Measure and Integration: Example Sheet 1

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1 The Cantor Set

It is easy to see that C_n consists of 2^n disjoint closed intervals of length 3^{-n}.

a) Prove that $\mathfrak{C}$ is compact and non-empty.

Since all the C_n are closed sets, and since an arbitrary intersection of closed sets is also closed. the set $\mathfrak{C}$ is closed. As it is a subset of $[0, 1]$ it is also bounded, hence compact. To see non-emptyness, note that clearly $0 \in \mathfrak{C}$. In fact, it is easy to see that the endpoints of every closed interval $C_{n,k}$ in the disjoint union $C_n = C_{n,1} \cup C_{n,2} \cup ... \cup C_{n,2^n}$ belong to $\mathfrak{C}$.

b) Prove that $\mathfrak{C}$ is totally disconnected, i.e. given x and y in $\mathfrak{C}$ with $x \neq y$ there is a $x < z < y$ with $z \notin \mathfrak{C}$.

Given $x < y$ we have $|x - y| = \delta > 0$. Choose n such that $3^{-n} < \delta$. Clearly x and y both have to be in C_n. The length of each of the disjoint closed intervals in C_n is 3^{-n}, so x and y have to lie in different connected components of $C_n = C_{n,1} \cup C_{n,2} \cup ... \cup C_{n,2^n}$ (from left to right). If x lies in $C_{n,k}$ then necessarily $k < 2^n$ and the open interval between $C_{n,k}$ and $C_{n,k+1}$ contains only points from $\mathfrak{C}^c$.

c) Prove that $\mathfrak{C}$ does not have isolated points.

We need to show that any δ-neighbourhood of an arbitrary $x \in \mathfrak{C}$ contains a point from $\mathfrak{C}$. Let hence $x \in \mathfrak{C}$ be given and $\delta > 0$ prescribed. Let n be such that $3^{-n} < \delta$. Clearly $x \in C_n$, so x sits in one of the connected components of $C_n = C_{n,1} \cup C_{n,2} \cup ... \cup C_{n,2^n}$, say $C_{n,k}$. But since the left and right endpoints of the $C_{n,k}$ are in $\mathfrak{C}$ and since $C_{n,k}$ has length 3^{-n} we can find at least one point $y \in \mathfrak{C} \cap C_{n,k}$ such that $|x - y| < 3^{-n} < \delta$.

d) Prove that $m_\star(\mathfrak{C}) = 0$.

Note that $\mathfrak{C} \subset C_n$ for any n. By the monotonicity property of the exterior measure, we have $m_\star(\mathfrak{C}) \leq m_\star(C_n)$. Now since C_n is a disjoint union of 2^n compact intervals of length 3^{-n}, we have $m_\star(C_n) = (2/3)^n$. We conclude that $m_\star(\mathfrak{C}) < \epsilon$ for any $\epsilon > 0$ and hence $m_\star(\mathfrak{C}) = 0$.

e) Show that we can write C_n as

$$C_n = \bigcup_{a_1,\dots,a_n \in \{0,2\}} \left[\sum_{k=1}^{n} a_k 3^{-k}, \sum_{k=1}^{n} a_k 3^{-k} + \frac{1}{3^n}\right].$$

We mentioned several times that C_n contains 2^n intervals of length 3^{-n}. To show the claim of the hint, we first note that the left endpoint of C_0 is 0 consistent with the formula. Assume now that for $n = N$ the 2^N left endpoints of the closed disjoint intervals in C_N are indeed given by the 2^N numbers arising as $\sum_{k=1}^{N} a_k 3^{-k}$ by different choices of $a_k \in \{0, 2\}$. We now look at C_{N+1}, which arises from C_N by deleting the mid-third intervals. Therefore, any of the 2^N connected component of C_N (which each had 1 left endpoint, say p_i) is decomposed into two different disjoint intervals with endpoints p_i and $p_i + 2\dot{3}^{-N-1}$. Therefore, the left endpoints of C_{N+1} are

$$\sum_{k=1}^{N} a_k 3^{-k} + 0 \quad \text{with } a_k \in \{0,2\} \quad \text{and} \quad \sum_{k=1}^{N} a_k 3^{-k} + 2\cdot 3^{-N-1} \text{with } a_k \in \{0,2\}$$

But the collection of these points can be written as $\sum_{k=1}^{N+1} a_k 3^{-k}$ with $a_k \in \{0,2\}$ and the proof by induction is completed. The overall claim then follows by noting that the intervals in C_n have length 3^{-n}.

f) Show that $x = \sum_{k=1}^{\infty} a_k 3^{-k}$ with $a_k \in \{0,2\} \Leftrightarrow x \in \mathfrak{C}$.

Let $x = \sum_{k=1}^{\infty} a_k 3^{-k}$ for some fixed collection of (a_k). We need to show that $x \in C_n$ for any n. But for any n we have

$$\sum_{k=1}^{n} a_k 3^{-k} \leq x \leq \sum_{k=1}^{n} a_k 3^{-k} + \sum_{k=n+1}^{\infty} 2\cdot 3^{-k} = \sum_{k=1}^{n} a_k 3^{-k} + \frac{1}{3^n}$$

which is precisely the statement that x is contained in one of the intervals of C_n.

Conversely, let $x \in \mathfrak{C}$, so x is in any C_n. We construct a (unique) sequence (a_k) (consisting of 0's and 2) such that we have

$$\sum_{k=1}^{n} a_k 3^{-k} \leq x \leq \sum_{k=1}^{n} a_k 3^{-k} + \frac{1}{3^n}\,. \tag{1}$$

for any n as follows. Clearly for any fixed N there are unique a_k with $k = 1, ..., N$ satisfying (1) with $n = N$ as x lies in exactly one of the disjoint intervals of C_N. Similarly, since $x \in C_{N+1}$, there are unique $\tilde{a}_k$ with $k = 1, ..., N+1$ satisfying (1) with $n = N+1$. We now show that $a_k = \tilde{a}_k$ for $k = 1, ..., n$, which (by induction) shows that the the coefficients a_k in (1) do not depend on n. To see this, note that if $x \in C_n$, then the interval of C_{n+1} of which x is an element can only be the left third or the right third of the interval of C_n of which x is an element of. So either

$$\sum_{k=1}^{N} a_k 3^{-k} + \frac{0}{3^{N+1}} \leq x \leq \sum_{k=1}^{N} a_k 3^{-k} + \frac{1}{3^{N+1}} \quad \text{or} \quad \sum_{k=1}^{N} a_k 3^{-k} + \frac{2}{3^{N+1}} \leq x \leq \sum_{k=1}^{N} a_k 3^{-k} + \frac{3}{3^{N+1}}\,.$$

In the first case,

$$\sum_{k=1}^{N+1} a_k 3^{-k} \leq x \leq \sum_{k=1}^{N+1} a_k 3^{-k} + \frac{1}{3^{N+1}}$$

holds with $a_{k+1} = 0$ while in the second the above holds with $a_{k+1} = 2$. This show ins particular, $a_k = \tilde{a}_k$ for $k = 1, ..., n$ as desired.

With (1) established for all n we observe that the sum on the left converges and that in fact the left hand side and the right hand side converge to the same value, giving $x = \sum_{k=1}^{\infty} a_k 3^{-k}$ as claimed.

g) Define the Cantor-Lebesgue function $F : \mathfrak{C} \to [0,1]$ as

$$F(x) = \sum_{k=1}^{\infty} \frac{b_k}{2^k} \quad \text{for } x = \sum_{k=1}^{\infty} a_k 3^{-k}, \text{ where } b_k = \frac{a_k}{2}.$$

Show that F is well-defined and in fact continuous on $\mathfrak{C}$. Show also that F is surjective. Conclude that $\mathfrak{C}$ is uncountable.

To show that F is well defined we need to show that the (a_k) in the expansion $x = \sum_{k=1}^{\infty} a_k 3^{-k}$ are unique. This is easy to see because suppose there were two sequences (a_k) and $(\tilde{a}_k)$ giving rise to the same x and differing at position N where $a_N = 0$ and $\tilde{a}_N = 2$ (or the other way round). Then we must have

$$0 + \sum_{N+1}^{\infty} a_k 3^{-k} = \frac{2}{3^N} + \sum_{N+1}^{\infty} \tilde{a}_k 3^{-k}$$

and in particular

$$0 + \sum_{N+1}^{\infty} 2 \cdot 3^{-k} \geq \frac{2}{3^N} \quad \text{which implies} \quad \frac{1}{3^N} \geq \frac{2}{3^N}, \quad \text{a contradiction.}$$

Surjectivity follows from the fact that every real number has a (non-unique) dyadic expansion. Surjectivity implies that the cardinality of $\mathfrak{C}$ is at least that of $[0,1]$ but since the latter is a subset of the former, they must have the same cardinality.

To show continuity at $x \in \mathfrak{C}$, we let (x_n) be an arbitrary sequence in $\mathfrak{C}$ with $x_n \to x$. We need to show that given $\epsilon > 0$ we can find N such that for all $n \geq N$ we have $|F(x) - F(x_n)| < \epsilon$.

We fix a K large such that $\sum_{k=K+1}^{\infty} \frac{1}{2^k} < \epsilon$. We let $x_n = \sum_{k=1}^{\infty} (a_n)_k 3^{-k}$ and $x = \sum_{k=1}^{\infty} a_k 3^{-k}$. (We already know that the $(a_n)_k$ and a_k are unique.) Since we have $x_n \to x$ we can choose an N such that we have $|x - x_n| < \frac{1}{3^{K+1}}$ for all $n \geq N$. It is easy to see that this implies $(a_n)_k = a_k$ for all $k = 1, ..., K$ and $n \geq N$ (get a contradiction using geometric series). With these choices, we have for all $n \geq N$

$$|F(x_n) - F(x)| = \Big| \sum_{k=1}^{\infty} \frac{(a_n)_k - a_k}{2} \frac{1}{2^k} \Big| = \Big| \sum_{K+1}^{\infty} \frac{(a_n)_k - a_k}{2} \frac{1}{2^k} \Big| \leq \sum_{K+1}^{\infty} \frac{1}{2^k} < \epsilon$$

as desired. Remark: F can actually be extended to a continuous function on all of $[0,1]$.

2 Fat Cantor Sets

a) Show that $m_\star\left(\hat{\mathfrak{C}}\right) = \frac{1}{2}$ and conclude that $\hat{\mathfrak{C}}$ is uncountable.

Note that $\hat{\mathfrak{C}}$ is measurable and $m(\hat{\mathfrak{C}}) = m_\star(\hat{\mathfrak{C}})$. It is also clear that $\hat{C}_k$ is measurable and we compute the measure of its complement in $[0,1]$ (the intervals we remove) for $k \geq 1$ to be

$$\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + ... + \frac{1}{2}\frac{1}{2^k} = \frac{1}{2}\sum_{j=1}^{k} \frac{1}{2^j} = \frac{1}{2} - \frac{1}{2^{k+1}}.$$

Hence $m\left(\hat{C}_k\right) = \frac{1}{2} + \frac{1}{2^{k+1}}$. We have $\hat{C}_k \supset \hat{C}_{k+1}$ and $m\left(\hat{C}_1\right) = 1$ and hence by a result from lectures $m\left(\cap_{k=1}^{\infty} \hat{C}_k\right) = \lim_{k\to\infty} m\left(\hat{C}_k\right) = \frac{1}{2}$. If $\hat{\mathfrak{C}}$ was countable we would necessarily have $m_\star(\mathfrak{C}) = 0$.

b) Show that $\hat{\mathfrak{C}}$ is again compact, totally disconnected and has no isolated points.

The proof is very similar to the first question and will not be repeated.

3 *Characterisation of Riemann integrable functions

Prove that a bounded function on an interval $[a,b]$ is Riemann integrable if and only if its set of discontinuities has measure zero.

This is a hard problem. You should look up the outline of the proof in Stein-Shakarchi (p.47) or elsewhere in the literature. You can use the result at the end of Exercise 4.

4 Limits of continuous functions $f : [0,1] \to \mathbb{R}$ (Exercise 10 of [SS])

In this exercise we construct the sequence of continuous functions promised in the first lecture.

Let $\hat{\mathfrak{C}}$ denote the Fat Cantor Set constructed in Exercise 2. We define the function F_1 to be a piecewise linear continuous function which is equal to 1 on the complement of the first interval $I_1 = [3/8, 5/8]$ removed from $[0,1]$ in the construction of $\hat{\mathfrak{C}}$ and zero at the point at the centre of I_1. Similarly we construct F_2 to be a piecewise linear continuous function which is equal to 1 on the complement of the open intervals removed at the second stage and zero at the centre of these intervals. Continuing in this way we can define F_n for all n and finally

$$f_n = F_1 \cdot F_2 \cdot \ldots \cdot F_n$$

a) Show that for all $n \geq 1$ and all $x \in [0,1]$ one has $0 \leq f_n(x) \leq 1$ and $f_n(x) \geq f_{n+1}(x)$. We conclude that $f_n(x)$ converges to a limit which we denote by $f(x)$.

All easy to see.

b) Show that the function f is discontinuous at every point of $\hat{\mathfrak{C}}$. Conclude that f is not Riemann integrable (despite the sequence $s_n = \int f_n$ converging).
Hint: Note that $f(x) = 1$ if $x \in \hat{\mathfrak{C}}$ and find a sequence of points $\{x_n\}$ so that $x_n \to x$ and $f(x_n) = 0$.

The last conclusion follows from Exercise 3 so we will focus on establishing discontinuity at the points $x \in \hat{\mathfrak{C}}$. We first define a sequence x_n with $|x_n - x| < \frac{1}{2^n}$ as follows. We know that each $\hat{C}_k$ consists of 2^k disjoint intervals of length L_k necessarily smaller than 2^{-k} (the actual length can be computed to be $L_k = \frac{2^k+1}{2 \cdot 4^k}$). Therefore, given n we know that x is contained in one of the intervals of $\hat{C}_n$ of length L_n. At the next stage, L_n is tri-sected and an interval of length 4^{n+1} is removed from the middle. We choose x_n to be the point in the middle of that interval. Clearly $|x - x_n| < L_n < 2^{-n}$ by construction. Moreover, since x_n sits at the middle point of an interval that gets removed, we have $F_{n+1}(x_n) = 0$ for all n and hence $\lim_{k\to\infty} f_k(x_n) = f(x_n) = 0$ for any n. This is the desired sequence which clearly proves that f is not continuous at x.

5 The outer Jordan content of a set (Exercise 14 of [SS])

The outer Jordan content $J_\star(E)$ of a bounded set E in $\mathbb{R}$ is defined by

$$J_\star(E) = \inf \sum_{j=1}^{N} |I_j|$$

where the infimum is taken over all *finite* coverings $E \subset \bigcup_{j=1}^{N} I_j$ by intervals I_j.

a) Prove that $J_\star(E) = J_\star\left(\overline{E}\right)$ for every set E. Here $\overline{E}$ denotes the closure of E.

Wlog we can assume that the intervals I_j of the finite covering are closed (why?).

But then, if E is covered by finitely many closed intervals I_i its closure $\overline{E}$ is also covered by those intervals because the closure is the *smallest* closed set containg E and the union of the finitely many closed intervals is a closed set containing E. It follows that $J_\star(E) \geq J_\star\left(\overline{E}\right)$. Since the other direction is clear by monotonicity, the claim follows.

b) Exhibit a countable subset $E \subset [0,1]$ such that $J_\star(E) = 1$ while $m_\star(E) = 0$.

Take $E = \mathbb{Q} \cap [0,1] \subset [0,1]$. Since the rationals are dense in $[0,1]$, their closure is all of $[0,1]$ which has outer Jordan content 1. But E is countable, so $m_\star(E) = 0$.