Measure and Integration: Example Sheet 3

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1 Properties of $\limsup$ and $\liminf$

a) Take the sequence (a_n) with $a_1 = 2$, $a_2 = -2$ and $a_n = 1$ if $n \geq 3$ is odd and $a_n = -1$ if $n \geq 4$ is even. The $\sup_{n\geq 1} a_n = 2$, $\inf_{n\geq 1} = -2$, $\limsup_{n\to\infty} a_n = 1$ and $\limsup_{n\to\infty} a_n = -1$.

b) We consider the cases $A = +\infty$, $A = -\infty$ and A a real number separately.

In the first case, we must have $\sup_{k\geq n} a_k = \infty$ for all n. We define a subsequence converging to $+\infty$ denoted $(a_{n(k)})$ as follows. We choose $n(1) = 1$ and generally $n(k) > n(k-1)$ to be such that $a_{n(k)} \geq k$.

In the second case, we have that $b_n = \sup_{k\geq n} a_k \geq a_n$ goes to $-\infty$ hence so does (a_n).

In the third case, we have $\inf_{n\geq 1} \sup_{k\geq n} a_k = \lim_{n\to\infty} \sup_{k\geq n} a_k = A$. Let $b_n = \sup_{k\geq n} a_k$. We define a subsequence $b_{n(j)}$ and a corresponding subsequence $a_{m(j)}$ as follows. We let $n(1) = 1$. Given $b_{n(j-1)}$, by the definition of the sup we can find an $m(j-1) \geq n(j-1)$ such that

$$0 \leq b_{n(j-1)} - a_{m(j-1)} \leq \frac{1}{j} \,. \tag{1}$$

We then let $n(j) = \max(n(j-1)+1, m(j-1))$. This defines two subsequences $b_{n(j)}$ and $a_{m(j)}$. Since $b_{n(j)}$ is a subsequence of the sequence (b_n) which converges to A, it is also converging to A. But then (1) implies that the subsequence $a_{m(j)}$ also converges to A.

For the second part let $(a_{n(j)})$ be a subsequence of (a_n) converging to $\tilde{A}$. Then $b_{n(j)} = \sup_{k\geq n(j)} a_k \geq a_{n(j)}$ and taking the limit as $j \to \infty$ yields $A \geq \tilde{A}$.

c) The first claim follows from $\sup_{k\geq n}(-a_k) = -\inf_{k\geq n} a_k$ for any n and taking the limit.

For the second claim, we distinguish two cases:

(1) One of the terms $\sup_{k\geq n} a_k$ and $\sup_{k\geq n} b_k$ is $-\infty$ for some $n = N$. Wlog let it be $\sup_{k\geq N} a_k = -\infty$. It follows that $a_n = -\infty$ for all $n \geq N$ and $\limsup a_n = -\infty$. We now distinguish two subcases:

(A) If $\sup_{k\geq M} b_k < \infty$ for some M then $b_n \leq C$ for all $n \geq M$ and some constant C, since $\sup_{k\geq n} b_k$ decreases in n. It follows that for $n \geq \max(M, N)$ we have $a_n + b_n \leq -\infty + C \leq -\infty$. Therefore $a_n + b_n = -\infty$ for large enough n. Since also $\limsup b_n < +\infty$ and $\limsup a_n = -\infty$ we verify the desired estimate.

(B) If $\sup_{k\geq n} b_k = +\infty$ for all n, then its limit is $+\infty$ and this is excluded by the assumptions that $\limsup a_n + \limsup b_n$ should not be $-\infty + \infty$.

(2) For any n, none of the summands in $\sup_{k\geq n} a_k + \sup_{k\geq n} b_k$ is $-\infty$.
It follows that

$$a_j + b_j \leq \sup_{k\geq n} a_k + \sup_{k\geq n} b_k \quad \text{for all } j \geq n, \tag{2}$$

with the left hand side being defined since it is never $+\infty - \infty$ by assumption on the sheet and the right hand side being defined for any n since none of the summands can be $-\infty$. We take the sup over all $j \geq n$ and obtain

$$\sup_{j \geq n} (a_j + b_j) \leq \sup_{k \geq n} a_k + \sup_{k \geq n} b_k$$

as the left hand side does not depend on j. Now the right hand side and the left hand side are both decreasing in n. Taking the limit $n \to \infty$ yields

$$\limsup_{n \to \infty} (a_n + b_n) \leq \lim_{n \to \infty} \left(\sup_{k \geq n} a_k + \sup_{k \geq n} b_k \right) = \limsup_{n \to \infty} a_k + \limsup_{n \to \infty} b_k$$

The last equality follows since the sequences in the bracket in the middle are both decreasing individually hence their limits exist individually. A quick check of the possible cases then establishes the last inequality noting that we are explicitly excluding the case that the invidual limits are $+\infty$ and $-\infty$ respectively.

2 G_δ and F_σ sets

a) We first claim that if F is closed, then $F = \bigcap_{n=1}^{\infty} U_n$ with $U_n = \{x \mid d(x, F) < \frac{1}{n}\}$. Since U_n is open (why?) this shows that F is a G_δ-set. To prove the claim we note that if $x \in F$, then $d(x, F) = 0 < \frac{1}{n}$ for all n, hence $x \in U_n$ for all n, hence $x \in \bigcap_{n=1}^{\infty} U_n$. Conversely, if $x \notin F$, then then since the set F is closed and $\{x\}$ is compact and since $\{x\}$ and F are disjoint, we must have $d(x, F) > \delta$ for some $\delta > 0$ by Question 2 from Example Sheet 2. Hence $x \notin U_n$ for some n, hence $x \notin \bigcap_{n=1}^{\infty} U_n$. This shows $F = \bigcap_{n=1}^{\infty} U_n$.

Given the above if U is an open set, then U^c is closed, hence an G_δ. Since the complement of a G_δ is an F_σ, we have shown that U is an F_σ.

b) Suppose the rationals were a G_δ, so $\mathbb{Q} = \bigcap_{n=1}^{\infty} U_n$ for U_n open. Then, since $\mathbb{Q}$ is dense in $\mathbb{R}$, every U_n must also be dense in $\mathbb{R}$. So we have writte $\mathbb{Q}$ as a countable intersection of dense open sets. The complement $\mathbb{Q}^c$ is therefore a countable union of closed nowhere dense (i.e. sets without interior points) sets. But since $\mathbb{Q}$ can itself be written as a countable union of closed nowhere dense sets we conclude that $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$ can be written as a countable union of nowhere dense sets. This contradicts the Baire Category theorem as $\mathbb{R}$ is a complete metric space.

c) Note that a slight modification of the argument in part b) also shows that the positive rationals $\mathbb{Q} \cap \{x \geq 0\}$ and also $\mathbb{Q} \cap \{x > 0\}$ cannot be a G_δ. Consider then the set

$$E = (\mathbb{Q} \cap \{x \geq 0\}) \bigcup (\mathbb{Q}^c \cap \{x < 0\})$$

i.e. the union of the non-negative rationals with the negative irrationals. Note that the complement is given by the union positive irrational numbers with negative rational ones.

If E was a G_δ, then so would be $E \cap \{x > 0\}$ in contradiction with the remark above. If E was an F_σ, then E^c would be a G_δ, hence the negative rational numbers $E^c \cap \{x < 0\}$ would be, which is a contradiction.

3 Measurable functions

a) Let $f : \mathbb{R} \to \mathbb{R}$ be monotone increasing. The decreasing case is similar or follows directly by considering $-f$ and noting that f is measurable iff $-f$ is.

It suffices to show that the sets $E_\alpha = \{x \mid f(x) > \alpha\}$ are Borel sets. We claim that E_α is either empty (hence Borel) or an interval of the forms $[x_\alpha, \infty)$, (x_α, ∞) or $(-\infty, \infty)$ which are also all Borel sets. Indeed, assuming E_α is non-empty, $x \in E_\alpha$ implies $[x, \infty) \subset E_\alpha$ since f is monotone. Consequently, defining $x_\alpha = \inf_x \{x \mid f(x) > \alpha\}$ we have one of the three aforementioned possibilities depending on whether $x_\alpha = -\infty$ or, in case x_α is finite, whether $x_\alpha \in E_\alpha$ or not.

b) We only need to show that $\{x \mid f(x) \geq q\}$ is measurable for every irrational q as then we have shown $\{x \mid f(x) \geq \alpha\}$ is measurable for any $\alpha \in \mathbb{R}$ which implies measurability of f.

Given an irrational q we can construct a monotone increasing sequence of rationals r_n with $r_n \nearrow q$ (this can be done for instance by cutting off the decimal expansions for q). But then

$$\{x \mid f(x) \geq q\} = \bigcap_{n=1}^{\infty} \{x \mid f(x) \geq r_n\}$$

and since the right hand side is measurable (being a countable intersection of sets which are measurable by assumption) so is the left hand side.

c) Let $g(x) = \limsup_{n\to\infty} f_n$ and $h(x) = \liminf_{n\to\infty} f_n$. We know that the sets $\{x \mid \limsup_{n\to\infty} f_n = +\infty\}$, $\{x \mid \limsup_{n\to\infty} f_n = -\infty\}$, $\{x \mid \liminf_{n\to\infty} f_n = +\infty\}$, $\{x \mid \liminf_{n\to\infty} f_n = -\infty\}$ are all measurable. Let us denote by E the (measurable) set obtained from $\mathbb{R}$ by removing these sets. We then have two finite valued measurable functions $\tilde{g} : E \to \mathbb{R}$, $\tilde{g}(x) = g|_E(x)$ and $\tilde{h} : E \to \mathbb{R}$, $\tilde{h}(x) = h|_E(x)$. We know that the difference of two finite valued measurable functions is measurable (shown in lectures) and hence in particular the set

$$\{x \in E \mid \tilde{g}(x) - \tilde{h}(x) = 0\} = \{x \in E \mid \tilde{g}(x) - \tilde{h}(x) \geq 0\} \bigcap \{x \in E \mid \tilde{g}(x) - \tilde{h}(x) \leq 0\}$$

is measurable. But this is precisely the set for which lim sup and lim inf are finite and agree.

4 Approximating measurable functions by continuous ones

Claim: Given a step function φ and $\epsilon > 0$ we can approximate φ by a continuous function of compact support g such that $m(\{x \mid \varphi(x) \neq g(x)\}) < \epsilon$.

Proof of Claim: We note that it suffices to prove this claim for the step function $\varphi = \chi_R$ of a single rectangle R, since a general φ consists of a *finite* linear combination of such step functions. To prove it for χ_R we let $R = [a_1, b_1] \times ... \times [a_d, b_d]$ where we can assume $a_i < b_i$ as otherwise R has measure 0 and χ_R will be approximated by the zero function up to a set of measure zero. [Note also that the boundary of R is a set of measure zero and hence that the following argument would work equally well if R is open.] For each i we define a continuous gluing function g_{i,ℓ_i} as follows. We let $0 < \ell_i < \frac{b_i - a_i}{2}$ and (draw a picture!)

$$g_{i,\ell_i}(x) = \begin{cases} 1 & \text{if } x \in [a_i + \ell_i, b_i - \ell_i], \\ 0 & \text{if } x \in [a_i, b_i]^c \\ \frac{x - a_i}{\ell_i} & \text{if } x \in [a_i, a_i + \ell_i) \\ \frac{b_i - x}{\ell_i} & \text{if } x \in (b_i - \ell_i, b_i]. \end{cases}$$

The product function $g(x) = \Pi_{i=1}^{d} g_{i,\ell_i}$ is then a continuous function which is identically 1 on the smaller rectangle $\tilde{R} = [a_1 + \ell_1, b_1 - \ell_1] \times ... \times [a_d + \ell_d, b_d - \ell_d] \subset R$ and zero outside R. Moreover it is immediate that given any $\epsilon > 0$ we can choose the ℓ_i sufficiently small such that $m(R \setminus \tilde{R}) = m(R) - m(\tilde{R}) < \epsilon$. But this shows the result because the set where g and χ_R do not agree is contained in $R \setminus \tilde{R}$.

Having established the claim, we can now finish the proof. In lectures, we showed that given a measurable function f, we can find a sequence of step functions (φ_n) such that $\varphi_n \to f$ holds for almost every x. By the claim we can find for any φ_n a function of compact support g_n such that $m(\{x \mid \varphi_n(x) \neq g_n(x)\}) < \frac{1}{2^n}$. Now we apply the Borel-Cantelli Lemma to the sequence of measurable sets $E_n := \{x \mid \varphi_n(x) \neq g_n(x)\}$. Indeed, since $\sum_n m(E_n) < \infty$ we know that the set of x which belong to infinitely many E_n has measure zero. If we call this set $\mathcal{N}$, then all $x \in \mathcal{N}^c$ belong to *finitely* many E_n and hence we must have, for all such $x \in \mathcal{N}^c$, that $\varphi_n(x) = g_n(x)$ holds for all $n \geq N_x$ for some N_x. Therefore, if we denote the measure zero set of x for which we do not have $\varphi_n(x) \to f(x)$ by $\tilde{\mathcal{N}}$ we we must have $\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \varphi_n(x) = f(x)$ for all $x \in \mathcal{N}^c \cap \tilde{\mathcal{N}}^c$.

5 The Cantor function revisited

a) We indeed note that if $x \in \mathfrak{C}^c = \bigcup_n U_n$ (the complement in $[0,1]$) then $x \in U_n$ for some n. The open interval U_n was removed from one of the 2^n disjoint intervals of C_n as the middle third in the construction of $\mathfrak{C}$. Since any interval of C_n can be parametrised as

$$\left[\sum_{k=1}^{n} a_k \frac{1}{3^k}, \sum_{k=1}^{n} a_k \frac{1}{3^k} + \frac{1}{3^n}\right]$$

for some sequence $(a_k)_{k=1}^{N}$ with $a_k \in \{0,2\}$, the middle third U_n is parametrised as

$$\left(a = \sum_{k=1}^{n} a_k \frac{1}{3^k} + \frac{1}{3^{n+1}}, b = \sum_{k=1}^{n} a_k \frac{1}{3^k} + \frac{2}{3^{n+1}}\right)$$

In order to evaluate the function F on the point a and b (which belong to the Cantor set) we write a and b in their expansions (cf. Example Sheet 1, Part 1f))

$$a = \sum_{k=1}^{n} a_k \frac{1}{3^k} + \sum_{k=n+2}^{\infty} \frac{2}{3^k} \quad \text{and} \quad b = \sum_{k=1}^{n} a_k \frac{1}{3^k} + \frac{2}{3^{n+1}}$$

so that we can compute

$$F(a) = \sum_{k=1}^{n} b_k \frac{1}{2^k} + \sum_{k=n+2}^{\infty} \frac{1}{2^k} \quad , \quad F(b) = \sum_{k=1}^{n} b_k \frac{1}{2^k} + \frac{1}{2^{n+1}} \, .$$

Computing the geometric series for $F(a)$ we easily see $F(a) = F(b)$.

To show f is continuous we first note that it is clearly continuous at all $x \in \mathfrak{C}^c$ as we defined f to be constant on the open intervals U_n. We hence let $x \in \mathfrak{C}$ and take a sequence $x_n \to x$ in $[0,1]$.

We define from the sequence (x_n) a sequence (y_n) in the Cantor set as follows.

$$y_n = x_n \quad \text{if } x_n \in \mathfrak{C}.$$

If $x_n \in C^c$ then we let

$$y_n = a_n \text{ if } x_n > x \quad , \quad y_n = b_n \text{ if } x_n < x$$

where (a_n, b_n) denotes the interval U_n in which x_n is contained. (Draw a picture to see what's happening here!). With this definition we easily see that

$$|y_n - x_n| \leq |x_n - x|$$

holds for (y_n). Hence $y_n \to x$. We also know that $f(y_n) = F(y_n) = F(x) = f(x)$ since we have shown that F is continuous on $\mathfrak{C}$ on example Sheet 1. But since $f(x_n) = f(y_n)$ by construction we have that $f(x_n) \to f(x)$ and hence continuity at $x \in \mathfrak{C}$.

b) Let $\mathcal{N} \subset [0,1]$ be the non-measurable subset constructed in lectures. Note that $f^{-1}(\mathcal{N})$ is a set in $[0,1]$. Since $f|_{\mathfrak{C}} = F$ is surjective on $[0,1]$ by Example Sheet 1 and $\mathcal{N} \subset [0,1]$ we conclude that

$$f\left(f^{-1}(\mathcal{N}) \cap \mathfrak{C}\right) = \mathcal{N} \, .$$

Therefore, $f^{-1}(\mathcal{N}) \cap \mathfrak{C}$ is a measurable set (being a subset of a set of measure zero, namely $\mathfrak{C}$) which gets mapped to a non-measurable set by a continuous function.

c) Let $g : [0,1] \to [0,1]$ be defined by

$$g(y) = \inf\{x \in [0,1] \mid f(x) = y\} \, .$$

The intuition is that g is a partial inverse of the Cantor-Lebesgue function f. (Draw a picture of the Cantor-Lebesgue function and visualise this inverse by drawing horizontal lines.)

Observation 1: The inf is achieved since f is continuous. Indeed, if $x = g(y)$, then by the definition of the inf there is a sequence x_n with $x_n \to x$ and $f(x_n) = y$. Hence $f(x) = y$ by continuity. In particular we have $y = f(g(y))$ for all $y \in [0, 1]$.

Observation 2: If $y_1 < y_2$ then $g(y_1) < g(y_2)$. Assume not and $g(y_1) \geq g(y_2)$. Apply f to the last inequality. Since f itself is monotone it preserves the inequality and Observation 1 leads to $y_1 \geq y_2$ which is a contradiction. This shows in particular (strict) monotonicity of g.

Observation 3: If $g(y_1) = g(y_2)$ then $y_1 = y_2$. Assume not and $y_1 < y_2$; then Observation 2 immediately produces a contradiction. Injectivity follows.

Finally, we show that g maps to the Cantor set $\mathfrak{C}$. Assume there was a $z \in \mathfrak{C}^c$ with $g(y) = z$. Then by Observation 1 we have $f(z) = y$ and by the definition of g this z is the smallest number with this property. But we now z is contained in an open set on which f is constant, so there exists a $\tilde{z} < z$ with $f(\tilde{z}) = f(z)$ contradicting definition of z as the infimum.