Measure and Integration: Example Sheet 9

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1 Equivalence of Carathéodory and Lebesgue measurable

If E is Lebesgue measurable and A is any set, we use the property of the exterior measure (Property 3 in Proposition 2.1 of the notes) that we can find for $\frac{1}{n}$ an open set $\mathcal{U}_n$ with $A \subset \mathcal{U}_n$ and

$$m_\star(\mathcal{U}_n) \geq m_\star(A) \geq m_\star(\mathcal{U}_n) - \frac{1}{n}$$

The set $G := \bigcap_n \mathcal{U}_n$ is then a G_δ-set, hence measurable and $A \subset G$. Moreover, the above inequality implies

$$m_\star(G) - \frac{1}{n} \leq m_\star(\mathcal{U}_n) - \frac{1}{n} \leq m_\star(A) \leq m_\star(G)$$

for any n and hence $m_\star(G) = m_\star(A)$. Since G is Lebesgue measurable and the Lebesgue measure is countably additive

$$m_\star(A) = m_\star(G) = m(G) = m(E \cap G) + m(E^c \cap G) = m_\star(E \cap G) + m_\star(E^c \cap G) \geq m_\star(E \cap A) + m_\star(E^c \cap A)$$

for any set A, which proves Carathéodory measurability of E after recalling that the $\leq$-direction is trivial from subadditivity of the exterior measure.

For the reverse direction, let E be Carathéodory measurable and first assume $m_\star(E) < \infty$. Then we use again the property of the exterior measure to find a sequence of $\mathcal{U}_n$ with $E \subset \mathcal{U}_n$ and $m_\star(\mathcal{U}_n) \geq m_\star(E) \geq m_\star(\mathcal{U}_n) - \frac{1}{n}$. Applying the Carathéodory condition with $A = \mathcal{U}_n$ we have for any n

$$m_\star(\mathcal{U}_n) = m_\star(\mathcal{U}_n \cap E) + m_\star(\mathcal{U}_n \cap E^c)$$

or

$$m_\star(\mathcal{U}_n \setminus E) = m_\star(\mathcal{U}_n) - m_\star(\mathcal{U}_n \cap E) = m_\star(\mathcal{U}_n) - m_\star(E) \leq \frac{1}{n}.$$

Hence for any $\epsilon > 0$ there is a $\mathcal{U}_n$ open such that $m_\star(\mathcal{U}_n \setminus E) < \epsilon$ which proves E is Lebesgue measurable.

We finally lift the assumption that $m_\star(E) < \infty$. For arbitrary E we define $E_n = E \cap B_n$ with B_n being the open ball around the origin of radius n. Since B_n is Lebesgue measurable, by the first part of the question it is Carathéodory measurable and since Carathéodory measurable sets form a σ-algebra, the sets E_n are all Carathéodory measurable if E is, and satisfy $m_\star(E_n) < \infty$. By what we have already shown the E_n are Lebesgue measurable and since $E = \bigcup_n E_n$ and the Lebesgue measurable sets form a σ-algebra we conclude E is Lebesgue measurable.

2 Completion of a measure space (Exercise 1 in [SS])

a) We only need to show $\overline{\mathcal{M}}$ is a σ-algebra as $\overline{\mathcal{M}}$ clearly contains $\mathcal{M}$ and all subsets of sets of measure zero (and the smallest such σ-algebra certainly needs to contain unions of such sets).

- The empty set is in $\overline{\mathcal{M}}$ as $\emptyset = \emptyset \cup \emptyset$ and $\emptyset \subset \emptyset$ has $\mu(\emptyset) = 0$.
- Countable Unions: For $(A_n) \in \overline{\mathcal{M}}$ with $A_n = E_n \cup Z_n$ and $E_n \in \mathcal{M}$, $Z_n \subset F_n$ with $F_n \in \mathcal{M}$ and $\mu(F_n) = 0$ we have

$$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} (E_n \cup Z_n) = \bigcup_{n=1}^{\infty} E_n \cup \bigcup_{n=1}^{\infty} Z_n \tag{1}$$

  Since $\bigcup Z_n \subset \bigcup F_n$ and $\mu\left(\bigcup F_n\right) = 0$ (a countable union of measure zero sets has measure zero) the set $\bigcup Z_n$ is a subset of a measure zero set. Since also $\bigcup_n E_n$ is in $\mathcal{M}$, the right hand side of (1) is in $\overline{\mathcal{M}}$ and we have shown that the union of countably many sets in $\overline{\mathcal{M}}$ is again in $\overline{\mathcal{M}}$.
- Complements: For $A = E \cup Z \in \overline{\mathcal{M}}$ with $Z \subset F$ and $\mu(F) = 0$ we write the complement as (use $F^c \subset Z^c$)

$$A^c = E^c \cap Z^c = ((E^c \cap F^c) \cup (E^c \cap F)) \cap Z^c = (E^c \cap F^c \cap Z^c) \cup (E^c \cap F \cap Z^c) = (E \cup F)^c \cup (E^c \cap F \cap Z^c)$$

  Clearly $(E \cap F)^c \in \mathcal{M}$ and $E^c \cap F \cap Z^c \subset F$ is a subset of a measure zero set, hence $A^c \in \overline{\mathcal{M}}$.
  Remark: If one assumes in addition that the sets E and F above are disjoint one can remove the E^c from the second bracket (as claimed in an earlier version of this file). Note that one can assume this without loss of generality, since if $A = E' \cup Z'$ with $Z' \subset F'$ and $\mu(F') = 0$ does not satisfy this, one writes $A = E' \cup (Z' \cap (E')^c)$ with $Z' \cap (E')^c \subset F' \cap (E')^c$ a subset of a measure zero set.
- Closure under intersections now follows from de Morgan's laws (and using closure under countable unions and complements).

b) We first check that $\overline{\mu}$ is well-defined (i.e. that different representation of an element in $\overline{\mathcal{M}}$ give the same measure. Indeed if $E_1 \cup Z_1 = E_2 \cup Z_2$ with $Z_i \subset F_i$ and $F_i \in \mathcal{M}$ with $\mu(F_i) = 0$, then we have $E_1 \subset E_1 \cup Z_1 = E_2 \cup Z_2 \subset E_2 \cup F_2$ and hence $\mu(E_1) \leq \mu(E_2)$. Similarly $E_2 \subset E_1 \cup F_1$ and $\mu(E_2) \leq \mu(E_1)$. Therefore $\overline{\mu}(E_1 \cup Z_1) = \mu(E_1) = \mu(E_2) = \overline{\mu}(E_2 \cup Z_2)$.

   It remains to show $\overline{\mu}$ is a measure. (It has to be complete since if $A = E \cup Z \in \overline{\mathcal{M}}$ (with $Z \subset F \in \mathcal{M}$ having $\mu(F) = 0$) has $\overline{\mu}(A) = \mu(E) = 0$, then, given a subset $B \subset A$, we have $B \subset A = E \cup Z \subset E \cup F$, and hence B is a subset of the measure zero set $E \cup F$, hence included in $\overline{\mathcal{M}}$.)

   We check countable additivity. Let $(A_n = E_n \cup Z_n)_{n=1}^{\infty}$ be a countable collection of disjoint sets in $\overline{\mathcal{M}}$. We then have, since the E_n are necessarily disjoint

$$\overline{\mu}\left(\bigcup_{n=1}^{\infty} A_n\right) = \overline{\mu}\left(\bigcup_{n=1}^{\infty} E_n \cup Z_n\right) = \overline{\mu}\left(\bigcup_{n=1}^{\infty} E_n \cup \bigcup_{n=1}^{\infty} Z_n\right) = \mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} \overline{\mu}(A_n) \,.$$

3 Elementary Families

We first note that it is sufficient to show that $\mathcal{A}$ is closed under finite unions and complements. The closure under finite intersections is then a simple consequence of de Morgan's laws. Since the union of finitely many elements in $\mathcal{A}$ is a finite union of elements in $\mathcal{E}$, to show closure of $\mathcal{A}$ under finite unions it suffices to show that an arbitrary finite union of elements in $\mathcal{E}$ is in $\mathcal{A}$. To show the latter we proceed according to the hint:

- $A, B \in \mathcal{E} \implies A \setminus B \in \mathcal{A}$. Indeed,

$$A \setminus B = A \cap B^c = A \cap \left(\cup_{n=1}^{N} \tilde{B}_n\right) = \cup_{n=1}^{N} \left(A \cap \tilde{B}_n\right)$$

  with the $\tilde{B}_n \in \mathcal{E}$ disjoint. Since the right hand side is a disjoint union of sets in $\mathcal{E}$ (by the intersection property of the elementary family) we conclude $A \setminus B \in \mathcal{A}$ as desired.

- $A, B \in \mathcal{E} \implies A \cup B \in \mathcal{A}$. Indeed, using the previous item,

$$A \cup B = (A \setminus B) \cup B = \left(\cup_{n=1}^{N} C_n\right) \cup B$$

with the C_n disjoint and their union disjoint from B. The right hand side is hence a disjoint union of sets in $\mathcal{E}$ and therefore in $\mathcal{A}$.

- $A_1, ..., A_N \in \mathcal{E} \implies \bigcup_{n=1}^{N} A_n \in \mathcal{A}$.

We prove this by induction. The base case $N = 2$ was shown above so let us assume the statement is true for $N - 1$ and consider

$$\bigcup_{n=1}^{N} A_n = \left(\bigcup_{n=1}^{N-1} A_n\right) \cup A_N = \left(\bigcup_{m=1}^{M} \tilde{A}_m\right) \cup A_N = \left(\bigcup_{m=1}^{M} \left(\tilde{A}_m \setminus A_N\right)\right) \cup A_N$$

where the $\tilde{A}_m$ are disjoint by the induction assumption. The last expression is a (finite) disjoint union of sets in $\mathcal{A}$, which by definition can be written as a (finite and still) disjoint union of sets in $\mathcal{E}$ and hence lies in $\mathcal{A}$.

It remains to show that $\mathcal{A}$ is closed under complements. We let $A = \bigcup_{n=1}^{N} A_n \in \mathcal{A}$ with the $A_n \in \mathcal{E}$ disjoint and consider

$$A^c = \bigcap_{n=1}^{N} (A_n)^c = \bigcap_{n=1}^{N} \bigcup_{m=1}^{N_m} \tilde{A}_{nm} \quad \text{with all } \tilde{A}_{nm} \in \mathcal{E}$$

where the last identity follows from the fact that the complement of a set in $\mathcal{E}$ can be written as a disjoint union of sets in $\mathcal{E}$. We now observe the set theoretic identity

$$\bigcap_{n=1}^{N} \bigcup_{m=1}^{M_m} \tilde{A}_{nm} = \bigcup_{m_1=1}^{M_1} \bigcup_{m_2=1}^{M_2} \cdots \bigcup_{m_N=1}^{M_N} \left(\bigcap_{n=1}^{N} \tilde{A}_{nm_j}\right) \tag{2}$$

which finishes the proof since the intersection of a finite number of sets in $\mathcal{E}$ is in $\mathcal{E}$ and we've shown that any finite union of sets in $\mathcal{E}$ is in $\mathcal{A}$. To verify the identity (2) observe that

$$x \in \bigcap_{n=1}^{N} \bigcup_{m=1}^{M_m} \tilde{A}_{nm} \iff \text{for any } n \text{ there exists an } m'_n \in \{1, 2, ..., M_n\} \text{ such that } x \in \tilde{A}_{nm'_n}$$

$$x \in \bigcup_{m_1=1}^{M_1} \bigcup_{m_2=1}^{M_2} \cdots \bigcup_{m_N=1}^{M_N} \left(\bigcap_{n=1}^{N} \tilde{A}_{nm_j}\right) \iff \text{there exists a tuple } (m'_1, m'_2, ...m'_n) \text{ such that } x \in \tilde{A}_{nm'_j} \text{ for all } n$$

from which the identity of the two sets is easily seen.

For the second part of the problem, we only need to show that the given subsets of the real line form an elementary family. By the first part of we can then conclude that finite disjoint unions of such sets form an algebra. We clearly have $\emptyset \in \mathcal{E}$ and

- $\emptyset \cap (a, \infty) = \emptyset$, $\emptyset \cap (a, b] = \emptyset$,
- $(a, \infty) \cap (a', b'] = \emptyset$ if $b' \leq a$ and $(a, \infty) \cap (a', b'] = (\max(a', a), b'] \in \mathcal{E}$ for $b' > a$.

For the complement we check

- $(\emptyset)^c = (-\infty, \infty) \in \mathcal{E}$,
- $(a, \infty)^c = \emptyset \in \mathcal{E}$ if $a = -\infty$ and $(a, \infty)^c = (-\infty, a] \in \mathcal{E}$ if $a > -\infty$,
- $(a, b]^c = (b, \infty) \in \mathcal{E}$ if $a = -\infty$ and $(a, b]^c = (-\infty, a] \cup (b, \infty)$ if $a > -\infty$, the latter being a union of two elements in $\mathcal{E}$.

4 Uniqueness of the extension of a premeasure

We start with the following observation: If $E = \bigcup_{n=1}^{\infty} E_n$ with $E_n \in \mathcal{A}$ then by the regularity properties of the measure (the proof is the same for general measures!) and the fact that the two measures agree on $\mathcal{A}$

$$\nu(E) = \lim_{n\to\infty} \nu\left(\bigcup_{j=1}^{n} E_j\right) = \lim_{n\to\infty} \mu\left(\bigcup_{j=1}^{n} E_j\right) = \mu(E) \, .$$

Hence μ and ν agree on countable unions of sets in $\mathcal{A}$.

We now use the hint and first assume $F \in \mathcal{M}(\mathcal{A})$ has finite measure.

We take any covering $F \subset \bigcup_{n=1}^{\infty} E_n$ with $E_n \in \mathcal{A}$ from which we deduce

$$\nu(F) \le \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} \mu_0(E_n)$$

and after taking the inf over all such coverings $\nu(F) \le \mu(F)$ by the definition of μ in Theorem 5.2.

To prove the reverse we fix $\epsilon > 0$ and take a covering $F \subset E = \bigcup_{n=1}^{\infty} E_n$ with $\mu(F) \ge \mu(E) - \epsilon$. Since $\mu(E) < \infty$ we have $\mu(E \setminus F) \le \epsilon$ and hence

$$\mu(F) \le \mu(E) = \nu(E) = \nu(F) + \nu(E \setminus F) \le \nu(F) + \mu(E \setminus F) \le \nu(F) + \epsilon \, ,$$

where we used the observation made at the beginning and the fact that we already showed the other direction for any set of finite measure. Since $\epsilon > 0$ was arbitrary we are done.

We finally use the assumption of σ-finiteness. We have $X = \bigcup_{n=1}^{\infty} E_n$ with $\infty > \mu(E_n) = \nu(E_n)$. We can wlog assume that the E_n are disjoint.[1] Let F be an arbitrary element of $\mathcal{M}(\mathcal{A})$. We then have

$$\mu(F) = \mu\left(\bigcup_{n=1}^{\infty} (F \cap E_n)\right) = \sum_{n=1}^{\infty} \mu(F \cap E_n) = \sum_{n=1}^{\infty} \nu(F \cap E_n) = \nu(F) \, .$$

Note that this proof also works in the situation discussed in the Remark.

5 Hausdorff measure

Recall that for any $E \subset \mathbb{R}^d$ we defined the exterior α-dimensional Hausdorff-measure of E as

$$m_\alpha^\star(E) := \lim_{\delta\to 0} \mathcal{H}_\alpha^\delta(E)$$

where

$$\mathcal{H}_\alpha^\delta(E) = \inf\Big\{ \sum_k (diam F_k)^\alpha \;\mid E \subset \bigcup_{k=1}^{\infty} F_k \;\; , \;\; diam F_k \le \delta \;\text{ for all } k. \Big\}$$

a) We have $\mathcal{H}_\alpha^\delta(\emptyset) = 0$ for any δ and hence $m_\alpha^\star(\emptyset) = 0$. We also have $\mathcal{H}_\alpha^\delta(E_1) \le \mathcal{H}_\alpha^\delta(E_2)$ for $E_1 \subset E_2$ and any δ (as any covering of E_2 is also one of E_1 and hence for E_1 we are taking the inf over set at least as large). The limit must therefore satisfy $m_\alpha^\star(E_1) \le m_\alpha^\star(E_2)$. To show subadditivity we let $(E_n)_{n=1}^{\infty}$ be a countable family of sets. We fix $\delta > 0$.

For every E_n we choose a countable covering F_{nk} with

$$\sum_k (diam(F_{nk}))^\alpha \ge \mathcal{H}_\alpha^\delta(E_n) \ge \sum_k (diam(F_{nk}))^\alpha - \frac{\epsilon}{2^n} \tag{3}$$

[1]If they are not disjoint, define $\tilde{E}_n = E_n \setminus \bigcup_{i=1}^{n-1} E_i$. Then the $\tilde{E}_n$ are disjoint, have finite measure and their union is again X.

The union $\bigcup_{n,k} F_{nk}$ then covers $\bigcup_{n=1}^{\infty} E_n$ and hence

$$\mathcal{H}_\alpha^\delta \left(\bigcup_{n=1}^{\infty} E_n \right) \leq \sum_{k,n} (diam(F_{nk}))^\alpha = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (diam(F_{nk}))^\alpha \leq \sum_{n=1}^{\infty} \mathcal{H}_\alpha^\delta(E_n) + \epsilon$$

with the first inequality holding by the definition of the inf, the second by the fact that the sum is independent of the rearrangement (cf. Example Sheet 2 and Exercise 6 below!) and the third by inserting (3). Since this holds for any $\epsilon > 0$ we have, for any $\delta > 0$

$$\mathcal{H}_\alpha^\delta \left(\bigcup_{n=1}^{\infty} E_n \right) \leq \sum_{n=1}^{\infty} \mathcal{H}_\alpha^\delta(E_n) \, .$$

Both sides are increasing in δ we can take the limit to obtain $m_\alpha^\star \left(\bigcup_{n=1}^{\infty} E_n \right) \leq \sum_{n=1}^{\infty} m_\alpha^\star(E_n)$ as desired.

b) The $\leq$ direction holds by subadditivity proven in a), so we only need to show $\geq$. Since A and B have positive distance, we fix any $\delta < \frac{1}{2} d(A, B)$ and consider a covering of $A \cup B$ by sets E_n with diameter smaller than δ. We then define

$$\tilde{E}_n = E_n \cap A \quad \text{and} \quad E_n' = E_n \cap B$$

By the choice of δ, E_n can only intersect either A or B (or none). In particular, the collection of sets $(\tilde{E}_n)$ and (E_n') are necessarily disjoint with $(\tilde{E}_n)$ covering A and (E_n') covering B and also $\tilde{E}_n \cup E_n' \subset E_n$. It follows that

$$\mathcal{H}_\alpha^\delta(A) + \mathcal{H}_\alpha^\delta(B) \leq \sum_n (diam(\tilde{E}_n))^\alpha + \sum_n (diam(E_n'))^\alpha \leq \sum_n (diam(E_n))^\alpha \, .$$

Taking the inf over all coverings E_n of $A \cup B$ we obtain, for any fixed $\delta < \frac{1}{2} d(A, B)$ the inequality

$$\mathcal{H}_\alpha^\delta(A) + \mathcal{H}_\alpha^\delta(B) \leq \mathcal{H}_\alpha^\delta(A \cup B) \leq m_\alpha^\star(A \cup B) \, .$$

Taking the limit as $\delta \to 0$ produces $m_\alpha^\star(A) + m_\alpha^\star(B) \leq m_\alpha^\star(A \cup B)$ as desired.

c) Consider the line segments $A = [-1, 1] \times \{\epsilon\}$ and $B = [-1, 1] \times \{-\epsilon\}$ in $\mathbb{R}^2$ for $0 < \epsilon < 1/2$. We have $d(A, B) = \epsilon$. Both A and B are contained in a ball of radius $1 + \epsilon$, which implies

$$\tilde{m}_1(A \cup B) \leq 1 + \epsilon \, .$$

On the other hand, we have

$$\tilde{m}_1(A) = 1 \quad \text{and} \quad \tilde{m}_1(B) = 1 \, . \tag{4}$$

Since $1 + 1 \geq 1 + \epsilon$ the equality in *b*) is indeed violated. [To establish (4), note that the $\leq$ direction follows by choosing small almost disjoint balls covering A. To get the lower bound, assume $\tilde{m}_1(A) \leq 1 - \eta$ for some $\eta > 0$. Then there exists a $\delta > 0$ and a covering of A by sets F_n of diameter smaller than δ with

$$\sum_n diam(F_n) < 1 - \frac{\eta}{2} \, .$$

If we denote the diameter of the sets F_n by δ_n, then $\sum_n \delta_n < 1 - \frac{\eta}{2}$. But a set of diameter δ_n can cover at most a closed interval of length δ_n of $[0, 1]$. Since $\sum_n \delta_n < 1 - \frac{\eta}{2}$, the F_n cannot cover an interval of length 1, which is in contradiction with the F_n covering $[0, 1]$. Hence $\tilde{m}_1(A) \geq 1$.]

6 An example of the general Fubini-Tonelli-theorem

As suggested in the hint we define the measure space $(X = \mathbb{N}, \mathcal{M} = \mathcal{P}(X), \mu)$ with $\mu : \mathcal{P}(X) \to [0,\infty]$ the counting measure, i.e. if $U = \{x_1, x_2, ..., x_n\}$ is any finite subset of $\mathbb{N}$, then $\mu(U) = n$, the number of elements in U while if U has infinitely many elements, then $\mu(U) = +\infty$. The measure μ is clearly σ-finite. Given a (trivially measurable as all subsets of X are measurable) extended valued, *non-negative* function $f : \mathbb{N} \to \overline{\mathbb{R}}$, we define its integral to be

$$\int_{\mathbb{N}} f d\mu := \sup_F \sum_{n \in F \subset \mathbb{N}} f(n) = \sum_{n=1}^{\infty} f(n)$$

with the sup taken over all finite subsets of $\mathbb{N}$.[2] A general function $f : \mathbb{N} \to \overline{\mathbb{R}}$ is called integrable if $\sum_{n=1}^{\infty} |f(n)| < \infty$ is convergent. Of course an integrable function is necessarily finite valued.

Next we define the (unique since μ is σ-finite) product measure $\mu \times \mu$ on $\mathcal{M} \otimes \mathcal{M}$ (which is easily seen to consist of all subsets of $X \times Y$, since the algebra $\mathcal{M} \boxtimes \mathcal{M}$ contains all points $(x, y) \in X \times Y$ and the σ-algebra generated from $\mathcal{M} \boxtimes \mathcal{M}$ all countable unions of one-point sets) as outlined in the notes. The integral of an extended valued *non-negative* function $f : X \times Y \to \overline{\mathbb{R}}$ is defined as

$$\int_{\mathbb{N}\times\mathbb{N}} f \, d(\mu \times \mu) := \sup_F \sum_{(m,n) \in F \subset \mathbb{N}\times\mathbb{N}} f(m, n) \, ,$$

where the sup is taken over all finite subsets of $\mathbb{N} \times \mathbb{N}$. The general Tonelli theorem now says that if $f : \mathbb{N} \times \mathbb{N} \to [0,\infty]$ is a non-negative function, then

$$\sup_F \sum_{(m,n)\in F\subset\mathbb{N}\times\mathbb{N}} f(m, n) = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} f(m, n) \right) = \sum_{m=1}^{\infty} \left(\sum_{n=1}^{\infty} f(m, n) \right) \tag{5}$$

holds in the extended sense. Using the notation $f(m, n) = x_{m,n}$ for the function f we obtain the identity on the example sheet.

[2] Note that this is precisely the definition resulting from the four stage procedure for defining the integral as done in class for the Lebesgue integral.