the accumulation-like distribution of the small scale fluctuations in C in the vortical regions.

Following Kolmogorov & Tikhomirov (1959) and others, this regions can be analysed more easily than by spectra using the 'box-counting' method to calculate the capacity D_{K} in such individual accumulation region. (The interface is 'covered' with N slices of side l; as l decreases, N increases as $l^{-D_{\mathrm{K}}}$, where $1 < D_{\mathrm{K}} \leqslant 1$.) This fact implies that the scalar C has a *locally* self-similar structure (as for a spiral, see Dupain *et al.* 1983). It can be argued that the average value of D_{K} within these vortical regions is equal to the average value over the whole flow field. Note that outside these regions $D_{\mathrm{K}} = 1$.

In many other simulations and in many experiments it has also been found that D_{K} and D'_{K} can be measured with good statistical repeatability, where the spectra either could not be measured with the same repeatability or did not have the self-similar form of 'power-law' spectra (see Sreenivasan 1991). The different forms of capacity or fractal dimension that have been used, for example by covering a line on a plane with square boxes, or a surface in three dimensions with cubes, may demonstrate different aspects of the self-similar structure (see Jimenez & Martel 1991; Sreenivasan & Meneveau 1986). This is consistent with the fact that there is no general simple relation between D_K and D'_K, as there is for Hausdorff dimensions ($D_H = D'_H + 1$, see Falconer 1985).

If a random variable has a 'power-law' spectrum with exponent $-2p$ and a well defined value of capacity D'_K, since both indicate that the variable has a self-similar structure, geometrical scaling would suggest a relation between these two measures. Such relations have been derived for particular classes of variable (Orey 1970; Vassilicos & Hunt 1991); for the case of a general random interface across which C changes from 0 to 1, with spectrum exponent $2p_C$ (i.e. $E_C \approx k^{-2p_C}$), a recent theoretical analysis, confirmed by numerical simulation, shows that

$$2p_C = 2 - D'_K \tag{2.17}$$

(Vassilicos & Hunt 1991; see also Kingdon 1987). Note that there is no general extension for higher dimensions of capacity (for D_K). This result shows how as p_C decreases, which (as §2*c* showed) leads to a more discontinuous or wiggly distribution, D'_K increases, which implies that a larger number of small boxes are needed to cover the convoluted interface.

In the simulations of Vassilicos (1989), the exponent p describing the spectrum of the velocity was varied. It was found that as p decreases, D_K increases, which from (2.17) implies that p_C also decreases. Since the vortical regions become more singular as p increases, our analysis of such regions shows why p_C also increases.

So if these two measures of a multiscale random variable are equivalent (at least in this case), why is it necessary to use any other variable than the spectra or correlation function? The important experimental reason is that D_K can be measured reliably and even locally, when p or p_C cannot. This can be explained using simple geometrical reasoning, by considering a function of the form of (2.13) near an accumulation point. The Fourier coefficient a_n corresponding to a wave number, say k_n, is dominated by the integrations over a large number of those oscillations of the function which have about the same wavelength as $2\pi/k_n$. Consequently the high wave-number spectrum of the signal can only reflect the self-similar structure of the function near the accumulation point if the number of oscillations of the function is much greater than *ca.* L_0/k_n. (Experimentally this implies that the Reynolds number must be very large.) By contrast, the capacity is defined simply by the rate at which the distance between oscillations of the function or the interface changes. (The detailed analysis for a spiral interface is given by Vassilicos & Hunt (1991), but one impressive fact is that D_K can be defined by a spiral consisting of only 2 turns, whereas the spectrum requires more than 50 turns!)

Thus these studies of interfaces in artificial and real turbulent flows suggest that if small-scale turbulence is investigated using measurements of capacity rather than spectra or correlations, then the self-similar structure will be detected at much lower values of the Reynolds number. Therefore the criteria in §2*b* may only be necessary for the existence of the inertial range spectra, whereas some significant aspects of the universal structure of small-scale turbulence may be apparent at much lower Reynolds number using other measures such as capacity. This would seem to explain the experimental findings that in many turbulent flows (such as wakes, jets, boundary layers and even clouds) approximately the same values of the capacities

D'_K and D_K are found, even when the spectra do not have an inertial range form, and also when the conditions for such a spectrum are not satisfied (Lovejoy 1982; Sreenivasan & Menevau 1986; Sreenivasan 1991).

3. Extensions and applications of Kolmogorov's hypotheses

(a) *Local statistics or interactions with large scales?*

A central idea of K41 is that the small-scale variations of quantities, such as velocity or temperature, in a nonlinear multi-dimensional system are on average determined by a dimensional parameter that characterizes the interactions between processes on different scales within a limited range. In the case of the velocity fluctuations in turbulence over the inertial range of scales, this parameter is ϵ, and for temperature it is $\epsilon_C/\epsilon^{\frac{1}{3}}$, where ϵ and ϵ_C are the rates of dissipation of velocity and concentration fluctuations.

When expressed in these general terms, it is not surprising that this scaling concept of K41 can be applied to other systems. A notable example in fluid mechanics is the derivation of the frequency spectra $E(\omega)$ of the height, h, of surface gravity waves; when the waves are in a state of statistical equilibrium, the only dimensional parameter for the whole wave system over a certain range of scales is gravitational acceleration so that $E(\omega) \propto g^2\omega^{-3}$ (Phillips 1966). Similar analyses have been applied to internal waves in stratified flows.

As we have seen, when the above idea is combined with the assumption that the small scales are isotropic and universal, certain statistics can be derived, such as second-order moments, which do describe some aspects of small-scale turbulence accurately and universally. In the next section we discuss how these results can be extended to include time dependent and lagrangian statistics.

Other statistics of the small scales may not satisfy local isotropy and universality, and are sensitive to external constraints. But their structure may still be largely determined by these small-scale dynamical interactions; in some cases, as we show in §3*c*, the response to external distortions and body forces has a universal form or leads to general approximate computational models, even if the resulting statistics are not universal!

Probably the most widespread practical application of K41 has been for calculating the effects of turbulence on various processes that may or may not interfere significantly with the flow. Kolmogorov (1949) himself contributed a paper on the break-up of drops in a turbulent flow, using the results of his own theory. This problem has continued to be actively studied, because of its importance in many chemical engineering processes as well as for environmental flows (see, for example, Batchelor 1979). There is only space here to list a few other applications: sound production, transmission of light and other radiation, mixing and reactions between species including combustion, fluctuating forces and pressures on surfaces, and many more. It would be interesting to see a detailed review of this wide and increasing range of application.

(b) *Timescales and lagrangian and eulerian frequency spectra*

In K41 there is an implication that the turbulence changes slowly on the natural timescale $\delta u(l)$ of the small eddies of length l. In that case there are two possible timescales $\tau(l)$ for the change of velocity of a fluid element moving with the local flow;

either $\tau(l) \sim l/\delta u(l)$ as the element moves a distance l 'inside' a simple 'eddy', or $\tau(l) \sim l/u_0$ because the particle is advected around the outside of a local eddy of length l, such as a vortex tube (see figure 3), with a velocity characteristic of the larger scales which is typically found outside the small eddies.

If the 'inside' trajectory model is the most significant, it follows that the lagrangian spectrum for fluid elements has the form

$$E_{\mathrm{L}}(\omega) = C_{\mathrm{L}}\,\epsilon\omega^{-2}. \tag{3.1}$$

If the 'outside' trajectory is more significant,

$$E_{\mathrm{L}}(\omega) \propto \epsilon^{\frac{2}{3}} u_0^{\frac{2}{3}}\, \omega^{-\frac{5}{3}}. \tag{3.2}$$

Comparing these two formulae it is noticeable that the former has a universal form, whereas the latter is dependent on the nature of the large-scale motions.

However, most subsequent models, following Batchelor (1947), postulate that the scaling analysis of K41 can be extended to show that the small scale motions are unsteady on the same 'inside' timescale $\tau(l) \approx l/\delta u(l)$, which implies that the velocities of fluid elements also tend to change mainly on this timescale.

There is some experimental evidence by Hanna (1980) that in high Reynolds atmospheric turbulence the lagrangian spectrum agrees with the universal form of (3.1), first suggested by Inoue (1951). The coefficient was found to be $C_{\mathrm{L}} \approx 0.6 \pm 0.3$. In a direct numerical simulation (where $Re \approx 100$), Yeung & Pope (1989) found $C_{\mathrm{L}} \approx 0.64$. In a recent simulation of turbulence as a set of random Fourier modes (see equation (2.16)), it was found by Fung *et al.* (1991*b*) that, if the small scales of the velocity field are changing on their local timescale, the lagrangian spectra agree with (3.1) (and the coefficient $C_{\mathrm{L}} = 0.8$). But if the small scales are 'frozen' as they are advected by the large scales, then the lagrangian spectrum is given by (3.2). Therefore, for these and subsequent reasons, it seems that the different length scales of turbulence are evolving on different timescales. It appears that despite these different timescales certain lagrangian quantities have surprisingly long lifetimes in turbulent flows (such as the amplitude of the strain rate on a fluid element (Pope 1990), or the relative velocity of pairs of fluid elements (Fung *et al.* 1991*b*)).

Since the eulerian high-frequency spectrum is determined by the advection of small-scale eddies (rather than fluid elements) by large-scale motions, its form is the same as that of (3.2) (see Tennekes (1975) and random mode simulations by Fung *et al.* (1991*a*)).

(*c*) *Interactions between small scales and large-scale dynamical effects*

When an external effect (such as a mean shear $\mathrm{d}U/\mathrm{d}z$, or a body force f per unit mass, or anisotropic and non-gaussian large scale eddies with velocity w_0 and length L_0) is applied to a turbuent flow at high Reynolds number, its influence on the velocity scales $\delta u(l)$ of length scale l depends on whether it leads to inertial or body forces that are of order $\delta u(l)^2/l$. In most cases the length and timescales of the external effect are large compared with those of the smallest eddy scales; consequently over sufficiently small scales it may only induce a perturbation to the isotropic small-scale turbulence, but at some larger critical length scale $(k_*/2\pi)^{-1}$ and timescale ω^{-1}, the external effect becomes dominant.

We shall mainly consider the case where the turbulence is homogeneous over the scales of interest, and then these mechanisms are reasonably well described by a number of models of turbulence spectra that account for transfer of energy between different scales (e.g. the EDQNM model reviewed by Lesieur (1987) or the RNG model (Frisch & Orszag 1989)). On the other hand it is also possible to explain and estimate these effects of external influences on isotropic small-scale turbulence quite simply by considering the linear responses acting over periods equal to the relaxation times $\tau_R(k)$ for each length scale k^{-1}, i.e.

$$\tau_R(k) \sim \left(\int k^2 E(k)\, dk \right)^{-\frac{1}{2}} \sim \epsilon^{-\frac{1}{3}} k^{-\frac{2}{3}} \tag{3.3}$$

in the inertial range. An example of the use of this approach was given by Townsend (1976, p. 100), who estimated the rate of transfer of energy from large to small scales.

When turbulence is generated by a mean shear flow dU/dz, at sufficiently high Reynolds number and at small length scales k^{-1} the fluctuating strain rate is of order $\epsilon^{\frac{1}{3}} k^{\frac{2}{3}}$ and is much greater than the strain rate for those scales smaller than the critical length scale

$$(k_*/2\pi)^{-1} \sim \epsilon^{\frac{1}{2}} (dU/dz)^{-\frac{3}{2}}. \tag{3.4}$$

Nevertheless, even when the motions have length scales smaller than this critical size, they are still under the influence of the mean shear flow. The degree to which this makes the turbulence slightly anisotropic can be estimated using the linear theory for the (rapid) distortion of the isotropic small-scale turbulence by the shear; for example the co-spectrum $E_{13}(k_1)$ of the Reynolds shear stress is simply related to the spectrum $E_{33}(k_1)$ of the vertical turbulence by

$$E_{13}(k_1) = (2/5)\, \tau_R(k_1)(dU/dz)\, E_{33}(k_1) \sim (dU/dz)\, \epsilon^{\frac{1}{3}} k_1^{-\frac{7}{3}} \tag{3.5}$$

for k within the inertial range (Wyngaard & Cote 1972; Bertoglio 1986). Note that the same approach shows that the shear produces a small decrease in $E_{33}(k_1)$ of order $(dU/dz)^2 k_1^{-3}$, and a similar order of increase in E_{11}. These results, which are consistent with the spectra of atmospheric turbulence measured by Kaimal *et al.* (1973), show that the differences between the turbulence in shear flows and the ideal turbulence predicted in K41*a* vanishes as $k_1/l_* \to \infty$ (Derbyshire & Hunt 1991). Note also that the form of the difference can be derived as a perturbation from the universal structure.

This concept of a turbulent flow having a relaxation time depending on the wave number has been applied to the practical problems of assessing the distances over which the different parts of the spectra adjust in turbulent boundary layers which flow over changes in surface roughness and elevation (Panofsky *et al.* 1982).

Body forces f can also distort the small scale spectra in a way that depends on the relation between f and the velocity u. For example in turbulent flows with a large-scale rotation Ω, because of the Coriolis effect, $f \propto \Omega |u|$ (see Ibbetson & Tritton 1975), and in turbulent flows of liquids with electrical conductivity σ in the presence of a magnetic flux density B_0, $f \propto \sigma B_0^2 |u|$ (at low magnetic Reynolds number). Thus, in these two cases, $f \propto |u|/T_B$, where T_B is the timescale of the body force, which is independent of the turbulence. However, for turbulence in a stably stratified flow, where the buoyancy frequency is N, $f \propto (t/T_B^2) \int u\, dt$, where $T_B = N^{-1}$. The integral implies that f is somewhat more dependent on the larger scales than in the other two types of body force.

If the Reynolds number is large enough (as defined by 2.2*b*) that the strain rates in the inertial range are much greater than those of the energy containing eddies, we can use the scaling analysis to estimate the effects of body forces on the smaller scale motions. Given this condition, then for these and many other types of body force the structure of the turbulence is only affected for length and timescales larger than

$$(k_*/2\pi)^{-1} \approx \epsilon^{-\frac{1}{2}} T_{\mathrm{B}}^{-\frac{3}{2}} \quad \text{and} \quad w_*^{-1} \approx T_{\mathrm{B}} \tag{3.6}$$

respectively. Good evidence for this argument comes from atmospheric and oceanic measurements, reviewed in this issue, where there are significant body forces present and yet the small-scale second-order spectra are isotropic and approximately agree with the predictions of K41*a*. We shall see that this does not necessarily imply agreement with the predictions for the higher order spectra. (Note also that in lower Reynolds number experiments, body forces affect the whole spectrum, because the small-scale strain rates are too low (Derbyshire & Hunt 1991; Van Atta 1991).)

A different kind of 'non-ideal' behaviour on the small scales occurs when there is no mean shear, but the random large scale motions, with velocity u are highly non-gaussian, for example when they are skewed as in turbulent convection. Applying the same linear distortion theory as for the shear flow, but replacing the shear by the random large-scale strain $\partial U_i/\partial x_j$, enables the 'skewness' co-spectrum $E_{33,3}(k_1)$ to be expressed in terms of ϵ, k_1 and a dimensionless parameter A indicating the skewness of the large scales, as

$$E_{33,3}(k_1) = A\epsilon k_1^{-2}. \tag{3.7}$$

(This mechanism can also be discussed in terms of energy transfer between triads of two high and one low wave numbers (see, for example, Brasseur & Corrsin 1987; Domarodski & Rogallo 1990).)

This form of spectra was measured in the convective boundary layer by Hunt *et al.* (1988), at the same time as the usual second moment spectra $E_{33}(k_1)$, which had the usual isotropic, inertial range form of K41*a*. The third moment spectrum (3.3) was found to persist in the same inertial range of wave numbers over two 'decades' at 22 m and 50 m, and over four 'decades' at 250 m. Over this range of heights, the lengthscale of the turbulence, the variances and the anisotropy changed, but the skewness S_3 of the vertical component remained at about 0.4, and also the value of ϵ did not change appreciably. The fact that the coefficient A did not change supports the physical analysis leading to (3.7). (Note that provided the large scales gaussian even if they are anisotropic, then $E_{33,3} \approx 0$ as in the non-neutral boundary layer turbulence measured by Van Atta & Chen (1970).)

A striking implication of (3.7) is that the small scales in the inertial range contribute as much to the third as to the second moments of the turbulence, and that the skewness of the small scales are also significant, i.e.

$$S(k_1) = E_{33,3}/E_{33}^{\frac{3}{2}} = O(1). \tag{3.8}$$

By contrast in isotropic flows $S_i(k_1) = 0$ for any value of $i = 1, 2, 3$, even though some third moments may be non-zero, as predicted by K41*b* and demonstrated experimentally by Anselmet *et al.* (1984).

Thus even when the second moments are isotropic and their strain rates are large, if the large scales are significantly non-gaussian, then by the small scales are also non-gaussian; physical reasons and experiments suggest that there may be general models or even formulae which describe the non-gaussian higher moments of the smaller scales (as in (3.8)). In such situations the specific predictions of K41*b* are not valid (Van Atta 1991).

The scaling analysis for small-scale turbulence that has been developed in many different ways since K41*a* can also be applied to inhomogeneous turbulence, for example in regions near rigid or flexible surfaces moving with the mean flow. These 'shear free turbulent boundary layers' occur at free surfaces of liquid flows, or at the ground and upper inversion layers in thermal convection. In all these flows the rate of dissipation ϵ does not vary significantly with distance from the surface. Here the only scale is the distance z to the surface, and the critical wave number $k_* = 1/z$. Therefore the spectrum for scales smaller than z are not significantly affected, but for those of the order or larger than z the vertical velocity is 'blocked' by the surface. Detailed analysis (Hunt & Graham 1978; Hunt 1984), using the spectrum of K41a for the turbulence far from the surface, showed that the variance of the normal (or vertical) velocity varies in proportion to $\epsilon^{\frac{2}{3}}z^{\frac{2}{3}}$. This result has shown how quite different inhomogeneous layers all have much the same structure. This analysis has been extended to the flow near density interfaces where the turbulence is inhomogeneous and generates internal wave motions (Carruthers & Hunt 1986). These examples have shown how increasingly complex turbulent flows can be treated as perturbations to the general structure proposed by Kolmogorov in 1941.

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