# Basics of Lorentzian geometry

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**Definition.** *Let $V$ be an $(n+1)$-dimensional vectorspace. A* **Lorentzian inner product** *$m$ is a map $m : V \times V \to \mathbb{R}$ which is*

- *bilinear*
- *non-degenerate* ( $m(V, W) = 0$ *for all* $W$ *implies* $V = 0$ )
- *symmetric*
- *maximal dimension of any subspace $W$ such that $m|_W$ is* positive definite *is $n$.*

Recall that by the classification theory of bilinear forms we can always find a basis $e_0, e_1, ..., e_n$ such that

- $m\left(e_0, e_0\right) = -1$
- $m\left(e_0, e_i\right) = 0$ for $i = 1, 2, ..., n$.
- $m\left(e_i, e_j\right) = \delta_{ij}$ for $i, j = 1, 2, ..., n$.

We want to study **Lorentzian manifolds.**

These are manifolds equippend with an extra structure, a **Lorentzian metric**.

**Definition.** *Let $\mathcal{M}$ be a smooth manifold. A Lorentzian metric on $\mathcal{M}$ is an assignment to each point $p$ of a Lorentzian inner product, i.e. a map*

$$g_p : T_p\mathcal{M} \times T_p\mathcal{M} \to \mathbb{R}$$

*such that $g_p$ depends smoothly on $p$.*

Pedestrian way to understand smooth dependence: Choose local coordinates near $p$, say $x^0, x^1, ..., x^n$. Then

$$\left[\frac{\partial}{\partial x^0}\right]_{\widetilde{p}}, \cdots, \left[\frac{\partial}{\partial x^n}\right]_{\widetilde{p}}$$

is a (coordinate) basis for each $T_{\widetilde{p}}\mathcal{M}$ for $\widetilde{p} \in \mathcal{U} \subset \mathcal{M}$. The metric is determined by its values on this basis, the components

$$g_{\alpha\beta}(\widetilde{p}) = g_{\widetilde{p}}\left(\left[\frac{\partial}{\partial x^\alpha}\right]_{\widetilde{p}}, \left[\frac{\partial}{\partial x^\beta}\right]_{\widetilde{p}}\right)$$

Smooth dependence: $g_{\alpha\beta}(\widetilde{p})$ are smooth as functions of $\widetilde{p}$ in $\mathcal{U}$.

Given the local coordinate basis

$$\left[\frac{\partial}{\partial x^0}\right]_{\widetilde{p}} , \; \cdots , \; \left[\frac{\partial}{\partial x^n}\right]_{\widetilde{p}}$$

from the last slide, we can introduce a dual basis of one form $dx^\alpha$ on the dual (cotangent space) $T_p^\star \mathcal{M}$ via the defining relation

$$dx^\alpha \left( \left[ \frac{\partial}{\partial x^\beta} \right] \right) = \delta_\beta^\alpha \, .$$

We can then write our Lorentzian metric as

$$g_p = \sum_{\alpha,\beta} g_{\alpha\beta}\,(p)\, dx^\alpha \otimes dx^\beta \tag{1}$$

By the Einstein summation convention, the sum is often omitted.

**Example 1: Minkowski space.** $\mathcal{M} = \mathbb{R}^{1+n}$ equipped with

$$\begin{aligned} \eta &= -dt \otimes dt + dx^1 \otimes dx^1 + \ldots + dx^n \otimes dx^n \\ &= -dt^2 + (dx^1)^2 + ... + (dx^n)^2 \end{aligned} \tag{2}$$

Changing to polar coordinates, say $n = 3$

$$\eta = -dt^2 + dr^2 + r^2 \left( d\theta^2 + \sin^2 \theta d\phi^2 \right)$$

Changing also to null-coordinates $t = u + v$ and $r = v - u$

$$\eta = -4dudv + r^2 (u, v) \left( d\theta^2 + \sin^2 \theta d\phi^2 \right)$$

**Example 2: Schwarzschild spacetime.**

The manifold is

$$\mathcal{M} = (-\infty, \infty) \times (-\infty, \infty) \times S^2$$

Fix $M > 0$. We define

$$g = -16M^2 \frac{2M}{r(U,V)} e^{-\frac{r(U,V)}{2M}} dU dV + r^2\left(U,V\right)\left(d\theta^2 + \sin^2\theta d\phi^2\right)$$

with $r = r\left(U,V\right)$ given implicitly by

$$-UV = \frac{1-\frac{2M}{r}}{\frac{2M}{r}e^{-r/2M}} = \left(r-2M\right)\frac{1}{2M}e^{\frac{r}{2M}} \ .$$

We shall obtain a simpler form and understand the geometry and the physics of this metric very soon!

Given a Lorentzian manifold $(\mathcal{M}, g)$ we have a notion of

**timelike/ spacelike/ null vectors** in each tangent space

and also of

**timelike/ spacelike/ null curves**.

Timelike curves corresponds to massive particles, null curves to massless particles.

If $\gamma$ is a causal (timelike or null) curve we define its length as

$$L(\gamma) = \int \sqrt{-g(\dot{\gamma}, \dot{\gamma})} dt$$

This is the proper time felt by a local observer w.r.t. his own clock.

A smooth **hypersurface** $F \subset \mathcal{M}$ is

- timelike if its normal is spacelike
- spacelike if its normal is timelike
- null if its normal is null

**Examples** (Minkowski):

- $t = 0$ is spacelike (normal is $\partial_t$)
- $u = 0$ is null: Normal is $n = \nabla u = (du)^\flat \sim \partial_v$ ( $n = \eta^{\alpha\beta}\partial_\alpha u \partial_\beta$)

## Isometries of Lorentzian manifolds

Recall that if $\phi$ is a diffeomorphism

$$\phi : (\mathcal{M}, g) \to \left(\widetilde{\mathcal{M}}, \widetilde{g}\right)$$

such that $\phi^\star \widetilde{g} = g$, then $\phi$ is an isometry. The pull back condition:

$$g_p\left(v, w\right) = \widetilde{g}_{\phi(p)}\left((d\phi)_p v, (d\phi)_p w\right)$$

for all $v, w \in T_p M$.

We'll be interested in the set of isometries from $(M, g)$ to itself. This set forms a Lie group which is can be generated by the Lie algebra of so-called Killing fields.

**Definition.** *A* **Killing field** $W$ *on* $(\mathcal{M}, g)$ *is a vectorfield on* $\mathcal{M}$ *such that*

$$\mathcal{L}_W g = 0$$

*or in arbitrary coordinates*

$$(\mathcal{L}_W g)_{\mu\nu} = W^\alpha \partial_a g_{\mu\nu} + \partial_\mu W^\alpha g_{\alpha\nu} + \partial_\nu W^\alpha g_{\mu\alpha}$$

Given a Killing field $W$ we can look at its integral curves $(\dot{\gamma} = W(\gamma(s)))$. They generate a one parameter group of diffeomorphisms $\phi_s$ which are isometries.

Illustrate this with an example: Minkowski space in $1+3$ dimensions, coordinates $\left(t, x^1, x^2, x^3\right)$.

- Clearly, $\partial_t$ is a Killing field. It generates the one-parameter group of isometries

  $$\phi_s : \left(t, x^1, x^2, x^3\right) \mapsto \left(t+s, x^1, x^2, x^3\right)$$

  (time)-translation invariance.

- Check $x^1\partial_t + t\partial_{x^1}$ is a Killing field. It generates

  $$\phi_s : \left(t, x^1, x^2, x^3\right) \mapsto \left(\cosh(s)\ t + \sinh(s)\ x^1, \sinh(s)\ t + \cosh(s)\ x^1, x^2, x^3\right)$$

  Lorentz boosts (hyperbolic rotations)

It total 3 rotations, 3 boosts, 4 translations $\implies$ 10-dimensional Poincare group

## The Connection

The idea: We would like to differentiate vectorfields on a manifold. Need to compare vectors in different tangent spaces. This is achieved by the covariant derivative operator $\nabla$. We include the axiomatic definition of the Levi-Civita connection on the next slide.

## The Connection

If $X, Y$ are vectorfields on $(\mathcal{M}, g)$, then $\nabla_X Y$ is a vectorfield on $(\mathcal{M}, g)$ ("the directional derivative of $Y$ in the direction $X$") defined uniquely by

1. $\nabla_X Y$ is linear in $Y$ over the reals
2. $\nabla_X Y$ is linear in $X$ over smooth functions ($\nabla_{fX} Y = f \nabla_X Y$)
3. product rule: $\nabla_X (fY) = \nabla_X(f) Y + f \nabla_X Y$
4. torsion-free (symmetric): $\nabla_X Y - \nabla_Y X - [X, Y] = 0$
5. metric compatible: For all vectorfields $X, Y, Z$ on $(\mathcal{M}, g)$

$$X\left(g(Y, Z)\right) = g\left(\nabla_X Y, Z\right) + g\left(Y, \nabla_X Z\right) \, .$$

One can compute the connection in a local coordinate chart

$$\nabla_{\frac{\partial}{\partial x^\alpha}} \frac{\partial}{\partial x^\beta} = \Gamma^\gamma{}_{\alpha\beta} \frac{\partial}{\partial x^\gamma}$$

with

$$\Gamma^\gamma{}_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta} \left(\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}\right)$$

the Christoffel symbols. These are not tensors!
In Minkowski, $\Gamma = 0$ in Cartesian coordinates $\left(t, x^1, ..., x^n\right)$ but already in polar coordinates $\Gamma \neq 0$!

So, if I give you a metric in some coordinates, you can compute the $\Gamma$. Note also

$$\nabla_X Y = \left(X\left(Y^\mu\right) + \Gamma^\mu{}_{\alpha\beta} X^\alpha Y^\beta\right) \partial_\mu$$

Note that $\nabla_X Y$ depends only on $X$ at the point $p$ and the values of $Y$ along the integral curve of $X$!

**Parallel transport and Geodesics**

The vectorfield $Y$ along the curve $\gamma$ is called **parallel** if $\nabla_{\dot\gamma} Y = 0$.

By the theory of ODEs we can construct the parallel transport of $Y$ along $\gamma$ given $Y$ at a point $p$. Indeed we just need to solve the system

$$\frac{d}{ds} Y^{\mu}\left(x(s)\right) + \Gamma^{\mu}{}_{\alpha\beta}\left(x(s)\right) \dot\gamma^{\alpha}(x(s)) Y^{\beta}(x(s)) = 0\,.$$

A curve $\gamma : I \to \mathcal{M}$ is a **geodesic** if $\nabla_{\dot\gamma}\dot\gamma = 0$. Again ODE theory:

**Proposition 1.** *Given a point $p \in \mathcal{M}$ and a vector $v \in T_p\mathcal{M}$ there is a unique maximal geodesic*

$$\gamma : (T_-, T_+) \to \mathcal{M}$$

*such that $\gamma(0) = p$, $\dot\gamma|_p = v$ where $-\infty \leq T_- \leq \infty$.*

**Proposition 2.** *Let $\gamma : I \to \mathcal{M}$ be a geodesic such that $\gamma(s_0)$ is timelike for some $s_0 \in I$ then $\dot{\gamma}(s)$ is timelike for all $s \in I$.*

*Proof.*

$$\dot{\gamma}\left(g\left(\dot{\gamma},\dot{\gamma}\right)\right) = 2g\left(\nabla_{\dot{\gamma}}\dot{\gamma},\dot{\gamma}\right) = 0$$

□

Hence geodesics preserve their type.

Geodesics correspond to feely falling particles.

The connection defines a notion of **curvature.** The formulae are the same as in the Riemannian case. The curvature measures the failure of parallel transport to commute. The Riemann curvature tensor:

$$R : T_p\mathcal{M} \times T_p\mathcal{M} \times T_p\mathcal{M} \to T_p\mathcal{M}$$

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

In coordinates

$$R^{\mu}{}_{\rho\sigma\tau} = \partial_\tau \Gamma^{\mu}{}_{\rho\sigma} - \partial_\sigma \Gamma^{\mu}{}_{\rho\tau} + \Gamma\Gamma - \Gamma\Gamma$$

The Ricci curvature tensor arises from contracting indices

$$Ric_{\sigma\tau} = R^{\mu}{}_{\sigma\mu\tau}$$

The Einstein vacuum equations impose $Ric_{\sigma\tau} = 0$.

**Summary**

1. We started from $\eta$ appearing in the wave equation.
2. Defined the notion of a Lorentzian inner product (LIP).
3. Lorentzian manifold: LIP in each tangent space
4. Killing fields and isometries (understood Minkowski)
5. Connection, Parallel Transport, Geodesics
6. Curvature, vacuum Einstein equations.