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Anomalous diffusion of isolated flow singularities and of fractal or spiral structures

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Structures with isolated singularities and structures with nonisolated singularities (fractal structures) arise in many different physical situations such as in vertical and turbulent flows, chaotic mixing, clouds, shear layers, lightning, snowflakes, and flames, to name but a few. Whereas the mathematical description of the instantaneous geometry of many fractal structures has been widely explored, less attention has been devoted to the evolution of such structures under the action of physical fluxes. Do space-filling properties, for example, affect the way in which physical fluxes act on the structure? Here we consider the effects of molecular diffusion on scalar fields with singularities that are in the field itself rather than in the field's environment. These singularities may be simple power-law singularities in which case the stronger the singularity the faster its early decay by diffusive attrition; or may be fractal or isolated spiral singularities, in which case early diffusive decay is accelerated by the space-filling properties of fractal or spiral structures, and the diffusive length scale is a simple function of the geometry and structure of the field.

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Let us first consider a one-dimensional on-off scalar field $\theta_0(x)$ characterized by sharp discontinuities on a set of points x that has fractal properties as in Fig. 1. In Fig. 1(a) this set of points is spiral and in Fig. 1(b) this set of points is fractal. The spiral set of points of discontinuity has only one accumulation point, whereas the fractal set of points of discontinuity has accumulation points arbitrarily close to any point of the set. The spiral singularity is an isolated singularity and the fractal set is a set on nonisolated singularities. Both fractal and spiral singularities may be characterized by a well-defined nonintegral fractal dimension D which measures the degree to which the structure is space filling. This fractal dimension is the Kolmogorov capacity [1] of the set of points of discontinuity of the scalar field $\theta_0(x)$ and is defined by covering all the points in the set with the minimum number of segments of length l. If the number $N(l)$ of such segments has a well-defined power-law dependence on l, that is

$$N(l)=N(L)\left(\frac{l}{L}\right)^{-D}, \tag{1}$$

in a range of length scales $l \ll L$ where L is some large scale defining the overall spatial extent of the fractal or spiral structure (Fig. 1), then the exponent D is the Kolmogorov capacity of the set of points of discontinuity on the structure and $0 \leqslant D \leqslant 1$. The structure is not space filling when $D=0$ and is totally space filling when $D=1$. It has been shown [1] that when this fractal dimension D is well defined, the Fourier power spectrum $\Gamma_0(k)$ of the one-dimensional on-off field $\theta_0(x)$ is such that

$$\Gamma_0(k) \sim k^{-2p} \tag{2}$$

in the limit of large wave numbers k, and

$$2p=2-D. \tag{3}$$

For simplicity, we assume here that $\Gamma_0(k)=C(kL)^{-2p}$ for $kL>1$ and $\Gamma_0(k)=0$ for $kL \leqslant 1$, where C is a constant. If such a fractal or spiral on-off scalar field $\theta_0(x)$ is somehow suddenly subjected to molecular diffusion with molecular diffusivity κ, then the initial field $\theta_0(x)$ [with initial power spectrum $\Gamma_0(k)$ at time $t=0$] will become $\theta(x,t)$ at time $t \geqslant 0$ according to

$$\frac{\partial}{\partial t}\theta(x,t)=\kappa\frac{\partial^2}{\partial x^2}\theta(x,t) \tag{4}$$

and $\theta(x,0)=\theta_0(x)$. Equation (4) is the well-known diffusion equation that is traditionally solved by means of the Fourier transform $\hat{\theta}(k,t)=(1/\sqrt{2\pi})\int dx\ \theta(x,t)e^{ikx}$. Defining the variance (or "average energy") $\overline{\theta^2}(t) =(1/L)\int \frac{1}{2}|\theta(x,t)|^2 dx$, it follows from Plancherel's identity

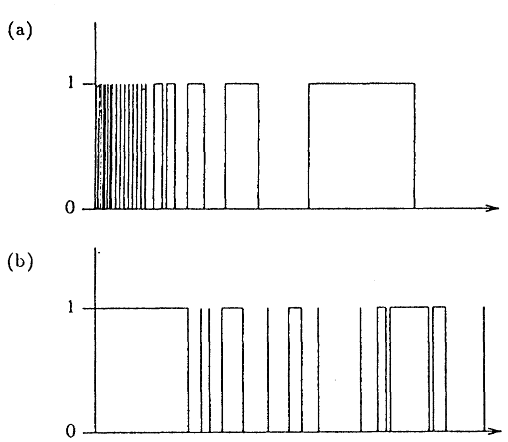

FIG. 1. (a) One-dimensional spiral on-off function $\theta_0(x)$ obtained, for example, by cutting through the two-dimensional spiral on-off function of Fig. 4. (b) One-dimensional fractal on-off function $\theta_0(x)$ obtained, for example, from the zero crossings of the Brownian sample function in Fig. 3.

that $\overline{\theta^2}(t)=\int\Gamma(k,t)dk$ where $\Gamma(k,t)=(1/2L)|\hat{\theta}(k,t)|^2$ and $\Gamma(k,0)=\Gamma_0(k)$. Equation (4) implies that

$$\frac{d}{dt}\overline{\theta^2}(t)=-2\kappa\int dk\, k^2\Gamma(k,t) \tag{5}$$

and

$$\Gamma(k,t)=\Gamma_0(k)e^{-2\kappa k^2 t}. \tag{6}$$

Note that $C=(1-D)L\overline{\theta^2}(0)$ if $D<1$, and Eqs. (5) and (6) lead to

$$\frac{d}{dt}\overline{\theta^2}(t)=-2\frac{\kappa}{L^2}(1-D)\overline{\theta^2}(0)\times\left(\frac{\kappa t}{L^2}\right)^{-(1+D)/2}\int_{\sqrt{\kappa t}/L}^{+\infty}dy\, y^D e^{-2y^2}, \tag{7}$$

where we have made use of the change of variables $y=k\sqrt{\kappa t}$. Hence, at the early stages of decay when $\kappa t/L^2 \ll 1$, the variance (or average energy) of a structure that is characterized by a fractal dimension D and is either fractal or spiral decays according to

$$\frac{\overline{\theta^2}(0)-\overline{\theta^2}(t)}{\overline{\theta^2}(0)}=\Gamma\left(\frac{D+1}{2}\right)\left(\frac{2\kappa t}{L^2}\right)^{(1-D)/2}+O\left(\frac{\kappa t}{L^2}\right) \tag{8}$$

where Γ is Euler's gamma function. The classical $t^{1/2}$ result is recovered when $D=0$, that is when the structure is either not fractal, not spiral, or spiral with $D=0$ (such as logarithmic spirals [1], for example). The closer D is to 1 the more space filling is the structure of sharp gradients. Even though the space-filling properties of spirals and fractals are qualitatively different (Fig. 2) they can be quantitatively measured by the same fractal dimension (or Kolmogorov capacity) D. From (8) we conclude that the more space filling the structure of singularities, the faster the early decay of the structure's average energy.

The integral or correlation length scale $\mathcal{L}(t)$ is a measure of the distance over which the function $\theta(x,t)$ is significantly correlated with itself. $\mathcal{L}(t)$ can be computed from the power spectrum as a weighted average in the following way [2]:

$$\mathcal{L}(t)=\int_0^\infty k^{-1}\frac{\Gamma(k,t)}{\overline{\theta^2}(t)}dk. \tag{9}$$

Firstly, note that the initial integral length scale $\mathcal{L}(0)=[(1-D)/(2-D)]L$ and tends to 0 as $D\to 1$. Fractal and spiral structures that are more space filling are autocorrelated over a smaller spatial extent. Secondly, inserting expressions (6) and (8) in Eq. (9) it turns out that

$$\mathcal{L}(t)=\mathcal{L}(0)\left[1+\Gamma\left(\frac{D+1}{2}\right)\left(\frac{2\kappa t}{L^2}\right)^{(1-D)/2}+O\left(\left(\frac{\kappa t}{L^2}\right)^{1-D}\right)\right] \tag{10}$$

in the limit of early times when $\kappa t/L^2 \ll 1$. It should not be surprising that $\mathcal{L}(t)$ is an increasing function of time because molecular diffusion smoothes out sharp gradients and is thereby a correlating mechanism. The excess length scale $\delta(t)=\mathcal{L}(t)-\mathcal{L}(0)$ is therefore a measure of the distance over which the effects of molecular diffusion are appreciable on the structure at time t. At the early stages of decay,

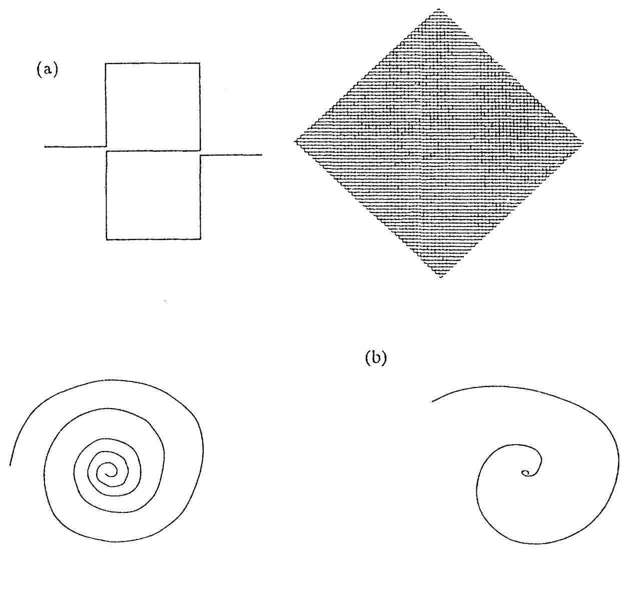

FIG. 2. (a) The Peano curve is a totally space-filling fractal curve. Its generating pattern has 9 straight line segments of length $\frac{1}{3}$ each. Subsequent iterations are obtained by replacing each straight line segment by the original generating pattern rescaled to be smaller by a $\frac{1}{3}$ factor. It becomes clear after a few iterations that the Peano curve comes arbitrarily close to any point inside a square of sidelength $\frac{1}{2}$ and diagonal length 1. The fractal dimension D of the Peano curve's intersection with a straight line is $D=1$. (b) A very space-filling spiral for which D is close to 1 (left) and a spiral that is much less space filling for which D is close to 0 (right).

$$\delta(t)\sim\mathcal{L}(0)\left(\frac{2\kappa t}{L^2}\right)^{(1-D)/2}=\frac{1-D}{2-D}\sqrt{2\kappa t}\left(\frac{\sqrt{2\kappa t}}{L}\right)^{-D}. \tag{11}$$

Molecular diffusion takes less time to smooth out sharp gradients that are packed together in a fractal or spiral way, than it takes to smooth out isolated discontinuities. In fact, the more space-filling the packing of discontinuities, the longer the spatial extent $\delta(t)$ over which diffusive attrition has rubbed off the fractal or spiral structure at an early time t. The classical result $\delta(t)\sim\sqrt{\kappa t}$ is recovered when $D=0$, and only when $D=0$ is the diffusive length scale $\delta(t)$ independent of the initial condition L.

Dimensional analysis leads to $\delta(t)=\delta(t,\kappa,L)=\sqrt{\kappa t}f(\sqrt{\kappa t}/L)$ but the function f cannot be determined from dimensional arguments. However, Eqs. (1) and (11) imply that $f(\sqrt{\kappa t}/L)\sim(\sqrt{2\kappa t}/L)^{-D}=N(\sqrt{\kappa t})/N(L)$ when $\sqrt{\kappa t}/L\ll 1$. In this particular context, fractal and spiral geometries with $D\neq 0$ correspond to Barenblatt's self-similarity of the second kind [3] for which the dependence on L is not lost in the limit where $L\to\infty$. Barenblatt's self-similarity of the first kind corresponds to $D=0$. The dependence of f and

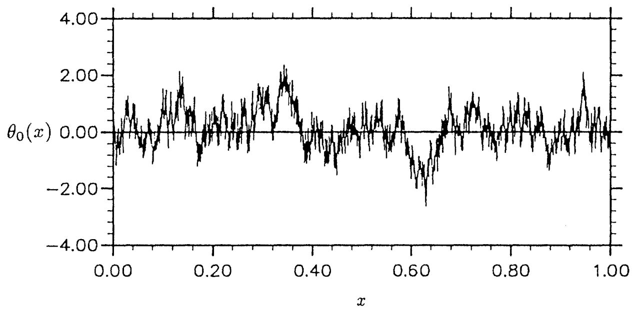

FIG. 3. A realization of a fractional Brownian motion.

$\delta(t)$ on the initial condition L when $D\neq 0$ comes from the fact that the ratio of the number of segments $N(\sqrt{\kappa t})/N(L)$ is multiplied by 2^D if L is replaced by $2L$ when $D\neq 0$, whereas $N(\sqrt{\kappa t})/N(L)$ is invariant under such a dilation when $D=0$.

$N(\sqrt{\kappa t})$ is the minimum number of segments of length $\sqrt{\kappa t}$ needed to cover the points of discontinuity where the effects of diffusion are mostly felt. Equations (1) and (11) imply

$$\delta(t)\sim\sqrt{\kappa t}N(\sqrt{\kappa t}) \tag{12}$$

for the earliest times of decay, which means that the diffusive length scale $\delta(t)$ is proportional to the total length of the fractal covering by segments of size $\sqrt{\kappa t}$. The closer D is to 1, the more space-filling the covering by segments of size $\sqrt{\kappa t}$, the larger the total diffused length $\sqrt{\kappa t}N(\sqrt{\kappa t})$, and therefore the faster the early decay by diffusive attrition.

Let us replace the initial field $\theta_0(x)$ with a realization of a fractional Brownian motion [4] (FBM) (Fig. 3). The high wave number power spectrum of FBM is given by (2) with

$$2p=3-2D, \tag{13}$$

where D is the fractal dimension of the zero crossings of $\theta_0(x)$. If we assume, as we did before for the sake of simplicity, that $\Gamma_0(k)=0$ for $kL\leqslant 1$ and $\Gamma_0(k)\sim(kL)^{-2p}$ for $kL>1$, then the above calculations can be reproduced to yield

$$\delta(t)\sim\frac{2-2D}{3-2D}L\left(\frac{2\kappa t}{L^2}\right)^{1-D}. \tag{14}$$

One way in which a realization of a FBM $\theta_0(x)$ (Fig. 3) differs from an on-off function (Fig. 1) is that [4] $\langle[\theta_0(x+r)-\theta_0(x)]^2\rangle\sim r^{2H}$ where the brackets $\langle\cdots\rangle$ signify an average over x and $H=1-D$. Hence, in the limit $\sqrt{\kappa t}\ll L$,

$$\delta(t)\sim\sqrt{\kappa t}N(\sqrt{\kappa t})\left(\frac{\sqrt{\kappa t}}{L}\right)^{H}. \tag{15}$$

It is instructive to compare Eqs. (12) and (15). The diffusive length scale $\delta(t)$ of a diffusing FBM realization is proportional to the total length of the fractal covering of zero crossings by segments of size $\sqrt{\kappa t}$, but is also proportional to the rms $\langle[\theta_0(x+\sqrt{\kappa t})-\theta_0(x)]^2\rangle^{1/2}\sim(\sqrt{\kappa t})^H$. The two effects multiply each other and as a result the early diffusive decay of a FBM realization is much faster than that of a spiral or a fractal on-off function. The closer D is to 1, the more space-filling is the covering of zero crossings by segments of size $\sqrt{\kappa t}$ *and* the smaller is the exponent $H=1-D$ so that the early decay by diffusive attrition is faster on *both* accounts.

The initial field with the simplest singularity would be the power-law singularity $\theta_0(x)\sim(x/L)^{-q}$ where $x\leqslant L$ (Fig. 4), setting $\theta_0(x)=0$ where $x>L$ and $2q<1$ for the initial variance $\overline{\theta^2}(0)$ to be finite. The initial high wave number power spectrum $\Gamma_0(k)$ may be defined if $q>0$ and is of the form (2) with $p=1-q$. By the exact same method as above we obtain for $\sqrt{\kappa t}\ll L$ that

$$\delta(t)\sim L\left(\frac{\kappa t}{L^2}\right)^{(1-2q)/2} \tag{16}$$

which we reinterpret as follows:

$$\delta(t)\sim\sqrt{\kappa t}\,\theta_0(\sqrt{\kappa t}). \tag{17}$$

The stronger the power-law singularity (that is, the larger the value of q), the faster the early diffusive decay of the field. It is instructive once again to contrast Eqs. (17) and (15).

There exist examples of two-dimensional fluid flows with a power-law singularity at the origin. Any steady flow with circular streamlines and azimuthal velocity $u=u(r)$ in polar coordinates (r,ϕ) is an exact steady and incompressible so-

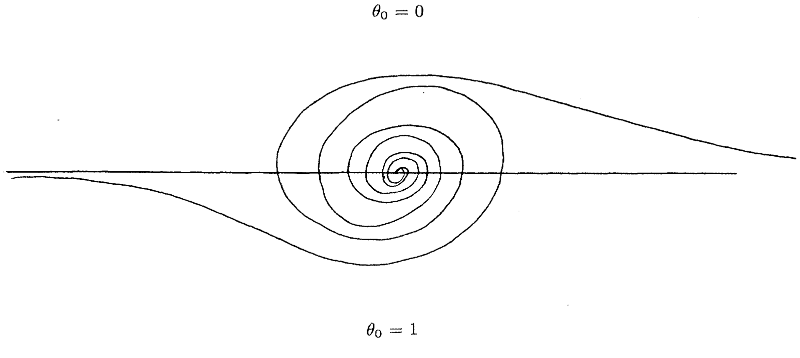

FIG. 4. The horizontal line represents the interface at time $t=0$ and the spiral represents the interface at a time t such that $0<t<t_0$. The value of the scalar field is 1 below the interface and 0 above. The center of the vortex is at the center of the spiral, and the vortex turns anticlockwise.

lution of the Euler equation provided that the pressure gradient dp/dr balances the centrifugal force $\rho(u^2/r)$, where ρ is the density of the fluid. Such flows may have a power-law singularity $u(r)\sim r^{-q}$. Flows outside a thin rotating circular cylinder with suction through the porous wall of the cylinder [5] have an azimuthal velocity $u=u(r)\sim r^{1-R}$ where R is a Reynolds number and are steady incompressible solutions of the Navier-Stokes equation. Spiral flow singularities can be generated on a two-dimensional on-off scalar tracer field (Fig. 4) by a steady azimuthal velocity field that has a power-law singularity $u(r)\sim r^{-q}$. Assuming, for the sake of argument, that at $t=0$ the interface between the region where $\theta=0$ and the region where $\theta=1$ is a straight line and that the singular velocity field $u(r)$ is centered at a point on that line, then the interface instantly adopts a spiral geometry under the action of the vortex (Fig. 4). We expect that the effect of molecular diffusion on the scalar field θ is significant only after the spiral has formed. Hence, the spiral structure of $\theta(r,\phi,t)$ is initially given by

$$\frac{\partial}{\partial t}\theta+\frac{u(r)}{r}\frac{\partial}{\partial\phi}\theta\approx 0 \tag{18}$$

which, considering the condition at $t=0$, admits the solution $\theta(r,\phi,t)=H(\phi-[u(r)/r]t-\pi)$ where H is the Heaviside on-off function. The spiral interface is, therefore, of the form $\phi\sim u(r)/r\sim r^{-1-q}$ and the Kolmogorov capacity of its point-intersections with a straight line cutting through the center of the spiral is [1] $D=(1+q)/(2+q)$. If, again for the sake of argument, we imagine a situation where the vortex is somehow suddenly removed at a time t_0 when the molecular diffusion has not had enough time to act significantly, then the evolution of the field θ at subsequent times $t\geq t_0$ is governed by

$$\frac{\partial}{\partial t}\theta=\kappa\nabla^2\theta \tag{19}$$

with initial condition $\theta_0(r,\phi)=\theta(r,\phi,t_0)=H(\phi-[u(r)/r]t_0)$. The method developed by Gilbert [6] to calculate the power spectrum of two-dimensional spiral singularities can be trivially reproduced for the power spectrum of the initial (at $t=t_0$) scalar field θ_0 leading to (2) with $2p=3-2D$. Solving (19) by two-dimensional Fourier methods, Eqs. (5) and (6) remain valid, and the method described in this paper [7] leads to

$$\delta(t)\sim L\left(\frac{\kappa t}{L^2}\right)^{1-D}\sim L\left[\frac{\sqrt{\kappa t}}{L}N(\sqrt{\kappa t})\right]^2 \tag{20}$$

for $\sqrt{\kappa t}\ll L$. A comparison between (12) and (20) suggests that the power 2 in the right hand side of (20) is nothing but the Euclidean dimensionality of the flow.

To summarize, the results (12), (15), (17), and (20) give the diffusive length scale's dependence on the geometry and structure of a singular tracer field in specific one- and two-dimensional situations. The diffusive length scale of one-dimensional spiral or fractal fields is proportional to the total length $\sqrt{\kappa t}N(\sqrt{\kappa t})$ of the covering of points where sharp gradients lie by segments of length $\sqrt{\kappa t}$. The more space-filling the sharp gradients, the larger the number $N(\sqrt{\kappa t})$ and the longer the diffusive length scale $\delta(t)$. To obtain $L\delta(t)$ for a two-dimensional spiral field, the total length $\sqrt{\kappa t}N(\sqrt{\kappa t})$ must be squared. If the initial field θ_0 has a simple power-law singularity, $\delta(t)\sim\sqrt{\kappa t}\,\theta_0(\sqrt{\kappa t})$, and if the fractal geometry of a field supports a random singular structure characterized by a Hurst exponent H (FBM), diffusive attrition is further accelerated and the purely geometrical estimate $\sqrt{\kappa t}N(\sqrt{\kappa t})$ must be multiplied by $(\sqrt{\kappa t}/L)^H$ for an accurate estimation of $\delta(t)$.

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[7] Equation (9) is exact for one-dimensional fields and under the condition of isotropy for fields of higher dimensionality. In view of the near isotropy of the two-dimensional spiral field, one can verify that $\mathscr{L}(t)$ remains proportional to the weighted average (9) and can therefore still be used to calculate $\delta(t)$. The large scale L is again introduced by setting $E_0(k)=0$ for $kL\leq 1$, and represents the initial size of the spiral.