1 Properties of \(\lim\sup\) and \(\lim\inf\)

a) Take the sequence \((a_n)\) with \(a_1 = 2\), \(a_2 = -2\) and \(a_n = 1\) if \(n \geq 3\) is odd and \(a_n = -1\) if \(n \geq 4\) is even.

The \(\sup_{n \geq 1} a_n = 2\), \(\inf_{n \geq 1} = -2\), \(\lim\sup_{n \to \infty} a_n = 1\) and \(\lim\sup_{n \to \infty} a_n = -1\).

b) We consider the cases \(A = +\infty\), \(A = -\infty\) and \(A\) a real number separately.

In the first case, we must have \(\sup_{k \geq n} a_k = \infty\) for all \(n\). We define a subsequence converging to \(+\infty\) denoted \((a_{n(k)})\) as follows. We choose \(n(1) = 1\) and generally \(n(k) > n(k - 1)\) to be such that \(a_{n(k)} \geq k\).

In the second case, we have that \(b_n = \sup_{k \geq n} a_k \geq a_n\) goes to \(-\infty\) hence so does \((a_n)\).

In the third case, we have \(\inf_{n \geq 1} \sup_{k \geq n} a_k = \lim_{n \to \infty} \sup_{k \geq n} a_k = A\). Let \(b_n = \sup_{k \geq n} a_k\). We define a subsequence \(b_{n(j)}\) and a corresponding subsequence \(a_{m(j)}\) as follows. We let \(n(1) = 1\). Given \(b_{n(j-1)}\), by the definition of the sup we can find an \(m\) \((j - 1) \geq n(j - 1)\) such that

\[
0 \leq b_{n(j-1)} - a_{m(j-1)} \leq \frac{1}{j}. \tag{1}
\]

We then let \(n(j) = \max(n(j-1) + 1, m(j-1))\). This defines two subsequences \(b_{n(j)}\) and \(a_{m(j)}\). Since \(b_{n(j)}\) is a subsequence of the sequence \((b_n)\) which converges to \(A\), it is also converging to \(A\). But then (1) implies that the subsequence \(a_{m(j)}\) also converges to \(A\).

For the second part let \((a_{n(j)})\) be a subsequence of \((a_n)\) converging to \(\tilde{A}\). Then \(b_{n(j)} = \sup_{k \geq n(j)} a_k \geq a_{n(j)}\) and taking the limit as \(j \to \infty\) yields \(A \geq \tilde{A}\).

c) The first claim follows from \(\sup_{k \geq n}(-a_k) = -\inf_{k \geq n} a_k\) for any \(n\) and taking the limit.

For the second claim, we distinguish two cases:

(1) One of the terms \(\sup_{k \geq n} a_k\) and \(\sup_{k \geq n} b_k\) is \(-\infty\) for some \(n = N\). Wlog let it be \(\sup_{k \geq N} a_k = -\infty\).

It follows that \(a_n = -\infty\) for all \(n \geq N\) and \(\lim\sup a_n = -\infty\). We now distinguish two subcases:

(A) If \(\sup_{k \geq M} b_k < \infty\) for some \(M\) then \(b_n \leq C\) for all \(n \geq M\) and some constant \(C\), since \(\sup_{k \geq n} b_k\) decreases in \(n\). It follows that for \(n \geq \max(M, N)\) we have \(a_n + b_n \leq -\infty + C \leq -\infty\). Therefore \(a_n + b_n = -\infty\) for large enough \(n\). Since also \(\lim\sup b_n < +\infty\) and \(\lim\sup a_n = -\infty\) we verify the desired estimate.

(B) If \(\sup_{k \geq n} b_k = +\infty\) for all \(n\), then its limit is \(+\infty\) and this is excluded by the assumptions that \(\lim\sup a_n + \lim\sup b_n\) should not be \(-\infty + \infty\).

(2) For any \(n\), none of the summands in \(\sup_{k \geq n} a_k + \sup_{k \geq n} b_k\) is \(-\infty\).

It follows that

\[
a_j + b_j \leq \sup_{k \geq n} a_k + \sup_{k \geq n} b_k \quad \text{for all } j \geq n, \tag{2}
\]
with the left hand side being defined since it is never \(+\infty - \infty\) by assumption on the sheet and the right hand side being defined for any \(n\) since none of the summands can be \(-\infty\). We take the sup over all \(j \geq n\) and obtain

\[
\sup_{j \geq n} (a_j + b_j) \leq \sup_{k \geq n} a_k + \sup_{k \geq n} b_k
\]
as the left hand side does not depend on \(j\). Now the right hand side and the left hand side are both decreasing in \(n\). Taking the limit \(n \to \infty\) yields

\[
\limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} \left( \sup_{k \geq n} a_k + \sup_{k \geq n} b_k \right) = \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n
\]
The last equality follows since the sequences in the bracket in the middle are both decreasing individually hence their limits exist individually. A quick check of the possible cases then establishes the last inequality noting that we are explicitly excluding the case that the individual limits are \(+\infty\) and \(-\infty\) respectively.

**2 \(G_\delta\) and \(F_\sigma\) sets**

a) We first claim that if \(F\) is closed, then \(F = \bigcap_{n=1}^\infty U_n\) with \(U_n = \{x \mid d(x, F) < \tfrac{1}{n}\}\). Since \(U_n\) is open (why?) this shows that \(F\) is a \(G_\delta\)-set. To prove the claim we note that if \(x \in F\), then \(d(x, F) = 0 < \tfrac{1}{n}\) for all \(n\), hence \(x \in U_n\) for all \(n\), hence \(x \in \bigcap_{n=1}^\infty U_n\). Conversely, if \(x \notin F\), then then since the set \(F\) is closed and \(\{x\}\) is compact and since \(\{x\} \cap F\) are disjoint, we must have \(d(x, F) > \delta\) for some \(\delta > 0\) by Question 2 from Example Sheet 2. Hence \(x \notin U_n\) for some \(n\), hence \(x \notin \bigcap_{n=1}^\infty U_n\). This shows \(F = \bigcap_{n=1}^\infty U_n\).

Given the above if \(U\) is an open set, then \(U^c\) is closed, hence an \(G_\delta\). Since the complement of a \(G_\delta\) is an \(F_\sigma\), we have shown that \(U\) is an \(F_\sigma\).

b) Suppose the rationals were a \(G_\delta\), so \(\mathbb{Q} = \bigcap_{n=1}^\infty U_n\) for \(U_n\) open. Then, since \(\mathbb{Q}\) is dense in \(\mathbb{R}\), every \(U_n\) must also be dense in \(\mathbb{R}\). So we have write \(\mathbb{Q}\) as a countable intersection of dense open sets. The complement \(\mathbb{Q}^c\) is therefore a countable union of closed nowhere dense (i.e. sets without interior points) sets. But since \(\mathbb{Q}\) can itself be written as a countable union of closed nowhere dense sets we conclude that \(\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c\) can be written as a countable union of nowhere dense sets. This contradicts the Baire Category theorem as \(\mathbb{R}\) is a complete metric space.

c) Note that a slight modification of the argument in part b) also shows that the positive rationals \(\mathbb{Q} \cap \{x \geq 0\}\) and also \(\mathbb{Q} \cap \{x > 0\}\) cannot be a \(G_\delta\). Consider then the set

\[
E = (\mathbb{Q} \cap \{x \geq 0\}) \bigcup (\mathbb{Q}^c \cap \{x < 0\})
\]
i.e. the union of the non-negative rationals with the negative irrationals. Note that the complement is given by the union positive irrational numbers with negative rational ones.

If \(E\) was a \(G_\delta\), then so would be \(E \cap \{x > 0\}\) in contradiction with the remark above. If \(E\) was an \(F_\sigma\), then \(E^c\) would be a \(G_\delta\), hence the negative rational numbers \(E^c \cap \{x < 0\}\) would be, which is a contradiction.

**3 Measurable functions**

a) Let \(f : \mathbb{R} \to \mathbb{R}\) be monotone increasing. The decreasing case is similar or follows directly by considering \(-f\) and noting that \(f\) is measurable iff \(-f\) is.

It suffices to show that the sets \(E_\alpha = \{x \mid f(x) > \alpha\}\) are Borel sets. We claim that \(E_\alpha\) is either empty (hence Borel) or an interval of the forms \([x_\alpha, \infty)\), \((x_\alpha, \infty)\) or \((-\infty, \infty)\) which are also all Borel sets. Indeed, assuming \(E_\alpha\) is non-empty, \(x \in E_\alpha\) implies \([x, \infty) \subset E_\alpha\) since \(f\) is monotone. Consequently, defining \(x_\alpha = \inf \{x \mid f(x) > \alpha\}\) we have one of the three aforementioned possibilities depending on whether \(x_\alpha = -\infty\) or, in case \(x_\alpha\) is finite, whether \(x_\alpha \in E_\alpha\) or not.

2
b) We only need to show that \( \{ x \mid f(x) \geq q \} \) is measurable for every irrational \( q \) as then we have shown \( \{ x \mid f(x) \geq \alpha \} \) is measurable for any \( \alpha \in \mathbb{R} \) which implies measurability of \( f \).

Given an irrational \( q \) we can construct a monotone increasing sequence of rationals \( r_n \) with \( r_n \not\nearrow q \) (this can be done for instance by cutting off the decimal expansions for \( q \)). But then

\[
\{ x \mid f(x) \geq q \} = \bigcap_{n=1}^{\infty} \{ x \mid f(x) \geq r_n \}
\]

and since the right hand side is measurable (being a countable intersection of sets which are measurable by assumption) so is the left hand side.

c) Let \( g(x) = \limsup_{n \to \infty} f_n \) and \( h(x) = \liminf_{n \to \infty} f_n \). We know that the sets \( \{ x \mid \limsup_{n \to \infty} f_n = +\infty \} \), \( \{ x \mid \limsup_{n \to \infty} f_n = -\infty \} \), \( \{ x \mid \liminf_{n \to \infty} f_n = +\infty \} \), \( \{ x \mid \liminf_{n \to \infty} f_n = -\infty \} \) are all measurable. Let us denote by \( E \) the (measurable) set obtained from \( \mathbb{R} \) by removing these sets. We then have two finite valued measurable functions \( \tilde{g} : E \to \mathbb{R} \), \( \tilde{g}(x) = g|_E(x) \) and \( \tilde{h} : E \to \mathbb{R} \), \( \tilde{h}(x) = h|_E(x) \). We know that the difference of two finite valued measurable functions is measurable (shown in lectures) and hence in particular the set

\[
\{ x \in E \mid \tilde{g}(x) - \tilde{h}(x) = 0 \} = \{ x \in E \mid \tilde{g}(x) - \tilde{h}(x) \geq 0 \} \cap \{ x \in E \mid \tilde{g}(x) - \tilde{h}(x) \leq 0 \}
\]

is measurable. But this is precisely the set for which \( \limsup \) and \( \liminf \) are finite and agree.

## 4 Approximating measurable functions by continuous ones

**Claim:** Given a step function \( \varphi \) and \( \epsilon > 0 \) we can approximate \( \varphi \) by a continuous function of compact support \( g \) such that \( m \{ \{ x \mid \varphi(x) \neq g(x) \} \} < \epsilon \).

**Proof of Claim:** We note that it suffices to prove this claim for the step function \( \varphi = \chi_R \) of a single rectangle \( R \), since a general \( \varphi \) consists of a finite linear combination of such step functions. To prove it for \( \chi_R \) we let \( R = [a_1, b_1] \times ... \times [a_d, b_d] \) where we can assume \( a_i < b_i \) as otherwise \( R \) has measure 0 and \( \chi_R \) will be approximated by the zero function up to a set of measure zero. [Note also that the boundary of \( R \) is a set of measure zero and hence that the following argument would work equally well if \( R \) is open.] For each \( i \) we define a continuous gluing function \( g_{i, \ell_i} \) as follows. We let \( 0 < \ell_i < \frac{b_i - a_i}{2} \) and (draw a picture!)

\[
g_{i, \ell_i}(x) = \begin{cases} 
1 & \text{if } x \in [a_i + \ell_i, b_i - \ell_i], \\
0 & \text{if } x \in [a_i, b_i]^c, \\
\frac{x - a_i}{\ell_i} & \text{if } x \in [a_i, a_i + \ell_i), \\
\frac{b_i - x}{\ell_i} & \text{if } x \in (b_i - \ell_i, b_i].
\end{cases}
\]

The product function \( g(x) = \prod_{i=1}^d g_{i, \ell_i} \) is then a continuous function which is identically 1 on the smaller rectangle \( \bar{R} = [a_1 + \ell_1, b_1 - \ell_1] \times ... \times [a_d + \ell_d, b_d - \ell_d] \subset R \) and zero outside \( R \). Moreover it is immediate that given any \( \epsilon > 0 \) we can choose the \( \ell_i \) sufficiently small such that \( m(R \setminus \bar{R}) = m(R) - m(\bar{R}) < \epsilon \). But this shows the result because the set where \( g \) and \( \chi_R \) do not agree is contained in \( R \setminus \bar{R} \).

Having established the claim, we can now finish the proof. In lectures, we showed that given a measurable function \( f \), we can find a sequence of step functions \( (\varphi_n) \) such that \( \varphi_n \to f \) holds for almost every \( x \). By the claim we can find for any \( \varphi_n \) a function of compact support \( g_n \) such that \( m \{ \{ x \mid \varphi_n(x) \neq g_n(x) \} \} < \frac{1}{2^n} \). Now we apply the Borel-Cantelli Lemma to the sequence of measurable sets \( E_n := \{ x \mid \varphi_n(x) \neq g_n(x) \} \). Indeed, since \( \sum_{n=1}^{\infty} m(E_n) < \infty \) we know that the set of \( x \) which belong to infinitely many \( E_n \) has measure zero. If we call this set \( \mathcal{N} \), then all \( x \in \mathcal{N}^c \) belong to finitely many \( E_n \) and hence we must have, for all such \( x \in \mathcal{N}^c \), that \( \varphi_n(x) = g_n(x) \) holds for all \( n \geq N_x \) for some \( N_x \). Therefore, if we denote the measure zero set of \( x \) for which we do not have \( \varphi_n(x) \to f(x) \) by \( \bar{\mathcal{N}} \) we must have \( \lim_{n \to \infty} g_n(x) = \lim_{n \to \infty} \varphi_n(x) = f(x) \) for all \( x \in \mathcal{N}^c \cap \mathcal{N}^c \).
5 The Cantor function revisited

a) We indeed note that if \( x \in \mathcal{C}^c = \bigcup_n U_n \) (the complement in \([0,1]\)) then \( x \in U_n \) for some \( n \). The open interval \( U_n \) was removed from one of the \( 2^n \) disjoint intervals of \( C_n \) as the middle third in the construction of \( \mathcal{C} \). Since any interval of \( C_n \) can be parametrised as

\[
\left[ \sum_{k=1}^n a_k \frac{1}{3^k}, \sum_{k=1}^n a_k \frac{1}{3^k} + \frac{1}{3^n} \right]
\]

for some sequence \((a_k)_{k=1}^N\) with \( a_k \in \{0, 2\} \), the middle third \( U_n \) is parametrised as

\[
\left( a = \sum_{k=1}^n a_k \frac{1}{3^k} + \frac{1}{3n+1}, b = \sum_{k=1}^n a_k \frac{1}{3^k} + \frac{2}{3n+1} \right)
\]

In order to evaluate the function \( F \) on the point \( a \) and \( b \) (which belong to the Cantor set) we write \( a \) and \( b \) in their expansions (cf. Example Sheet 1, Part 1f))

\[
a = \sum_{k=1}^n a_k \frac{1}{3^k} + \sum_{k=n+2}^\infty \frac{2}{3^k} \quad \text{and} \quad b = \sum_{k=1}^n a_k \frac{1}{3^k} + \frac{2}{3n+1}
\]

so that we can compute

\[
F(a) = \sum_{k=1}^n b_k \frac{1}{2^k} + \sum_{k=n+2}^\infty \frac{1}{2^k} \quad , \quad F(b) = \sum_{k=1}^n b_k \frac{1}{2^k} + \frac{1}{2n+1}.
\]

Computing the geometric series for \( F(a) \) we easily see \( F(a) = F(b) \).

To show \( f \) is continuous we first note that it is clearly continuous at all \( x \in \mathcal{C}^c \) as we defined \( f \) to be constant on the open intervals \( U_n \). We hence let \( x \in \mathcal{C} \) and take a sequence \( x_n \to x \) in \([0,1]\).

We define from the sequence \((x_n)\) a sequence \((y_n)\) in the Cantor set as follows.

\[
y_n = x_n \quad \text{if} \quad x_n \in \mathcal{C}.
\]

If \( x_n \in C^c \) then we let

\[
y_n = a_n \quad \text{if} \quad x_n > x \quad , \quad y_n = b_n \quad \text{if} \quad x_n < x
\]

where \((a_n, b_n)\) denotes the interval \( U_n \) in which \( x_n \) is contained. (Draw a picture to see what’s happening here!). With this definition we easily see that

\[
|y_n - x_n| \leq |x_n - x|
\]

holds for \((y_n)\). Hence \( y_n \to x \). We also know that \( f(y_n) = F(y_n) = F(x) = f(x) \) since we have shown that \( F \) is continuous on \( \mathcal{C} \) on example Sheet 1. But since \( f(x_n) = f(y_n) \) by construction we have that \( f(x_n) \to f(x) \) and hence continuity at \( x \in \mathcal{C} \).

b) Let \( \mathcal{N} \subset [0,1] \) be the non-measurable subset constructed in lectures. Note that \( f^{-1}(\mathcal{N}) \) is a set in \([0,1]\). Since \( f|_{\mathcal{C}} = F \) is surjective on \([0,1]\) by Example Sheet 1 and \( \mathcal{N} \subset [0,1] \) we conclude that

\[
f (f^{-1}(\mathcal{N}) \cap \mathcal{C}) = \mathcal{N}.
\]

Therefore, \( f^{-1}(\mathcal{N}) \cap \mathcal{C} \) is a measurable set (being a subset of a set of measure zero, namely \( \mathcal{C} \)) which gets mapped to a non-measurable set by a continuous function.

c) Let \( g : [0,1] \to [0,1] \) be defined by

\[
g(y) = \inf\{x \in [0,1] \mid f(x) = y\}.
\]
The intuition is that $g$ is a partial inverse of the Cantor-Lebesgue function $f$. (Draw a picture of the Cantor-Lebesgue function and visualise this inverse by drawing horizontal lines.)

Observation 1: The inf is achieved since $f$ is continuous. Indeed, if $x = g(y)$, then by the definition of the inf there is a sequence $x_n$ with $x_n \to x$ and $f(x_n) = y$. Hence $f(x) = y$ by continuity. In particular we have $y = f(g(y))$ for all $y \in [0, 1]$.

Observation 2: If $y_1 < y_2$ then $g(y_1) < g(y_2)$. Assume not and $g(y_1) \geq g(y_2)$. Apply $f$ to the last inequality. Since $f$ itself is monotone it preserves the inequality and Observation 1 leads to $y_1 \geq y_2$ which is a contradiction. This shows in particular (strict) monotonicity of $g$.

Observation 3: If $g(y_1) = g(y_2)$ then $y_1 = y_2$. Assume not and $y_1 < y_2$; then Observation 2 immediately produces a contradiction. Injectivity follows.

Finally, we show that $g$ maps to the Cantor set $\mathcal{C}$. Assume there was a $z \in \mathcal{C}^c$ with $g(y) = z$. Then by Observation 1 we have $f(z) = y$ and by the definition of $g$ this $z$ is the smallest number with this property. But we now $z$ is contained in an open set on which $f$ is constant, so there exists a $\tilde{z} < z$ with $f(\tilde{z}) = f(z)$ contradicting definition of $z$ as the infimum.