Measure and Integration: Example Sheet 9

Fall 2016 [G. Holzegel]

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1 Equivalence of Carathéodory and Lebesgue measurable (Exercise 3 in [SS])

Recall the exterior Lebesgue measure $m_\star$ defined at the beginning of the course. Prove that a set $E \subset \mathbb{R}^d$ is Carathéodory measurable if and only if it is Lebesgue measurable.
HINT: If E is Lebesgue measurable and A is any set, choose a G_δ set G with $A \subset G$ and $m_\star(A) = m(G)$. Conversely, if E is Carathéodory measurable and $m_\star(E) < \infty$, choose a G_δ set with $E \subset G$ and $m_\star(E) = m_\star(G)$. Then $G - E$ has exterior measure zero.

2 Completion of a measure space (Exercise 1 in [SS])

Let $(X, \mathcal{M}, \mu)$ be a measure space. One can define the completion of this space as follows. Let $\overline{\mathcal{M}}$ be the collection of sets of the form $E \cup Z$ where $E \in \mathcal{M}$ and $Z \subset F$ with $F \in \mathcal{M}$ and $\mu(F) = 0$ (so we are adjoining subsets of sets of measure zero). Also, define $\overline{\mu}(E \cup Z) = \mu(E)$.

a) $\overline{\mathcal{M}}$ is the smallest σ-algebra containing $\mathcal{M}$ and all subsets of elements of $\mathcal{M}$ of measure zero.

b) The function $\overline{\mu}$ is a measure on $\overline{\mathcal{M}}$ and this measure is complete.

3 Elementary Families

Here is a small technical result that is useful when checking whether a given collection of subsets is an algebra.

We define an **elementary family** to be a collection $\mathcal{E}$ of subsets of X such that

- $\emptyset \in \mathcal{E}$,
- if $E, F \in \mathcal{E}$, then $E \cap F \in \mathcal{E}$,
- if $E \in \mathcal{E}$ then E^c is a finite disjoint union of members of $\mathcal{E}$.

Prove that if $\mathcal{E}$ is an elementary family, then the collection $\mathcal{A}$ of **finite disjoint unions** of members of $\mathcal{E}$ forms an algebra. Deduce that finite disjoint unions of the following subsets of the real line form an algebra: $\emptyset, (a, \infty), (a, b]$ with $-\infty \leq a < b < \infty$. The latter was claimed in class.
HINT: First show $A, B \in \mathcal{E}$ implies $A \setminus B \in \mathcal{A}$ and $A \cup B = A \setminus B \cup B \in \mathcal{A}$. Now use induction to deduce that finite unions (not necessarily disjoint) of sets in $\mathcal{E}$ are in $\mathcal{A}$. Finally, show that $\mathcal{A}$ is closed under complements. Conclude.

4 Uniqueness of the extension of a premeasure

Let X be a non-empty set, $\mathcal{A}$ an algebra of subsets on X and $\mu_0 : \mathcal{A} \to [0,\infty]$ a premeasure. Recall that Theorem 5.2 in the notes (Hahn-extension) shows the existence of a measure $\mu : \mathcal{M}(\mathcal{A}) \to [0,\infty]$ extending μ_0. In general, μ is not unique. However if μ_0 is σ-finite, i.e. if one can express $X = \bigcup_{n=1}^{\infty} E_n$ for a countable collection of sets (E_n) in $\mathcal{A}$ with $\mu_0(E_n) < \infty$ for all n, then μ is unique in the following sense: If $\nu : \mathcal{M}(\mathcal{A}) \to [0,\infty]$ is another extension of μ_0, then $\mu(E) = \nu(E)$ holds for all $E \in \mathcal{M}(\mathcal{A})$. Prove this assertion. HINT: First show this on sets of finite measure, then use σ-finiteness.

REMARK: As we observed in the proof of Theorem 5.2, there may be larger σ-algebras $\tilde{M}$ to which one can extend μ_0, so the σ-algebra one extends to is generally not unique (even in the σ-finite case). Uniqueness holds once one fixes the σ-algebra one extends to, i.e. if $\mu : \tilde{\mathcal{M}} \to [0,\infty]$ and $\nu : \tilde{\mathcal{M}} \to [0,\infty]$ both extend μ_0, then μ and ν agree on the σ-algebra $\tilde{\mathcal{M}}$. Your proof above hopefully gives you this statement as well.

5 Hausdorff measure

a) Prove that the Hausdorff exterior measure defined in class is indeed an exterior measure.
HINT: For the subadditivity property first fix $\delta > 0$ and use the standard $\frac{\epsilon}{2^j}$-trick.

b) Prove that the Hausdorff exterior measure is finitely additive on sets of positive distance, i.e. $d(A,B) > 0$ implies that $m_\alpha^\star(A \cup B) = m_\alpha^\star(A) + m_\alpha^\star(B)$.

c) Suppose one defined a one-dimensional "exterior measure" $\tilde{m}_1$ as

$$\tilde{m}_1(E) = \inf \left\{ \sum_{k=1}^{\infty} diam\ F_k \ \ | \ \ E \subset \bigcup_{k=1}^{\infty} F_k \right\}.$$

Note this is similar to the 1-dimensional Hausdorff measure but without any restriction on the size of the diameters of the covering. Show that $\tilde{m}_1$ fails to be finitely additive in the sense of b).
HINT: Consider two horizontal unit intervals in $\mathbb{R}^2$ with distance ϵ.

REMARK: The finite additivity property in b) makes the $m_\alpha^\star$ a so-called metric exterior measure. (Recall this property was also key for us in the Lebesgue case!) This property ensures that applying Carathéodory's theorem, the Borel sets of $\mathbb{R}^d$ are measurable (see Section 1.2 of Chapter 6 in Stein-Shakarchi).

6 An example of the general Fubini-Tonelli-theorem

Let $x_{m,n}$ be a doubly infinite sequence of **non-negative** real numbers. On Example Sheet 2 you proved the formula

$$\sum_{m,n=1}^{\infty} x_{m,n} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{m,n} = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} x_{m,n}$$

Prove this formula using the general Tonelli theorem.

HINT: Defining the correct set-up where the theorem applies is part of the solution. You may want to start by defining a measure space setting $X = \mathbb{N}$, $\mathcal{M} = \mathcal{P}(X)$ and $\mu : \mathcal{P}(X) \to [0,\infty]$ the counting measure. How is integration of a function on $(X, \mathcal{M}, \mu)$ defined? Now define the product measure and apply the general Tonelli to conclude the result.