Measure and Integration: Example Sheet 2

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1 Tonelli's Theorem for sequences

We first establish $\sup_F \sum_F x_{m,n} \leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{m,n}$. We can assume that the right hand side is finite since if it is $+\infty$ the inequality holds trivially. We first verify the hint by noting that

$$\sum_F x_{m,n} \leq \sum_{m=1}^{M} \sum_{n=1}^{N} x_{m,n} \leq \sum_{m=1}^{M} \sum_{n=1}^{\infty} x_{m,n} \leq \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{m,n}$$

since the elements of any fixed finite subset F satisfy $(m \leq M, n \leq N)$ for some M and N. Taking the sup over all F immediately yields one of the desired directions.

We now show $\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} x_{m,n} \leq \sup_F \sum_F x_{m,n}$. We can assume that the right hand side is finite as otherwise the inequality is trivial. We start from the fact that

$$\sum_{m=1}^{M} \sum_{n=1}^{N} x_{m,n} \leq \sup_F \sum_F x_{m,n} \tag{1}$$

holds for any fixed M and N. Fixing M, we can take the limit as $N \to \infty$ (of the increasing sequence on the left) to obtain

$$\sum_{m=1}^{M} \sum_{n=1}^{\infty} x_{m,n} \leq \sup_F \sum_F x_{m,n} \tag{2}$$

for any M. Taking the limit $M \to \infty$ we obtain the desired result.

2 Distances between sets

a) One example is given by the following subsets of $\mathbb{R}$: Set

$$E = \mathbb{N} = \{1, 2, 3, ...\} \text{ and } F = \{1 + \frac{1}{2}, 2 + \frac{1}{4}, ..., n + \frac{1}{2n}, ...\}$$

E and F are both closed and disjoint from one another with the distance being equal to zero.

b) Let now E be compact and F closed. For each point x in E we can define its distance $d(x, F) = \delta_x$. Since E is compact, the cover of E by balls $E \subset \bigcup_x B_{\frac{\delta_x}{2}}(x)$ has a finite subcover, say $\bigcup_{n=1}^{N} B_{\frac{\delta_{x_n}}{2}}(x_n)$.

Now an arbitrary point $x \in E$ sits in (at least) one $B_{\frac{\delta_{x_n}}{2}}(x_n)$, hence $d(x, x_n) \leq \frac{\delta_{x_n}}{2}$. Therefore for this x

$$d(F, x) \geq d(F, x_n) - d(x_n, x) \leq \delta_{x_n} - \frac{\delta_{x_n}}{2} = \frac{\delta_{x_n}}{2} \geq \frac{1}{2} \min(\delta_{x_1}, ..., \delta_{x_N})$$

Since x was arbitrary, we have shown that $d(F, E) \geq \frac{1}{2} \min(\delta_{x_1}, ..., \delta_{x_N}) > 0$.

3 Approximation of measurable sets

a) We proved on Sheet 1 that the Fat Cantor set $\hat{\mathfrak{C}}$ does not have any interior points. It follows that there are no open (in $\mathbb{R}$) subsets of $\hat{\mathfrak{C}}$ except for the empty set. But since $m\left(\hat{\mathfrak{C}} \setminus \emptyset\right) = m\left(\hat{\mathfrak{C}}\right) = 1/2$, we see that the property cannot hold for $\epsilon < \frac{1}{2}$.

b) Given $\epsilon > 0$ we first find a closed set F with $m(E \setminus F) < \frac{\epsilon}{2}$ (as proven in lectures). We then define the sequence of compact sets

$$K_n = F \cap Q_n \subset F$$

where Q_n is a closed cube of side length n centred at the origin (note that K_n is closed and bounded). The sequence K_n increases to F and the regularity properties of the measure imply that $m(K_n) + m(F \setminus K_n) = m(F) = \lim_{n\to\infty} m(K_n)$. In particular, there exists an N such that $m(F \setminus K_N) < \frac{\epsilon}{2}$.

$$m(E \setminus K_N) = m(E \setminus F \cup (F \setminus K_N)) \leq (E \setminus F) + m(F \setminus K_N) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon\,.$$

Hence K_N is the desired compact set.

c) We know that there exists a countable union of closed cubes F with $E \subset F := \bigcup_{n=1}^{\infty} Q_n$ and

$$m_\star(E) > m_\star(F) - \frac{\epsilon}{2} \tag{3}$$

Note we can replace $m_\star$ by m because both E and F are clearly measurable. The sequence $F_n = \bigcup_{i=1}^{n} Q_n$ increases to F and by the regularity properties of the measure $m(F_n) \to m(F) < \infty$. In particular, there exists an N such that $m(F \setminus F_N) < \frac{\epsilon}{2}$. Then

$$m(E \Delta F_N) = m(E \setminus F_N) + m(F_N \setminus E) \leq m(F \setminus F_N) + m(F \setminus E) < \frac{\epsilon}{2} + m(F) - m(E) < \epsilon,$$

where (3) has been used in the last step.

4 The Borel-Cantelli Lemma

We first establish the hint. Suppose $x \in E = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n} E_k$. Then x is in $\bigcup_{k \geq n} E_k$ for ANY $n \geq 1$. This clearly implies that x must lie in infinitely many E_k as if x were only in finitely many E_k, the last one being E_K say, this would contradict that $x \in \bigcup_{k \geq K+1} E_k$. Conversely, if $x \notin \bigcap_{n=1}^{\infty} \bigcup_{k \geq n} E_k$, then there must exist an N such that $x \notin \bigcup_{k \geq N} E_k$. But this implies that x can only lies in finitely many E_k (the largest being E_{N-1}).

The fact that E is measurable now follows immediately from the hint as E_k is measurable and countable unions and intersections of measurable sets are measurable.

To establish that $m(E) = 0$ we look at the sequence of measurable sets $F_N = \bigcap_{n=1}^{N} \bigcup_{k \geq n} E_k$ which decreases to E and $m(F_1) \leq \sum_{k=1}^{\infty} m(E_k) < \infty$ by assumption. Consequently, the regularity property of the measure implies that $m(E) = \lim_{N \to \infty} m(F_N)$. But $m(F_N) \leq \sum_{k=N}^{\infty} m(E_k)$ and the right hand side goes to zero as it is the tail end of a converging sum.

5 Failure of additivity on all sets for the exterior measure

a) Let E be a measurable subset of the non-measurable subset $\mathcal{N}$. As suggested by the hint, we consider the translated sets $E_k = E + r_k \subset \mathcal{N}_k$ (this is clear from the definition of $\mathcal{N}_k = \mathcal{N} + r_k$ given in the construction of $\mathcal{N}$ in class). We have

$$\bigcup_k E_k \subset \bigcup_k \mathcal{N}_k \subset [-1, 2]$$

the last inclusion having been shown in lectures. Moreover, since the $\mathcal{N}_k$ are pairwise disjoint, so must be the E_k. But then by countable additivity and monotonicity (note E_k and $\bigcup_k E_k$ are measurable)

$$\sum_{k=1}^{\infty} m\left(E_k\right) \leq 3\,.$$

Using $m\left(E_k\right) = m\left(E\right)$ by translation invariance of the Lebesgue measure we conclude $m\left(E\right) = 0$.

b) Suppose for contradiction that $m_\star\left(I \setminus \mathcal{N}\right) = 1 - \delta$ for some $0 < \delta \leq 1$. Then there exists an open set $\mathcal{U}$ with $I \setminus \mathcal{N} \subset \mathcal{U}$ and $m_\star\left(\mathcal{U}\right) < 1 - \frac{\delta}{2}$ (Property 3 of the exterior measure). It follows that $\mathcal{U}^c \cap I \subset \mathcal{N}$ and by part a) that $m\left(\mathcal{U}^c \cap I\right) = 0$. Hence

$$1 - \frac{\delta}{2} > m_\star\left(\mathcal{U}\right) = m\left(\mathcal{U}\right) = m\left(\mathcal{U}\right) + m\left(\mathcal{U}^c \cap I\right) \geq m\left(\mathcal{U} \cup \left(\mathcal{U}^c \cap I\right)\right) = m\left(I \cup \mathcal{U}\right) \geq m\left(I\right) = 1\,,$$

a contradiction.

c) Assuming for contradiction *equality* in the formula, we would have (by part b)) that

$$m_\star\left(\mathcal{N}\right) + 1 = m_\star\left(I\right) = 1$$

and hence $m_\star\left(\mathcal{N}\right) = 0$. But this is a contradiction with $\mathcal{N}$ being non-measurable as sets of exterior measure zero are measurable and have measure zero.

6 Fun-Stuff

We define a sequence of decreasing measurable sets $A_n \subset [0,1)$ (decreasing to the desired subset A) inductively as follows.

To define A_1, we remove from $[0,1)$ one interval of length $1/10$, namely $[0.4, 0.5)$. This removes all numbers which have a four as the first digit in their decimal expansion.

Now A_1 can be thought of as a union of nine intervals of length $1/10$, namely $\bigcup_n [0.n, 0.n)$ where the union is over $n = 0, 1, 2, 3, 5, 6, 7, 8, 9$. To obtain A_2 from A_1 we repeat the procedure for each of these intervals. Namely, we remove from each of these intervals one interval of length $1/100$ namely the numbers which have a four as the second digit of their decimal expansion (i.e. from $[0.n, 0.n)$ above we remove $[0.n4, 0, n5)$ etc). Clearly, then

$$m\left(A_1\right) = 9^0\left(1 - \frac{1}{10}\right)$$

$$m\left(A_2\right) = 9\left(\frac{1}{10} - \frac{1}{100}\right)$$

$$m\left(A_3\right) = 9 \cdot 9\left(\frac{1}{100} - \frac{1}{1000}\right)$$

$$m\left(A_n\right) = \left(9/10\right)^n$$

Since A_n is a decreasing sequence decreasing to the desired set, we conclude $m\left(A\right) = 0$.