Eulerian-Lagrangian aspects of a steady multiscale laminar flow

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One key feature for the understanding and control of turbulent flows is the relation between Eulerian and Lagrangian statistics. This Brief Communication investigates such a relation for a laminar quasi-two-dimensional multiscale flow generated by a multiscale (fractal) forcing, which reproduces some aspects of turbulent flows in the laboratory, e.g., broadband power-law energy spectrum and Richardson's diffusion. We show that these multiscale flows abide with Corrsin's estimation of the Lagrangian integral time scale, T_L, as proportional to the Eulerian (integral) time scale, $L_E/u_{\rm rms}$, even though Corrsin's approach was originally constructed for high Reynolds number turbulence. We check and explain why this relation is verified in our flows. The Lagrangian energy spectrum, $\Phi(w)$, presents a plateau at low frequencies followed by a power-law energy spectrum $\Phi(w) \sim w^{-\alpha}$ at higher ones, similarly to turbulent flows. Furthermore, $\Phi(\omega)$ scales with L_E and $u_{\rm rms}$ with $\alpha > 1$. These are the key elements to obtain such a relation [$\Phi(w) \sim \epsilon w^{-2}$ is not necessary] as in our flows the dissipation rate varies as $\epsilon \sim u_{\rm rms}^3 / L_E {\rm Re}_\lambda^{-1}$. To complete our analysis, we investigate a recently proposed relation [M. A. I. Khan and J. C. Vassilicos, Phys. Fluids **16**, 216 (2004)] between Eulerian and Lagrangian structure functions, which uses pair-diffusion statistics and the implications of this relation on $\Phi(\omega)$. Our results support this relation, $\langle [\mathbf{u_L}(\mathbf{t}) - \mathbf{u_L}(\mathbf{t}+\tau)]^2 \rangle = \langle [\mathbf{u_E}(\mathbf{x}) - \mathbf{u_E}(\mathbf{x} + \sqrt{\Delta^2(\tau)}\mathbf{e})]^2 \rangle$, which leads to $\alpha = \gamma/2(p-1)+1$. This Eulerian-Lagrangian relation is striking as in the present flows it is imposed by the multiscale distribution of stagnation points, which are an Eulerian property. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2754348]

The relations between Eulerian and Lagrangian statistics of turbulent flows are of central importance since they hold the key to the understanding of many phenomena such as turbulent diffusion and pair separation. Since Taylor's contribution,[2] it has been known that the turbulent diffusion is proportional to the time integral of the velocity Lagrangian autocorrelation function. Corrsin[3] introduced the proportionality of Eulerian, $L_E/u_{\rm rms}$, and Lagrangian, T_L, integral time scales for high Reynolds number turbulence (L_E being the Eulerian integral length scale and $u_{\rm rms}$ the turbulent intensity). Corrsin's estimation relies on two properties of homogeneous turbulence: first, a kinetic energy dissipation rate per unit of mass, ϵ, varying according to $\epsilon \sim u_{\rm rms}^3/L_E$; second, a Lagrangian energy spectrum,$\Phi(\omega)$, possessing a plateau for $\omega < \omega_L$ and $\Phi(\omega) \sim \epsilon\omega^{-2}$ for $\omega > \omega_L$. From $u_{\rm rms}^2 \sim \int \Phi(\omega) d\omega$, one then obtains $\omega_L \sim \epsilon / u_{\rm rms}^2$; and from $T_L \sim \Phi(\omega_L)/u_{\rm rms}^2$, which follows from integrating the Lagrangian autocorrelation function expressed as the Fourier transform of $\Phi(\omega)$, it then follows that $T_L \sim L_E/u_{\rm rms}$.

If they exist, it is important to know which key features of the Eulerian fields are responsible for such close relations between Eulerian and Lagrangian statistics. Rossi *et al.*[4] have generated a steady laminar multiscale flow with turbulent-like properties such as a broadband power-law energy spectrum and Richardson's diffusion.[5,6] Following the work of Fung *et al.*[7] and Davila and Vassilicos,[8] these turbulent properties are controlled by the multiscale distribution of stagnation points generated by a multiscale (fractal) forcing. If such flows are really turbulent like and withhold some key properties of the multiscale structure of turbulent flows, it is interesting to analyze the Eulerian-Lagrangian relations of such flows.

Does $T_L \sim L_E/u_{\rm rms}$ hold in our turbulent-like laminar multiscale flow?

A recent approach complementary to Corrsin's estimation for Eulerian-Lagrangian relation has been given by Khan and Vassilicos.[1] They compare Lagrangian and Eulerian second-order structure functions. We use this approach to complete our analysis of our turbulent-like flow by studying the relation between the Eulerian energy wavenumber spectrum, the Lagrangian energy frequency spectrum, and two-particle dispersion.

A horizontal shallow layer of brine (NaCl, 158 g/l, thickness $H=5$ mm) is forced by a fractal distribution of opposite pairs of Lorentz forces. These electromagnetic (EM) forces are generated by an electric current through the brine and permanent magnets of various horizontal sizes (10, 40, and 160 mm) placed under the bottom wall, which supports the brine.

The two-component velocity field $\mathbf{u}(\mathbf{x},t)$ at the free surface of the brine layer generated by these fractal EM forces has been measured by particle image velocimetry (PIV), using a 15-Hz, 12-bit, 2048×2048 pixel^2 camera. The flow is

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measured in a large square frame (which covers all magnets) of size $L_{PIV}=813.4$ mm, which is small compared to the size of the tank (1700×1700 mm^2). The physical length of one pixel is about 0.3972 mm. The correlation windows have 16×16 pixels (search window 42×42 pixels), and the overlap in each direction is of 9 pixels. This leads to a measurement grid containing 287×287 velocity vectors. For full details on the rig and experiments, see Ref. 4, which also shows that the flow at the free surface of the brine is quasi-two-dimensional (Q2D).

With steady EM forcing the flows are stationary after an initial transient following the sudden switch on of the forces. A Reynolds number $\mathrm{Re}_{2D}=u_{rms}L_{PIV}/\nu$ can be defined based on L_{PIV}, the root mean square of the PIV velocity field, u_{rms} (which is controlled by varying the intensity of the electric current), and the kinematic viscosity of the brine, ν. In this Brief Communication we present results obtained for 11 different values of Re_{2D} from 600 to 9900. Despite the large values of these Reynolds number, the flows are laminar as they present no instabilities and the fluid velocity values are never larger than a few cm/s.

Integral length scales, L_E, are obtained from the spatial autocorrelation of the velocity fields and their values slowly increase from about 16 cm to about 20 cm as Re_{2D} increases from 600 to 9900. This increase of L_E reflects the slight increase of the multiscale flow's larger eddies as a result of the EM forcing overcoming the bottom wall's friction over a larger portion of space.

The Lagrangian trajectories $d/dt\mathbf{x}(t,\mathbf{x}_0)=\mathbf{u_L}(t,\mathbf{x}_0)=\mathbf{u}(\mathbf{x}(\mathbf{t},\mathbf{x}_0),\mathbf{t})$ and their statistics, are calculated starting from random initial positions $\mathbf{x}_0$ at a time $t=0$ well after the initial transient caused by the sudden switch on of the electric field. These trajectories are integrated until $t=6L_E/u_{rms}$. Fluid elements are tracked in highly resolved PIV fields. We export positions every $\Delta t=6/1023(L_E/u_{rms})$ to obtain 1024 positions tracked during $6L_E/u_{rms}$. The integration time is 11 times smaller than Δt and is much smaller than the time resolution of the PIV measurements.

Figure 1(a) shows the statistic of one-particle dispersion normalized by the integral length scale, $\overline{(\mathbf{x}-\mathbf{x}_0)^2}/L_E$. A first ballistic motion regime, where $t<0.1(L_E/u_{rms})$ and $\overline{(\mathbf{x}-\mathbf{x}_0)^2}/L_E\sim t^2$, is later on followed by a Taylor regime,[2] where $t>2(L_E/u_{rms})$ and $\overline{(\mathbf{x}-\mathbf{x}_0)^2}/L_E\sim t^1$. Furthermore, it can be noticed that all these curves are superimposed over the entire range of Reynolds number considered in this Brief Communication.

Figure 1(b) gives the Lagrangian correlation function, $R(t)=\langle\mathbf{u_L}(0)\cdot\mathbf{u_L}(\mathbf{t})\rangle$, where $\mathbf{u_L}(\mathbf{t})$ is the Lagrangian velocity of the fluid element at time t. The Lagrangian correlation functions are very similar over the entire range of Reynolds numbers until $t\simeq L_E/u_{rms}$. For longer times, the Lagrangian correlation functions present different harmonics. Figure 1(c) is very similar to the results obtained in turbulent flows (e.g., Ref. 9) with a logarithmic decrease of $R(t)$ followed by a sudden dropoff. The main differences with turbulent flows appear for long times with the oscillations regime. These oscillations are induced by the steadiness (constant forcing) of the present quasi-two-dimensional flows, leading to closed trajectories (fitting streamlines) of various lengths. We would expect these oscillations to disappear for time-dependent forcing. Nevertheless, it is interesting to see that these oscillations have no significant effect on the Brownian (uncorrelated) time dependence of single-particle dispersion; see Fig. 1(a).

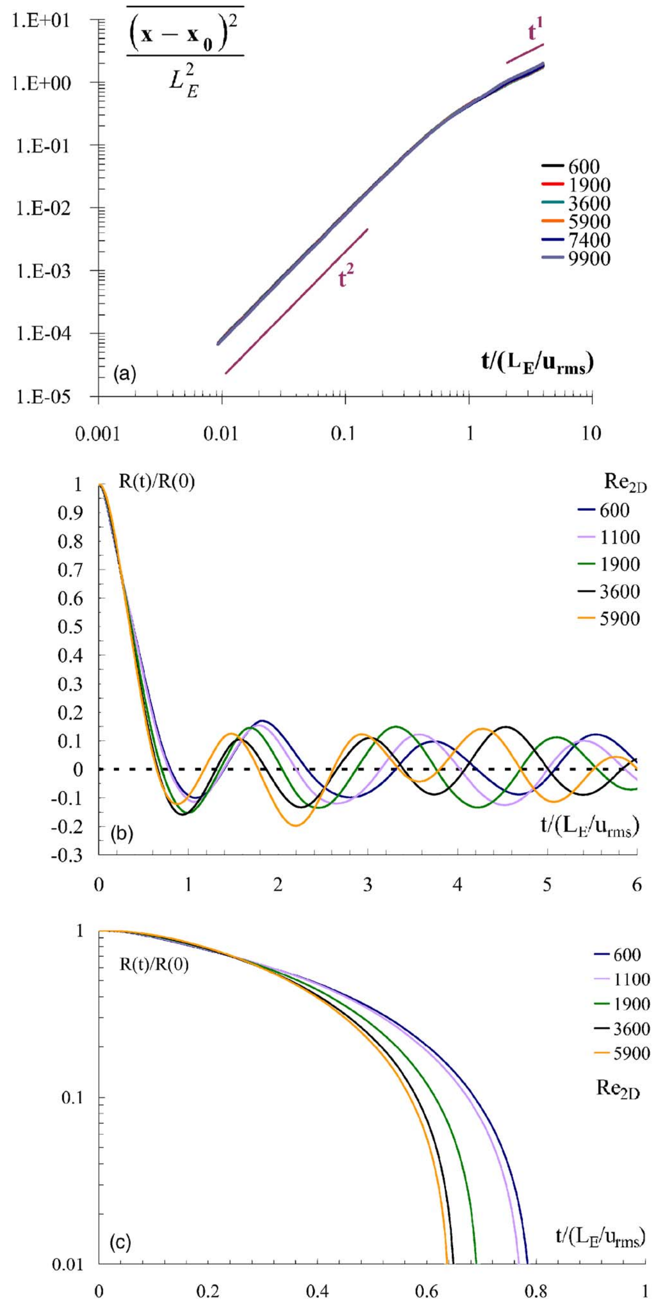

FIG. 1. (Color online) (a) One-particle dispersion for various Reynolds numbers, Re_{2D} from 600 to 9900; [(b), (c)] Lagrangian correlation function for various Reynolds numbers, Re_{2D}: (b) linear-linear plot, (c) semi-log plot.

From the Lagrangian correlation function we estimate the Lagrangian correlation time by $T_L(t)=1/R(0)\int_0^t R(t)dt$. Figure 1(b) illustrates the evolution of $R(t)/R(0)$ with time, tu_{rms}/L_E. As shown in Fig. 2(a), when $t>(L_E/u_{rms})$, $T_L(t)$ has reached a regime where it oscillates [due to oscillations of $R(t)$] around an average value, $\overline{T_L}=\overline{T_L(t)}$, which is defined by averaging over the time range $L_E/u_{rms}\leqslant t\leqslant 6L_E/u_{rms}$. These

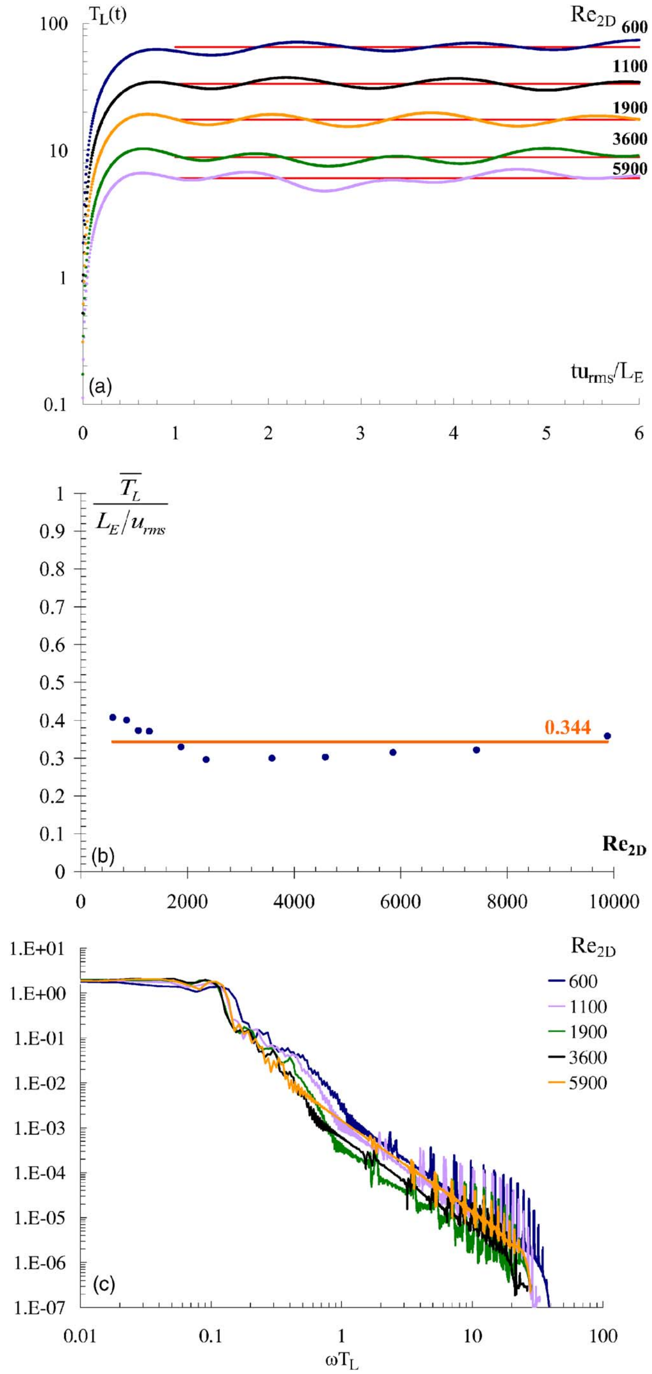

FIG. 2. (Color online) (a) Plot of the Lagrangian integral time scale, T_L according to the duration of fluid element tracking $tu_{\rm rms}/L_E$; (b) $T_L/(L_E/u_{\rm rms})$ vs Reynolds numbers Re_{2D}; (c) Lagrangian energy spectra, $2\pi\phi(\omega)/u_{\rm rms}^2\hat{T}_L$ vs ωT_L.

values of $\overline{T_L}$ are illustrated by straight (red) lines in Fig. 2(a). Later on, $\overline{T_L}$ is called T_L, and $T_L(t=6L_E/u_{\rm rms})$ is called $\hat{T}_L$.

Corrsin[3] states that a turbulent flow with a broad power-law energy spectrum should exhibit a constant ratio between the Eulerian integral time scale, $L_E/u_{\rm rms}$, and the Lagrangian correlation time, T_L. As this family of multiscale flows possesses a broadband power-law energy spectrum (see Ref. 4), we test Corrsin's statement on these flows. To do that we plot the ratio $T_L/(L_E/u_{\rm rms})$ in Fig. 2(b). This ratio is found to be close to a constant over the considered range of Reynolds number, Re_{2D}, i.e., more than one decade, in agreement with Corrsin's estimation.

Why do these steady multiscale laminar flows abide with Corrsin's estimation, which has been obtained for high Reynolds number turbulence? The power-law shape of the Eulerian energy spectrum being already demonstrated,[4] we now investigate (for these flows) the validity of the main hypotheses behind Corrsin's approach, which concern Lagrangian energy spectra and dissipation.

The energy spectra related to the Lagrangian correlation function, $2\pi\phi(\omega)/(u_{\rm rms}^2\hat{T}_L)$, are plotted in Fig. 2(c) for various Reynolds numbers, Re_{2D}. For each Reynolds number, the Lagrangian energy spectrum presents a plateau over one decade in frequency for $\omega T_L<0.1$. This plateau is important as it adds one new element of similarity between our steady multiscale flows and turbulent flows. For example, Refs. 9–11 find such a plateau in their respective turbulent flows: laboratory experiment, direct numerical simulation, and high Reynolds number oceanic environment. For higher frequencies, the energy spectra can be approximated as power-law functions for about one decade with $\phi(\omega)\sim(\omega T_L)^{-\alpha}$, where α varies with the Reynolds number. We should mention that for energy values lower than 10^{-4} the noise of the PIV measurement is dominant (accuracy at 1%; see Ref. 4) leading to spikes and a w^{-2} power law for large values of ωT_L.

Corrsin's estimation assumes that the dissipation varies like $\epsilon\sim u_{\rm rms}^3/L_E$. We compute the dissipation using the PIV velocity fields and considering: $\epsilon=2\nu\langle S_{ij}S_{ij}\rangle$ with $S_{ij}=\frac{1}{2}(\partial u_i/\partial x_j+\partial u_j/\partial x_i)$ and $\lambda=\sqrt{\nu u_{\rm rms}^2/\epsilon}$. Figure 3(a) gives the rescaled dissipation of these multiscale flows, $\epsilon^*=\epsilon/(u_{\rm rms}^3/L_E)$ versus the Reynolds number based on Taylor's microscale, λ, $\mathrm{Re}_\lambda=u_{\rm rms}\lambda/\nu$. ϵ^* evolves like $1/\mathrm{Re}_\lambda$ over more than one decade, $22\leqslant\mathrm{Re}_\lambda\leqslant555$. This relation in $1/\mathrm{Re}_\lambda$ could be expected for laminar flows, where $\epsilon\sim u_{\rm rms}^2$. Nevertheless, the present case is not trivial. When $\mathrm{Re}_\lambda<100$ ($\mathrm{Re}_{2D}<3000$), $\epsilon^*\sim\mathrm{Re}_\lambda^{-(1+0.02\pm0.02)}$, the streamlines remain effectively unchanged with changing Reynolds number because the flow is dominated by the bottom friction, $\epsilon\sim u_{\rm rms}^{1.81\pm0.1}$, $L_E\sim u_{\rm rms}^{0.056\pm0.02}$, and $\lambda\sim u_{\rm rms}^{0.093\pm0.02}$. When $\mathrm{Re}_\lambda>100$, $\epsilon^*\sim\mathrm{Re}_\lambda^{-(1+0.096\pm0.02)}$, the strain rates have a clear multiscale structure evolving with Re_λ as explained in Ref. 6, $\epsilon\sim u_{\rm rms}^{1.51\pm0.1}$, $L_E\sim u_{\rm rms}^{0.13\pm0.02}$, and $\lambda\sim u_{\rm rms}^{0.24\pm0.02}$.

These multiscale laminar flows have a prescribed power-law Eulerian energy wavenumber spectrum but $\epsilon^*\sim\mathrm{Re}_\lambda^{-1}\neq cte$, over more than one decade. Nevertheless, $T_L\sim L_E/u_{\rm rms}$. The reason for this is that $\Phi(\omega)$ scales with $u_{\rm rms}$ and L_E, and that it has a plateau at small ω followed by $\Phi(\omega)\sim u_{\rm rms}L_E(\omega L_E/u_{\rm rms})^{-\alpha}$ at higher ω with $\alpha>1$ (see Fig. 2).

How can we estimate α in our flow? To complete our description of the considered laminar multiscale flows and give an estimation for α, we use a relation between Eulerian and Lagrangian statistics proposed by Khan and Vassilicos.[1] They introduced relation (1) between Eulerian and Lagrangian structure functions of order 2, where $\mathbf{u_L}$ is the Lagrangian velocity of a fluid element, $\mathbf{u}$ is the Eulerian velocity, $\mathbf{e}$ represents normalized spatial vectors, and $\overline{\Delta^2}(t)$ is the mean-

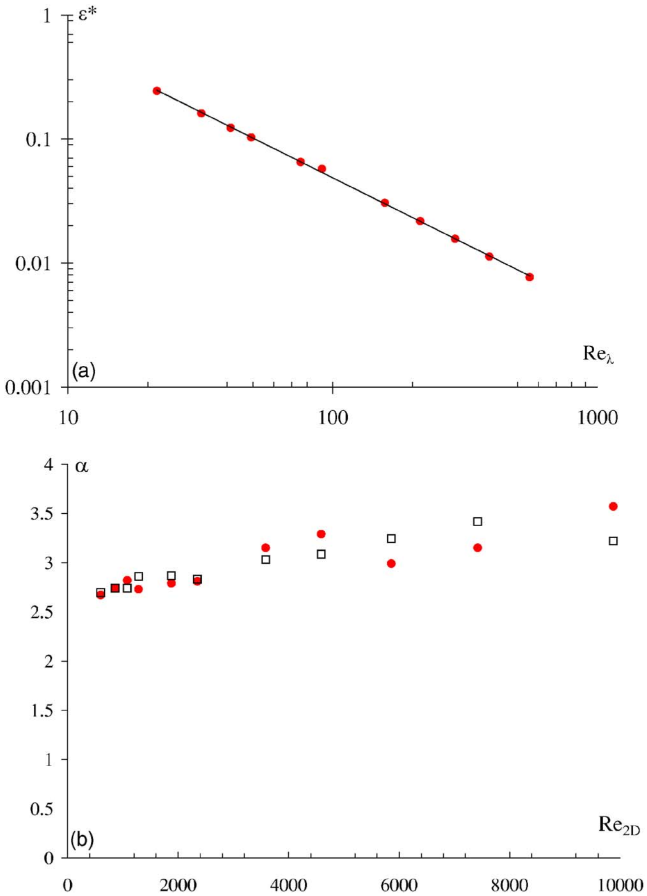

FIG. 3. (Color online) (a) $\epsilon^*=\epsilon/(u_{\mathrm{rms}}^3/L_E)$ as function of the Reynolds number, $\mathrm{Re}_\lambda=u_{\mathrm{rms}}\lambda/\nu$. The line corresponds to the best power-law fit $\epsilon^* \sim \mathrm{Re}_\lambda^{-1.063}$; (b) ● are the values of α extracted from the best power-law fit of the Lagrangian energy spectra; □ are the values of α estimated using relation (2).

square separation, at time t, between two fluid elements starting at time $t=0$ with an initial displacement smaller than the smallest length scale of the flow,

$$\langle[\mathbf{u_L}(\mathbf{t})-\mathbf{u_L}(\mathbf{t}+\tau)]^2\rangle = \langle[\mathbf{u}(\mathbf{x})-\mathbf{u}(\mathbf{x}+\sqrt{\overline{\boldsymbol{\Delta}^2}(\tau)}\mathbf{e})]^2\rangle. \quad (1)$$

This leads to the relation given in Eq. (2), where γ is the exponent of the Richardson-like power-law function of time for two-particle dispersion ($\overline{\Delta^2}\sim t^\gamma$) and p is the exponent of the power-law Eulerian energy wavenumber spectrum, $E(k) \sim k^{-p}$:

$$\alpha=\frac{\gamma}{2}(p-1)+1. \quad (2)$$

Pair statistics are initialized with initial separations $\Delta_0 =1$ pixel, which is about 25 times smaller than the size of the smallest magnet and therefore smaller than all the length scales of the flow by the size of the smallest magnets. Statistics such as mean-square pair separations are sensitive to the choice of Δ_0 but the turbulent diffusivity $d/dt\overline{\Delta^2}$ is much less sensitive as shown recently in Ref. 12. In addition, $d/dt\overline{\Delta^2}$ allows us to clearly identify different dispersion regimes such as the expected initial ballistic dispersion $\overline{\Delta^2} \sim t^2$, the final Brownian dispersion $\overline{\Delta^2}\sim t$, and a nontrivial Richardson-like regime in an intermediate range of times between the ballistic and the Brownian regimes (see Refs. 4 and 6). Rossi *et al.*[6] have shown the sensitivity to γ to the Reynolds number Re_{2D} and the multiscale distribution of stagnation points as well as their scale rate distribution. It should be noticed that the values of α and γ are estimated over the same decade in time. As the energy spectra of this family of flow can be approximated by power-law function $E(k)\sim k^{-2.5}$ (see Ref. 4), we take $p=2.5$.

The values of α extracted from the best power-law fit of the Lagrangian energy spectra are clearly interlaced with the values of α estimated using relation (2) as shown in Fig. 3(b). The small increase in differences with the increase of the Reynolds number are well under an uncertainty of ±5% for the measurement of each value of α, γ, and p.

It is extremely interesting and encouraging to find such a relation between Eulerian and Lagrangian structure functions of order 2 in these steady multiscale laminar flows as, until now, this relation has only been tested in kinematic simulation with a strong time dependency.[1]

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